Local Kinematic Analysis of Lower Pair Linkages

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Some terms…

- Finite DOF, full cycle mobility
- Instantaneous DOF
- Overconstrained mechanisms
- Paradoxical mechanisms
- Mechanism Geometry
- Screw systems
- Polynomial systems, algebraic geometry

- Kinematic topology
- Classes/Families of mechanisms
- Chebychev-Kutzbach-Grübler (CKG) formula
- Planar/spherical/spatial mechanisms

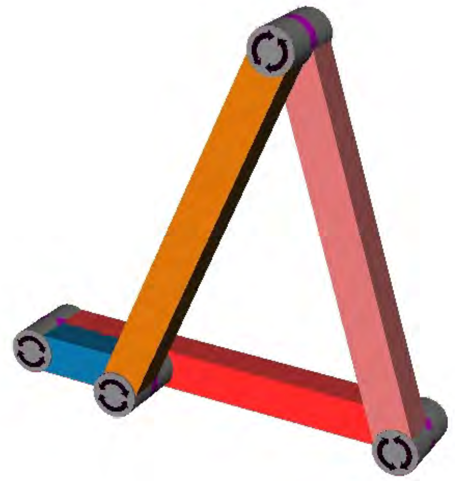
- Generic DOF
- Stable mappings
- Combinatorial algorithms
- Pebble game

1. Phenomenology
   - Mobility
   - Constraint-, C-Space-, Input-, Output-Singularities
   - A Model for the Mechanism Kinematics

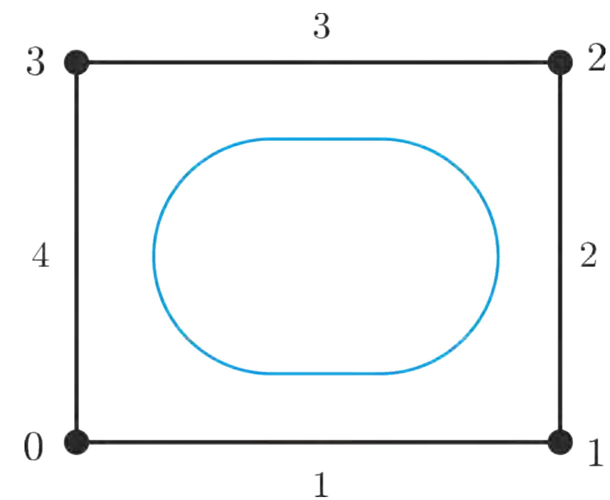
2. Kinematics Modeling
   - Screws and the Product-of-Exponentials
   - Geometric Jacobian
   - Higher-Order Kinematics

3. Local analysis of C-Space and ‘singular set’
   - Tangent space, Tangent cone
   - ‘Kinematic tangent cone’

4. Further Topics
   - Singularities of Non-Holonomic Systems
   - Escapement from singularities
   - Combinatorial algorithm: Pebble Game



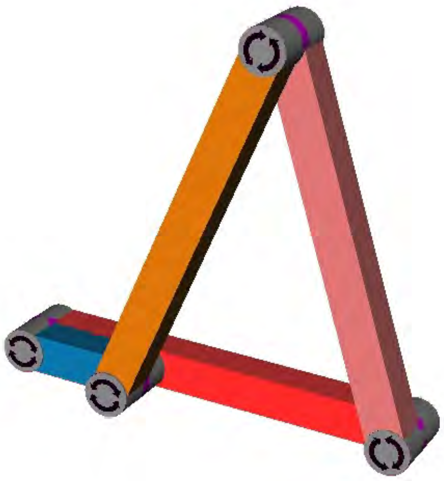
- One kinematic loop (topology)
- Parallel joint axes (geometry)
- Rigid links (physical properties)



Loop constraints: $h(\mathbf{q}) = \mathbf{0}$

Coordinates $\mathbf{q}$:

- Joint variables (angles, translations)
- ‘Absolute coordinates’



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- Parallel joint axes (geometry)
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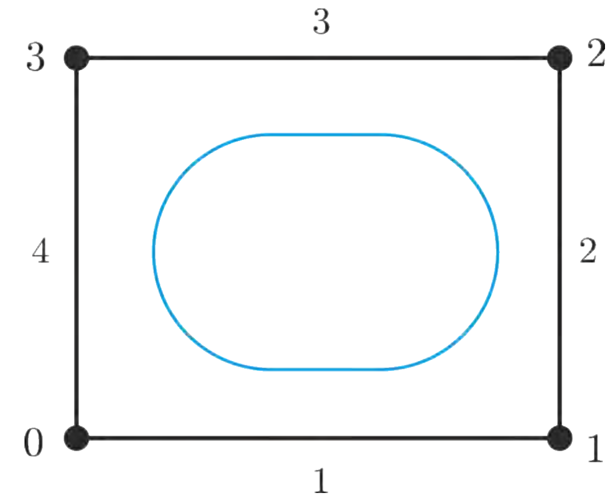
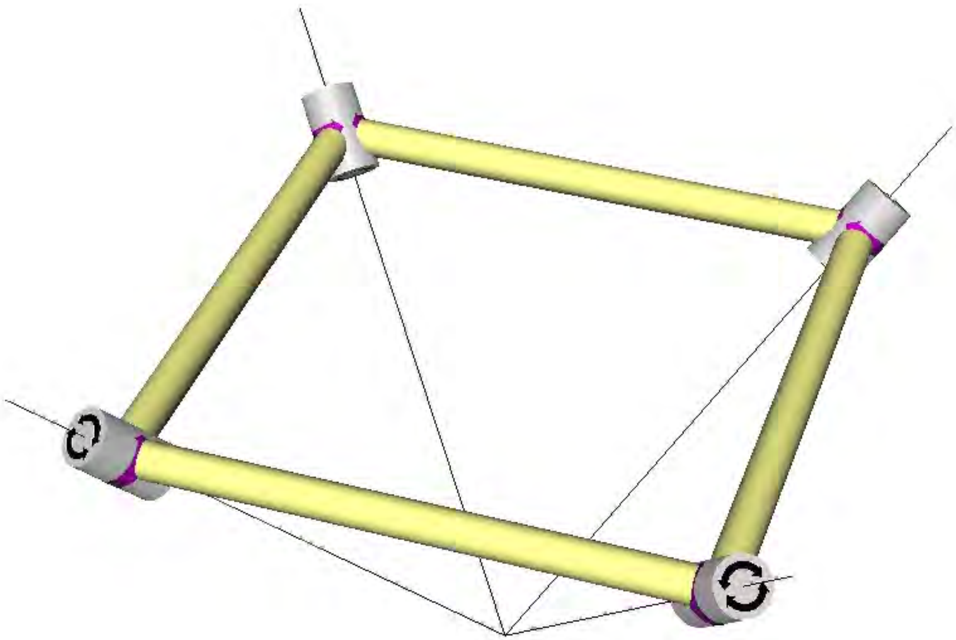
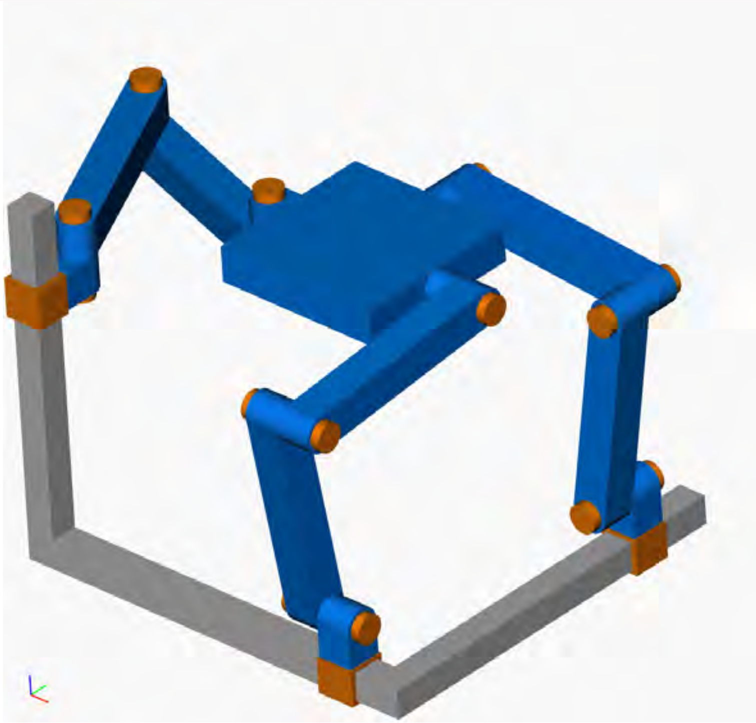
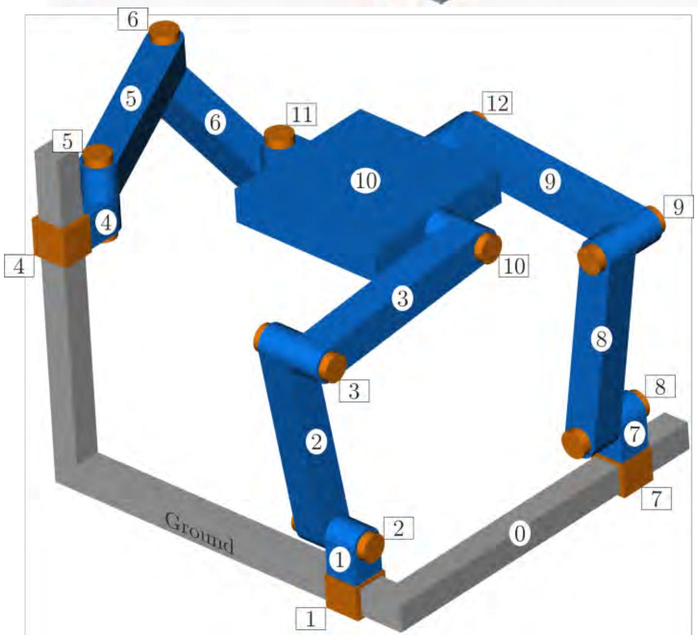


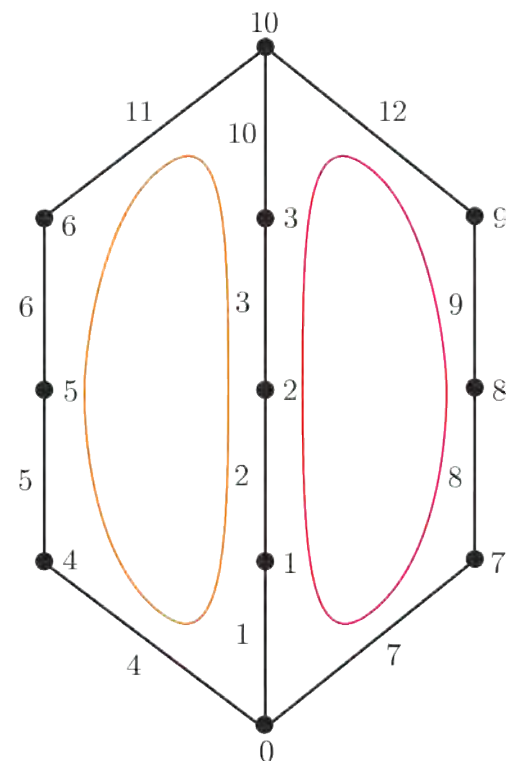


- One kinematic loop (topology)
- Axes intersect at one point (geometry)
- Rigid links (physical properties)

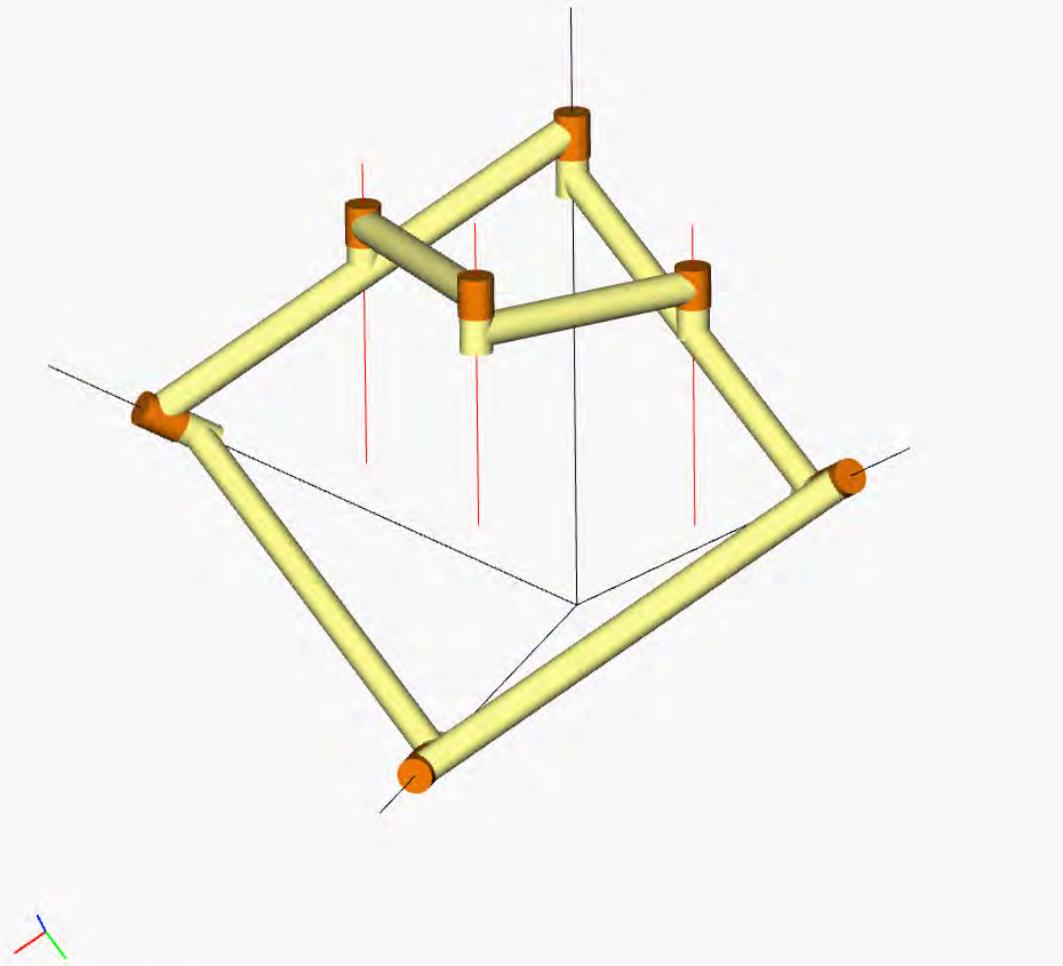




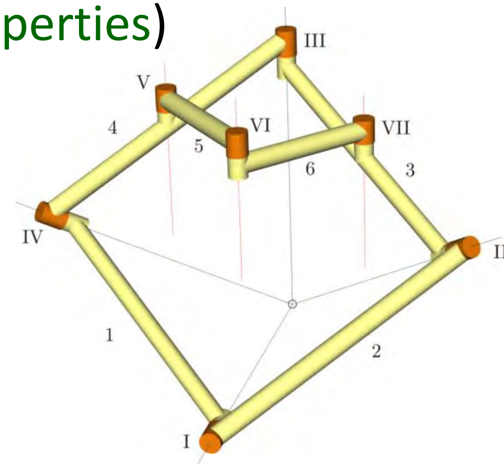
- Multiloop kinematics (topology)
- Intersection of planar motions (geometry)
- Rigid links (physical properties)

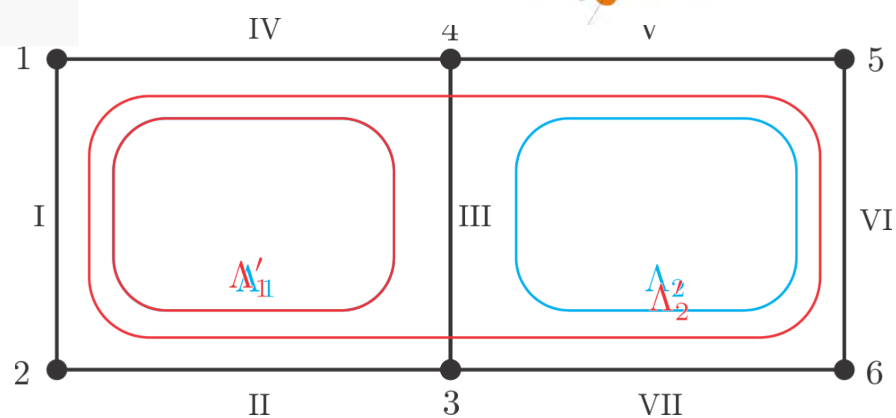


Phenomenology



- Multiloop kinematics (topology)
- Spherical + planar motion (geometry)
- Rigid links (physical properties)





What are Singularities?

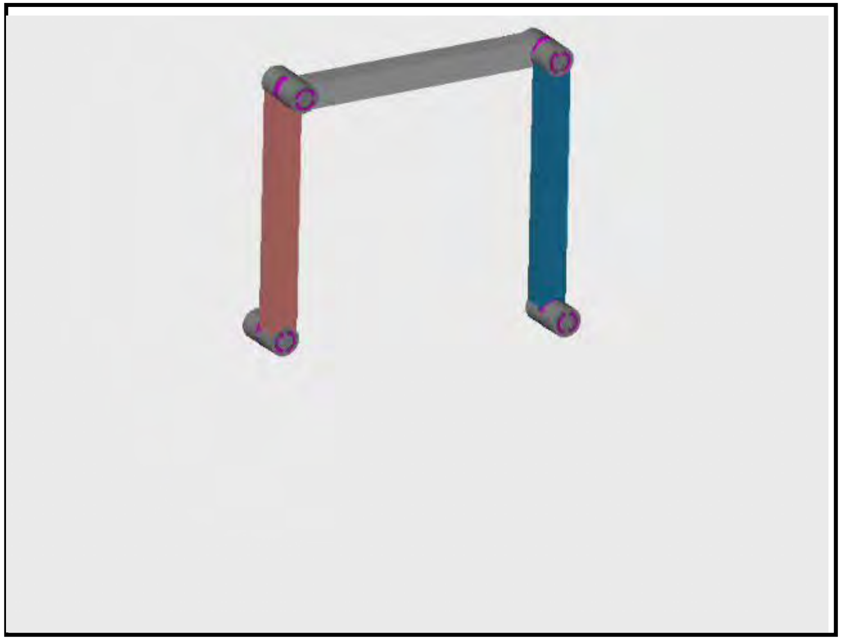
- Singularity = ‘Critical situation’

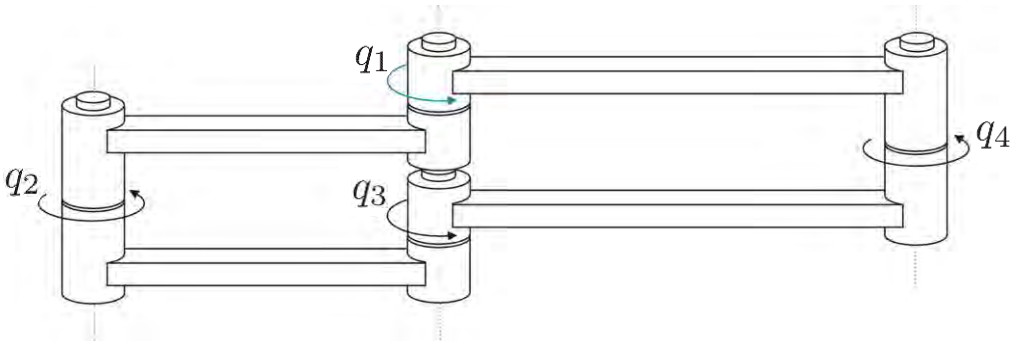
Is this a critical situation?

