Dual Numbers

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There are three main ways to represent two real variables:

- Complex numbers: $a + ib,\ i^2 = -1$ (complex numbers of hyperbolic type, Algebraic Field)
- Dual numbers: $a + \epsilon b,\ \epsilon^2 = 0$ (complex numbers of elliptic type, Algebraic Ring (there is no inverse for $a = 0$))
- Double numbers: $a + jb,\ j^2 = +1$ (complex numbers of parabolic type, Algebraic Field)

In particular we will analyze the connection between dual numbers and rigid body kinematics:

$$\hat{a} = a + \epsilon b \tag{1}$$

$$\epsilon^2 = 0 \tag{2}$$

$$(a_1 + \epsilon b_1)(a_2 + \epsilon b_2) = a_1 a_2 + \epsilon(a_1 b_2 + b_1 a_2) \tag{3}$$

$$f(a + \epsilon b) = f(a) + \epsilon b f'(a) \tag{4}$$

$$(a + \epsilon b)^{-1} = \frac{1}{a} - \epsilon \frac{b}{a^2} \tag{5}$$

$$e^{a+\epsilon b} = e^a e^{\epsilon b} = e^{a(1+\epsilon b)} = e^a + \epsilon b e^a \tag{6}$$

$$\sin(\hat{\theta}) = \sin(\theta + \epsilon d) = \sin(\theta) + \epsilon d \cos\theta \tag{7}$$

$$\cos(\hat{\theta}) = \cos(\theta + \epsilon d) = \cos(\theta) - \epsilon d \sin\theta \tag{8}$$

$$a + \epsilon b = a(1 + \epsilon \frac{b}{a}) = a e^{\epsilon \frac{b}{a}} \tag{9}$$

A vector or a matrix in complex space is treated the same as the real vectors and matrices:

$$\hat{\mathbf{a}} = \begin{bmatrix} a_1 + \epsilon b_1 \\ a_2 + \epsilon b_2 \\ a_3 + \epsilon b_3 \end{bmatrix} \tag{10}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} a_{11} + \epsilon b_{11} & a_{12} + \epsilon b_{12} & a_{13} + \epsilon b_{13} \\ a_{21} + \epsilon b_{21} & a_{22} + \epsilon b_{22} & a_{23} + \epsilon b_{23} \\ a_{31} + \epsilon b_{31} & a_{32} + \epsilon b_{32} & a_{33} + \epsilon b_{33} \end{bmatrix} \tag{11}$$

$$\hat{\mathbf{a}} = \mathbf{a} + \epsilon \mathbf{b};\ \ \hat{\mathbf{c}} = \mathbf{c} + \epsilon \mathbf{d} \tag{12}$$

$$\hat{\mathbf{A}} = \mathbf{A} + \epsilon \mathbf{B};\ \ \hat{\mathbf{C}} = \mathbf{C} + \epsilon \mathbf{D} \tag{13}$$

$$\hat{\mathbf{a}}^T \hat{\mathbf{c}} = \mathbf{a}^T \mathbf{c} + \epsilon(\mathbf{b}^T \mathbf{c} + \mathbf{a}^T \mathbf{d}) \tag{14}$$

$$\hat{\mathbf{A}} \hat{\mathbf{C}} = \mathbf{AC} + \epsilon(\mathbf{BC} + \mathbf{AD}) \tag{15}$$

$$\hat{\mathbf{A}} \hat{\mathbf{a}} = \mathbf{Aa} + \epsilon(\mathbf{Ab} + \mathbf{Ba}) \tag{16}$$

$$\hat{\mathbf{A}}^{-1} = \mathbf{A}^{-1} - \epsilon \mathbf{A}^{-1} \mathbf{B} \mathbf{A}^{-1} \tag{17}$$

$$\det(\hat{\mathbf{A}}) = \det(\mathbf{A})(1 + \epsilon\ tr(\mathbf{A}^{-1}\mathbf{B})) \tag{18}$$

There are three main Lemmas which are governing the Dual Number theory that we will use:

1. For any 3x3 orthogonal dual matrix $\hat{\mathbf{A}} = \mathbf{A} + \epsilon \mathbf{B}$ if $\mathbf{B} \neq 0 \rightarrow\ rank(\mathbf{B}) = 2$. Moreover:

$$\mathbf{A}^T \mathbf{B} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \in so(3) \tag{19}$$

2. For any 3x3 orthogonal dual matrix $\hat{\mathbf{A}} = \mathbf{A} + \epsilon \mathbf{B}$ each column or each row of $\hat{\mathbf{A}}$ is a unit screw

3. The unit screw is invariant under the orthogonal dual number transformation in 3D space

Now let's consider the normal DH matrix with the Dual DH matrix for a single joint:

$$\mathbf{A}_{k-1}^{k} = \begin{bmatrix} c\theta_k & -s\theta_k & 0 & 0 \\ s\theta_k & c\theta_k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_k & -s\alpha_k & 0 \\ 0 & s\alpha_k & c\alpha_k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} c\hat{\theta}_k & -s\hat{\theta}_k & 0 \\ s\hat{\theta}_k & c\hat{\theta}_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\hat{\alpha}_k & -s\hat{\alpha}_k \\ 0 & s\hat{\alpha}_k & c\hat{\alpha}_k \end{bmatrix} \tag{21}$$

The real part of any dual number transformation describes the rotation while the dual part contains the complete information about the translation, but what is the mapping between the dual part and the usual translation vector $\in \mathbb{R}^3$?

$$p_j^i = (p_x, p_y, p_z) \tag{22}$$

$$(p_j^i)^\wedge = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \in so(3) \tag{23}$$

$$\hat{\mathbf{A}}_j^i = \hat{\mathbf{B}}_j^i + \epsilon \hat{\mathbf{C}}_j^i \tag{24}$$

$$(p_j^i)^\wedge = \hat{\mathbf{C}}_j^i (\hat{\mathbf{B}}_j^i)^T = \hat{\mathbf{C}}_j^i \hat{\mathbf{B}}_i^j \tag{25}$$

Each column of $\hat{\mathbf{A}}_j^i$ is a unit screw which moment is the corresponding column of $\mathbf{C}_j^i$:

$$(p_j^i)^\wedge \mathbf{B}_j^i = \mathbf{C}_j^i \tag{26}$$

$$\mathbf{c}_j^i = p_j^i \times \mathbf{b}_j^i \tag{27}$$