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A Metric for Spatial Displacements using Biquaternions on SO(4)

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Abstract

In this paper we use the fact that 4×4 homogeneous transforms can be viewed as limiting cases of rotations in four dimensional Euclidean space, E^4, to construct a metric for spatial displacements. For each spatial displacement, we compute an associated four-dimensional rotation and determine the associated biquaternion representation. We then use the standard Euclidean metric for these eight-dimensional vectors, in order to obtain a bi-invariant metric on $SO(4)$. The result is an induced metric on $SE(3)$ that is bi-invariant to a specific degree of approximation.

As examples we determine the distance between two specified displacements, and find the reference frame "equidistant" between two given frames for various positions of the global reference frame.

1 Introduction

Many interesting problems in robotics depend on the measurement of distances between reference frames, in particular, motion approximation, obstacle avoidance, and automated assembly. Because there is no bi-invariant Riemannian metric on the manifold $SE(3)$ of spatial displacements, (Park 1995), solutions to these problems depend on the choice of reference frame used to formulate the problem.

The desire for solutions that are frame independent has lead to the study of various ways to define the distance between two rigid bodies. Kazerounian and Rastegar (1992) compute a weighted volume integral to determine the average distance between corresponding points of two positions of a given rigid body. Martinez and Duffy (1995) show that this metric meets all the requirements of a bi-invariant metric, and go on to determine several equivalent metrics that are less computationally demanding. These metrics depend on the shape of the body, which means the addition or subtraction of material from the moving body may result in different computed distances between two positions.

Ravani and Roth (1983) use the distance between points in a Clifford Algebra representing planar displacements to design mechanisms that move a workpiece through a set of goal positions. As in robotic motion interpolation, this technique yields solutions that depend on the choice of a global reference frame. Larochelle and McCarthy (1994) sought to control this frame dependence of Ravani's technique by replacing the planar positions with associated 3×3 rotations of $SO(3)$, in quaternion form. The design procedure yields a frame invariant solution in $SO(3)$ that is mapped back to the plane. The degree of approximation in this map is directly related to the deficiency in bi-invariance.

In this paper, we use the results of Ge (1994) to extend Larochelle and McCarthy's work to spatial displacements. Homogeneous transforms are considered as approximations to 4×4 rotation matrices in $SO(4)$ for which the movement of the fourth, or W, axis away from its original position is small (McCarthy 1983). We use Ge's construction to obtain the biquaternion representing a 4×4 rotation. The distance between two rotations in E^4 is the magnitude of the difference between their two biquaternions, considered as vectors in R^8. Because rotations in E^4 approximate displacements in E^3, this induces a metric on spatial displacements. We use this metric to measure distances to specified positions, and determine the frame equidistant to two given frames.

2 Spatial Motion and Rotations in E^4

Recall that by introducing homogeneous coordinates in E^3, we can assemble the 3×3 rotation matrix $A(\theta,\phi,\psi)$ and the 3×1 translation vector $\vec{d}=(a,b,c)^T$ into the compact form of a 4×4 homogeneous transform (Paul 1979). Matrices of this form comprise the matrix group SE(3), or special Euclidean group. The angles θ,ϕ and ψ are the longitude, latitude and roll of the moving frame, thus $A(\theta,\phi,\psi) = Rot(y,\theta)Rot(x,-\phi)Rot(z,\psi)$. We can think of this 4×4 matrix as performing a displacement in the $W=1$ hyperplane of E^4 (McCarthy 1990). From this point of view, we obtain an equivalent transformation using the $W=R$ hyperplane:

$$\left\{\begin{array}{c} X \\ Y \\ Z \\ R \end{array}\right\} = \left[\begin{array}{ccc|c} & & & a/R \\ & A(\theta,\phi,\psi) & & b/R \\ & & & c/R \\ \hline 0 & 0 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} x \\ y \\ z \\ R \end{array}\right\} \quad (1)$$