- Joint variables: $\mathbf{q} \in \mathbb{V}^n$
- Constraints: $h(\mathbf{q}) = \mathbf{0}$

$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$$

- Configuration space: $V := \{\mathbf{q} \in \mathbb{V}^n | h(\mathbf{q}) = \mathbf{0}\}$ (Parameter space!)
- Differential DOF: $\delta_{\mathrm{diff}}(\mathbf{q}) := n - \operatorname{rank} \mathbf{J}(\mathbf{q})$ (Instantaneous DOF)
- Local (finite) DOF: $\delta_{\mathrm{loc}}(\mathbf{q}) := \dim_{\mathbf{q}} V$





$$\mathbf{q} = (q_1, q_2, q_3, q_4)^T \in \mathbb{V}^4, \ \ n = 4$$

Some examples: 4-bar linkage

- Joint variables: $\mathbf{q} \in \mathbb{V}^n$
- Constraints: $h(\mathbf{q}) = \mathbf{0}$

$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$$

- Configuration space: $V := \{\mathbf{q} \in \mathbb{V}^n | h(\mathbf{q}) = \mathbf{0}\}$
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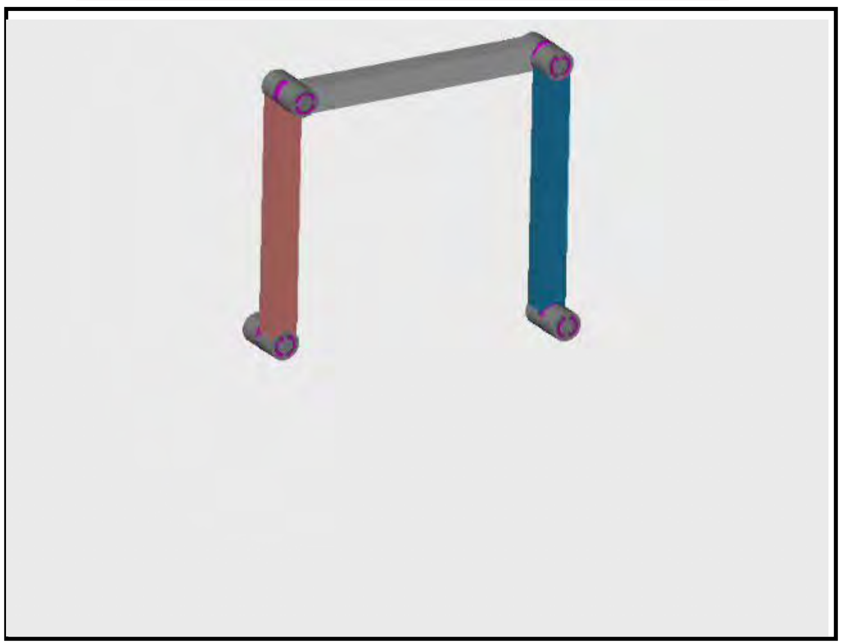
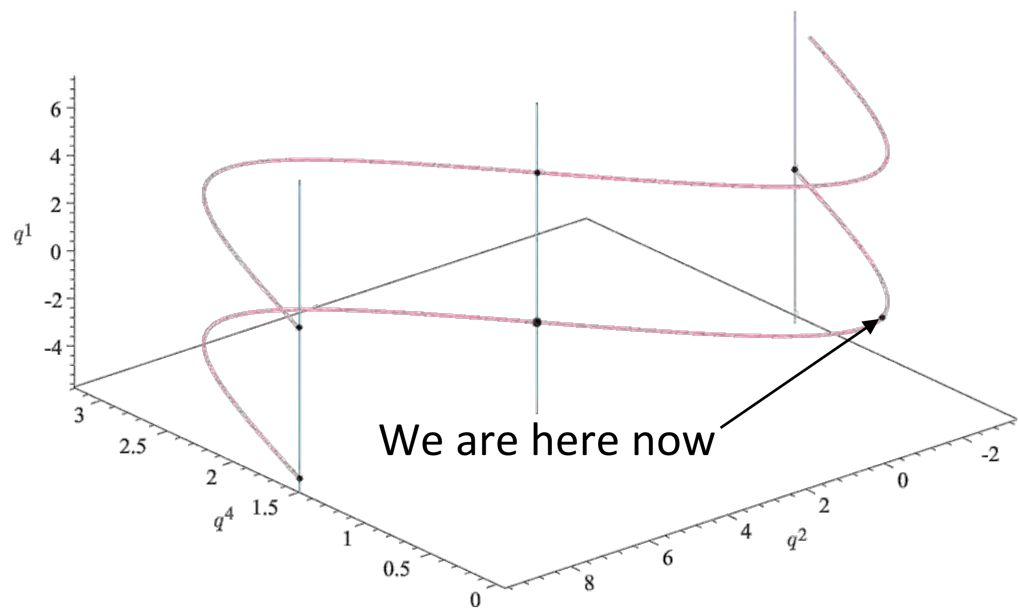
Rank drops at bifurcation point



$\delta_{\mathrm{diff}} = 1, 2$ $\delta_{\mathrm{loc}} = 1$

C-space is non-smooth at bifurcation point



- Joint variables: $\mathbf{q} \in \mathbb{V}^n$
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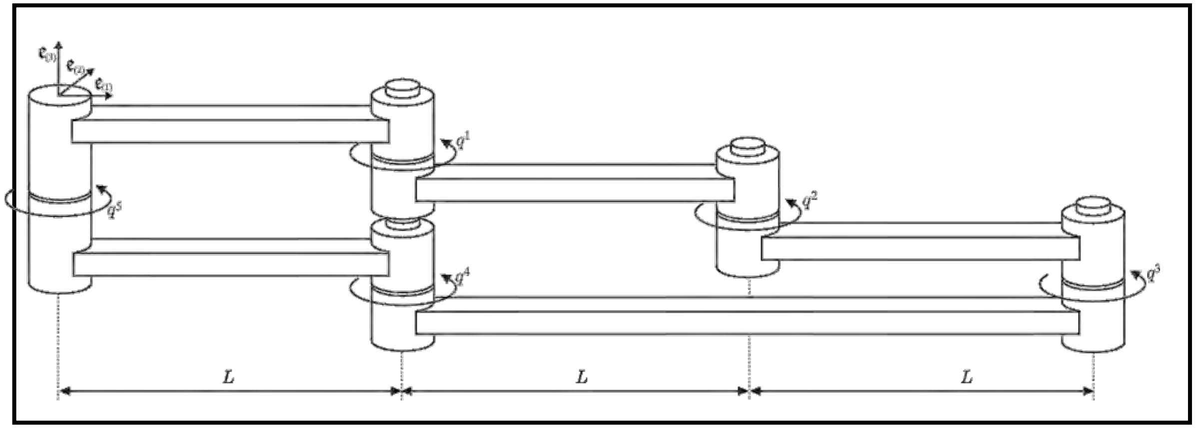
Rank drops at bifurcation point



$$\mathbf{q} = (q_1, q_2, q_3, q_4, q_5) \in \mathbb{V}^5$$
$$n = 5$$

$\delta_{\text{diff}} = 2, 3$ $\delta_{\text{loc}} = 2$

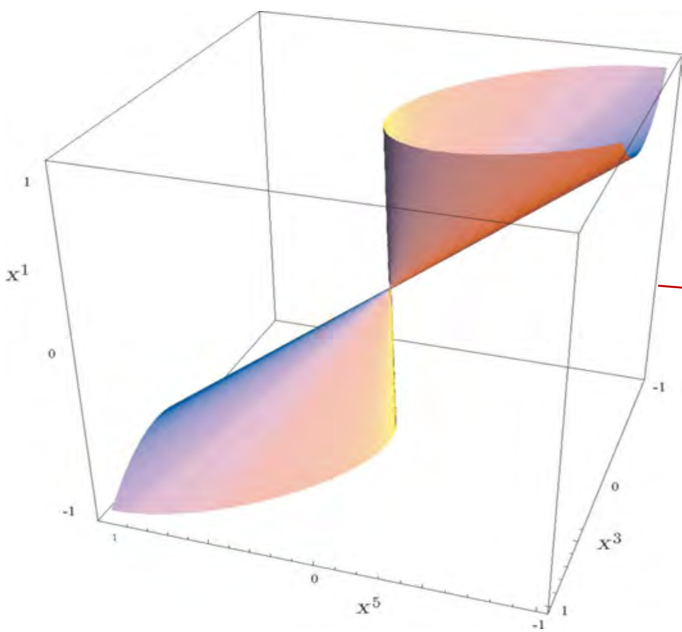
Some examples: 5-bar linkage

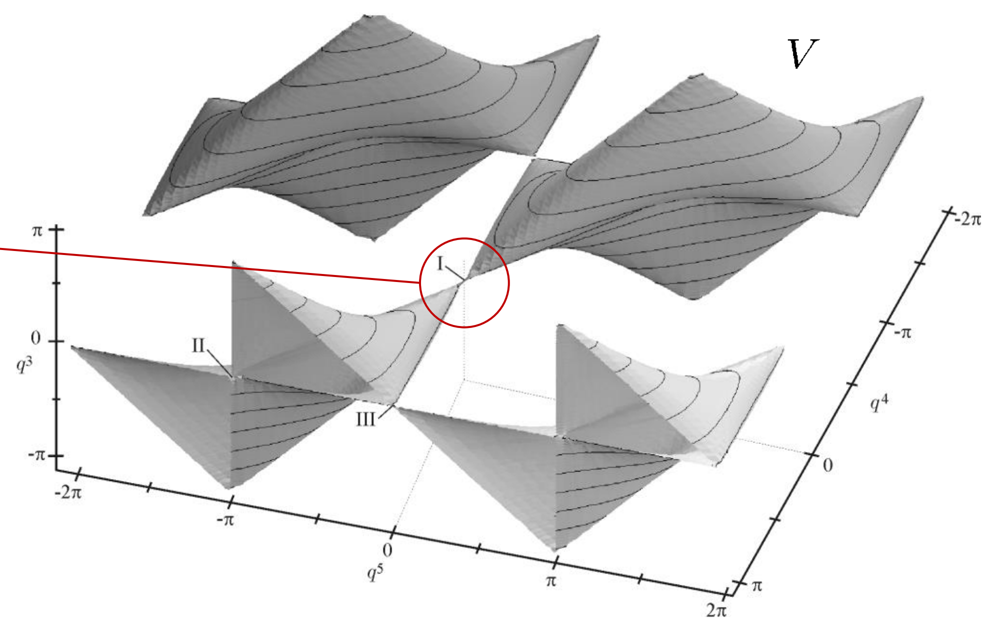
- Joint variables: $\mathbf{q} \in \mathbb{V}^n$
- Constraints: $h(\mathbf{q}) = \mathbf{0}$

$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$$

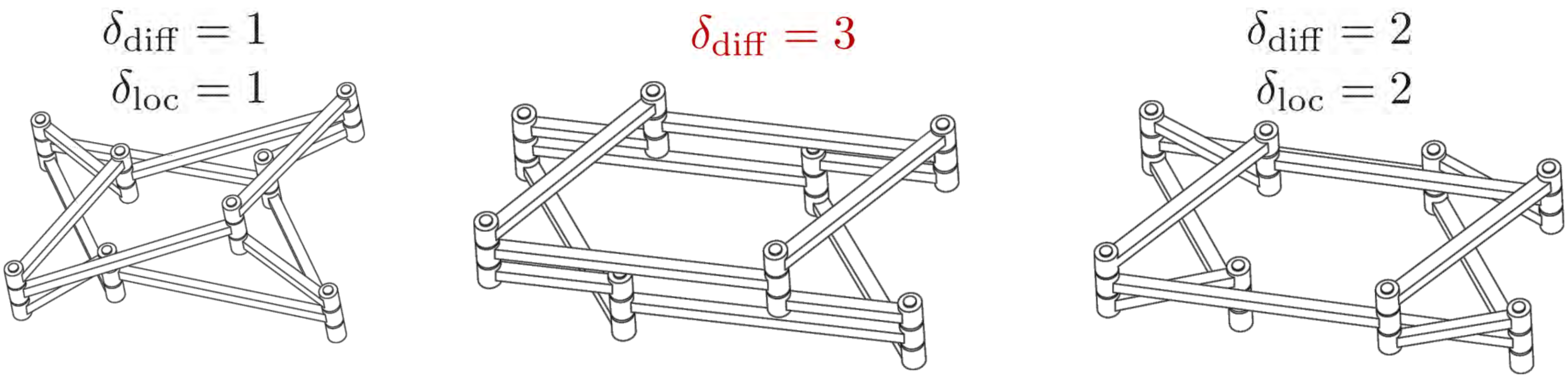
- Configuration space: $V := \{\mathbf{q} \in \mathbb{V}^n | h(\mathbf{q}) = \mathbf{0}\}$
  (Parameter space!)
- Differential DOF: $\delta_{\text{diff}}(\mathbf{q}) := n - \operatorname{rank} \mathbf{J}(\mathbf{q})$
  (Instantaneous DOF)
- Local (finite) DOF: $\delta_{\text{loc}}(\mathbf{q}) := \dim_{\mathbf{q}} V$

C-space is non-smooth at bifurcation point

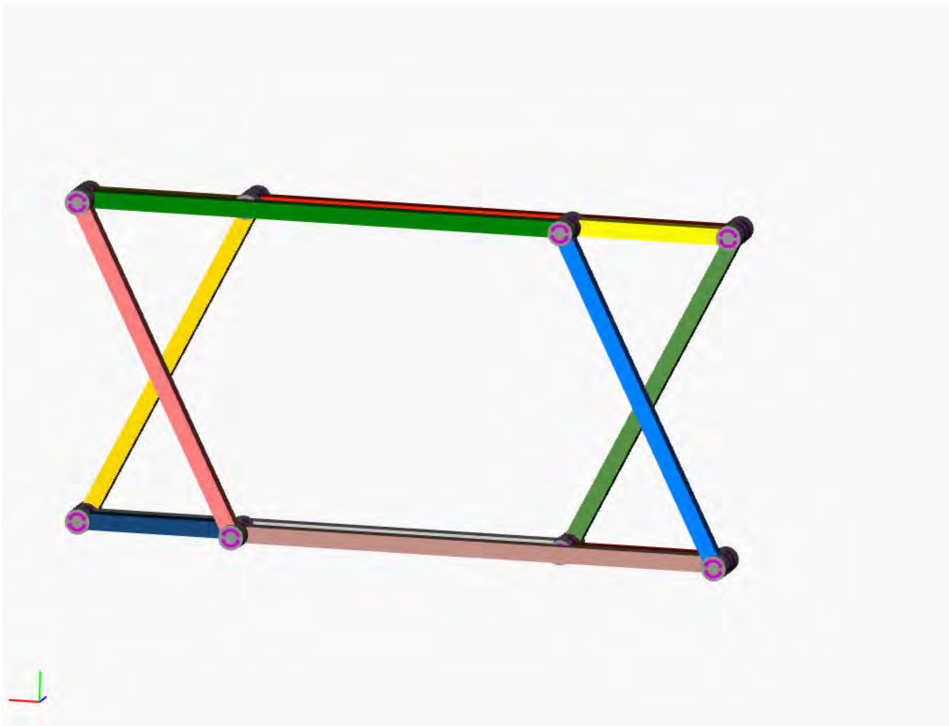


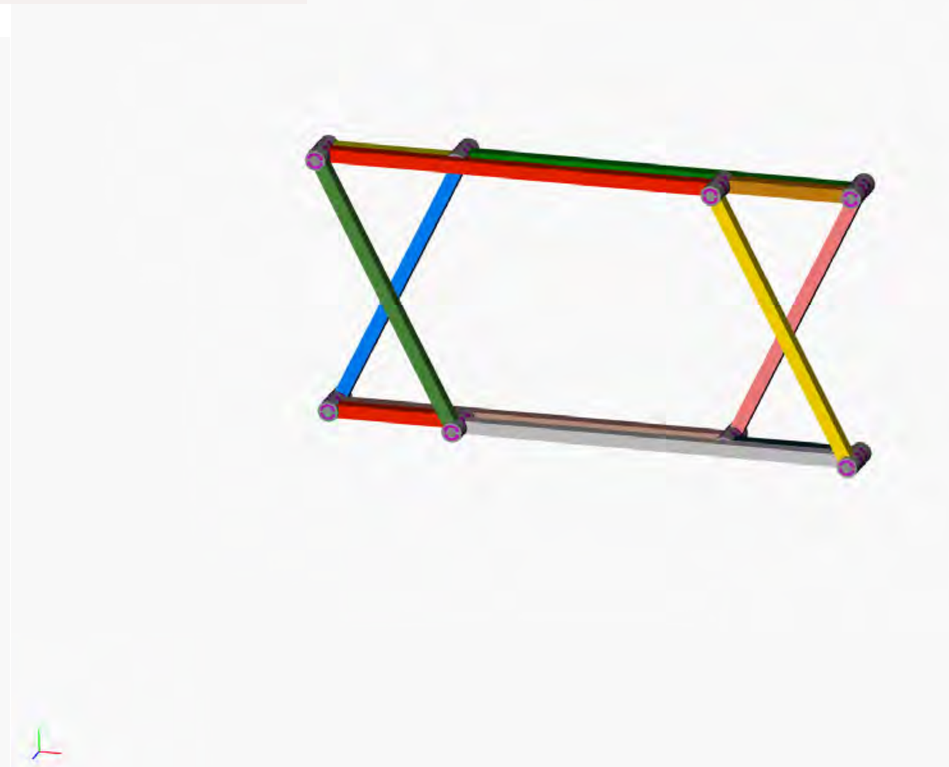


Some examples: Wunderlich's kinematotropic linkage



Rank drops at bifurcation point





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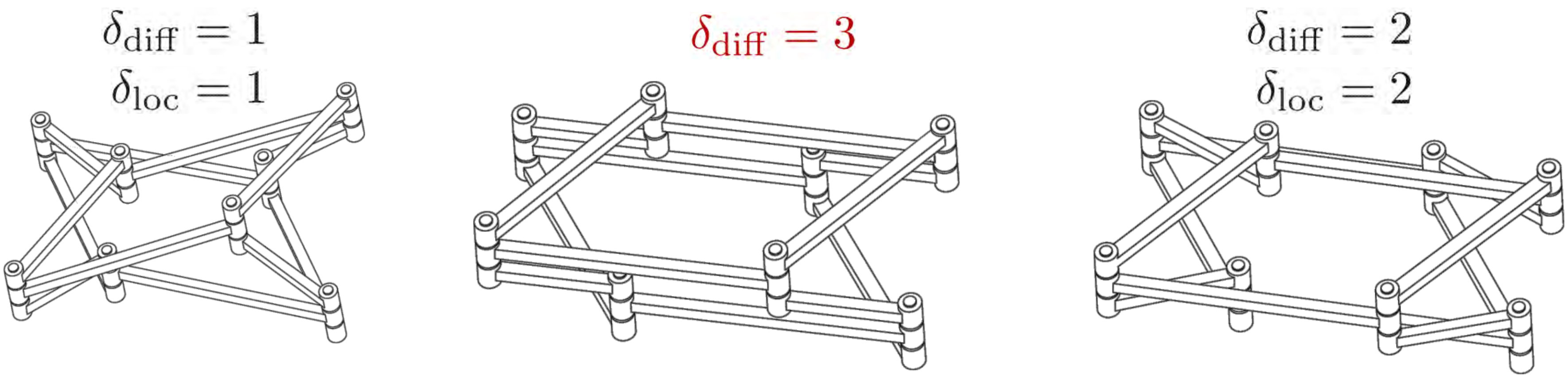
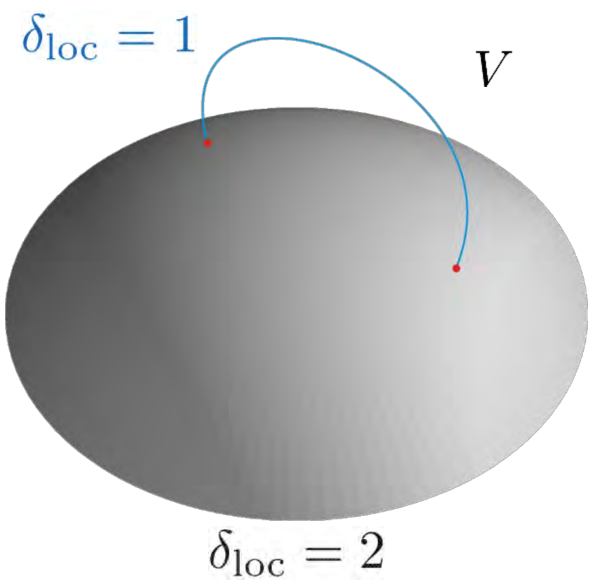


