Probability And Statistics

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Abstract
This is a brief summary about the course of Probability and Statistics conducted at the Engineering School of Universidad Panamericana. The objective is to help the student understand basic theoretical concepts reviewed in class and to reinforce specific subjects the student may have missed. For any mistake found in this document, please feel free to reach out. This document is compiled for educational purposes only.

0 Set Theory

Definition A set is a collection of elements. Let: $S = \{\theta_1, \theta_2, ..., \theta_n\}$ Where θ_n is the nth element of the set. A set can contain all kinds of elements such as: all tropical fruits, the first 35 Fibonacci numbers etc. Some of them are finite or others are infinite or uncountable. Understanding the abstraction of what is a set will be fundamental for this course. On a very harsh stroke we could define Probability as: **the measure of a size of a set.** Chan (2021)

Notation

- $\theta \in S$:Element θ is in set S.
- $M \subset S$: Set M is a subset of set S.
- $A \cup B$: Union of sets A and B
- $A \cap B$: Intersection of sets A and B
- $S = \emptyset$: Set S is null or empty.
- $S = \Omega$: S is the Universal set
- S^c: Compliment of set S

0.1 Set properties

Commutative:

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

Associative:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DeMorgan's Law:

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

0.2 Set Operations

Union: Elements from A or B are combined into one set.
Example:

$$A \subset S \, and \, B \subset S$$
$$S = \{All \, NFL \, Teams\}$$
$$A = \{All \, teams \, in \, the \, AFC \, West\} = \{LAC, DEN, KC, LV\}$$
$$B = \{All \, teams \, in \, CA\} = \{LAC, LAR, SF\}$$
$$A \cup B = \{LAC, LAR, SF, DEN, KC, LV\}$$

Intersection: Only elements in common from A and B are selected into one set.
Example:

$$A \subset S \, and \, B \subset S$$
$$S = \{All \, NFL \, Teams\}$$
$$A = \{All \, teams \, in \, the \, AFC \, West\} = \{LAC, DEN, KC, LV\}$$
$$B = \{All \, teams \, in \, CA\} = \{LAC, LAR, SF\}$$
$$A \cap B = \{LAC\}$$

Difference: Takes all the elements from A but not the ones that are in common with B. Or viceversa
Example:

$$A \subset S \, and \, B \subset S$$
$$S = \{All \, NFL \, Teams\}$$
$$A = \{All \, teams \, in \, the \, NFC \, East\} = \{NYG, DAL, WAS, PHI\}$$
$$B = \{All \, teams \, with \, at \, least \, 3 \, Super \, Bowls\} = \{SF, DAL, NYG, WAS, LV, DEN, GB, PIT, NE\}$$
$$B \setminus A = \{NE, PIT, GB, SF, DEN, LV\}$$

Sample Space: We will define the sample space from now on as S and this is going to be the most important set on the experiments we will make. The sample space is the collection of all possible outcomes of an experiment. As we know, there are finite and infinite sets.

0.3 Counting principles

Suppose we ought to perform any experiment. As we said before, some sample spaces are countable and some of them are uncountable. There are some methods for counting this sample spaces.

Permutations Useful for calculating sample spaces in which replacement matters. For example the possible batting orders for a baseball team is influenced by the first at bat. To calculate you just need to do the numbers factorial.

$$n! = n(n-1)(n-2)\dots(2)(1)$$

Combinations Whenever we want to make groups out of a bigger set and count the possible outcomes of doing this experiment we will need combinations. We can say all the possible combinations of doing a Powerball or Melate.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

It can be read as from r set take n elements.

1 Probability 101

Definition: Although there are many sources that define probability in a different way. Let's stick to Devore (2019) definition which states: We refer to **probability** as the study of randomness and uncertainty in any given situation in which many outcomes can occur. We always use probability in our everyday activities. Even unconsciously we calculate probabilities, but that matter will be left for another course. *See: Kahneman (2011)*

Notation: We need to understand that **probability** calculates **events** within **sample spaces**. For example the event that a die is rolled the sample space is all the possible outcomes $\{1,2,3,4,5,6\}$ and the probability of each event is $\frac{1}{6}$. *For the next steps, we will refer as the probability of an event as $Pr(x)$*

Experiments: The most common experiments we will contemplate and create examples form this are:

- Drawing a card from the traditional French deck of 52 playing cards $S = \{4A, 4K, 4Q, 4J, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$
- Flipping a fair coin $S = \{H, T\}$
- Tossing a fair die $S = \{1, 2, 3, 4, 5, 6\}$

These will be used for further examples since all of the events have equal probability.