Now consider the 4×4 rotation constructed as follows. Let $A(\theta,\phi,\psi)$ be the upper left 3×3 submatrix of the 4×4 rotation matrix $[K]$ representing a rotation of the $X-Y-Z$ subspace of E^4. Note that $[K]$ keeps the W axis fixed and therefore has a 1 in the fourth diagonal location. Rotations of angle α in $W-X$ plane, β in $W-Y$ plane, and γ in the $W-Z$ plane combine to define the 4×4 matrix:

$$J(\alpha,\beta,\gamma) = \left[\begin{array}{cccc} c\alpha & 0 & 0 & s\alpha \\ -s\beta s\alpha & c\beta & 0 & s\beta c\alpha \\ -s\gamma c\beta s\alpha & -s\gamma s\beta c\gamma & c\gamma & s\gamma c\beta c\alpha \\ -c\gamma c\beta s\alpha & -s\beta c\gamma & -s\gamma & c\gamma c\beta c\alpha \end{array}\right] \quad (2)$$

where c and s represent the cosine and sine functions, respectively. A general 4×4 rotation is now given by the product:

$$[D] = [J(\alpha,\beta,\gamma)][K(\theta,\phi,\psi)]. \quad (3)$$

The spatial displacement (1) is obtained as a limiting case of (3), by introducing the relations,

$$\tan\alpha = \frac{a}{R},\quad \tan\beta = \frac{b}{R},\quad \text{and,}\quad \tan\gamma = \frac{c}{R}, \quad (4)$$

and calculating the Taylor series expansion:

$$\left\{\begin{array}{c} X \\ Y \\ Z \\ W \end{array}\right\} = \left[\begin{array}{ccc|c} & & & a/R \\ & A(\theta,\phi,\psi) & & b/R \\ & & & c/R \\ \hline 0 & 0 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} x \\ y \\ z \\ w \end{array}\right\} +$$

$$1/R\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline e_1 & e_2 & e_3 & 0 \end{array}\right]\left\{\begin{array}{c} x \\ y \\ z \\ w \end{array}\right\} + O(1/R^2). \quad (5)$$

Comparing this result to (1), we see that, to the second order in $\frac{1}{R}$, this 4×4 rotation is identical to a homogeneous transform representing a displacement in the $W=R$ hyperplane. Note that the first order term contributes components in the W direction, which is orthogonal to the $W=R$ hyperplane. Thus by increasing R, displacements in E^3 can be approximated as closely as desired using rotations in E^4.

3 Biquaternions and SO(4)

We now follow Ge (1994) and determine the pair of quaternions, known as a biquaternion, associated with a rotation matrix $[D]$. First note that the matrix $[D]$ can be obtained as the matrix exponential of a 4×4 skew-symmetric matrix $[M]$ (Curtis 1984):

$$[M] = \left[\begin{array}{cccc} 0 & -z_3 & z_2 & w_1 \\ z_3 & 0 & -z_1 & w_2 \\ -z_2 & z_1 & 0 & w_3 \\ -w_1 & -w_2 & -w_3 & 0 \end{array}\right]. \quad (6)$$

Ge (1994) decomposes $[M]$ into two skew-symmetric matrices by interchanging the w_i and z_i elements to define the matrix $[M]'$. He then constructs

$$[M^+] = ([M]+[M]')/2 \quad (7)$$
$$[M^-] = ([M]-[M]')/2 \quad (8)$$

which have the form

$$[M^+] = \mu\left[\begin{array}{cccc} 0 & -s_3 & s_2 & s_1 \\ s_3 & 0 & -s_1 & s_2 \\ -s_2 & s_1 & 0 & s_3 \\ -s_1 & -s_2 & -s_3 & 0 \end{array}\right] = \mu[S], \quad (9)$$

and

$$[M^-] = \nu\left[\begin{array}{cccc} 0 & -t_3 & t_2 & -t_1 \\ t_3 & 0 & -t_1 & -t_2 \\ -t_2 & t_1 & 0 & -t_3 \\ t_1 & t_2 & t_3 & 0 \end{array}\right] = \nu[T], \quad (10)$$

where μ and ν are computed so that $\Sigma s_i^2 = \Sigma t_i^2 = 1$. The result is

$$[M] = \mu[S] + \nu[T]. \quad (11)$$

It can be shown by direct computation that $[S]$ and $[T]$ commute, which means the rotation matrix $[D]$ can be represented by the product of rotations:

$$[D] = e^{[M]} = e^{\mu[S]+\nu[T]} = e^{\mu[S]}e^{\nu[T]}. \quad (12)$$

Ge (1994) shows that the matrices $[G]^+ = e^{\mu[S]}$ and $[H^*]^- = e^{\nu[T]}$ have the form:

$$[G]^+ = \left[\begin{array}{cccc} c\mu & -s_3\,s\mu & s_2\,s\mu & s_1\,s\mu \\ s_3\,s\mu & c\mu & -s_1\,s\mu & s_2\,s\mu \\ -s_2\,s\mu & s_1\,s\mu & c\mu & s_3\,s\mu \\ -s_1\,s\mu & -s_2\,s\mu & -s_3\,s\mu & c\mu \end{array}\right], \quad (13)$$

and

$$[H^*]^- = \begin{bmatrix} c\nu & -t_3\,s\nu & t_2\,s\nu & -t_1\,s\nu \\ t_3\,s\nu & c\nu & -t_1\,s\nu & -t_2\,s\nu \\ -t_2\,s\nu & t_1\,s\nu & c\nu & -t_3\,s\nu \\ t_1\,s\nu & t_2\,s\nu & t_3\,s\nu & c\nu \end{bmatrix}. \quad (14)$$

where the + and − indicate the particular construction of the matrices from the quaternion components (McCarthy 1990). These matrices yield the unit quaternions:

$$\begin{aligned} \mathbf{G} &= s_1 \sin\mu\, i + s_2 \sin\mu\, j + s_3 \sin\mu\, k + \cos\mu \\ \mathbf{H}^* &= -t_1 \sin\nu\, i - t_2 \sin\nu\, j - t_3 \sin\nu\, k + \cos\nu, \end{aligned} \quad (15)$$

where $\mathbf{H}^*$ is the conjugate of $\mathbf{H}$, obtained by negating the vector part. Thus the matrix $[D]$ is associated with the unit quaternions $\mathbf{G}$ and $\mathbf{H}$, which we call a biquaternion $\hat{\mathbf{G}} = (\mathbf{G}, \mathbf{H})$.

The rotation in E^4 of a point $\vec{x} = (x, y, z, w)^T$ to coordinates $\vec{X}$ in the fixed frame by $[D]$ can be written as

$$\vec{X} = e^{[M]}\vec{x} = [G]^+[H^*]^-\vec{x}. \quad (16)$$

This is equivalent to the quaternion product

$$\mathbf{X} = \mathbf{G}\mathbf{x}\mathbf{H}^*. \quad (17)$$

where $\mathbf{x} = xi + yj + zk + w$ and $\mathbf{X} = Xi + Yj + Zk + W$.

Given $[D]$, the skew symmetric matrix $[M]$ may not be readily available. Cayley's formula (Bottema and Roth 1990) can be used to deternmine the skew-symmetric matrix $[B]$:

$$[B] = [D - I][D + I]^{-1}. \quad (18)$$

which can be used to determine the biquaternion $\hat{\mathbf{G}}$. Using Ge's decomposition, we obtain $[B]'$, and

$$\begin{aligned} k_1[S] &= ([B] + [B]')/2 \\ k_2[T] &= ([B] - [B]')/2 \end{aligned} \quad (19)$$

where k_1 and k_2 are again computed so that $\Sigma s_i^2 = \Sigma t_i^2 = 1$. The constants k_1 and k_2 are

$$\begin{aligned} k_1 &= (\tan(\lambda/2) + \tan(\zeta/2))/2 \\ k_2 &= (\tan(\lambda/2) - \tan(\zeta/2))/2. \end{aligned} \quad (20)$$

where $e^{\pm i\lambda}$ and $e^{\pm i\zeta}$ are the eigenvalues of the matrix $[D]$. See the Appendix for the proof of these equations. The angles λ and ζ are related to the quaternion angles μ and ν by:

$$\mu = \frac{\lambda + \zeta}{2} \quad \text{and} \quad \nu = \frac{\lambda - \zeta}{2}. \quad (21)$$

See Ge (1994).