Rank drops at bifurcation point



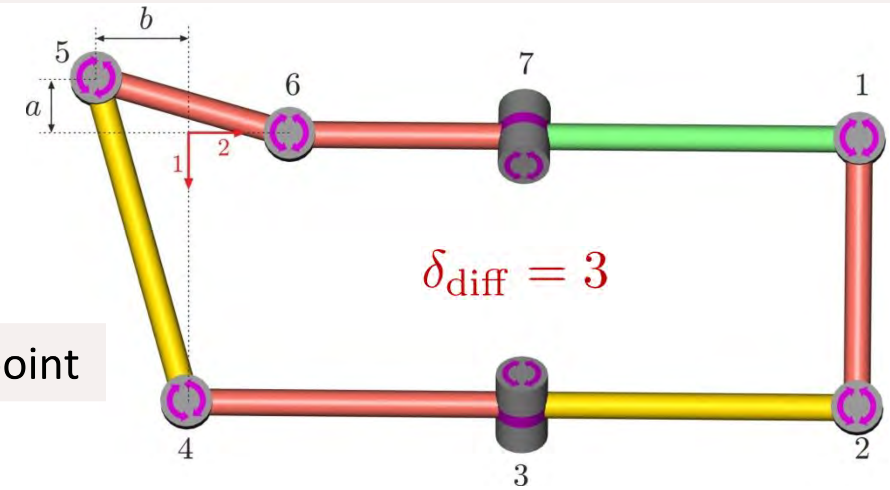
C-space is non-smooth at bifurcation point

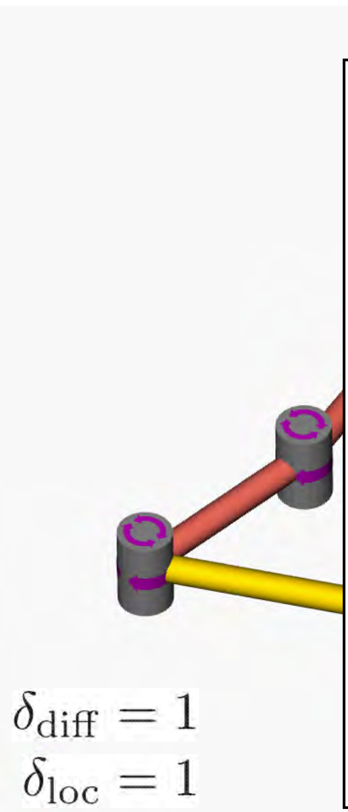
The local picture near $\mathbf{q}$

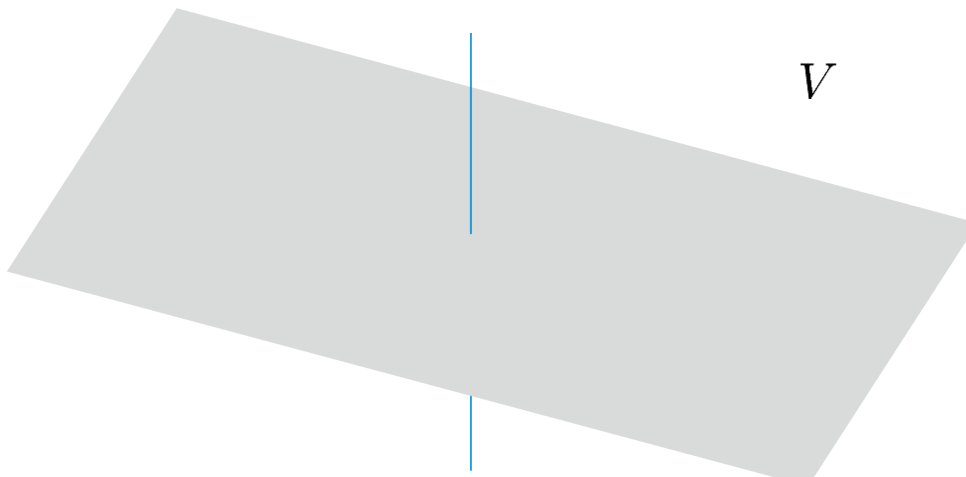
X. Kong, M. Pfurner,

Mech. Mach. Theory, 2015

Rank drops at bifurcation point







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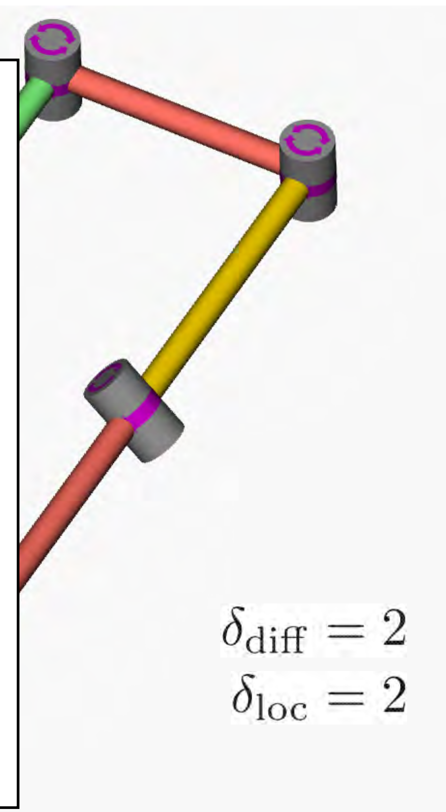
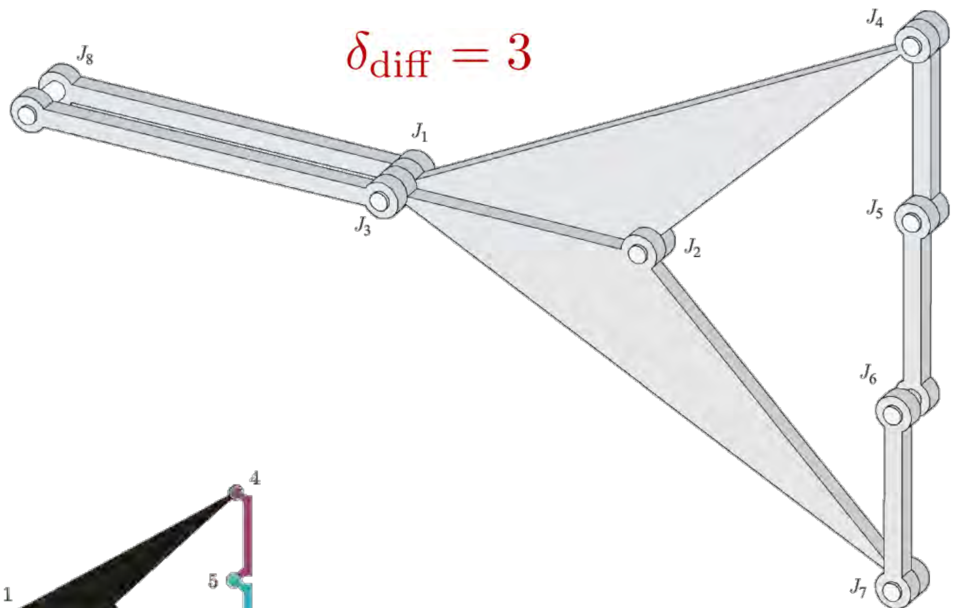
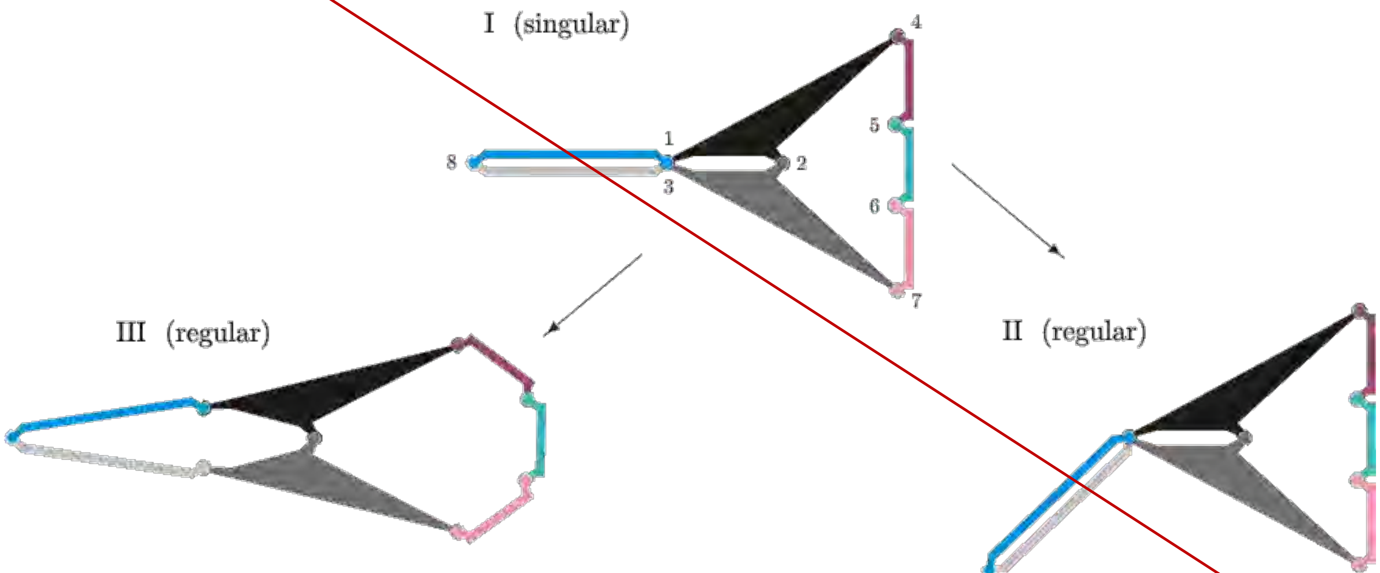


Rank drops at bifurcation point and stays constant

$$\delta_{\text{diff}}(\mathbf{q}) = n - \operatorname{rank} \mathbf{J}(\mathbf{q})$$
$$= \dim \ker \mathbf{J}(\mathbf{q})$$

$\delta_{\text{diff}} = 3$



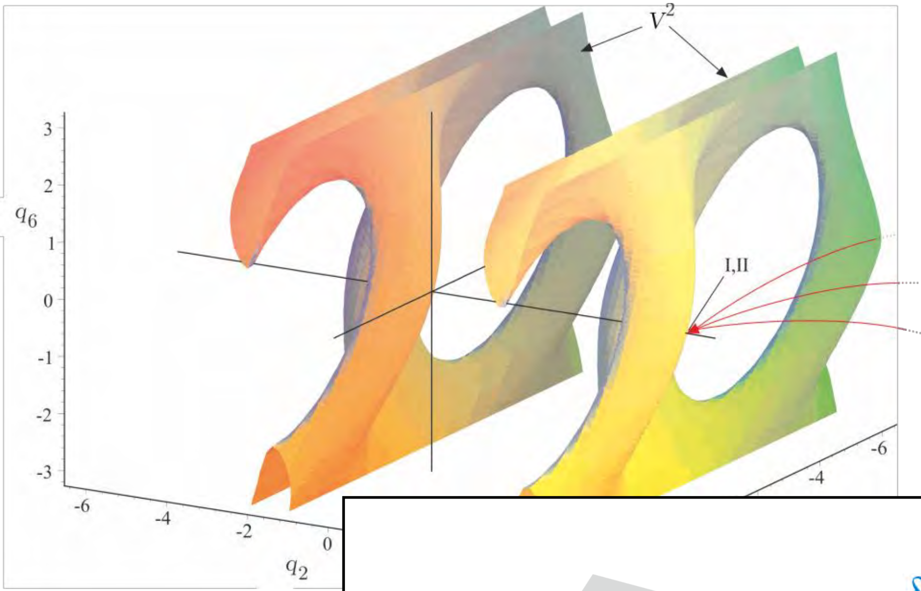


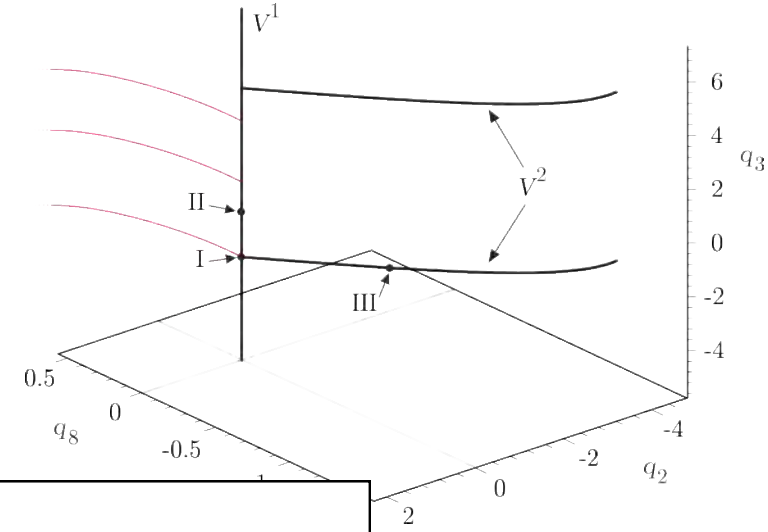
$\delta_{\text{diff}} = 2$

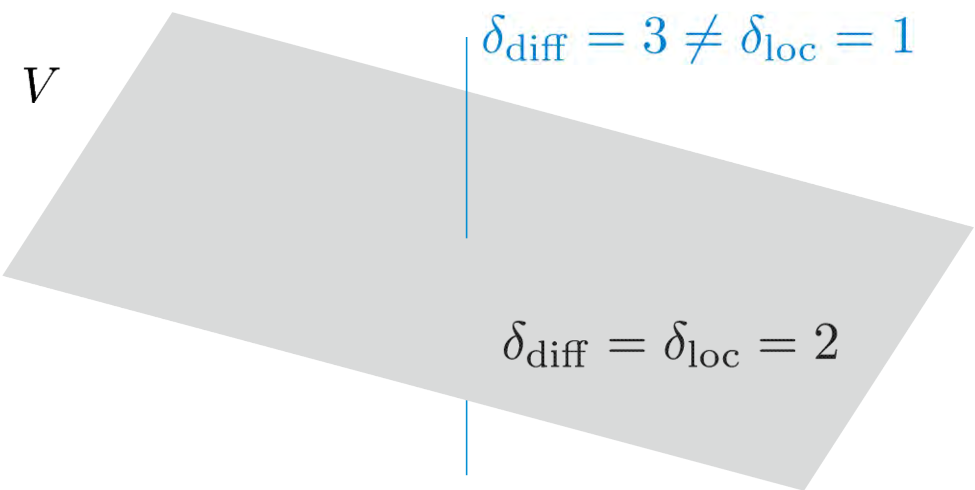
$\delta_{\text{loc}} = 2$ $\quad T_{\mathbf{q}}V = \ker \mathbf{J}(\mathbf{q})$

$\delta_{\text{diff}} = 3$

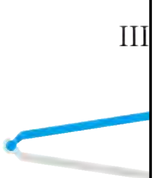
$\delta_{\text{loc}} = 1$ $\quad T_{\mathbf{q}}V \neq \ker \mathbf{J}(\mathbf{q})$





