INTRODUCTION TO LIE GROUPS AND LIE ALGEBRAS APPLICATIONS IN ROBOTICS

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In this presentation I am trying to merge (I am not adding **anything** but some extended proofs) the notions about Lie Groups and Algebras presented here:

- ”Lie Groups and Lie Algebras in Robotics”, Jon Selig
- ”Geometric Fundamentals of Robotics”, Jon Selig
- ”Mathematical Introduction to Robotic Manipulation”, Murray, Li, Sastry
- ”Introduction to Lie Groups and Lie Algebras”, Alexander Kirillov

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INTRO TO DIFFERENTIAL GEOMETRY

Euclidean Space

- a set of points satisfying certain relationships, expressible in terms of distance and angle

Vector Space

- a vector space over a field is a set of vectors with a binary operator and a multiplication by elements of the field (usually $\mathbb{R}$) being defined

- **Ring**: it is one of the fundamental algebraic structures used in abstract algebra. It consists of a set equipped with two binary operations that generalize the arithmetic operations of addition and multiplication
- **Field**: it is one of the fundamental algebraic structures used in abstract algebra. It is a nonzero commutative division ring, or equivalently a ring whose nonzero elements form an abelian group under addition and multiplication. Examples of fields are the field of real numbers and the field of complex numbers

- In mathematics, a manifold is a topological space that locally resembles Euclidean space near each point. More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic (a mapping between groups that preserves the group structure $\approx$ the groups have the same number of ”holes”) to the Euclidean space of dimension n
- In mathematics, a differentiable manifold is a type of manifold that is locally similar enough to a linear space (we can do calculus though)

INTRO TO GROUP THEORY

A group is an algebraic structure consisting of a set of elements equipped with a **binary multiplication** that combines any two elements to form a third element, the operation satisfies four conditions called the group axioms, namely **closure, associativity, identity** and **invertibility**. If the group operator is commutative the group is called Abelian.

A **Lie Group** has to satisfy also these properties:

- The set of group elements G must form a differentiable manifold. A manifold is a smooth space with no singularities on which local coordinates can be used to do calculus
- The group operation must be a differentiable map
- The map from a group element to its inverse must be a differentiable mapping $\rightarrow$ the map is an homeomorphism

The Lie Groups of interest in the description of the rigid body motion are:

- **GL**(n, **R**) is the General Linear Group of degree n over the field of real numbers, it is often represented as **GL**(n) omitting the field (if it is **R**), it represents every non singular $n \times n$ matrices, in it is defined the matrix multiplication
- **SL**(n) is the Special Linear Group, a subgroup of **GL**(n) characterized by matrices with unitary determinant
- **O**(n) is the Orthogonal Group, a subgroup of **GL**(n)characterized by matrices such that $R^T R = I$ (follows that det(R) = ± 1)
- **SO**(n) is the Special Orthogonal Group, a subgroup of **GL**(n) characterized by matrices with unitary determinant and mutually orthonormal columns ($R^T R = I$) $\rightarrow$ Rotation Matrices
- **so**(n) is a subgroup of **GL**(n) and represents all the $n \times n$ skew-symmetric matrices ($S^T = -S$), it is the Lie Algebra of SO(n) and it is a vector spaces over the reals.

- **SE**(n) is the Special Euclidean Group, it can be represented by $(n+1) \times (n+1)$ matrices and it represents the group of rigid body transformations, in particular a generic rigid body transformation $g \in \mathbf{SE}(n) = \mathbb{R}^n \rtimes \mathbf{SO}(n)$
- **se**(n) is the generalization of the **so**(n), it is the group of the $(n+1) \times (n+1)$ matrices $(v, \hat{\omega}) : v \in \mathbb{R}^n, \hat{\omega} \in so(n).$ The elements of se(3) are the Lie Algebra of SE(3) and are called twists. Every twist spans over $\mathbb{R}^6$ so we can reduce the 4×4 matrix with a new operator called vee (6 by 1 vector representation with the Plucker notation)

However it's important to notice that **not every Lie Group is a matrices group** (i.e. the quaternions form a Lie Group AND they are not matrices)

RIGID BODY MOTIONS

Any **Rigid Body Transformation** is an element of SE(3) and it is the map $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, in particular it has some useful properties:

- the length is preserved
- the cross product between vector in the body is preserved (and the inner product as a consequence as well)

