GG1: Closed Form Inverse Kinematics

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Abstract

In this short report is described the inverse kinematics of the GG1 robot. A geometric insight is shown and the closed form solution is obtained.

Formulation

The end effector positions and orientations are represented with a $\mathbb{R}^3$ element and the Euler Angles $Z \rightarrow Y' \rightarrow X$" convention (also know as Yaw-Pitch-Roll, Fig. 1) for the $SO(3)$ part.

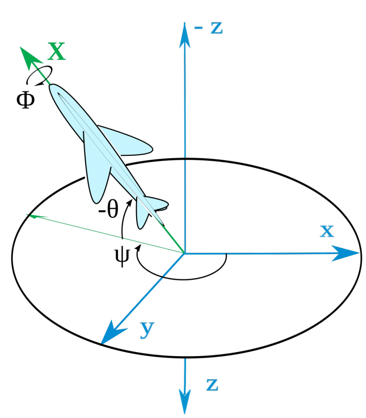

Figure 1: Representation of the Yaw,Pitch,Roll convention

The Cartesian end effector position is represented as $p_x,\ p_y,\ p_z$ while the orientation with $\psi,\ \theta,\ \phi$. The plane (x, y) part of the end effector pose is determined just by the Prismatic joint (d_1) and the first Rotational joint (R_1), as presented in Fig. 2, thus we can decompose the kinematic analysis into two different parts:

1. d_1, R_5, and R_1 are addressed using the (x, y) view
2. $R_{2,3,4}$ are found using the view $(z, \sqrt{p_x^2 + p_y^2}\ )$

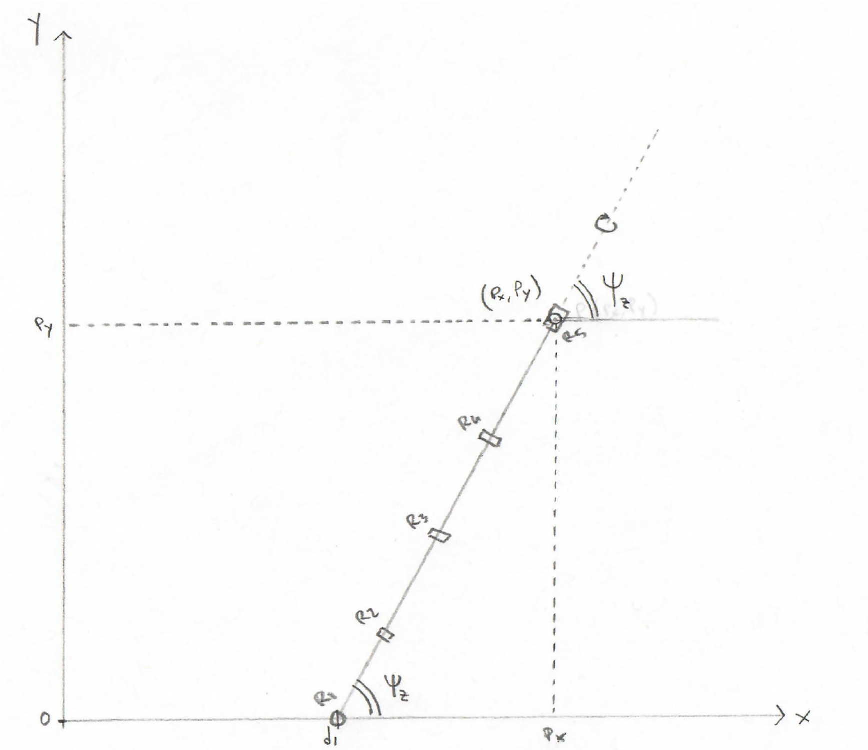

Figure 2: x, y plane

Calculation of d_1, R_5, and R_1

The value of the prismatic variable d_1 is found using the trigonometric properties of the robot in the x, y plane (Fig. 2):

$$\tan(\psi) = \frac{p_y}{\tilde{p}_x} \tag{1}$$

$$\tilde{p}_x = \frac{p_y}{\tan(\psi)} \tag{2}$$

$$d_1 = p_x - \frac{p_y}{\tan(\psi)} \tag{3}$$

With $\tilde{p}_x = p_x - d_1$. The last rotational joint variable R_5 is automatically the Roll angle, thus:

$$R_5 = \phi \tag{4}$$

The first Rotational joint can be easily found as well because of the peculiar architecture of the robot and the chosen rotation representation convention:

$$R_1 = \psi \tag{5}$$

Calculation of R_1, R_2, and R_3

In order to get the last three joint variables $R_{2,3,4}$ we may consider the architecture of the robot and notice that these joints can perform rotations in one plane only (the one parallel to the z-axis and defined by the point d_1 on the x-axis and the end effector position (p_x, p_y)), as presented in Fig. 3. The joint angles are relative because they are easier to handle this way, basically we can send the joint variables directly to the motors without further calculations. The value of the fourth rotational joint R_4 is easy to obtain just doing geometrical considerations:

$$\alpha - R_2 + R_3 - \frac{\pi}{2} \tag{6}$$

$$\theta = R_4 - \alpha \tag{7}$$

$$R_4 = \theta + R_2 + R_3 - \frac{\pi}{2} \tag{8}$$

On the other hand the equation for the partial forward kinematics is:

$$p_{xy} = l_1 s(R_2) + l_2 s(\pi - R_2 - R_3) + l_3 s(R_2 + R_3 - R_4) \tag{9}$$

$$p_z = l_1 c(R_2) + l_2 c(\pi - R_2 - R_3) + l_3 c(R_2 + R_3 - R_4) \tag{10}$$

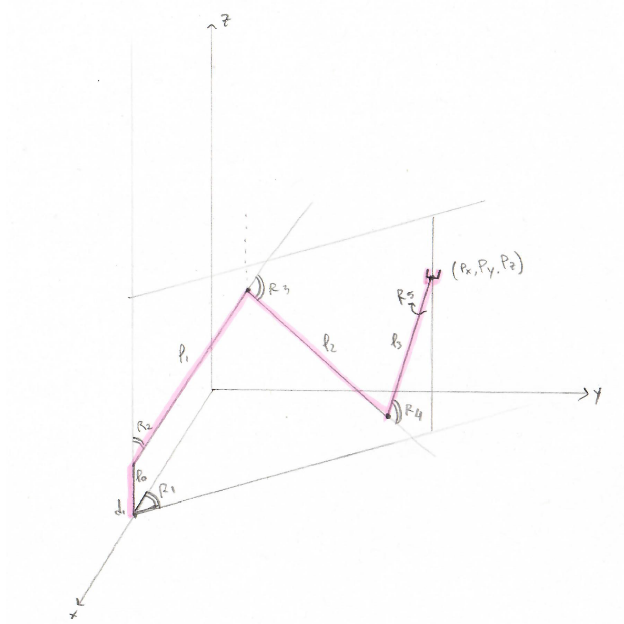

Figure 3: $z, \sqrt{p_x^2 + p_y^2}$ plane

and after some simple calculations gives:

$$\sqrt{p_x^2 + p_y^2} = l_1 s(R_2) + l_2 s(R_2 + R_3) + l_3 c(\theta) \tag{11}$$

$$p_z = l_1 c(R_2) - l_2 c(R_2 + R_3) + l_3 s(\theta) \tag{12}$$

if we express the position of the fourth rotational joint (let's call it L):

$$L_{x,y} = l_1 s(R_2) + l_2 s(R_2 + R_3) = p_{x,y} - l_3 c(\theta) \tag{13}$$

$$L_z = l_1 c(R_2) - l_2 c(R_2 + R_3) = p_z - l_3 s(\theta) \tag{14}$$

and if we square and take the sum:

$$L_{x,y}^2 + L_z^2 = l_1^2 + l_2^2 - 2l_1 l_2 c(R_3) \tag{15}$$

$$p_{x,y}^2 + p_z^2 + l_3^2 - 2l_3(p_{x,y}c(\theta) + p_z s(\theta)) = l_1^2 + l_2^2 - 2l_1 l_2 c(R_3) \tag{16}$$

$$R_3 = cos^{-1}\left(\frac{p_{x,y}^2 + p_z^2 + l_3^2 - 2l_3(p_{x,y}c(\theta) + p_z s(\theta)) - l_1^2 - l_2^2}{2l_1 l_2}\right) \tag{17}$$

$$\tag{18}$$

Or if we use the atan formula:

$$R_3 = ATAN2\left(\frac{\pm\sqrt{1-b^2}}{b}\right) \tag{19}$$

$$b = -\frac{p_{x,y}^2 + p_z^2 + l_3^2 - 2l_3(p_{x,y}c(\theta) + p_z s(\theta)) - l_1^2 - l_2^2}{2l_1 l_2} \tag{20}$$

R_3 can assume 2 possible values since $\pm R_3$ would provide the same result.

The last joint variable R_2 can be calculated using Eq. (11) or (12):

$$p_z = l_1 c(R_2) - l_2 c(R_2)c(R_3) + l_2 s(R_2)s(R_3) + l_3 s(\theta) \tag{21}$$

$$l_1 c(R_2) - l_2 c(R_2)c(R_3) + l_2 s(R_2)s(R_3) = p_z - l_3 s(\theta) \tag{22}$$

Using the tangent half-angle substitution:

$$c(\theta) = \frac{1-t^2}{1+t^2} \tag{23}$$

$$s(\theta) = \frac{2t}{1+t^2} \tag{24}$$

we can get a 2^{nd} order equation in t:

$$t^2(l_2 c(R_3) - l_1 - p_z + l_3 s(\theta)) + t(2l_2 s(R_3)) + l_1 - l_2 - p_z + l_3 s(\theta) \tag{25}$$

which provides 2 solutions, again it is convenient to use the atan rule again:

$$R_2 = ATAN2\left(\frac{b}{a}\right) \pm ATAN2\left(\frac{\sqrt{a^2+b^2-c^2}}{c}\right) \tag{26}$$

$$a = l_1 - l_2 c(R_3) \tag{27}$$

$$b = l_2 s(R_3) \tag{28}$$

$$c = p_z - l_3 s(\theta) \tag{29}$$

$$\tag{30}$$