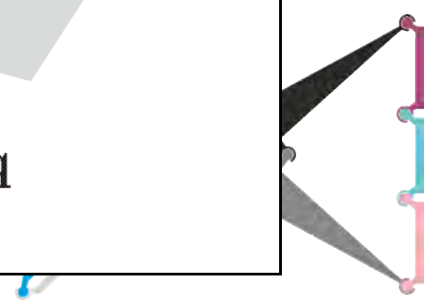


The local picture near $\mathbf{q}$



$\delta_{\text{diff}} = 2$

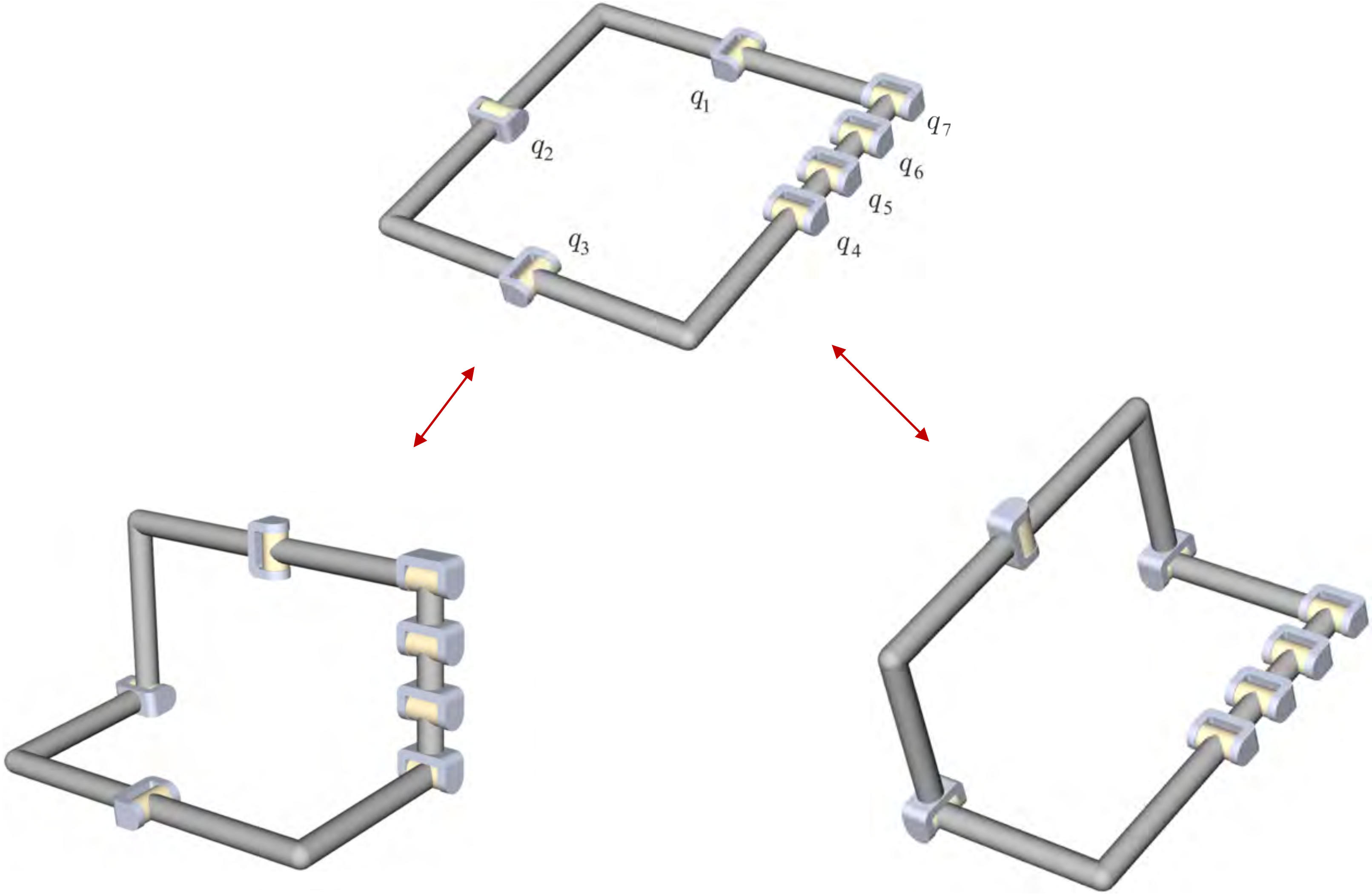
$\delta_{\text{loc}} = 2$



$\delta_{\text{diff}} = 3$

$\delta_{\text{loc}} = 1$

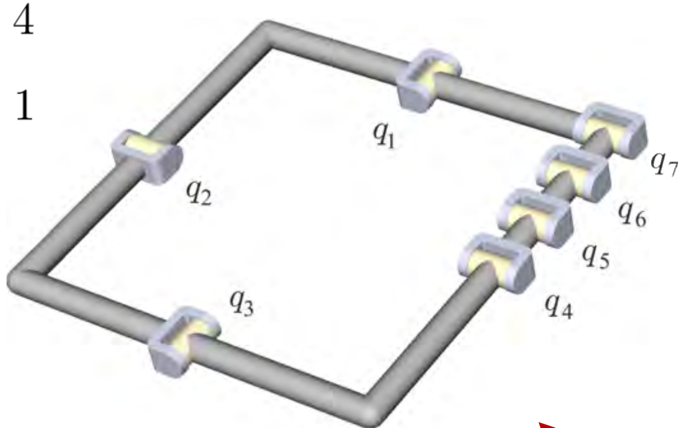
Some examples: Underconstrained linkage



Some examples: Underconstrained (shaky) linkage

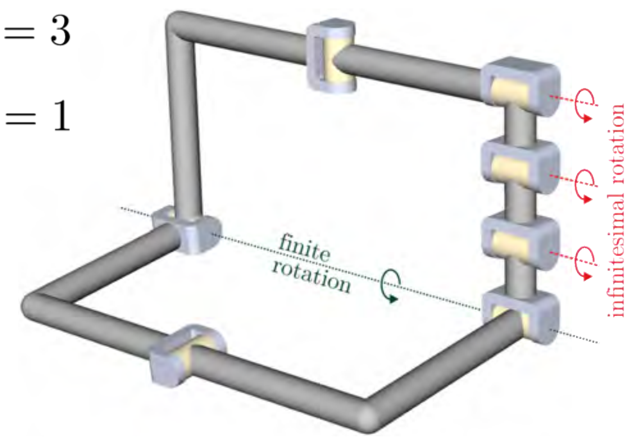
$$\delta_{\text{diff}} = 4$$

$$\delta_{\text{loc}} = 1$$



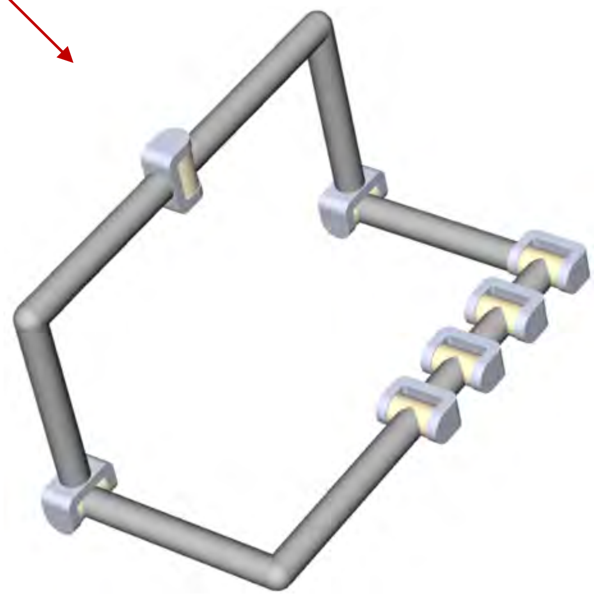
$$\delta_{\text{diff}} = 3$$

$$\delta_{\text{loc}} = 1$$



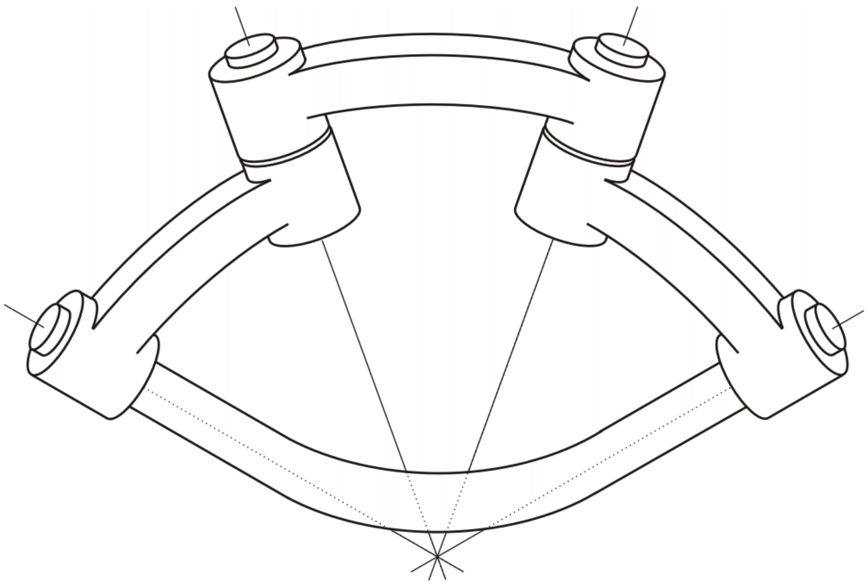
$$\delta_{\text{diff}} = 3$$

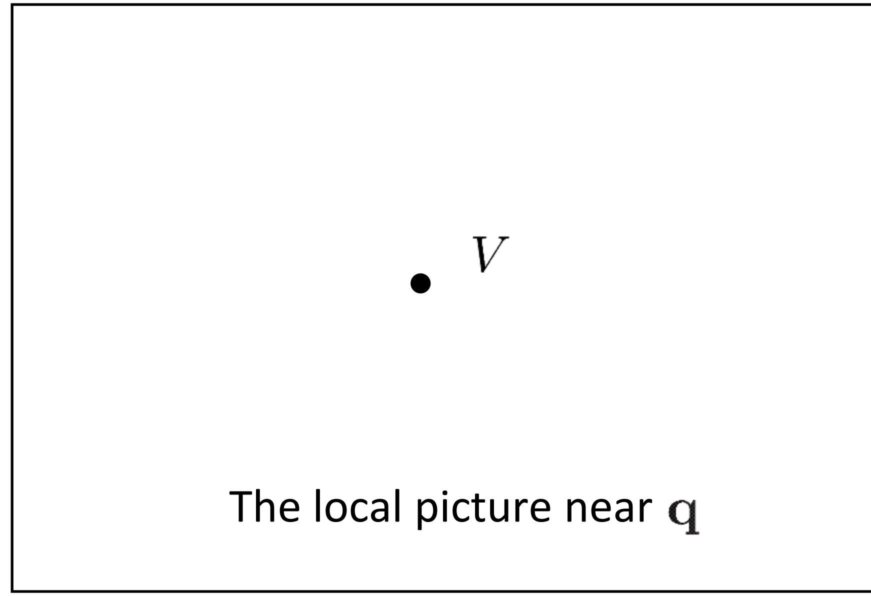
$$\delta_{\text{loc}} = 1$$



Some examples: Immobile & shaky 4R linkages

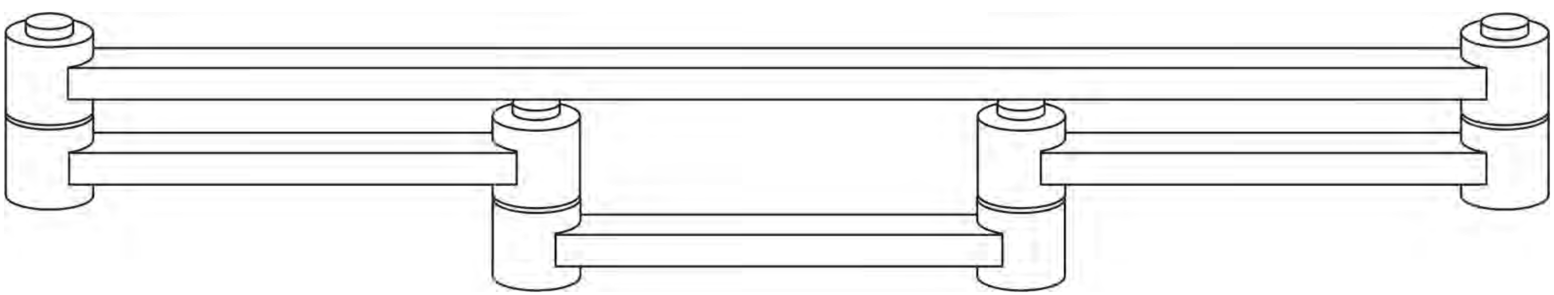
All joint axes are in a common plane



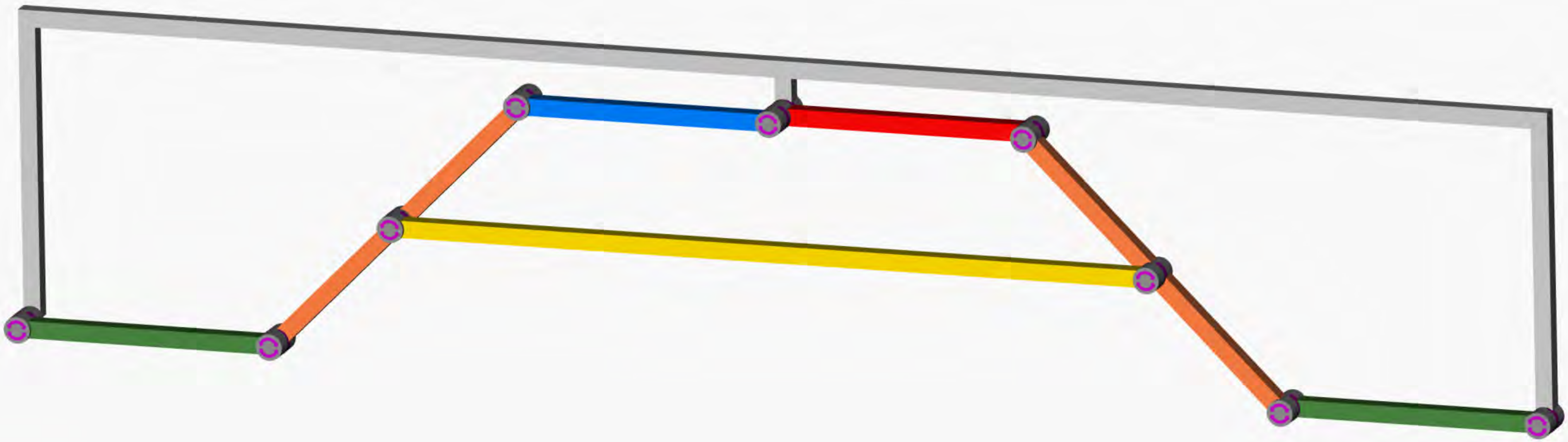


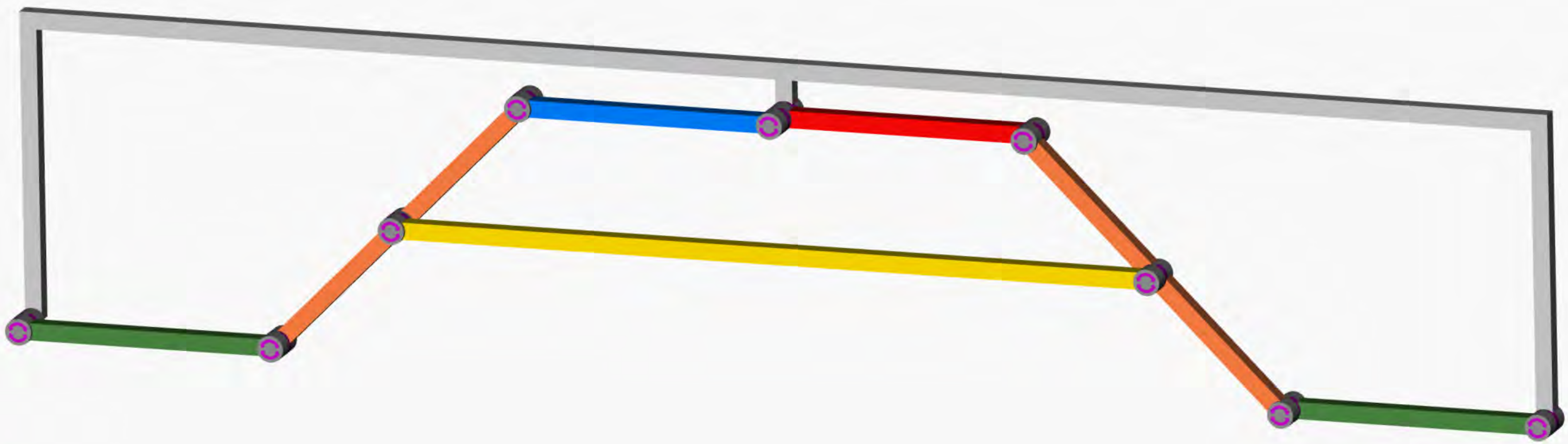
$$\delta_{\text{diff}} = 2$$
$$\delta_{\text{loc}} = 0$$

'Shaky' linkages



Some examples: ‚Double Watt‘ Linkage





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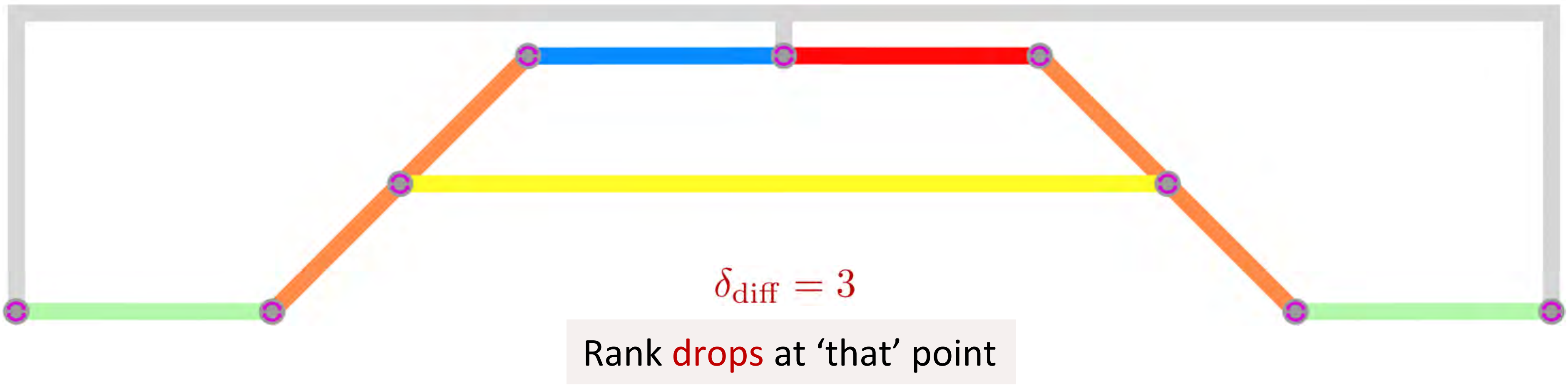
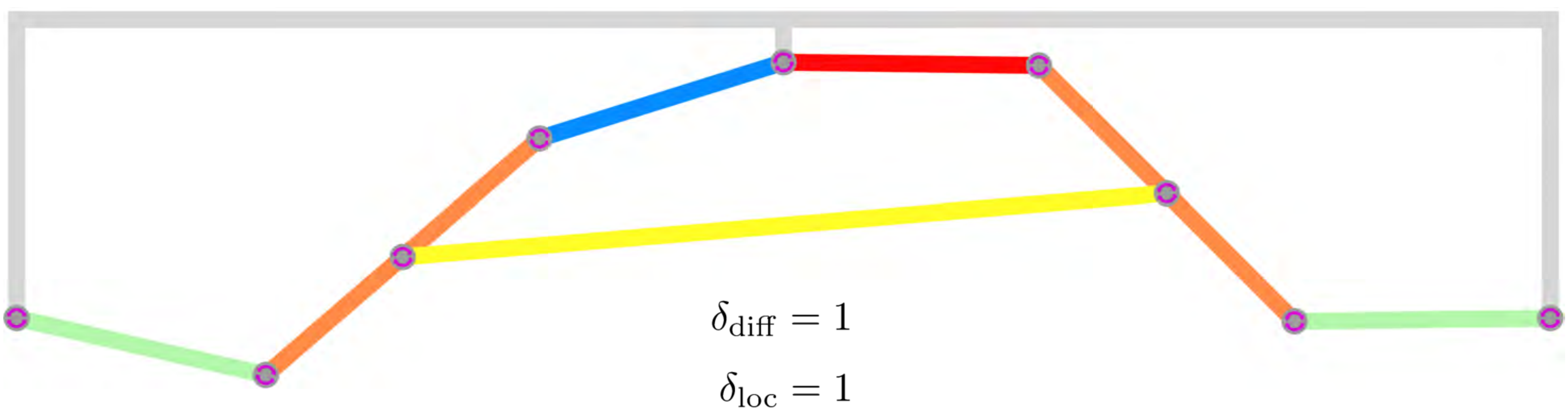
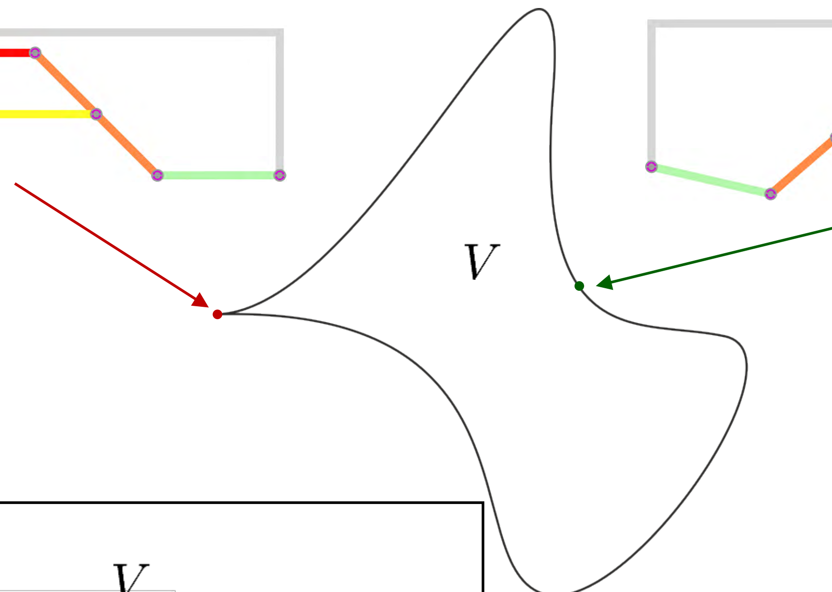
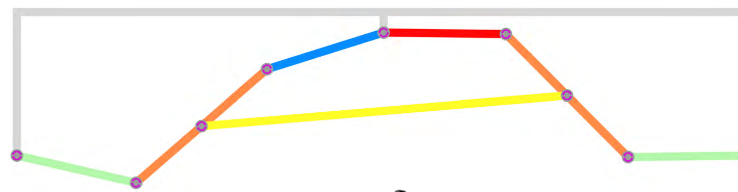