4 Coordinates of Planes in E^4

Biquaternions transform the coordinates of two dimensional planes through the origin in E^4 in a manner equivalent to the transformation of line coordinates in E^3 using dual quaternions. To see this we introduce the coordinates of planes in E^4, obtained as the wedge product of two independent vectors in the plane. This is conveniently done by assembling the components of the two vectors into a 4×2 matrix and computing the six 2×2 minors, known as the Grassmann coordinates of the plane (Harris 1992). Let $\vec{x} = (x_1, x_2, x_3, x_4)^T$ and $\vec{y} = (y_1, y_2, y_3, y_4)^T$ be the two vectors. We obtain the Grassmann coordinates $p_{ij} = x_i y_j - x_j y_i$, and assemble them into the pair of vector quaternions $\mathbf{p} = p_{41}i + p_{42}j + p_{43}k$ and $\mathbf{p}^\circ = p_{23}i + p_{31}j + p_{12}k$.

We now follow Ge (1994) and note that these coordinates can be related to the product of $\vec{x}$ and $\vec{y}$ as quaternions by the equation:

$$\begin{aligned} \mathbf{y}\mathbf{x}^* - \mathbf{x}\mathbf{y}^* &= 2(\mathbf{p} + \mathbf{p}^\circ) \\ \mathbf{x}^*\mathbf{y} - \mathbf{y}^*\mathbf{x} &= 2(\mathbf{p} - \mathbf{p}^\circ). \end{aligned} \quad (22)$$

We now define the following vector quaternions

$$\mathbf{u} = (\mathbf{p} + \mathbf{p}^\circ)/2 \quad \mathbf{v} = (\mathbf{p} - \mathbf{p}^\circ)/2, \quad (23)$$

which Coxeter (1957) calls "Study coordinates."

The rotation of coordinates in E^4 determined by $[D]$ defines a transformation of plane coordinates p, p°, and equivalently of u and v. By (17) we determine the transformation of coordinates of the plane spanned by $\vec{x}$ and $\vec{y}$ by the biquaternion $\hat{\mathbf{G}} = (\mathbf{G}, \mathbf{H})$ to be:

$$\begin{aligned} \mathbf{Y}\mathbf{X}^* - \mathbf{X}\mathbf{Y}^* &= \mathbf{G}(\mathbf{y}\mathbf{x}^* - \mathbf{x}\mathbf{y}^*)\mathbf{G}^* \\ \mathbf{X}^*\mathbf{Y} - \mathbf{Y}^*\mathbf{X} &= \mathbf{H}(\mathbf{x}^*\mathbf{y} - \mathbf{y}^*\mathbf{x})\mathbf{H}^*. \end{aligned} \quad (24)$$

This calculation uses the fact that $\mathbf{G}\mathbf{G}^* = \mathbf{H}\mathbf{H}^* = 1$. Thus, combining the previous three equations we obtain

$$\begin{aligned} \mathbf{U} &= \mathbf{G}\mathbf{u}\mathbf{G}^* \\ \mathbf{V} &= \mathbf{H}\mathbf{v}\mathbf{H}^*. \end{aligned} \quad (25)$$

Clifford (1873) introduced the two symbols η and ξ that satisfy the relations

$$\eta^2 = \eta, \quad \xi^2 = \xi, \quad (26)$$

$$\xi\eta = 0, \text{and}\, \eta\xi = 0, \quad (27)$$

in order to facilitate computations with biquaternions:

$$\hat{\mathbf{G}} = \xi\mathbf{G} + \eta\mathbf{H}. \quad (28)$$

Using our definition of plane coordinates, we construct

$$\hat{\mathbf{u}} = \xi\mathbf{u} + \eta\mathbf{v} = (\xi + \eta)\frac{\mathbf{p}}{2} + (\xi - \eta)\frac{\mathbf{p}^\circ}{2}, \quad (29)$$

and obtain the plane coordinate transformation equations

$$\hat{\mathbf{U}} = \hat{\mathbf{G}}\hat{\mathbf{u}}\hat{\mathbf{G}}^*. \quad (30)$$

5 A Metric on SO(4)

We now turn to the problem of measuring the distance between rotations in E^4. Ravani and Roth (1983) define the distance between two 3×3 rotation matrices, $[Q]$ and $[R]$, as the magnitude of the difference between their associated quaternions, $\mathbf{Q}$ and $\mathbf{R}$, considered as four dimensional vectors. We extend this idea to rotations in E^4 and compute the magnitude of the difference of the biquaternions representing two rotations, treating them as eight dimensional vectors.