Theoretically any rigid body transformation can be decomposed in a rotation, a translation, and a reflection (that is physically impossible). A rigid body transformation can be represented with the matrix:

$$A = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix}_{4\times 4} \rightarrow \tilde{\mathbf{p}}' = A\tilde{\mathbf{p}} \tag{1}$$

with $\tilde{\mathbf{p}}$ the homogeneous representation of $\mathbf{p}$

Any rigid body motion is composed by a rotation ($\in SO(3)$) and a translation ($\in \mathbb{R}^3$)

$$\begin{pmatrix} \mathbf{p}' \\ 1 \end{pmatrix} = \begin{pmatrix} R & \mathbf{t} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = \begin{pmatrix} R\mathbf{p} + \mathbf{t} \\ 1 \end{pmatrix} \tag{2}$$

since the rotation acts first and then the translation is applied the group of rigid body motions $SE(3)$ is obtained as the semi-direct product of $SO(3)$ and $\mathbb{R}^3$ ($SE(3) = SO(3) \rtimes \mathbb{R}^3$).

This action of the semi-direct product arises from the product of two consecutive transformations: $(R_1, \mathbf{t_1})(R_2, \mathbf{t_2}) = (R_1R_2, R_2\mathbf{t_1} + \mathbf{t_2})$ and not as in the direct product $(R_1R_2, \mathbf{t_1} + \mathbf{t_2})$ [1].

The inverse of a transformation matrix is given by:

$$A^{-1} = \begin{pmatrix} R^T & -R^T\mathbf{t} \\ 0 & 1 \end{pmatrix} \tag{3}$$

[1] Remember that the group operation of $SO(3)$ is the matrix multiplication (R_1R_2) and for the vector space $\mathbb{R}^3$ the operator is the sum of vectors ($\mathbf{t_1} + \mathbf{t_2}$).

FINITE SCREW MOTION

Chasles's Theorem: "every rigid body transformation (except for pure translation) is given by a finite screw motion, that is a rotation around an axis (**Screw Axis**) together with a translation along the same axis". The finite screw motion around a line through the origin is:

$$A(\theta) = \begin{pmatrix} R_{\hat{x}}(\theta) & \frac{\theta p}{2\pi}\hat{x} \\ 0 & 1 \end{pmatrix} \tag{4}$$

where $\hat{x}$ is the screw axis, θ is the angle of rotation, p is the pitch of the motion (the ratio between the translation and the rotation about the screw axis), and $R_{\hat{x}}(\theta)$ is the rotation (since $R_{\hat{x}}(\theta)$ is about $\hat{x}$ $\rightarrow R_{\hat{x}}(\theta)\hat{x} = \hat{x}$, thus $\hat{x}$ is an eigenvector of the matrix $R_{\hat{x}}$)

If a screw motion is not represented with a line through the origin we can use conjugation to translate in to the origin and move it back:

$$\begin{pmatrix} \mathsf{I} & \mathbf{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{R}_{\hat{x}}(\theta) & \frac{\theta p}{2\pi}\hat{\mathbf{x}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathsf{I} & \mathbf{u} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathsf{R}_{\hat{x}}(\theta) & \frac{\theta p}{2\pi}\hat{\mathbf{x}} + (\mathsf{I} - \mathsf{R}_{\hat{x}}(\theta))\mathbf{u} \\ 0 & 1 \end{pmatrix} \tag{5}$$

and this is the classic representation of a screw motion, and thanks to Chasles's theorem ($\mathsf{R}_{\hat{x}}(\theta) = \mathsf{R}_{\hat{x}}$ for a nicer spacing):

$$\begin{pmatrix} \mathsf{R} & \mathbf{t} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathsf{R}_{\hat{x}} & \frac{\theta p}{2\pi}\hat{\mathbf{x}} + (\mathsf{I} - \mathsf{R}_{\hat{x}})\mathbf{u} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathsf{R}_{\hat{x}} & \delta\hat{\mathbf{x}} + (\mathsf{I} - \mathsf{R}_{\hat{x}})\mathbf{u} \\ 0 & 1 \end{pmatrix} \tag{6}$$

where δ is the translation along the screw axis, the equation results in these two constraints:

$$\mathsf{R} = \mathsf{R}_{\hat{x}}(\theta) \qquad \mathbf{t} = (\mathsf{I} - \mathsf{R}_{\hat{x}}(\theta))\mathbf{u} + \delta\hat{\mathbf{x}}$$

It has been proved that (since $||\hat{x}|| = 1$):

$$R - R^T = 2\sin\theta \begin{pmatrix} 0 & -\hat{x}_z & \hat{x}_y \\ \hat{x}_z & 0 & -\hat{x}_x \\ -\hat{x}_y & \hat{x}_x & 0 \end{pmatrix} \tag{7}$$

and

$$\mathrm{Tr}(R) = 1 + 2\cos\theta \tag{8}$$

the method above fails in the zero pitch case (pure translation) since the axis cannot be addressed:

$$2\sin\theta = 0 \tag{9}$$

however in such a case the reader will see that $R = I$. With Eq. [7][8][9] we can obtain the values of $\theta, \hat{\mathbf{x}}$.