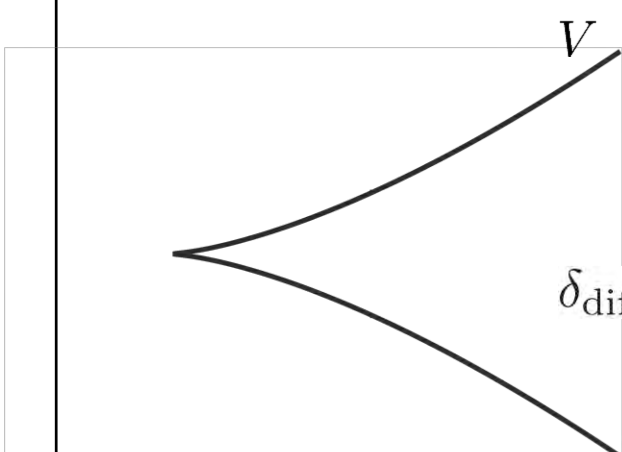
$\delta_{\mathrm{diff}} = 3$





$\delta_{\mathrm{diff}} = 1$

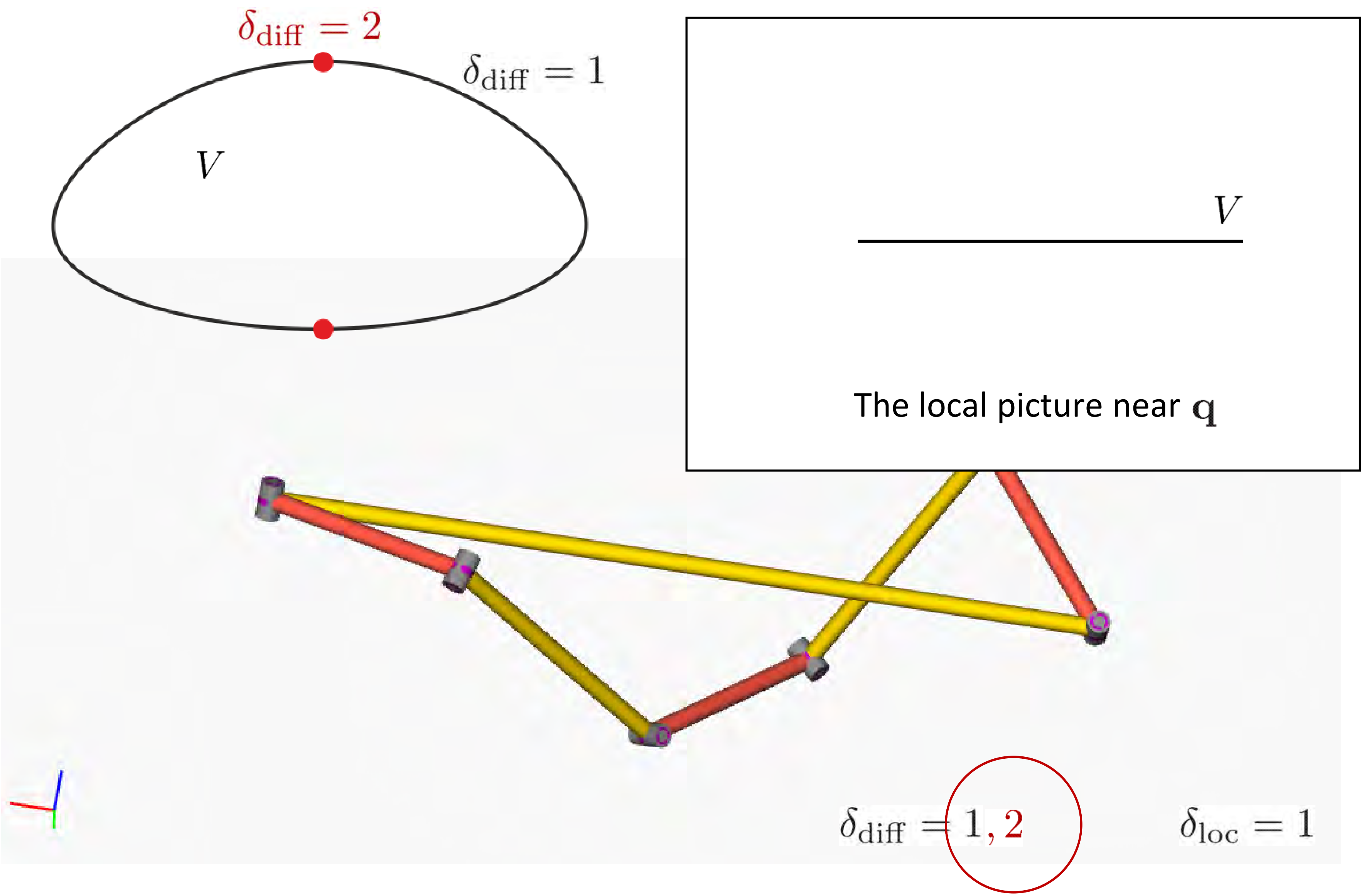
Cusp singularity:



$\delta_{\mathrm{diff}} = \delta_{\mathrm{loc}} = 1$

The local picture near $\mathbf{q}$

Some examples: Goldberg 6R linkage



Configuration space is a smooth 1-dim manifold

Increased instantaneous mobility but c-space is smooth

What are Singularities?

- Singularity = ‘Critical situation’

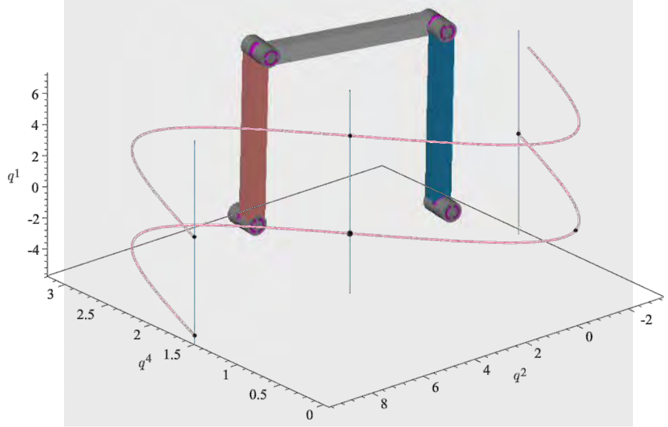
Is this a critical situation?

Definition 1:

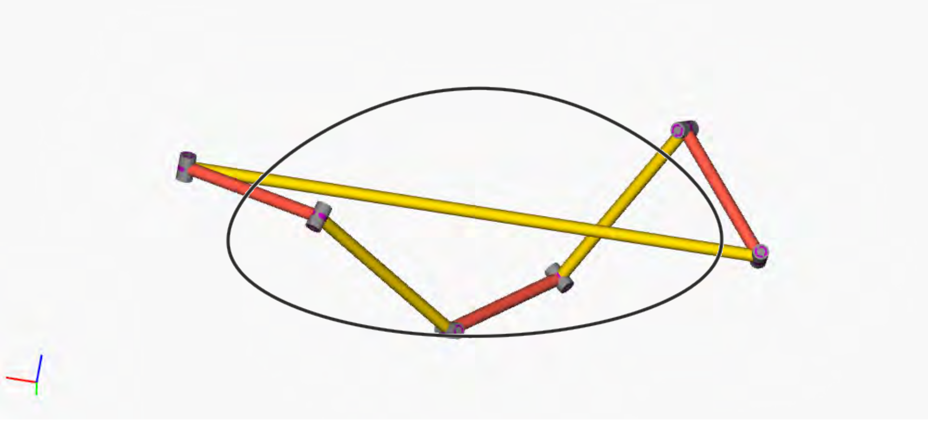
A kinematic singularity is a configuration $\mathbf{q} \in V$ where the differential (instantaneous) DOF is not constant in any neighborhood $U(\mathbf{q})$ of $\mathbf{q}$.

Simply: $\mathbf{q} \in V$ is singular if $\operatorname{rank} \mathbf{J}$ changes at $\mathbf{q} \in V$

V has bifurcation points



V is smooth



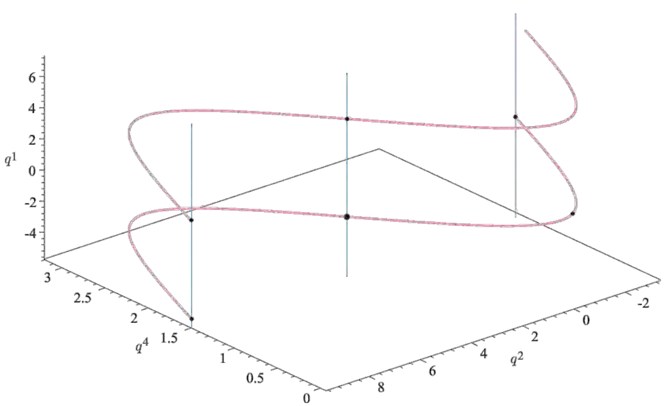
What are Singularities?

Definition 2:

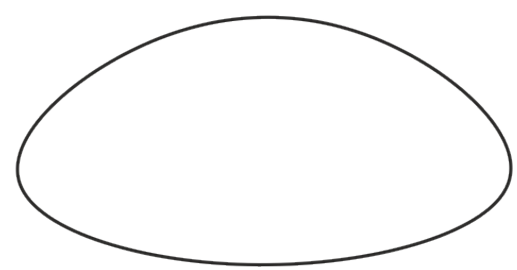
A C-space singularity is a configuration $\mathbf{q} \in V$ where V is not a smooth manifold.

- This is a topological property of V.
- C-space singularity → kinematic singularity

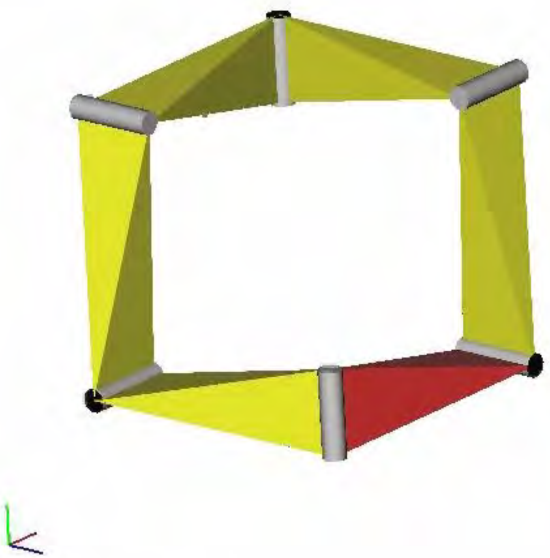
<u>V has bifurcation points</u>

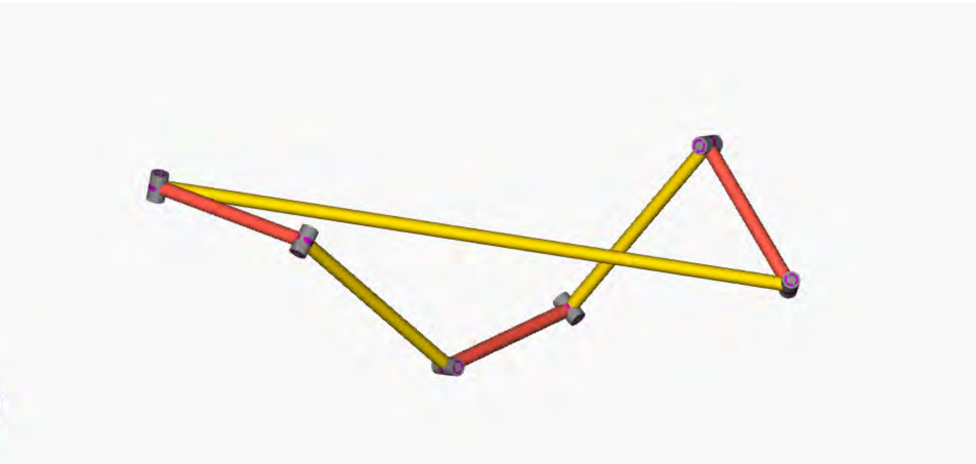


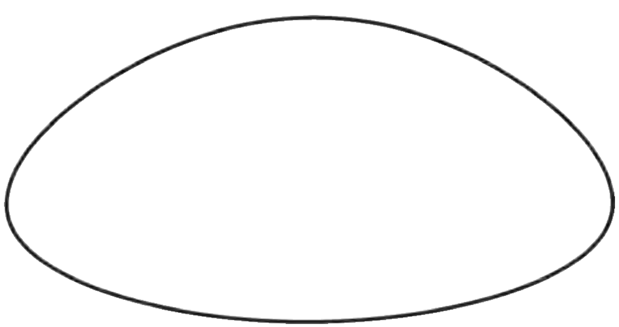
<u>V is smooth</u>



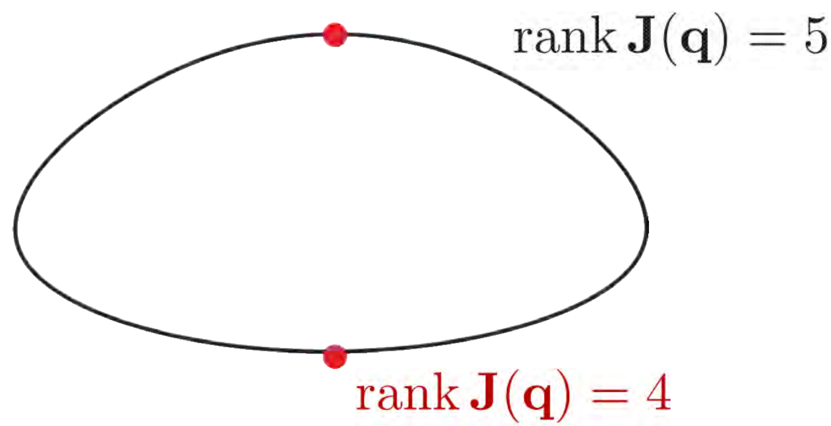
- What if $\mathbf{J}$ is singular all the time?
- Then we have an overconstrained linkage.



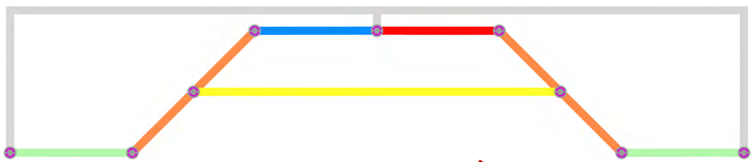


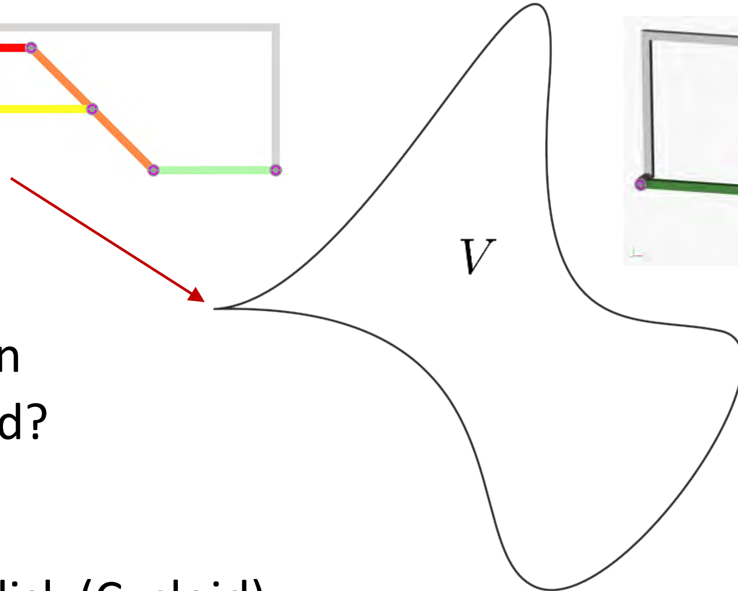


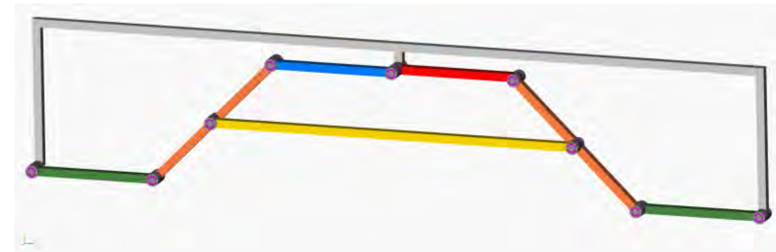
$\operatorname{rank} \mathbf{J}(\mathbf{q}) = 5 \quad \forall \mathbf{q} \in V$



How likely are Singularities?







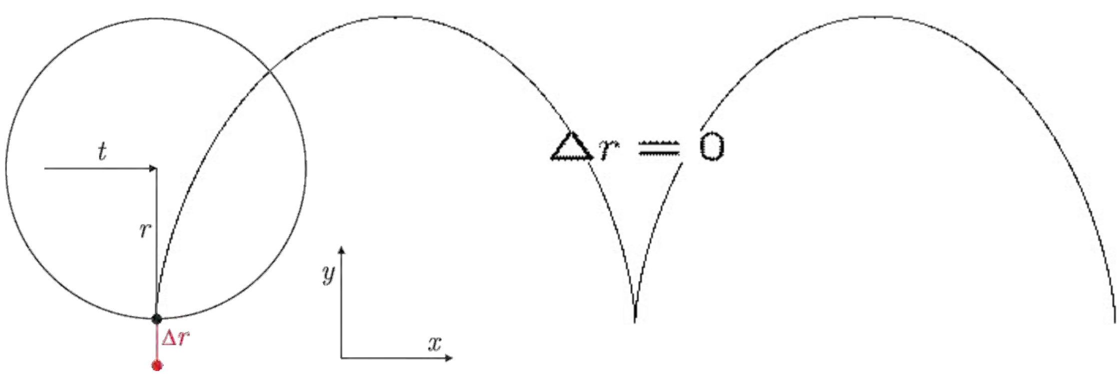
What happens when geometry is changed?

- Example: Rolling disk (Cycloid)

$$
\begin{aligned}
f :\ & \mathbb{R} \to \mathbb{R}^2 \\
& t \mapsto (x, y)
\end{aligned}
$$

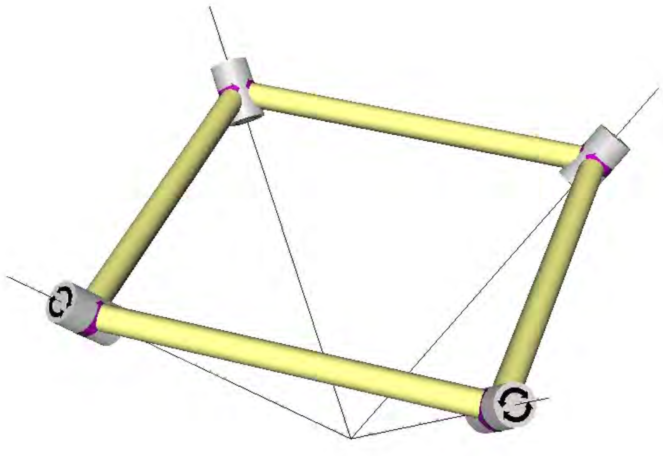
$$
\begin{aligned}
F_f :\ & \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \\
& (t, \Delta r) \mapsto (x, y)
\end{aligned}
$$

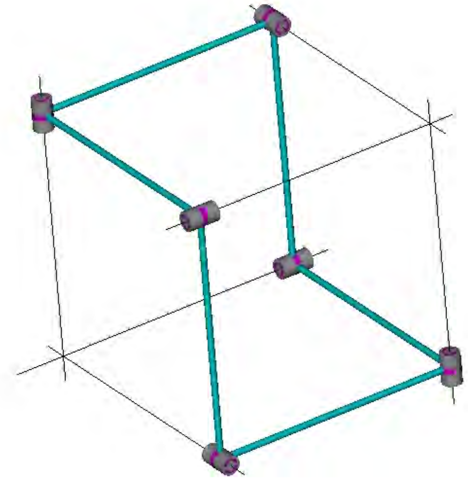
Deformation of the mapping (homotopy)

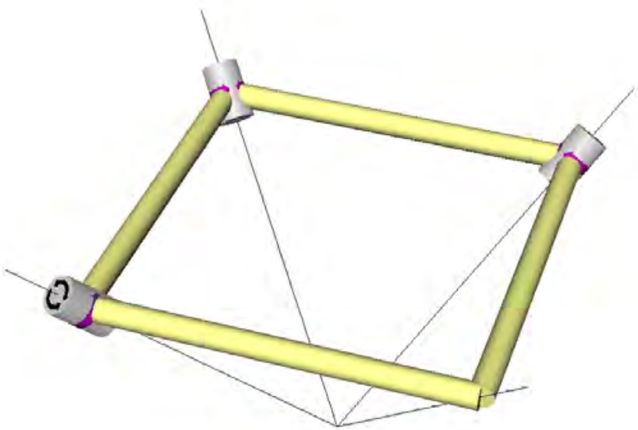


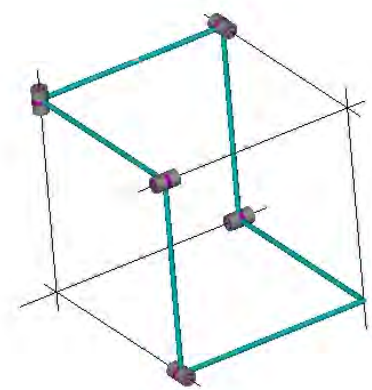
- Cusp is not ‘stable’
- Bifurcation is ‘stable’

How likely are overconstrained linkages?