1.1 Axioms of Probability

1. All probabilities must be denoted with a number between 0 and 1. *Uncountable set*

$$0 \leq \ Pr(x) \ \leq 1$$

2. Let S be the sample space

$$Pr(S) = 1$$

   This means that all the events that are plausible within the sample space need to sum 1.

3. On mutually exclusive events the probability of all the events is the sum of each probabilities.

$$Pr\left(\bigcup_{i=1}^{\infty} x_i\right) = \sum_{i=1}^{\infty} Pr(x_i)$$

1.2 Propositions from Axioms

Definition: With the axioms of probability already stated we can talk about some propositions we can infer from this axioms.

1. Let

$$Pr(X) < 1$$

   therefore

$$Pr(X) + Pr(X^c) = 1$$

   we can conclude that we can also calculate the probability of an event by doing

$$1 - Pr(X^c)$$

2. For any event $A \subset S$ $Pr(A) \leq 1$

3. For any two events A and B there are certain approximations we need to make in order to avoid double counting. Here it will come real handy your sets theory knowledge. Let:

$$Pr(A \cup B) = Pr(A) + Pr(B) - P(A \cap B)$$

4. For three events A, B and C

$$Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) - Pr(A \cap B \cap C)$$

Sample Spaces Having Equally Likely Outcomes We will review some brief examples to understand the experiments that have equally likely outcomes.
Example 1: If two dice are rolled, what is the probability that the sum of the faces will equal 7? **(Catan problem)**
We can define all possible outcomes are equally likely by assuming both dice are fair. Therefore we have 36 possible combinations. The tuples that can yield 7 are: $(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$ this means there are $\frac{6}{36} = \frac{1}{6}$ of chances the sum of the dices is 7.

2 Conditional Probability and Bayes' Theorem

Conditional probability quantifies the likelihood of an event occurring given that another event has already occurred. Formally, the conditional probability of event A given event B is defined as:

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}$$

provided that $Pr(B) > 0$. Here, $Pr(A \cap B)$ represents the probability that both events A and B occur, and $Pr(B)$ is the probability of event B.

2.0.1 Example

Consider a standard deck of 52 playing cards. Suppose we want to find the probability that a card drawn is a King (A), given that it is a face card (B). There are 12 face cards in a deck (4 Kings, 4 Queens, 4 Jacks), so:

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)} = \frac{4/52}{12/52} = \frac{4}{12} = \frac{1}{3}$$

This means that if a face card is drawn, there is a one-third chance that it is a King.

2.1 Bayes' Theorem

Bayes' Theorem provides a way to update the probability estimate for an event based on new evidence. It relates the conditional and marginal probabilities of events and is stated as:

$$Pr(A \mid B) = \frac{Pr(B \mid A) \cdot Pr(A)}{Pr(B)}$$

where:

- $Pr(A \mid B)$ is the posterior probability: the probability of event A occurring given that B is true.
- $Pr(B \mid A)$ is the likelihood: the probability of event B occurring given that A is true.
- $Pr(A)$ is the prior probability: the initial probability of event A.
- $Pr(B)$ is the marginal probability: the total probability of event B.

2.1.1 Derivation

Starting from the definition of conditional probability:

$$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)} \quad \text{and} \quad Pr(B \mid A) = \frac{Pr(A \cap B)}{Pr(A)}$$

Rearranging the second equation:

$$Pr(A \cap B) = Pr(B \mid A) \cdot Pr(A)$$

Substituting this into the first equation:

$$Pr(A \mid B) = \frac{Pr(B \mid A) \cdot Pr(A)}{Pr(B)}$$

2.1.2 Example

Suppose a certain disease affects 1% of a population. A diagnostic test has a 99% accuracy rate, meaning it correctly identifies 99% of cases both when the disease is present and when it is not. If a person tests positive, we can use Bayes' Theorem to determine the probability that they actually have the disease.