If we multiply $\mathbf{t} = (\mathrm{I} - \mathrm{R}_{\hat{\mathrm{x}}}(\theta))\mathbf{u} + \delta\hat{\mathbf{x}}$ with the screw axis ($\hat{\mathbf{x}}$) we get:

$$\mathbf{t} \cdot \hat{\mathbf{x}} = 0 + \delta\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} \tag{10}$$

$$\delta = \mathbf{t} \cdot \hat{\mathbf{x}} \tag{11}$$

since $\mathbf{u}$ is a generic point on the line and δ is a fixed value we can take $\mathbf{u}$ as the distance between the screw axis and the origin, i.e $\mathbf{u} \perp \hat{\mathbf{x}} \rightarrow$ the dot product is zero.

Now the values of $\theta, \hat{\mathbf{x}}$ and δ have been addressed, thus we are able to find the equation of the line $\mathbf{u}$ (it is a line since $(\mathrm{I} - \mathrm{R}_{\hat{\mathrm{x}}}(\theta))$ is not full rank).

Find the screw axis, the rotation angle, and the pitch of the following transformation:

$$\begin{pmatrix} \frac{2+\sqrt{3}}{4} & \frac{2-\sqrt{3}}{4} & \frac{1}{2\sqrt{2}} & \frac{-1}{6\sqrt{2}} \\ \frac{2-\sqrt{3}}{4} & \frac{2+\sqrt{3}}{4} & \frac{-1}{2\sqrt{2}} & \frac{5}{6\sqrt{2}} \\ \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & \frac{2-\sqrt{3}}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. $R - R^T = 2\sin\theta \cdot S(\hat{x})$ we get $\hat{x}$
2. $Tr(R) = 1 + 2\cos\theta$ we get θ
3. $\mathbf{t} \cdot \hat{x}$ we get the translation δ
4. $\frac{\delta}{\theta}$ and we get the pitch p
5. Substituting the values in $(I - R_{\hat{x}}(\theta))\mathbf{u} = \mathbf{t} - \delta\hat{x}$ we get the equation of the line $\mathbf{u}$

At the end of the nineteenth century Franz Reuleaux described what he called ”lower pairs”, these were pairs of surfaces which can move relative to each other while remaining in surface contact. He took these to be idealisations for the most basic of mechanical joints. To find these surfaces, consider the surfaces invariant under one-parameter (1-dimensional) subgroups:

- The pitch zero subgroups correspond to rotations about a line. A surface invariant under such a subgroup is just a surface of rotation
- Infinite pitch subgroups correspond to translations in a fixed direction, so any surface of translation is invariant under such a subgroup
- The subgroups with finite, non-zero pitch have helicoidal surfaces as invariants

LIE ALGEBRAS

Here we can think of the Lie algebra of a Lie group as the tangent space at the identity element ($\approx$ the Lie algebra of a Lie Group is a sort of linearised version of the group). To find the Lie algebra elements we take a curve in the group and find the derivative at the identity.

$$A(\theta) = \begin{pmatrix} R & \frac{\theta p}{2\pi}\hat{\chi} + (I - R)\mathbf{u} \\ 0 & 1 \end{pmatrix}_{4\times 4} \quad (12)$$

$$S = \frac{\partial A(\theta)}{\partial \theta}\Big|_{\theta=0} \quad (13)$$

$$\frac{\partial \mathrm{A}(\theta)}{\partial \theta}\Big|_{\theta=0} = \begin{pmatrix} \mathrm{S}(\omega)\mathrm{R} & \frac{\omega \mathrm{p}}{2\pi}\boldsymbol{\omega} + (\mathrm{S}(\omega)\mathrm{R})\mathbf{u} \\ 0 & 0 \end{pmatrix}\Big|_{\theta=0} \tag{14}$$

$$\mathrm{S} = \begin{pmatrix} \mathrm{S}(\omega)\mathrm{I} & \frac{\omega \mathrm{p}}{2\pi}\boldsymbol{\omega} + (\mathrm{S}(\omega)\mathrm{I})\mathbf{u} \\ 0 & 0 \end{pmatrix} \tag{15}$$

$$\mathrm{S} = \begin{pmatrix} \Omega & \frac{\omega \mathrm{p}}{2\pi}\boldsymbol{\omega} + \Omega\mathbf{u} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix}_{4\times 4} \tag{16}$$