Tasks:

- Determine the finite local DOF and mobility
  - Determine feasible finite motions
  - Approximate local c-space geometry
- Determine kinematic singularities
  - Check whether diff. DOF is constant in a neighborhood
  - Stratification of V by (co)rank

Methods:

- Screws, Lie-groups
- Higher-order analysis
- Tangent cones
- Distributions

1. Phenomenology
   - Mobility
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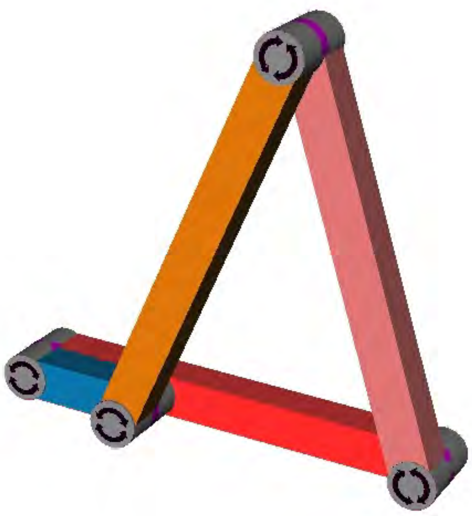
Definition [IFToMM]:

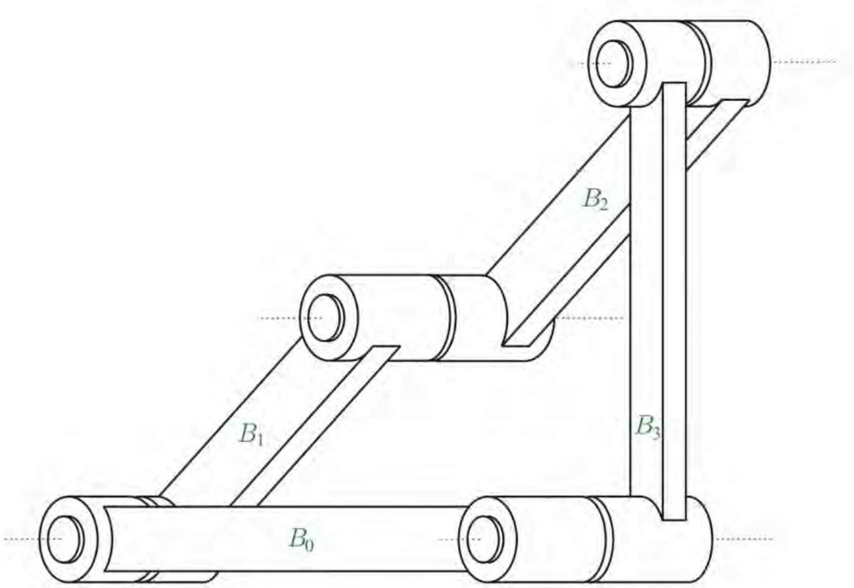
A *Mechanism* is a system of bodies designed to convert motions of one or several bodies into constrained motions of other bodies.



Definition [IFToMM]:

A *Mechanism* is a system of bodies designed to convert motions of one or several bodies into constrained motions of other bodies.

1. System of constrained bodies



Definition [IFToMM]:

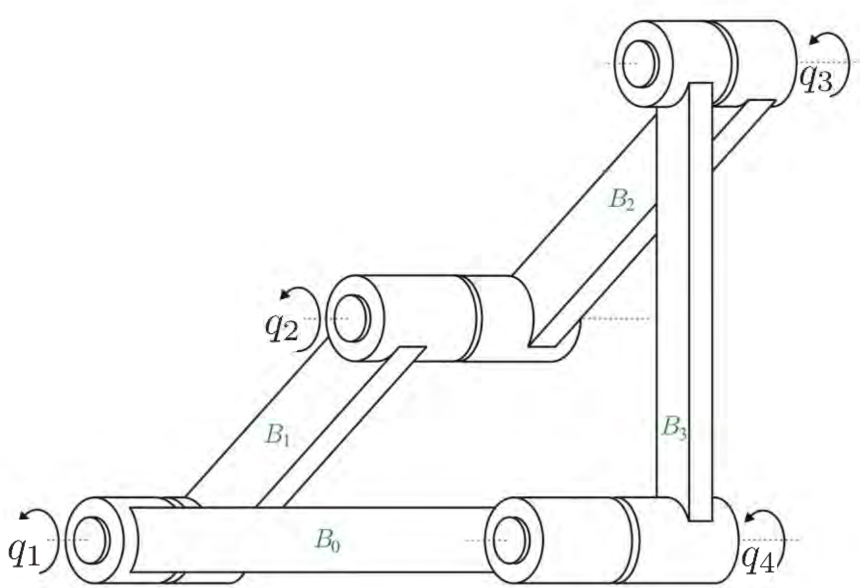
A *Mechanism* is a system of bodies designed to convert motions of one or several bodies into constrained motions of other bodies.

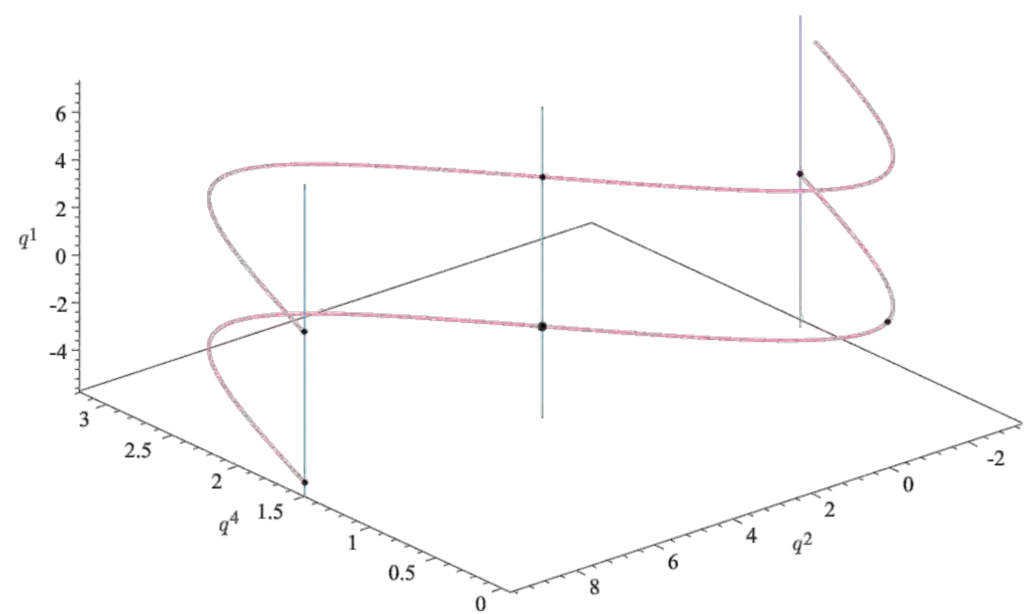
1. System of constrained bodies

- Joint variables: $\mathbf{q} \in \mathbb{V}^n$
- Constraints: $h(\mathbf{q}) = \mathbf{0}$
- Constraint mapping: $h : \mathbb{V}^n \to \mathbb{R}^r$

Configuration space:

$$V := \{\mathbf{q} \in \mathbb{V}^n | h(\mathbf{q}) = \mathbf{0}\}$$





Definition [IFToMM]:

A *Mechanism* is a system of bodies designed to convert motions of one or several bodies into constrained motions of other bodies.

1. System of constrained bodies

2. Input motion

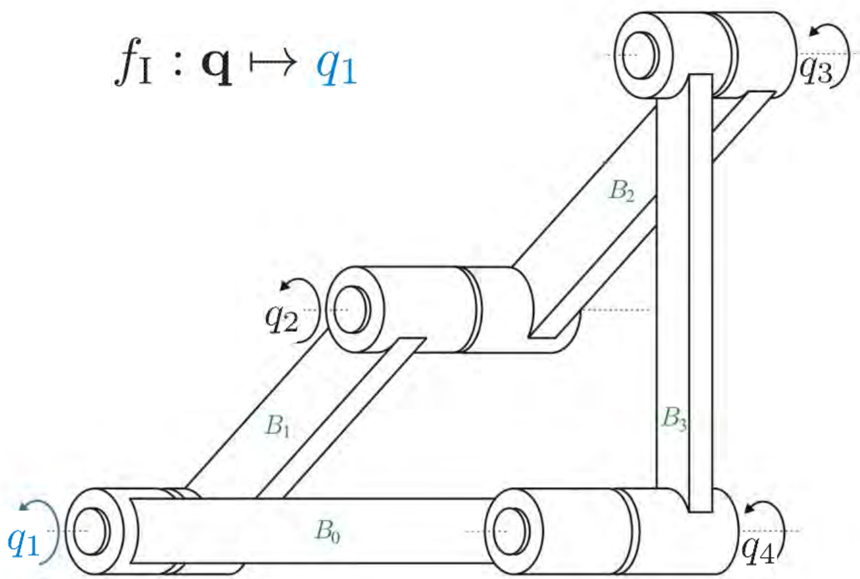
Configuration space:

$$V := \{\mathbf{q} \in \mathbb{V}^n | h(\mathbf{q}) = \mathbf{0}\}$$

Input mapping:

$$f_\mathrm{I} : V \to \mathbb{R}^m$$

$$f_\mathrm{I} : \mathbf{q} \mapsto q_1$$



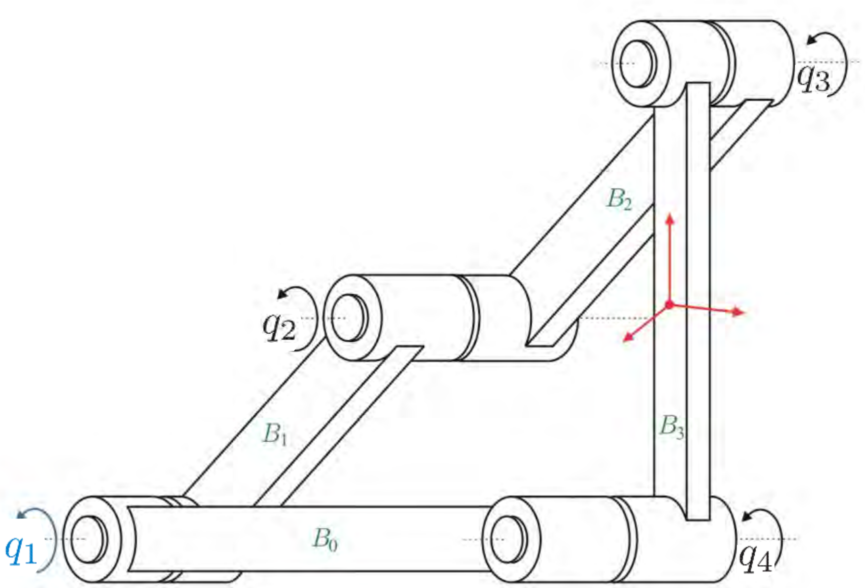
Definition [IFToMM]:

A *Mechanism* is a system of bodies designed to convert motions of one or several bodies into constrained motions of other bodies.

1. System of constrained bodies

2. Input motion

3. Output motion



Configuration space:

$$V := \{\mathbf{q} \in \mathbb{V}^n | h(\mathbf{q}) = \mathbf{0}\}$$

Input mapping:

$$f_{\mathrm{I}} : V \to \mathbb{R}^m$$

Output mapping:

$$f_{\mathrm{O}} : V \to SE(3)$$

Kinematic model

$$\begin{tikzcd}
 & {\color{green}V} \arrow[dl, "f_{\mathrm{I}}"'] \arrow[dr, "f_{\mathrm{O}}"] & \\
{\color{blue}\mathbb{V}^m} \arrow[rr, "f_{\mathrm{FK}}"] & & {\color{red}SE(3)}
\end{tikzcd}$$

Configuration space

$$V := h^{-1}(\mathbf{0})$$

- admissible configurations
- analytic variety possessing singularities
- intrinsic to the kinematics

Input space

$$\mathcal{I} := f_{\mathrm{I}}(V) \subset \mathbb{V}^m$$

- admissible inputs
- not necessarily a subspace of $\mathbb{V}^n$

Workspace

$$\mathcal{W} := f_{\mathrm{O}}(V) \subset SE(3)$$

- admissible outputs
- possible not connected

Kinematic model

Parameter space

~~Configuration space~~

$$V := h^{-1}(\mathbf{0})$$

- admissible configurations
- analytic variety possessing singularities
- intrinsic to the kinematics

Configuration space

- Abstract representation of configuration
- Coordinate free

$$\begin{tikzcd}
\mathfrak{C} \arrow[rr, "\varphi"] & & V \arrow[dl, "f_{\mathrm{I}}"'] \arrow[dr, "f_{\mathrm{O}}"] & \\
& \mathbb{V}^m \arrow[rr, "f_{\mathrm{FK}}"] & & SE(3)
\end{tikzcd}$$

Input space

$$\mathcal{I} := f_{\mathrm{I}}(V) \subset \mathbb{V}^m$$

- admissible inputs
- not necessarily a subspace of $\mathbb{V}^n$

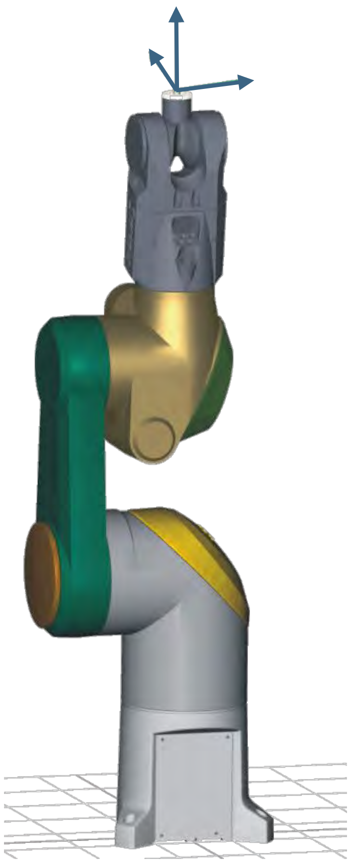
Workspace

$$\mathcal{W} := f_{\mathrm{O}}(V) \subset SE(3)$$

- admissible outputs
- possible not connected

Special case: serial manipulator

$$\textcolor{blue}{\mathcal{I}} = \textcolor{green}{V} \xrightarrow{f_{\mathrm{O}} = f_{\mathrm{FK}}} \textcolor{red}{SE(3)}$$



Kinematic model

C-space singularities

Input singularities

Output singularities

$$\begin{tikzcd}
 & V \arrow[dl, "f_{\mathrm{I}}"'] \arrow[dr, "f_{\mathrm{O}}"] & \\
\mathbb{V}^m \arrow[rr, "f_{\mathrm{FK}}"] & & SE(3)
\end{tikzcd}$$

$$\mathbf{0} = h(\mathbf{q}) \quad \Rightarrow \quad V = h^{-1}(\mathbf{0}) \quad \text{(analytic, algebraic) variety}$$

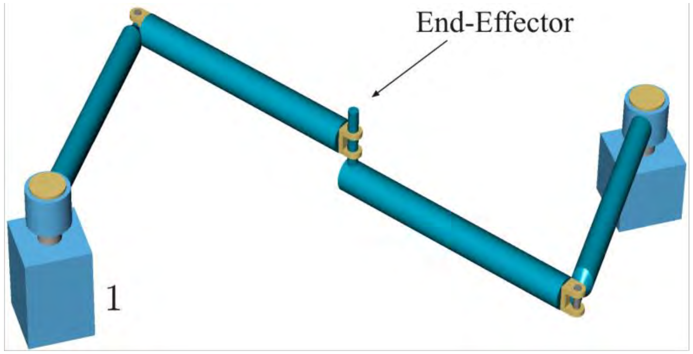
$$\boldsymbol{\theta} = f_{\mathrm{I}}(\mathbf{q})$$

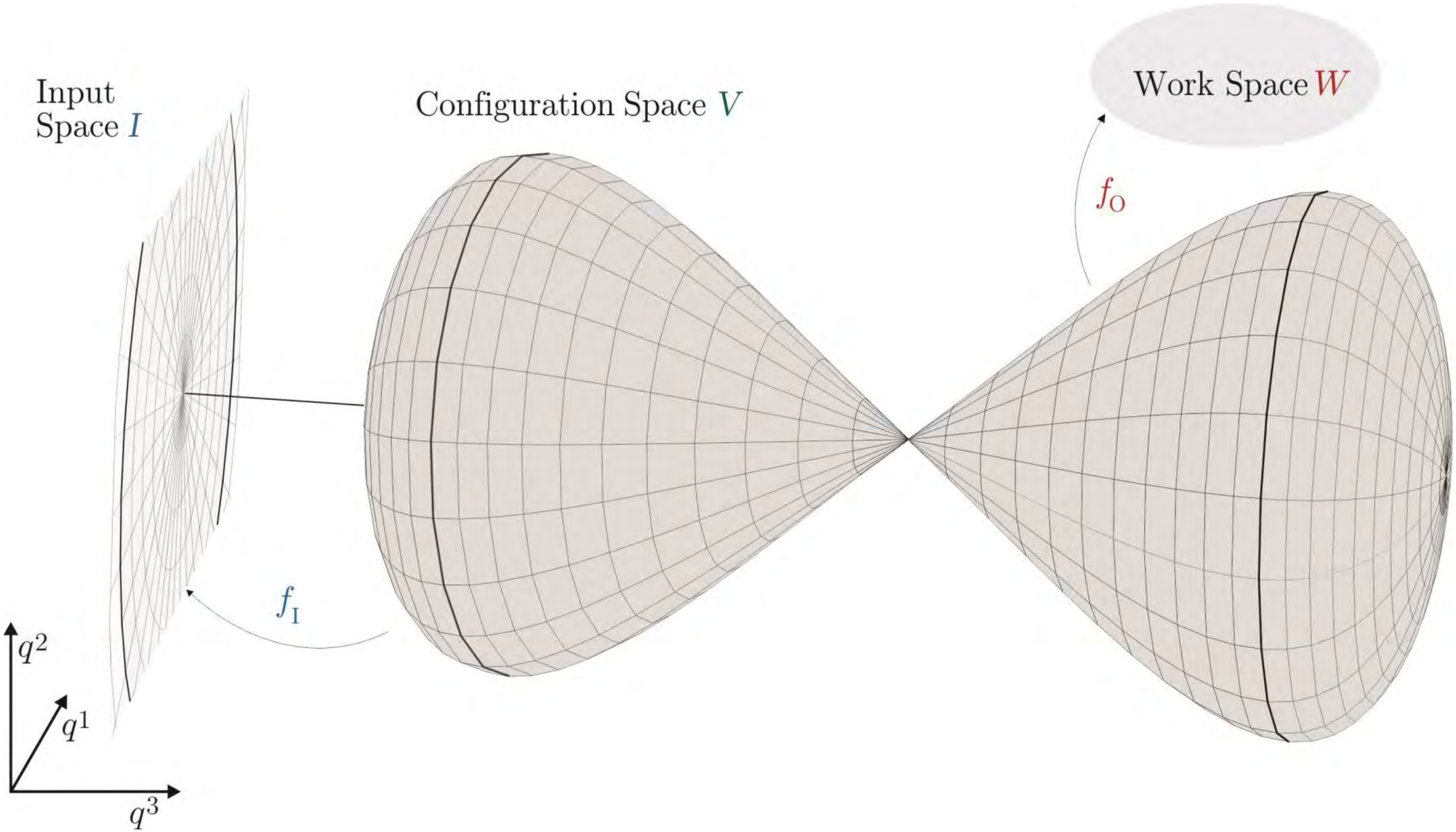
$$\mathbf{C} = \underbrace{f_{\mathrm{O}}(\mathbf{q})}_{\text{analytic mappings}}$$