Let D represent having the disease, and T represent testing positive. We know:

- $Pr(D) = 0.01$
- $Pr(\neg D) = 0.99$

- $Pr(T \mid D) = 0.99$
- $Pr(T \mid \neg D) = 0.01$

We want to find $Pr(D \mid T)$:

$$Pr(D \mid T) = \frac{Pr(T \mid D) \cdot Pr(D)}{Pr(T)}$$

First, calculate $Pr(T)$:

$$Pr(T) = Pr(T \mid D) \cdot Pr(D) + Pr(T \mid \neg D) \cdot Pr(\neg D)$$

$$Pr(T) = 0.99 \cdot 0.01 + 0.01 \cdot 0.99 = 0.0198$$

Now, apply Bayes' Theorem:

$$Pr(D \mid T) = \frac{0.99 \cdot 0.01}{0.0198} \approx 0.5$$

This result indicates that even with a positive test result, there is only a 50% chance that the person actually has the disease, highlighting the importance of considering base rates in medical testing.

3 Random Variables

In probability theory, a **random variable** is a function that assigns numerical values to the outcomes of a random phenomenon. Formally, a random variable is defined as a measurable function from a sample space Ω to the real numbers $\mathbb{R}$. Random variables are fundamental in defining probability distributions and facilitating the analysis of stochastic processes.

3.1 Discrete Random Variables

A **discrete random variable** takes on a countable number of distinct values. Each value has an associated probability, and the sum of these probabilities equals one. The function that maps each value to its probability is called the **probability mass function** (PMF).

3.1.1 Example: Number of Poké Balls Needed to Catch a Pokémon

Consider the random experiment of attempting to catch a wild Pokémon in the Pokémon universe. Let the random variable X represent the number of Poké Balls thrown until the Pokémon is caught. Assuming each throw is independent and has a constant probability p of success, X follows a geometric distribution:

$$Pr(X = k) = (1-p)^{k-1}p, \quad k = 1, 2, 3, \ldots$$

This model helps trainers understand the expected number of attempts required to catch a Pokémon.

3.2 Continuous Random Variables

A **continuous random variable** can take on any value within a continuous range. Unlike discrete random variables, the probability of the variable taking on any specific value is zero; instead, probabilities are assigned to intervals of values and are described by the **probability density function** (PDF).

3.2.1 Example: Height of a Pokémon

Suppose we are interested in the height of a particular species of Pokémon, such as Pikachu. Let the random variable Y denote the height of a randomly selected Pikachu. If the heights are continuously distributed over an interval, Y is a continuous random variable. The PDF $f_Y(y)$ describes the likelihood of Pikachu heights within specific ranges:

$$Pr(a \leq Y \leq b) = \int_a^b f_Y(y)\, dy$$

This allows researchers to calculate the probability that a Pikachu's height falls within a certain range.

3.3 Expected Value and Variance

The **expected value** (mean) of a random variable provides a measure of its central tendency, while the **variance** measures the dispersion of its possible values around the mean.

3.3.1 Example: Combat Power (CP) of Caught Pokémon

Let Z represent the Combat Power (CP) of a randomly caught Pokémon. The expected value $E[Z]$ gives the average CP one might anticipate, and the variance $\text{Var}(Z)$ indicates the variability in CP among caught Pokémon. These metrics are valuable for trainers aiming to assess and compare the strength of their Pokémon.

3.4 Applications in Pokémon Statistics

Understanding random variables enables trainers to model and predict various aspects of Pokémon encounters and attributes. For instance:

- **Catch Success Rates**: Modeling the number of attempts needed to catch different Pokémon species.
- **Attribute Distributions**: Analyzing the spread of characteristics like height, weight, and CP.
- **Battle Outcomes**: Estimating probabilities of winning based on Pokémon statistics and move effectiveness.

3.5 Covariance and Correlation

Understanding the relationship between two random variables is fundamental in statistics. Two primary measures to quantify this relationship are **covariance** and **correlation**. This section provides an overview of these concepts, their definitions, properties, and distinctions.

3.6 Covariance

Covariance measures how two random variables change together. Specifically:

- A **positive covariance** indicates that as one variable increases, the other tends to increase.
- A **negative covariance** suggests that as one variable increases, the other tends to decrease.

3.6.1 Definition

For two random variables X and Y with expected values $E(X) = \mu_X$ and $E(Y) = \mu_Y$, the covariance is defined as:

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

This can also be expressed as:

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

This formula indicates that covariance is the expected product of the deviations of each variable from their respective means Statistics (2023).