The element S of the Lie Algebra is called **twist**. These matrices form a 6-dimensional vector space and it is often useful to write the elements of the space as six-dimensional vectors (3 parameters for Ω and 3 parameters for $\mathbf{v}$) using the **vee** operator:

$$\mathrm{s} = \mathrm{S}^{\vee} = \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}_{6\times 1} \tag{17}$$

We can represent (the group element is an abstract entity and we can represent it in many ways) group elements by matrices, or more generally linear transformations, the group product is modeled by matrix multiplication and inverses in the group by matrix inversion.

Any Lie group acts linearly on its Lie algebra and such a representation is called Adjoint:

$$S' = ASA^{-1} \tag{18}$$

where: $S', S \in se(3)$ and $A \in SE(3)$ (A is the 4x4 matrix representation of an element of the group $g \to A = A(g)$).

THE ADJOINT 2

For the 6-dimensional vector representation we can write:

$$s' = \mathrm{Ad}(g)s \tag{19}$$

$$\mathrm{Ad}(g) = \begin{pmatrix} R & 0 \\ TR & R \end{pmatrix}_{6\times 6} \tag{20}$$

$$T = t^{\wedge} = \begin{pmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{pmatrix}_{3\times 3} \quad \to T \in so(3) \tag{21}$$

This adjoint action of the group on its Lie algebra can be extended to an action of the Lie algebra on itself, Take S_2 to be an element of the Lie algebra, as a 4×4 matrix. Now suppose $A(\theta)$ is a one-parameter subgroups corresponding to a Lie algebra element S_1, that is $\left.\frac{\partial A(\theta)}{\partial \theta}\right|_{\theta=0} = S_1$.

Now if we differentiate the adjoint action of the group, AS_2A^{-1} and set $\theta = 0$ we get the Lie brackets operator (the product in the group turns into Lie bracket in the Lie algebra):

$$[S_1, S_2] = \frac{\partial(AS_2A^{-1})}{\partial\theta} = \frac{\partial A}{\partial\theta}S_2A^{-1} + A\frac{\partial S_2}{\partial\theta}A^{-1} + AS_2\frac{\partial A^{-1}}{\partial\theta} \tag{22}$$

the term in the middle goes to zero and the rest becomes (evaluated at $\theta = 0$):

$$[S_1, S_2] = S_1S_2 - S_2S_1 \tag{23}$$

And in terms of 6-dimensional vectors we will use the:

$$[\mathbf{s_1}, \mathbf{s_2}] = \mathrm{ad}(\mathbf{s_1})\mathbf{s_2} = \begin{pmatrix} \Omega_1 & 0 \\ V_1 & \Omega_1 \end{pmatrix}_{6\times 6} \begin{pmatrix} \omega_2 \\ \mathbf{v_2} \end{pmatrix}_{6\times 1} = \tag{24}$$

$$\begin{pmatrix} \Omega_1\omega_2 \\ V_1\omega_2 + \Omega_1\mathbf{v_2} \end{pmatrix} = \begin{pmatrix} \omega_1 \times \omega_2 \\ v_1 \times \omega_2 + \omega_1 \times \mathbf{v_2} \end{pmatrix} \tag{25}$$

THE EXPONENTIAL MAP

The mapping sends elements of Lie Algebras to the Lie Group, close to the origin the map is a homeomorphism (the group structure is preserved). In general the exponential of a matrix can be obtained using the following expansion:

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \dots \tag{26}$$

the series converges and it represents the element of the Lie Group corresponding to the X element of the Lie Algebra. Elements of the group generally don't commute, however if the element of the algebra is characterized by the same screw axis and the same pitch (the so called one-parameter subgroup of SE(3), where $A(\theta) = e^{\theta S}$) the exponentials can commute:

$$e^{\theta_1 S} e^{\theta_2 S} = e^{(\theta_1 + \theta_2) S} \tag{27}$$

Thus we can associate a screw to every single dof joint (prismatic and rotational)!