Let the biquaternions associated with the matrices $[Q], [R] \in SO(4)$ be $\hat{\mathbf{Q}} = (\mathbf{Q}, \mathbf{S})$ and $\hat{\mathbf{R}} = (\mathbf{R}, \mathbf{T})$. We define the distance between these two rotations to be the length of the eight dimensional vector $\hat{\mathbf{Q}} - \hat{\mathbf{R}}$, or

$$d([Q],[R]) = ((\mathbf{Q}-\mathbf{R})^T(\mathbf{Q}-\mathbf{R})+(\mathbf{S}-\mathbf{T})^T(\mathbf{S}-\mathbf{T}))^{1/2}. \tag{31}$$

Here we treat each unit quaternion as a four dimensional vector, thus the transpose indicates converting the column vector into a row vector.

To see that this metric is left invariant, we apply a fixed frame transformation defined by the rotation $[D]$, or equivalently $\hat{\mathbf{G}} = (\mathbf{G}, \mathbf{H})$. This transforms the biquaternion $\hat{\mathbf{Q}}$ to $\hat{\mathbf{G}}\hat{\mathbf{Q}} = (\mathbf{GQ}, \mathbf{HS})$. The biquaternion $\hat{\mathbf{R}}$ is transformed similarly. Therefore, the difference $\hat{\mathbf{Q}} - \hat{\mathbf{R}}$ in the new coordinate frame becomes

$$\begin{aligned} \hat{\mathbf{G}}\hat{\mathbf{Q}} - \hat{\mathbf{G}}\hat{\mathbf{R}} &= (\mathbf{GQ} - \mathbf{GR}, \mathbf{HS} - \mathbf{HT}) \\ &= ([G]^+(\mathbf{Q} - \mathbf{R}), [H]^+(\mathbf{S} - \mathbf{T})) \end{aligned} \tag{32}$$

The last step uses the fact that multiplication by the quaternions $\mathbf{G}$ and $\mathbf{H}$ is equivalent to multiplication by the orthogonal matrices $[G]^+$ and $[H]^+$, constructed according to (13). Substituting this into (31) we obtain:

$$\begin{aligned} d([DQ],[DR])^2 &= (\mathbf{GQ} - \mathbf{GR})^T(\mathbf{GQ} - \mathbf{GR}) + \\ &\quad (\mathbf{HS} - \mathbf{HT})^T(\mathbf{HS} - \mathbf{HT}), \\ &= (\mathbf{Q} - \mathbf{R})^T[G]^{+T}[G]^+(\mathbf{Q} - \mathbf{R}) + \\ &\quad (\mathbf{S} - \mathbf{T})^T[H]^{+T}[H]^+(\mathbf{S} - \mathbf{T}), \\ &= d([Q],[R])^2. \end{aligned} \tag{33}$$

The last step uses the identities $[G]^{+T}[G]^+ = [I]$ and $[H]^{+T}[H]^+ = [I]$, since these matrices are orthogonal.

The proof that this metric is right invariant is essentially the same.

6 Examples

To demonstrate the application of this metric to spatial displacements take as an example the positions presented in Section 2 of Martinez and Duffy (1995) (see Fig. 1). Consider a frame M_1 which has been displaced from the fixed frame F by a rotation of $\psi = 5°$ about the z axis; and another frame M_2 which is a pure translation of $\vec{d} = (5, 3, 2)^T$ relative to F.

To apply the metric, we first must choose the distance R to the hyperplane in which the transformations take place. By examining the Taylor's series expansion for cosine used in (5), we see that the error ϵ induced is of order L^2/R^2 where L is a maximum length characterizing the workspace. This gives the relation

$$R = L/\epsilon^{\frac{1}{2}}. \tag{34}$$

This allows us to choose R that yields a specific degree of approximation. For this example, let $L = 5$, the largest of the components of $\vec{d}$. If the error is to be less that $\epsilon = 0.01$, then R is computed to be $(5^2/0.01)^{\frac{1}{2}} = 50$.