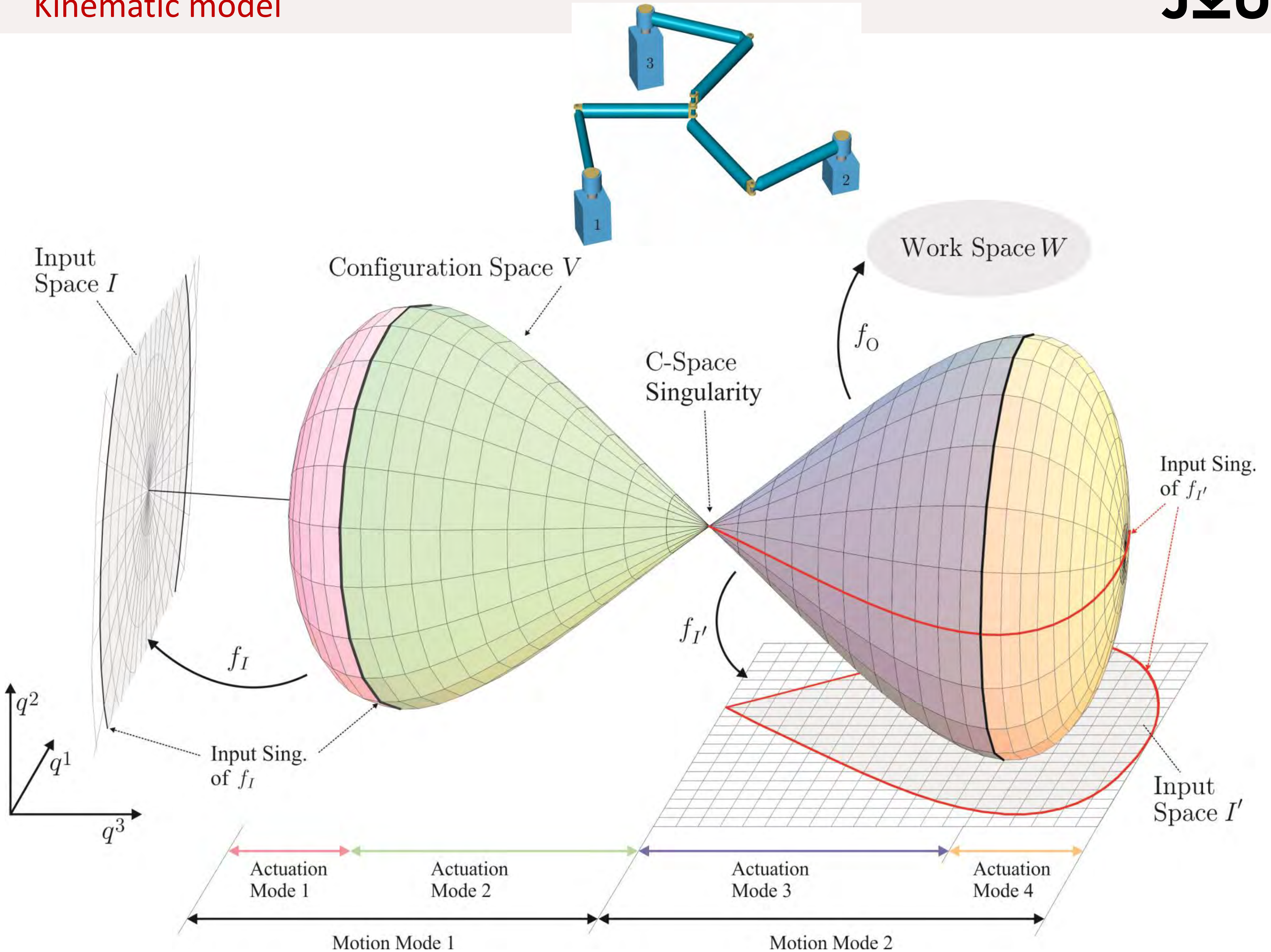
Sard's theorem:

Let $f : M \rightarrow N$ be a smooth map of manifolds M and N, where only N has boundary. Almost all points of N are regular values of $f : M \rightarrow N$ and $\partial f : \partial M \rightarrow N$.

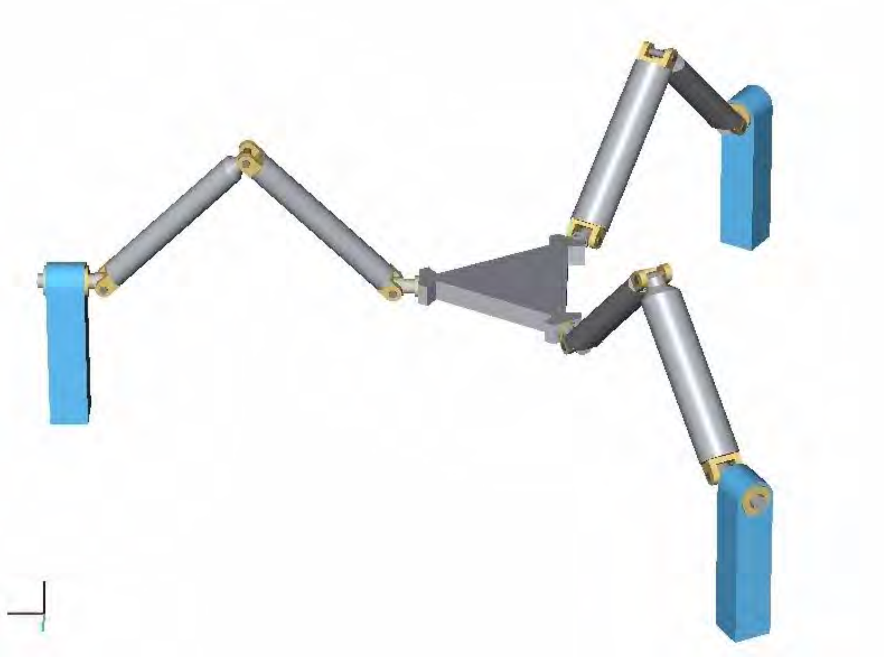
That is, the set of critical values of f has measure zero in N.


End-Effector
1
2


Input
Space I
Configuration Space V
Work Space W
f_O
f_I
q^2
q^1
q^3


3
2
1
Work Space W
Input Space I
Configuration Space V
f_O
C-Space Singularity
Input Sing. of $f_{I'}$
$f_{I'}$
f_I
q^2
q^1
q^3
Input Sing. of f_I
Input Space I'
Actuation Mode 1
Actuation Mode 2
Actuation Mode 3
Actuation Mode 4
Motion Mode 1
Motion Mode 2

Kinematic model



Lock-Up mode

Planar mode

1. Phenomenology
   - Mobility
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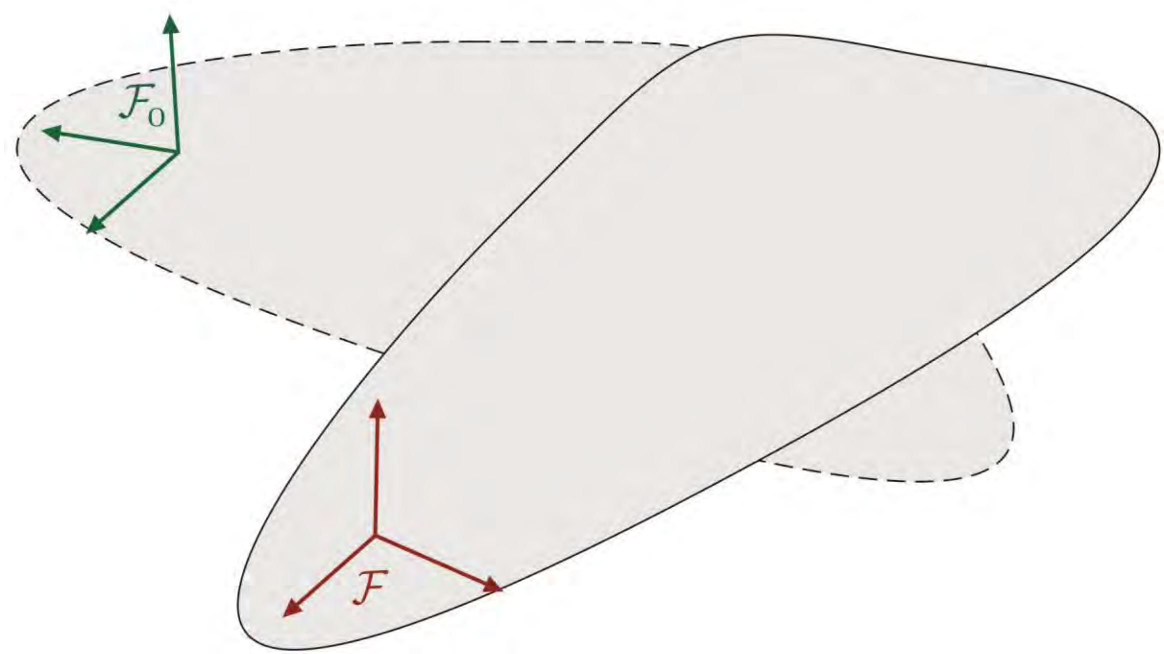
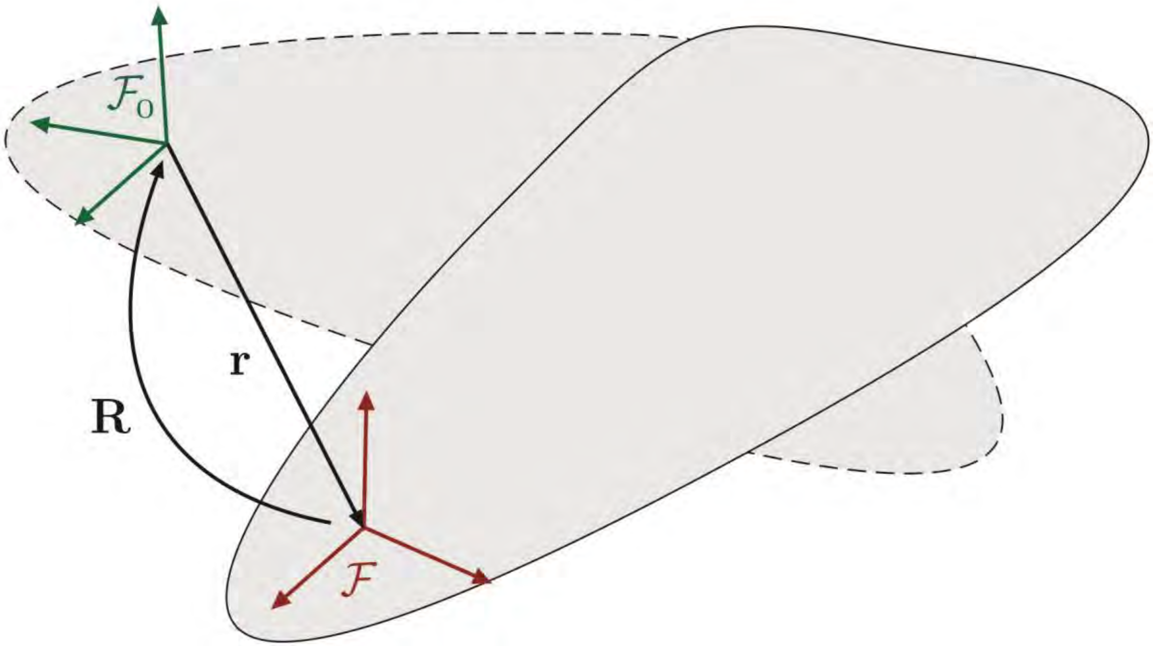
- Frame transformation $\mathcal{F}$ → $\mathcal{F}_0$



- Frame transformation $\mathcal{F}$ → $\mathcal{F}_0$

$\mathbf{R} \in SO(3)$ – rotation

$\mathbf{r} \in \mathbb{R}^3$ – translation

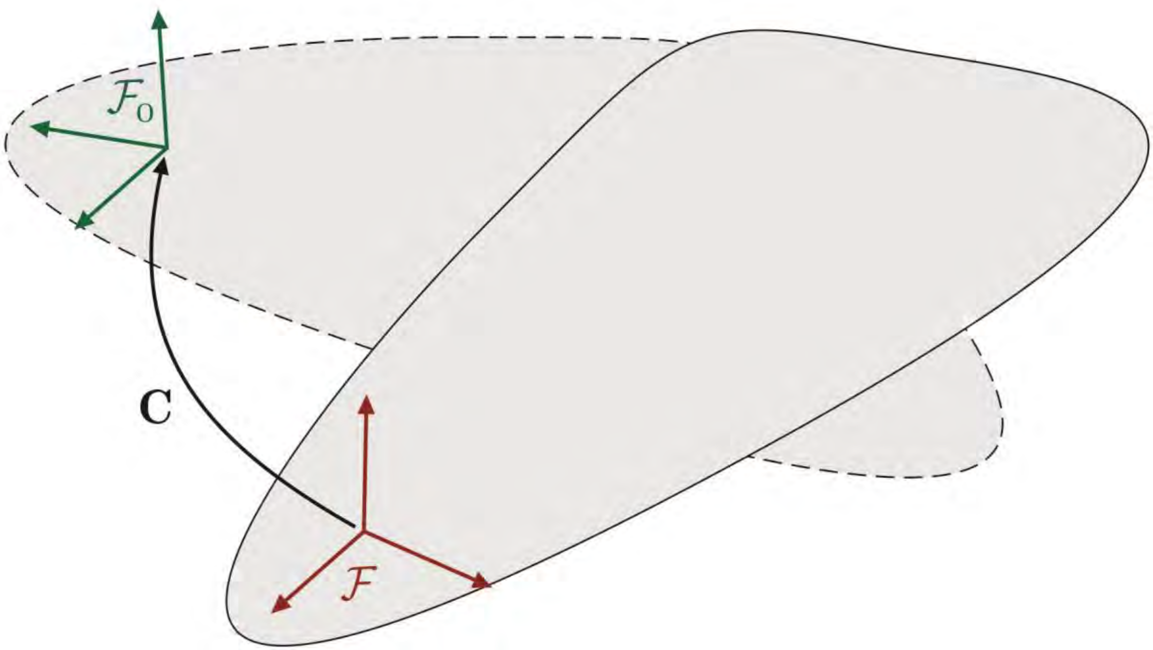


- Frame transformation $\mathcal{F} \rightarrow \mathcal{F}_0$

$$\boxed{\text{Configuration of } \mathcal{F}: \quad \mathbf{C} = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix} \in SE(3)}$$

'Homogenous matrix'

$$SE(3) = SO(3) \ltimes \mathbb{R}^3$$



Chasles' theorem:

Any rigid body displacement can be achieved by a screw motion, i.e. a rotation about a constant axis together with a translation along this axis.

Mozzi-Cauchy theorem:

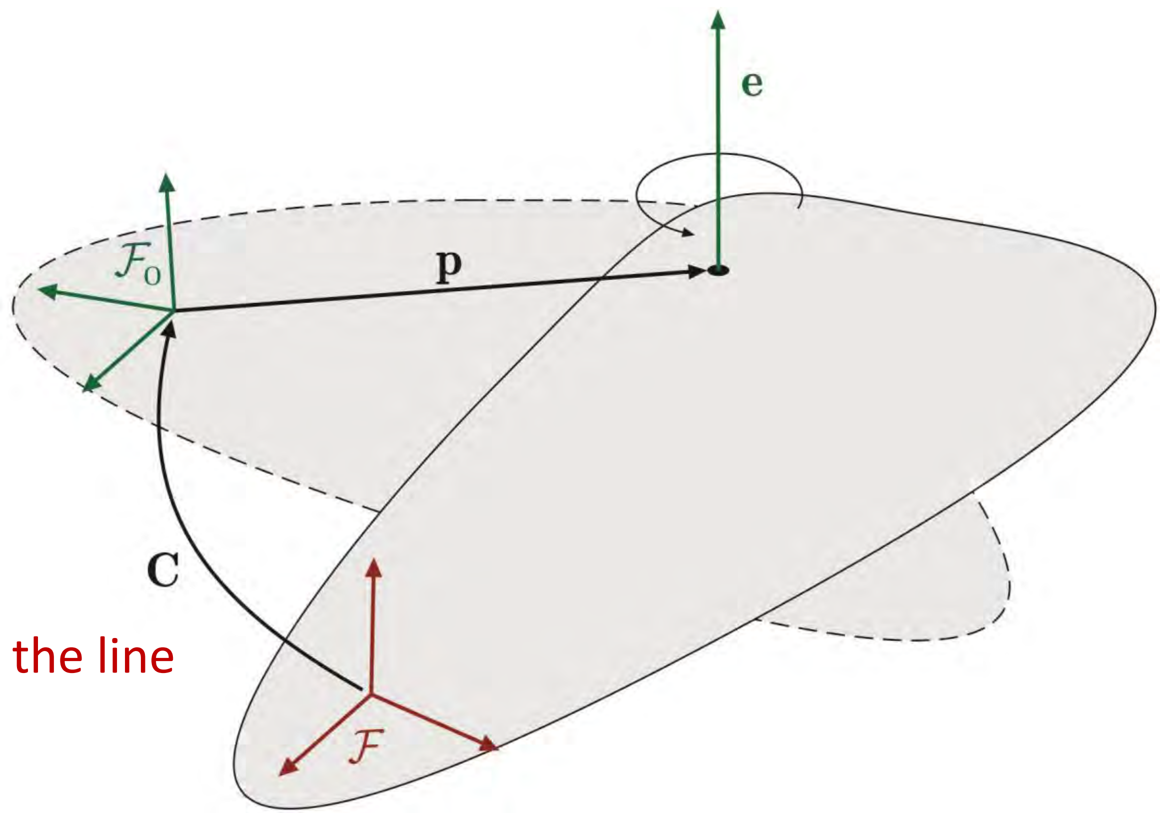
For any rigid body motion there exists an instantaneous screw axis.