3.6.2 Properties of Covariance

1. **Variance Relationship:** The covariance of a variable with itself is its variance:

$$\text{Cov}(X, X) = \text{Var}(X)$$

2. **Symmetry:** Covariance is symmetric:

$$\text{Cov}(X, Y) = \text{Cov}(Y, X)$$

3. **Linear Transformation:** For constants a and b:

$$\mathrm{Cov}(aX, bY) = ab\,\mathrm{Cov}(X, Y)$$

4. **Sum Property:** The covariance of the sum of two variables with a third variable is the sum of their covariances:

$$\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$$

5. **Independence:** If X and Y are independent, their covariance is zero:

$$\text{If } X \perp Y, \text{ then } \mathrm{Cov}(X, Y) = 0$$

However, a covariance of zero does not necessarily imply independence Statistics (2023).

3.7 Correlation

Correlation standardizes covariance, providing a dimensionless measure of the linear relationship between two variables. It indicates both the strength and direction of the relationship, ranging from -1 to 1.

- A **correlation of 1** implies a perfect positive linear relationship.
- A **correlation of -1** implies a perfect negative linear relationship.
- A **correlation of 0** indicates no linear relationship.

3.7.1 Definition

The correlation coefficient, denoted as ρ_{XY}, is defined as:

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

where σ_X and σ_Y are the standard deviations of X and Y, respectively. This normalization ensures that correlation is unitless and facilitates comparison between different pairs of variables Statistics (2023).

3.7.2 Properties of Correlation

1. **Boundedness:** The correlation coefficient always lies between -1 and 1:

$$-1 \leq \rho_{XY} \leq 1$$

2. **Symmetry:** Correlation is symmetric:

$$\rho_{XY} = \rho_{YX}$$

3. **Scale Invariance:** Correlation is unaffected by changes in location and scale of the variables. For constants a and b (with $a, b \neq 0$):

$$\rho_{aX+c, bY+d} = \rho_{X,Y}$$

4. **Independence:** While a correlation of zero suggests no linear relationship, it does not imply independence unless the variables are jointly normal.

3.8 Relation Between Covariance and Correlation

Correlation is essentially the covariance normalized by the product of the standard deviations of the variables:

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

This standardization addresses the scale dependency inherent in covariance, allowing for a consistent measure of linear association irrespective of the units of the variables.

3.9 Distinguishing Covariance and Correlation

- **Scale Dependence:** Covariance values are influenced by the units of the variables, making direct comparisons difficult. Correlation, being dimensionless, allows for straightforward comparison across different datasets.
- **Interpretability:** While covariance indicates the direction of a relationship, its magnitude lacks a standardized interpretation. Correlation provides both direction and a quantifiable measure of the relationship's strength.

3.10 Examples of Covariance and Correlation

3.10.1 Taco Sales and Temperature

One possible application of covariance and correlation is analyzing the relationship between temperature and taco sales. Consider the following dataset:

Day	Temperature (°C)	Taco Sales
1	28	120
2	32	150
3	25	90
4	30	135
5	27	105

Table 1: Temperature vs Taco Sales

The covariance between temperature and taco sales is calculated as:

$$\text{Cov}(X, Y) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}) \tag{1}$$

where X represents temperature and Y represents taco sales.

If $\text{Cov}(X, Y) > 0$, this indicates that hotter days are associated with more taco sales, while $\text{Cov}(X, Y) < 0$ would suggest an inverse relationship. Similarly, the correlation coefficient is computed as:

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \tag{2}$$

which normalizes the covariance.

3.10.2 FIFA Ultimate Team Player Prices and Pace

Another application is in FIFA Ultimate Team, where we analyze the relationship between a player's pace rating and their market price in coins.

Player	Pace	Market Price (Coins)
Mbappé	97	1,200,000
Haaland	89	600,000
Modrić	74	100,000
Walker	94	250,000
Van Dijk	83	300,000

Table 2: Pace vs Market Price of FIFA Ultimate Team Players

The covariance and correlation between pace and market price can be computed similarly. If the correlation is high ($\rho \approx 1$), it indicates that players with higher pace tend to be more expensive in the transfer market. If $\rho \approx 0$, it means there is no strong relationship between pace and price.

3.11 Expected Values and Variance of Random Variables

Parameter	Discrete	Continuous
$\mu = E[X]$	$\sum_{i=1}^{n} x_i Pr(X = x_i)$	$\int xf(x)dx$
$\sigma^2 = Var(x)$	$\sum_{i=1}^{n} (x - \mu)^2 Pr(X = x_i)$	$\int (x - \mu)^2 f(x)dx$
$\sigma_{XY} = Cov(X, Y)$	$\sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j (PrX = x_i, Y = y_j) -$ $\sum_{i=1}^{n} x_i Pr(X = x_i) \sum_{j=1}^{m} y_j Pr(Y = y_j)$	$\int \int xyf(x, y)dydx -$ $\int xf(x)dx \int yf(y)dy$

4 Discrete Distributions

4.1 Bernoulli's Distribution

The Bernoulli distribution is the most fundamental discrete probability distribution. It models an experiment that has exactly two possible outcomes: success (typically coded as 1) and failure (coded as 0). A classic example is a single coin toss, where we can define "success" as the coin landing heads. In this case, the outcome is either a win (success) or a loss (failure), making it a perfect illustration of a Bernoulli trial.