The vector representation of a twist is

$$\mathrm{s} = \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix} \tag{28}$$

with

$$\mathbf{v} = -\omega \times \mathbf{u} + \frac{\mathrm{p}}{2\pi}\omega = \mathbf{u} \times \omega + \frac{\mathrm{p}}{2\pi}\omega \tag{29}$$

and for rotational and prismatic joints it turns out to be different:

Rotational Joints

$$\begin{pmatrix} \omega \\ \mathbf{u} \times \omega \end{pmatrix} \tag{30}$$

Prismatic Joints

$$\begin{pmatrix} 0 \\ \hat{\mathbf{x}} \end{pmatrix} \tag{31}$$

Through calculations is it possible to prove that:

$$g = e^S = I + S + \frac{1}{|\omega|^2}(1 - \cos|\omega|)S^2 + \frac{1}{|\omega|^3}(|\omega| - \sin|\omega|)S^3 \qquad (32)$$

and for the adjoint representation:

$$\begin{aligned} \mathrm{Ad}(g) = e^{\mathrm{ad}(s)} = I &+ \frac{1}{2|\omega|}(3\sin|\omega| - |\omega|\cos|\omega|)\mathrm{ad}(s) + \\ &+ \frac{1}{2|\omega|^2}(4 - 4\cos|\omega| - |\omega|\sin|\omega|)\mathrm{ad}(s)^2 + \\ &+ \frac{1}{2|\omega|^3}(\sin|\omega| - |\omega|\cos|\omega|)\mathrm{ad}(s)^3 + \\ &+ \frac{1}{2|\omega|^4}(2 - 2\cos|\omega| - |\omega|\sin|\omega|)\mathrm{ad}(s)^4 \end{aligned}$$

DERIVATIVE OF EXPONENTIALS

Let's suppose we have a path in SE(3) as an exponential $g(t) = e^{X(t)}$, how can we find the time derivative and what does it represent? If the screw stays the same the calculations are easy (the elements of the Lie Algebra commute) but what if the screw changes (as soon as the time goes on the screw axis and the pitch change)?

$$\frac{dX}{dt} = \dot{X} \tag{33}$$

$$\frac{d}{dt}e^X = \dot{X} + \frac{1}{2!}(\dot{X}X + X\dot{X}) + \frac{1}{3!}(\dot{X}X^2 + X^2\dot{X} + X\dot{X}X) + \cdots \tag{34}$$

but $\dot{X}$ doesn't commute with X, thus Hausdorff introduced an useful notation:

$$\left(\frac{d}{dt}e^X\right)e^{-X} = \left(\dot{X} + \frac{1}{2!}(\dot{X}X + X\dot{X}) + \frac{1}{3!}(\dot{X}X^2 + X\dot{X}X + X^2\dot{X}) + \cdots\right)\cdot$$
$$\cdot\left(I - X - \frac{1}{2!}X^2 - \frac{1}{3!}X^3 + \cdots\right)$$

the product above can be expanded as:

$$\dot{X} - \dot{X}X + \frac{1}{2!}\dot{X}X^2 - \frac{1}{3!}\dot{X}X^3 + \frac{1}{2!}(\dot{X}X + X\dot{X}) - \frac{1}{2!}(\dot{X}X + X\dot{X})X +$$
$$+\frac{1}{2!2!}(\dot{X}X + X\dot{X})X^2 - \frac{1}{2!3!}(\dot{X}X + X\dot{X})X^3 + \frac{1}{3!}(\dot{X}X^2 + X\dot{X}X + X^2\dot{X}) + \cdots$$

considering just the terms of zero, first, and second order:

$$\begin{aligned}\left(\frac{d}{dt}e^X\right)e^{-X} &= \dot{X} + \frac{1}{2!}(\dot{X}X + X\dot{X} - 2\dot{X}X) + \frac{1}{3!}(3\dot{X}X^2 - 3\dot{X}X^2 + \\ &\quad -3X\dot{X}X + \dot{X}X^2 + X\dot{X}X + X^2\dot{X}) + \cdots = \\ &= \dot{X} + \frac{1}{2!}\left[X, \dot{X}\right] + \frac{1}{3!}\left[X, [X, \dot{X}]\right] + \cdots = X_d\end{aligned}$$

so finally we can express the derivative of the exponential map as:

$$\frac{d}{dt}e^X = X_d e^X$$

with $X_d \in se(3)$, $X \in se(3)$, and $e^X \in SE(3)$.