Now to obtain $[D_1] \in SO(4)$ corresponding to M_1, we determine the matrices $[K(\theta, \phi, \psi)]$ and $[J(\alpha, \beta, \gamma)]$ from (3). We let θ and ϕ denote the longitude and latitude of the z-axis of the M_1 frame relative to the fixed frame F and the parameter ψ is the roll of M_1 about the z-axis. Thus, we obtain $\theta = \phi = 0°$ and $\psi = 5°$ for the orientation of this frame. The angles $\alpha = \beta = \gamma = 0°$ correspond to the translation $\vec{d} = (0, 0, 0)^T$ of the origin of the frame according to (4). The result is the 4×4 rotation matrix $[D_1]$.

For M_2 we have the orientation angles $\theta = \psi = \phi = 0°$, and construct the angles $\alpha = \arctan(5/50) = 5.7°, \beta = \arctan(3/50) = 3.4°, \gamma = \arctan(2/50) = 2.3°$ from the translation data. Equation 3 yields the 4×4 matrix $[D_2]$.

We determine the biquaternions $\hat{\mathbf{G}}_1$ and $\hat{\mathbf{G}}_2$ associated with $[D_1]$ and $[D_2]$ to obtain:

$$\begin{aligned} \mathbf{G}_1 &= 0.0436k + 0.999 \\ \mathbf{H}_1 &= 0.0436k + 0.999 \\ \mathbf{G}_2 &= 0.0493i + 0.0310j + 0.0185k + 0.998 \\ \mathbf{H}_2 &= -0.0505i + -0.0290j + -0.0215k + 0.998. \end{aligned}$$

The results of the distance computations are presented in Table 1. For purposes of comparison, we compute the distance measurements using the three values $R = 25$, $R = 50$, and $R = 75$. The distance, Δ_1, to M_1 from F does not vary with R since it has no translation component. As R is increased, the distance, Δ_2, to M_2 from F decreases since the rotation angles associated with the translation terms decrease, thus decreasing the influence of the translation. Decreasing R increases the influence of the translation terms. The conclusion is the parameter R is a physical realization of the weighting term often used to construct metrics combining rotations and translations in space (Park 1995).

Table 1

Radius	Δ_1	Δ_2
25	0.0617	0.1741
50	0.0617	0.0872
75	0.0617	0.0581

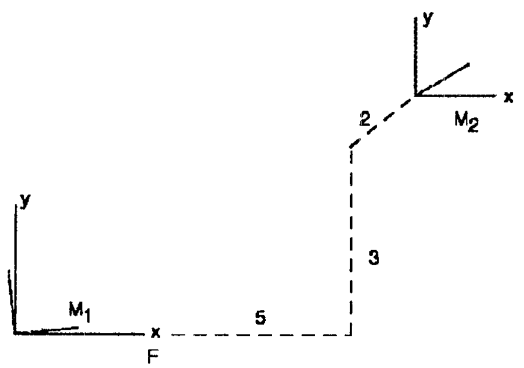

Figure 1: Positions M_1 and M_2 (Martinez and Duffy, 1994)

Another example of the use of this metric is the computation of a displacement in E^3 which is equidistant from two given displacements. Let the first frame M_1 be coincident with the fixed frame F, so $\alpha = \beta = \gamma = \theta = \phi = \psi = 0°$. M_2 is located at $\vec{d} = (0, 2.5, 1)^T$ and rotated by 45° about the z-axis (see Fig. 2). Our workspace has a maximum dimension of 2.5 units in x, y or z. To obtain an error of $\epsilon = 0.01$ we compute $R = 25$ using (34). For M_2 this yields $\alpha = 0°, \beta = 1.43°, \gamma = 0.573°$ and $\theta = \phi = 0°, \psi = 45°$. The associated biquaternions are:

$$
\begin{aligned}
\mathbf{G}_1 &= 1.0 \\
\mathbf{H}_1 &= 1.0 \\
\mathbf{G}_2 &= 0.018i + 0.047j + 0.401k + 0.915 \\
\mathbf{H}_2 &= -0.020i + -0.046j + 0.364k + 0.930.
\end{aligned}
$$

The midpoint of the biquaternions $\hat{\mathbf{G}}_1$ and $\hat{\mathbf{G}}_2$ considered as vectors in R^8 yields an eight-dimensional vector $\hat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2)$, which is not composed two unit quaternions. We normalize the quaternions $\mathbf{A}_1$ and $\mathbf{A}_2$ to obtain the biquaternion $\hat{\mathbf{G}}_a$ which represents the "mid-point" rotation in E^4. The result is

$$
\begin{aligned}
\mathbf{G}_a &= 0.009i + 0.024j + 0.205k + 0.978 \\
\mathbf{H}_a &= -0.010i - 0.023j + 0.185k + 0.982.
\end{aligned}
$$