The Bernoulli Distribution has a probability density function (p.d.f.) of:

$$Pr(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if x=0} \end{cases} \tag{3}$$

4.2 Binomial Distribution

5 Statistics

5.1 Bessel's Correction

We know that whenever we estimate the variance of a population based on a sample we do:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - x\bar{)}^2 \tag{4}$$

When calculating variance, we assess how each data point deviates from the mean. In a sample, the sample mean $\bar{x}$ serves as an estimate of the population μ. However, using $\bar{x}$ introduces a dependency among deviations: the sum of deviations from the sample mean always equals zero. This constraint reduces the degrees of freedom in our data from n to $n - 1$. As a result, if we were to divide by n our calculation would systematically underestimate the true population variance. Dividing by $n - 1$ corrects this bias, ensuring that our sample variance is an accurate reflection of the population variance. Hardy (2002)

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A. Probability Distributions Reference

Distribution	**Notation**	**Support**	**Function**	$E[X]$	**Var**$[X]$
Uniform (Discrete)	$X \sim \text{Uniform}\{x_1, \ldots, x_n\}$	$x \in \{x_1, \ldots, x_n\}$	$\frac{1}{N}$	$\frac{1}{N}\sum_{i=1}^{N} x_i$	$\frac{1}{N}\sum_{i=1}^{N}(x_i - E[X])^2$
Bernoulli	$X \sim \text{Bernoulli}(p)$	$x \in \{0, 1\}$	$p^x(1-p)^{1-x}$	p	$p(1-p)$
Binomial	$X \sim \text{Binomial}(n, p)$	$x \in \{0, 1, \ldots, n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Geometric	$X \sim \text{Geom}(p)$	$x \in \{0, 1, 2, \ldots\}$	$p(1-p)^x$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$
Poisson	$X \sim \text{Poisson}(\lambda)$	$x \in \{0, 1, 2, \ldots\}$	$\frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ
Uniform (Continuous)	$X \sim \text{Uniform}(a, b)$	$a \leq x \leq b$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$X \sim N(\mu, \sigma^2)$	$-\infty < x < \infty$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$	μ	σ^2
Exponential	$X \sim \text{Exp}(\lambda)$	$x \geq 0$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Beta	$X \sim \text{Beta}(\alpha, \beta)$	$0 < x < 1$ $\alpha > 0$ $\beta > 0$	$\frac{x^{\alpha-1}(1-x)^{\beta}-1}{B(\alpha, \beta)}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Note: The Beta function has the following relationship with the Gamma function: $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