Thus we can rewrite as

$$\frac{d}{dt}e^X = X_d e^X \rightarrow X_d = \frac{d}{dt}e^X(e^X)^{-1} \tag{35}$$

$$S_d = \frac{dA}{dt}A^{-1} \tag{36}$$

At any time after the starting point of the motion we will have:

$$\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = A(t)\begin{pmatrix} \mathbf{p_0} \\ 1 \end{pmatrix} \tag{37}$$

and differentiating wrt time:

$$\begin{pmatrix} \dot{\mathbf{p}} \\ 0 \end{pmatrix} = \dot{A}A^{-1}\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} = S_d\begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix} \tag{38}$$

with S_d being an element of the Lie Algebra the velocity of the point $\dot{p}$ becomes:

$$\dot{\mathbf{p}} = \omega_{\mathbf{d}} \times \mathbf{p} + \mathbf{v_d} \tag{39}$$

ROBOTICS

Consider a point attached to the end effector of a serial manipulator, in the robot's home position the point has position vector p and every subsequent position will have the formulation:

$$\begin{pmatrix} \mathsf{p}' \\ 1 \end{pmatrix} = \mathsf{K}(\theta) \begin{pmatrix} \mathsf{p} \\ 1 \end{pmatrix} \tag{40}$$

If we are considering a 6 dof serial robot the velocity of the end effector point p depends on the joint variables and it is:

$$\begin{pmatrix} \dot{\mathsf{p}} \\ 0 \end{pmatrix} = \frac{\mathrm{d}\mathsf{K}(\theta)}{\mathrm{dt}} \begin{pmatrix} \mathsf{p} \\ 1 \end{pmatrix} = \frac{\mathrm{d}\theta}{\mathrm{dt}} \frac{\mathrm{d}\mathsf{K}(\theta)}{\mathrm{d}\theta} \begin{pmatrix} \mathsf{p} \\ 1 \end{pmatrix} = \dot{\theta}^{\mathsf{T}} \nabla \mathsf{K}(\theta) \begin{pmatrix} \mathsf{p} \\ 1 \end{pmatrix} \tag{41}$$

with:

$$\nabla \mathsf{K}(\theta) = \left[\frac{\partial \mathsf{K}}{\partial \theta_1} \ \frac{\partial \mathsf{K}}{\partial \theta_2} \ \cdots \ \frac{\partial \mathsf{K}}{\partial \theta_6} \right]^{\mathsf{T}} \tag{42}$$

$$\dot{\theta} = \left[\dot{\theta}_1 \ \dot{\theta}_2 \ \cdots \ \dot{\theta}_6 \right]^{\mathsf{T}} \tag{43}$$

and evaluating everything at the home position:

$$K(\theta) = e^{\theta_1 S_1} e^{\theta_2 S_2} \cdots e^{\theta_6 S_6} \qquad (44)$$

$$\left.\frac{\partial K}{\partial \theta_i}\right|_{\theta=0} = e^{\theta_1 S_1} e^{\theta_2 S_2} \cdots S_i e^{\theta_i S_i} \cdots e^{\theta_6 S_6} = S_i K(\theta)\Big|_{\theta=0} = S_i \qquad (45)$$

and finally we get the point velocity (still in the home position):

$$\begin{pmatrix} \dot{p} \\ 0 \end{pmatrix} = \left(\dot{\theta}_1 S_1 + \dot{\theta}_2 S_2 + \cdots + \dot{\theta}_6 S_6\right) \begin{pmatrix} p \\ 1 \end{pmatrix} \qquad (46)$$

If we don't want to use the representation wrt the home position:

$$\mathbf{s}_i = e^{\theta_1 \mathrm{ad}(\mathbf{s}_1^0)} e^{\theta_2 \mathrm{ad}(\mathbf{s}_2^0)} \cdots e^{\theta_{i-1} \mathrm{ad}(\mathbf{s}_{i-1}^0)} \mathbf{s}_i^0 \qquad (47)$$

$$\frac{\partial K}{\partial \theta_i} = S_i K(\theta) \qquad (48)$$

and here is the velocity screw of the robot's end effector:

$$\begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}_{6\times1} = \mathbf{s_1}\dot{\theta}_1 + \mathbf{s_2}\dot{\theta}_2 + \cdots + \mathbf{s_6}\dot{\theta}_6 \quad (49)$$

$$\begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}_{6\times1} = \mathrm{J}\dot{\theta} \quad (50)$$

$$\mathrm{J} = (\mathbf{s}_1|\mathbf{s}_2|\cdots|\mathbf{s}_6|)_{6\times6} \quad (51)$$

with J the screw Jacobian of the manipulator.

Questions?