Converting $\hat{\mathbf{G}}_a$ to a spatial displacement, we obtain the location of the frame A, given in Table 2. We examine its frame dependence, by recalculating this "mid-point" displacement using fixed frames F_1 and F_2 located at opposite corners of our workspace. Let the origin of F_1 be defined by the translation vector $\vec{d} = (2.5, 2.5, 2.5)^T$, then the mid-point frame is A^1 listed in Table 2. The same calculation with the fixed frame F_2 with origin located at $\vec{d} = (-2.5, -2.5, -2.5)^T$, yields the mid-point displacement A^2. The difference in the coordinates of A, A^1, and A^2 demonstrates the level of coordinate frame dependence of this metric.

Table 2

	a	b	c
A	0.249	1.250	0.500
A^1	0.251	1.256	0.513
A^2	0.242	1.213	0.439
	θ	ϕ	ψ
A	0.011°	−0.028°	22.5°
A^1	−0.050°	−0.011°	22.2°
A^2	0.034°	−0.005°	21.6°

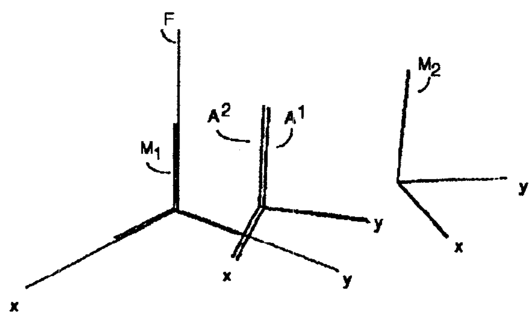

Figure 2: Equidistant displacements A^1 and A^2.

7 Conclusions

In this paper, we approximate a spatial displacement by a 4×4 rotation matrix and compute the associated biquaternion (Ge 1994). The standard Euclidean distance in R^8 can be used to define a metric on biquaternions that is bi-invariant on $SO(4)$. The result is a metric on the set of spatial displacements $SE(3)$ for which the deficiency in bi-invariance can be specified. It also provides a physical interpretation for the weighting of translation and rotation terms typical to Riemannian metrics of $SE(3)$.

Acknowledgements

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Appendix

To prove (19) and (20) begin with Cayley's Formula for the rotation matrix $[D]$:

$$[D] = [I - B]^{-1}[I + B]. \tag{35}$$

Substitute $[B] = k_1[S] + k_2[T]$, using (20), into (35) and expand using Mathematica. Collecting terms we obtain:

$$\begin{aligned}[D] \;=\; & (\cos^2(\frac{\lambda}{2})\cos^2(\frac{\zeta}{2}) - \sin^2(\frac{\lambda}{2})\sin^2(\frac{\zeta}{2}))[I] + \\ & (\sin^2(\frac{\lambda}{2})\cos^2(\frac{\lambda}{2}) - \sin^2(\frac{\zeta}{2})\cos^2(\frac{\zeta}{2}))[S][T] + \\ & (\sin(\frac{\lambda}{2})\cos(\frac{\lambda}{2}) - \sin(\frac{\zeta}{2})\cos(\frac{\zeta}{2}))[T] + \\ & (\sin(\frac{\lambda}{2})\cos(\frac{\lambda}{2}) + \sin(\frac{\zeta}{2})\cos(\frac{\zeta}{2}))[S]. \end{aligned} \tag{36}$$

Next expand the matrix exponential (12) and use the fact that $[S]^2 = [T]^2 = -[I]$ to obtain the expression:

$$\begin{aligned}[D] &= e^{[M]} = e^{(\mu[S]+\nu[T])} = e^{\mu[S]}e^{\nu[T]} \\ &= \cos\mu\cos\nu[I] + \sin\mu\sin\nu[S][T] + \\ &\quad \cos\mu\sin\nu[T] + \sin\mu\cos\nu[S]. \end{aligned} \tag{37}$$

The coefficents of (36) and (37) can be shown to be identical using the relationship given in (21) between the quaternion angles μ and ν and the rotation angles λ and ζ.