B. Standard Normal Distribution Table ($Pr(X \leq x) = \Phi(z)$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

C: Chi Squared Distribution Table

	α										
df	0.995	0.990	0.975	0.950	0.900	0.100	0.050	0.025	0.010	0.005	0.001
1	0.0000	0.0002	0.0010	0.0039	0.0158	2.7055	3.8415	5.0239	6.6349	7.8794	10.8276
2	0.0100	0.0201	0.0506	0.1026	0.2107	4.6052	5.9915	7.3778	9.2103	10.5966	13.8155
3	0.0717	0.1148	0.2158	0.3518	0.5844	6.2514	7.8147	9.3484	11.3449	12.8382	16.2662
4	0.2070	0.2971	0.4844	0.7107	1.0636	7.7794	9.4877	11.1433	13.2767	14.8603	18.4668
5	0.4117	0.5543	0.8312	1.1455	1.6103	9.2364	11.0705	12.8325	15.0863	16.7496	20.5150
6	0.6757	0.8721	1.2373	1.6354	2.2041	10.6446	12.5916	14.4494	16.8119	18.5476	22.4577
7	0.9893	1.2390	1.6899	2.1673	2.8331	12.0170	14.0671	16.0128	18.4753	20.2777	24.3219
8	1.3444	1.6465	2.1797	2.7326	3.4895	13.3616	15.5073	17.5345	20.0902	21.9550	26.1245
9	1.7349	2.0879	2.7004	3.3251	4.1682	14.6837	16.9190	19.0228	21.6660	23.5894	27.8772
10	2.1559	2.5582	3.2470	3.9403	4.8652	15.9872	18.3070	20.4832	23.2093	25.1882	29.5883
11	2.6032	3.0535	3.8157	4.5748	5.5778	17.2750	19.6751	21.9200	24.7250	26.7568	31.2641
12	3.0738	3.5706	4.4038	5.2260	6.3038	18.5493	21.0261	23.3367	26.2170	28.2995	32.9095
13	3.5650	4.1069	5.0088	5.8919	7.0415	19.8119	22.3620	24.7356	27.6882	29.8195	34.5282
14	4.0747	4.6604	5.6287	6.5706	7.7895	21.0641	23.6848	26.1189	29.1412	31.3193	36.1233
15	4.6009	5.2293	6.2621	7.2609	8.5468	22.3071	24.9958	27.4884	30.5779	32.8013	37.6973
16	5.1422	5.8122	6.9077	7.9616	9.3122	23.5418	26.2962	28.8454	31.9999	34.2672	39.2524
17	5.6972	6.4078	7.5642	8.6718	10.0852	24.7690	27.5871	30.1910	33.4087	35.7185	40.7902
18	6.2648	7.0149	8.2307	9.3905	10.8649	25.9894	28.8693	31.5264	34.8053	37.1565	42.3124
19	6.8440	7.6327	8.9065	10.1170	11.6509	27.2036	30.1435	32.8523	36.1909	38.5823	43.8202
20	7.4338	8.2604	9.5908	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968	45.3147
21	8.0337	8.8972	10.2829	11.5913	13.2396	29.6151	32.6706	35.4789	38.9322	41.4011	46.7970
22	8.6427	9.5425	10.9823	12.3380	14.0415	30.8133	33.9244	36.7807	40.2894	42.7957	48.2679
23	9.2604	10.1957	11.6886	13.0905	14.8480	32.0069	35.1725	38.0756	41.6384	44.1813	49.7282
24	9.8862	10.8564	12.4012	13.8484	15.6587	33.1962	36.4150	39.3641	42.9798	45.5585	51.1786
25	10.5197	11.5240	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9279	52.6197
26	11.1602	12.1981	13.8439	15.3792	17.2919	35.5632	38.8851	41.9232	45.6417	48.2899	54.0520
27	11.8076	12.8785	14.5734	16.1514	18.1139	36.7412	40.1133	43.1945	46.9629	49.6449	55.4760
28	12.4613	13.5647	15.3079	16.9279	18.9392	37.9159	41.3371	44.4608	48.2782	50.9934	56.8923
29	13.1211	14.2565	16.0471	17.7084	19.7677	39.0875	42.5570	45.7223	49.5879	52.3356	58.3012
30	13.7867	14.9535	16.7908	18.4927	20.5992	40.2560	43.7730	46.9792	50.8922	53.6720	59.7031
31	14.4578	15.6555	17.5387	19.2806	21.4336	41.4217	44.9853	48.2319	52.1914	55.0027	61.0983
32	15.1340	16.3622	18.2908	20.0719	22.2706	42.5847	46.1943	49.4804	53.4858	56.3281	62.4872
33	15.8153	17.0735	19.0467	20.8665	23.1102	43.7452	47.3999	50.7251	54.7755	57.6484	63.8701
34	16.5013	17.7891	19.8063	21.6643	23.9523	44.9032	48.6024	51.9660	56.0609	58.9639	65.2472
35	17.1918	18.5089	20.5694	22.4650	24.7967	46.0588	49.8018	53.2033	57.3421	60.2748	66.6188
36	17.8867	19.2327	21.3359	23.2686	25.6433	47.2122	50.9985	54.4373	58.6192	61.5812	67.9852
37	18.5858	19.9602	22.1056	24.0749	26.4921	48.3634	52.1923	55.6680	59.8925	62.8833	69.3465
38	19.2889	20.6914	22.8785	24.8839	27.3430	49.5126	53.3835	56.8955	61.1621	64.1814	70.7029
39	19.9959	21.4262	23.6543	25.6954	28.1958	50.6598	54.5722	58.1201	62.4281	65.4756	72.0547
40	20.7065	22.1643	24.4330	26.5093	29.0505	51.8051	55.7585	59.3417	63.6907	66.7660	73.4020