Screw coordinates:

$$\mathbf{X} = \begin{pmatrix} \mathbf{e} \\ \mathbf{p} \times \mathbf{e} + h\mathbf{e} \end{pmatrix} = \begin{pmatrix} \mathbf{e} \\ \mathbf{p} \times \mathbf{e} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ h\mathbf{e} \end{pmatrix}$$

Translation along the line

Plücker line coordinates



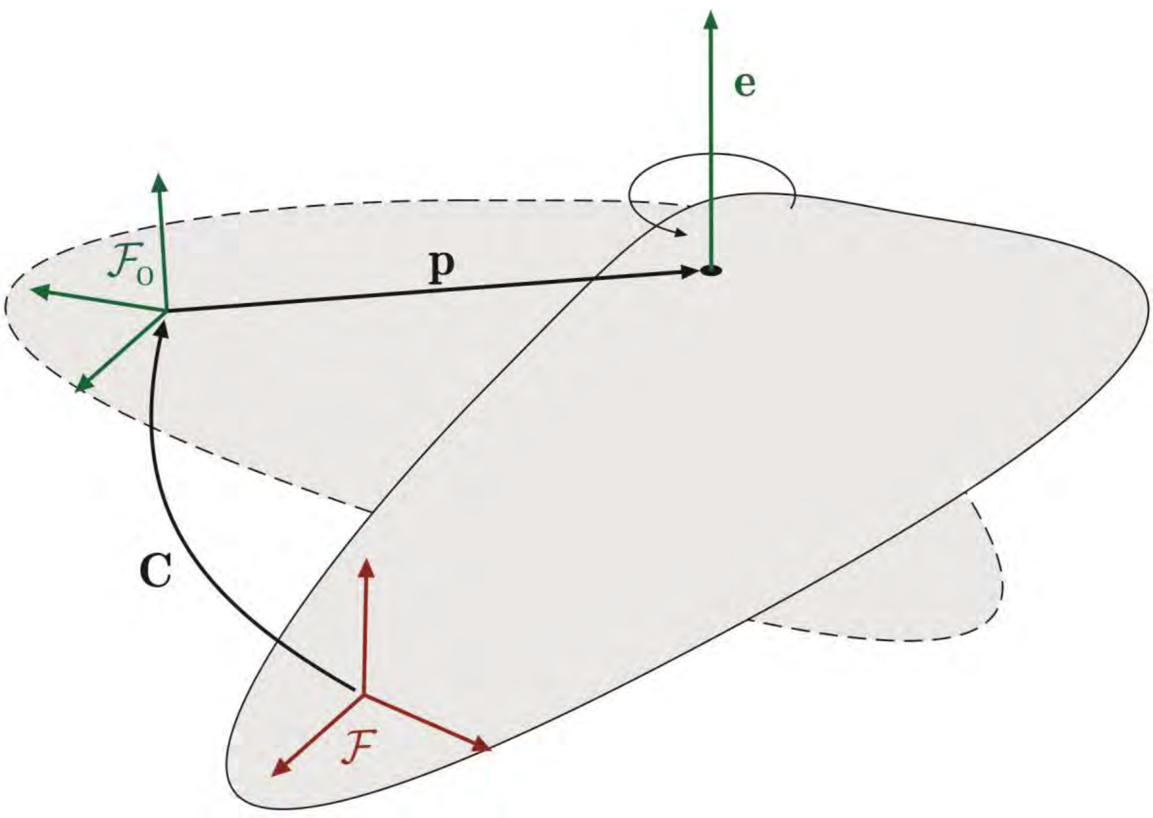
- Rigid body motion in terms of screw coordinates: $\mathbf{C} = \exp(\varphi \mathbf{X})$

$$\boxed{\exp(\varphi \mathbf{X}) = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix} \qquad \begin{aligned} \mathbf{R} &= \exp(\varphi \tilde{\mathbf{e}}) \\ \mathbf{r} &= (\mathbf{I} - \mathbf{R})\mathbf{p} + \varphi h \mathbf{e} \end{aligned}}$$

Screw coordinates:

$$\mathbf{X} = \begin{pmatrix} \mathbf{e} \\ \mathbf{p} \times \mathbf{e} + h\mathbf{e} \end{pmatrix}$$

- Unit vector along screw axis: $\mathbf{e}$
- Position vector to any point on the axis: $\mathbf{p}$
- Pitch: h



- Rigid body motion in terms of screw coordinates: $\mathbf{C} = \exp(\varphi \mathbf{X})$

$$\exp(\varphi \mathbf{X}) = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix} \qquad \begin{aligned} \mathbf{R} &= \exp(\varphi \tilde{\mathbf{e}}) \\ \mathbf{r} &= (\mathbf{I} - \mathbf{R})\mathbf{p} + \varphi h \mathbf{e} \end{aligned}$$

Screw coordinates:

$$\mathbf{X} = \begin{pmatrix} \mathbf{e} \\ \mathbf{p} \times \mathbf{e} + h\mathbf{e} \end{pmatrix}$$

$$\exp(\varphi \tilde{\mathbf{e}}) = \mathbf{I} + \sin \varphi \tilde{\mathbf{e}} + (1 - \cos \varphi)\, \tilde{\mathbf{e}}^2$$

Euler-Rodrigues formula

- Unit vector along screw axis: $\mathbf{e}$
- Position vector to any point on the axis: $\mathbf{p}$
- Pitch: h

- Rigid body motion in terms of screw coordinates: $\mathbf{C} = \exp(\varphi \mathbf{X})$

$$\exp(\varphi \mathbf{X}) = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix}$$

$$\mathbf{R} = \exp(\varphi \widetilde{\mathbf{e}})$$

$$\mathbf{r} = (\mathbf{I} - \mathbf{R})\mathbf{p} + \varphi h \mathbf{e}$$

Why exp mapping?

$$\mathbf{e} := \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \in \mathbb{R}^3 \quad \Rightarrow \quad \widetilde{\mathbf{e}} := \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix} \in so(3)$$

$$\begin{aligned} \exp(\varphi \widetilde{\mathbf{e}}) &= \mathbf{I} + \widetilde{\mathbf{e}}\varphi + \frac{1}{2}\widetilde{\mathbf{e}}^2\varphi^2 + \frac{1}{3!}\widetilde{\mathbf{e}}^3\varphi^3 + \frac{1}{4!}\widetilde{\mathbf{e}}^4\varphi^4 + \dots \\ &= \mathbf{I} + \sin\varphi\,\widetilde{\mathbf{e}} + (1 - \cos\varphi)\,\widetilde{\mathbf{e}}^2 \end{aligned}$$

- Remember the nice property of the exp mapping!

- Rigid body motion in terms of screw coordinates: $\mathbf{C} = \exp(\varphi \mathbf{X})$

$$\exp(\varphi \mathbf{X}) = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix} \qquad \begin{aligned} \mathbf{R} &= \exp(\varphi \tilde{\mathbf{e}}) \\ \mathbf{r} &= (\mathbf{I} - \mathbf{R})\mathbf{p} + \varphi h \mathbf{e} \end{aligned}$$

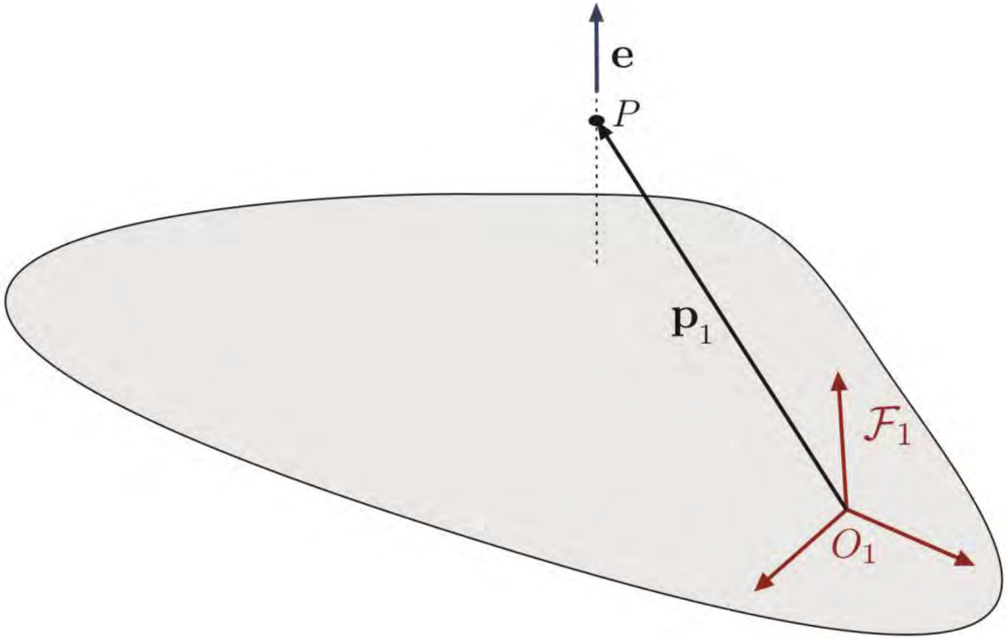
Why exp mapping?

$$\mathbf{X} = \begin{pmatrix} \mathbf{e} \\ \mathbf{p} \times \mathbf{e} + h\mathbf{e} \end{pmatrix} =: \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix} \in \mathbb{R}^6 \quad \Rightarrow \quad \widehat{\mathbf{X}} := \begin{pmatrix} \widetilde{\boldsymbol{\xi}} & \boldsymbol{\eta} \\ \mathbf{0} & 0 \end{pmatrix} \in se(3)$$

$$\exp(\varphi \mathbf{X}) = \mathbf{I} + \widehat{\mathbf{X}}\varphi + \frac{1}{2}\widehat{\mathbf{X}}^2\varphi^2 + \frac{1}{3!}\widehat{\mathbf{X}}^3\varphi^3 + \frac{1}{4!}\widehat{\mathbf{X}}^4\varphi^4 + \dots$$

- Screw coordinates for motion of $\mathcal{F}_1$

$$^{1}\mathbf{X} = \begin{pmatrix} ^{1}\mathbf{e} \\ ^{1}\mathbf{p}_1 \times {}^{1}\mathbf{e} + {}^{1}\mathbf{e}h \end{pmatrix}$$



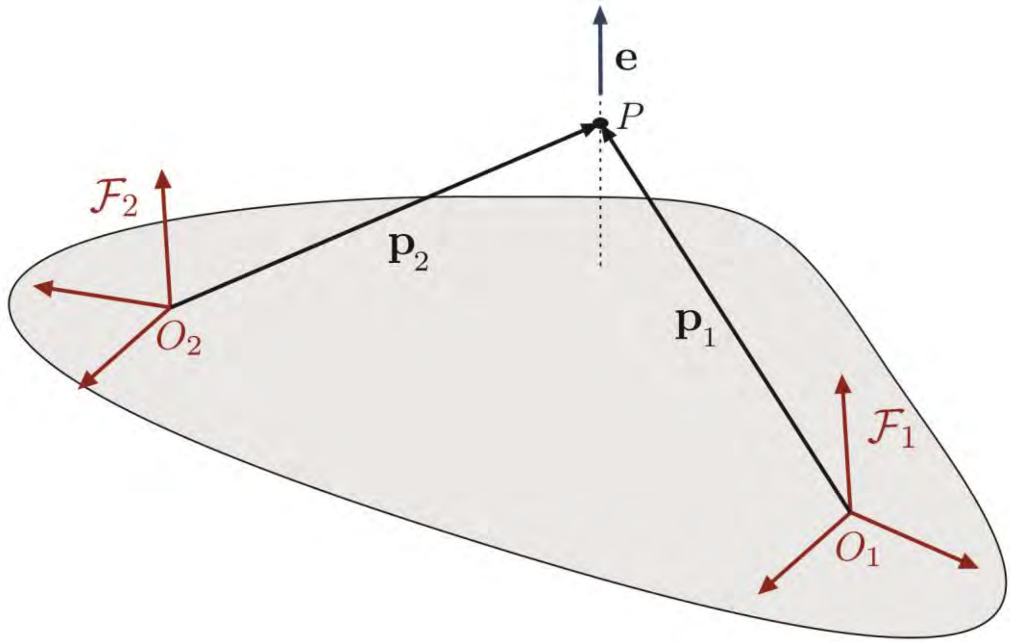
$^{1}\mathbf{e}, {}^{1}\mathbf{p}_1$ – Coordinate vectors resolved in frame $\mathcal{F}_1$

- Screw coordinates for motion of $\mathcal{F}_1$

$$^1\mathbf{X} = \begin{pmatrix} ^1\mathbf{e} \\ ^1\mathbf{p}_1 \times {}^1\mathbf{e} + {}^1\mathbf{e}h \end{pmatrix}$$

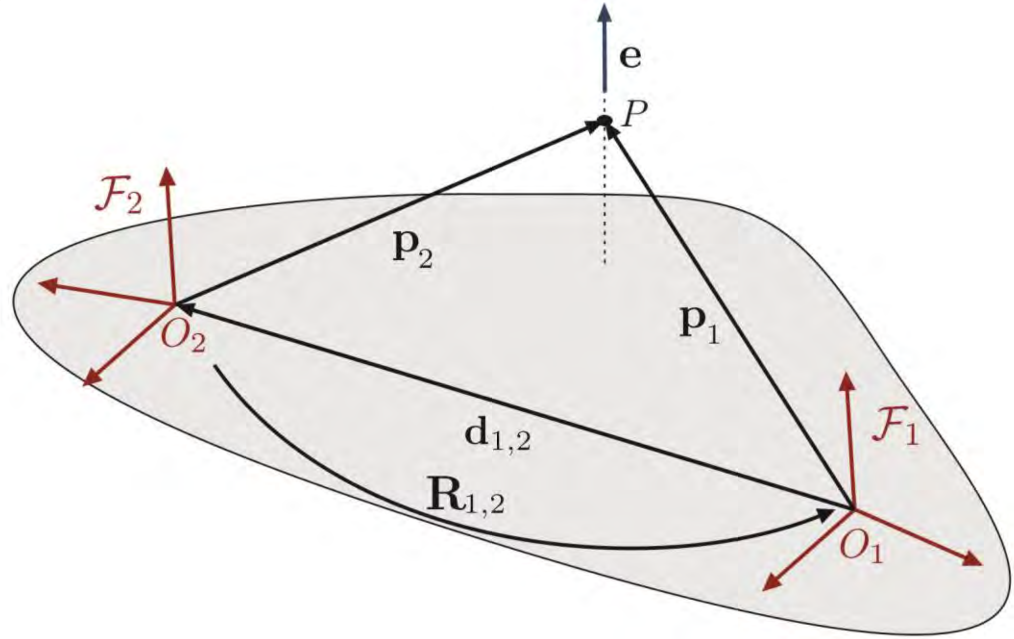
- Screw coordinates for motion of $\mathcal{F}_2$

$$^2\mathbf{X} = \begin{pmatrix} ^2\mathbf{e} \\ ^2\mathbf{p}_2 \times {}^2\mathbf{e} + {}^2\mathbf{e}h \end{pmatrix}$$



$^1\mathbf{e}$, $^1\mathbf{p}_1$ – Coordinate vectors resolved in frame $\mathcal{F}_1$

$^2\mathbf{e}$, $^2\mathbf{p}_2$ – Coordinate vectors resolved in frame $\mathcal{F}_2$



- Screw coordinates for motion of $\mathcal{F}_1$

$$^{1}\mathbf{X} = \begin{pmatrix} {}^{1}\mathbf{e} \\ {}^{1}\mathbf{p}_1 \times {}^{1}\mathbf{e} + {}^{1}\mathbf{e}h \end{pmatrix}$$

- Screw coordinates for motion of $\mathcal{F}_2$

$$^{2}\mathbf{X} = \begin{pmatrix} {}^{2}\mathbf{e} \\ {}^{2}\mathbf{p}_2 \times {}^{2}\mathbf{e} + {}^{2}\mathbf{e}h \end{pmatrix}$$

- Transformation $\mathcal{F}_2 \rightarrow \mathcal{F}_1$

$$\boxed{{}^{1}\mathbf{X} = \mathbf{Ad}_{\mathbf{S}_{1,2}}\,{}^{2}\mathbf{X} \qquad \text{with } \mathbf{S}_{1,2} = \begin{pmatrix} \mathbf{R}_{1,2} & \mathbf{r}_{1,2} \\ \mathbf{0} & 1 \end{pmatrix}}$$

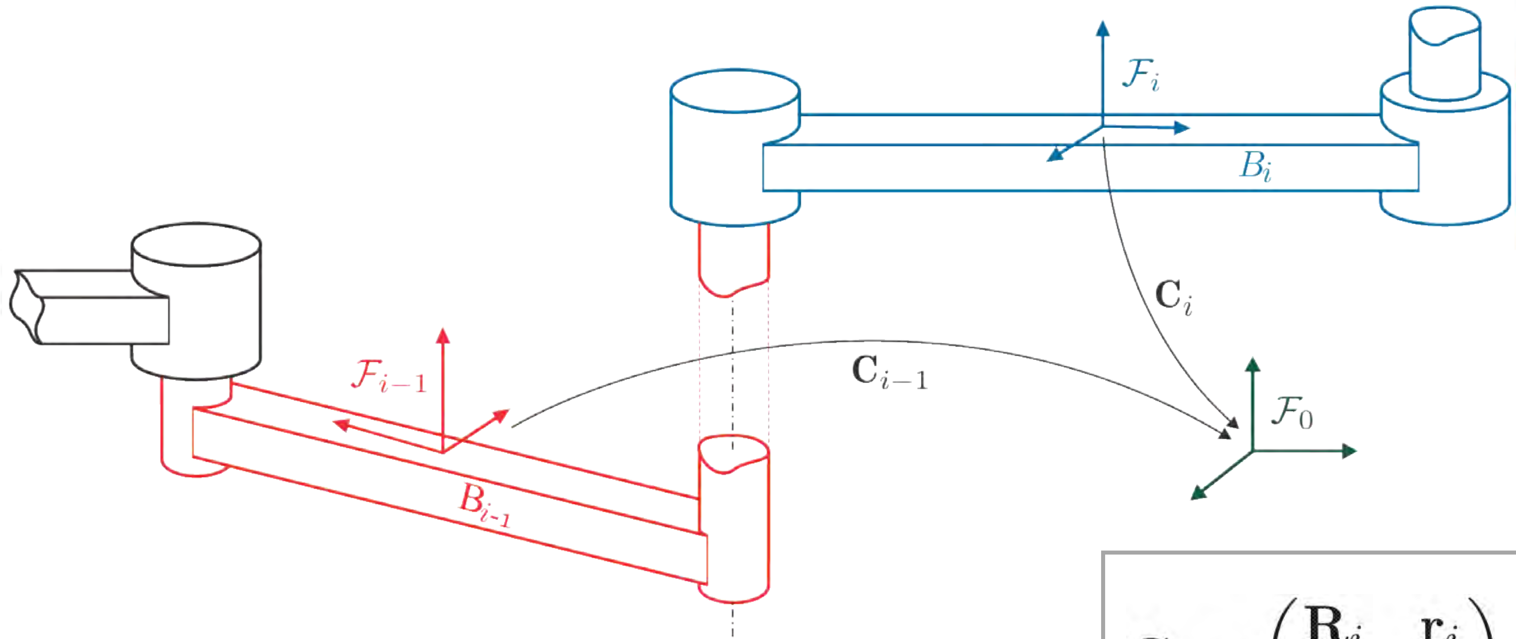
Adjoined mapping on $SE(3)$: $\mathbf{Ad}_{\mathbf{C}} = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \widetilde{\mathbf{r}}\mathbf{R} & \mathbf{R} \end{pmatrix}$ for $\mathbf{C} = \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix}$

‘Motion tensor’

Transforms between different representations of a screw

- Body-fixed reference frame $\mathcal{F}_i$ at body i
- Body-fixed reference frame $\mathcal{F}_{i-1,i}$ at body i

⇨ Relative configuration of body i w.r.t. body i-1: $\mathbf{C}_{i-1,i} := \mathbf{C}_{i-1}^{-1}\mathbf{C}_i$

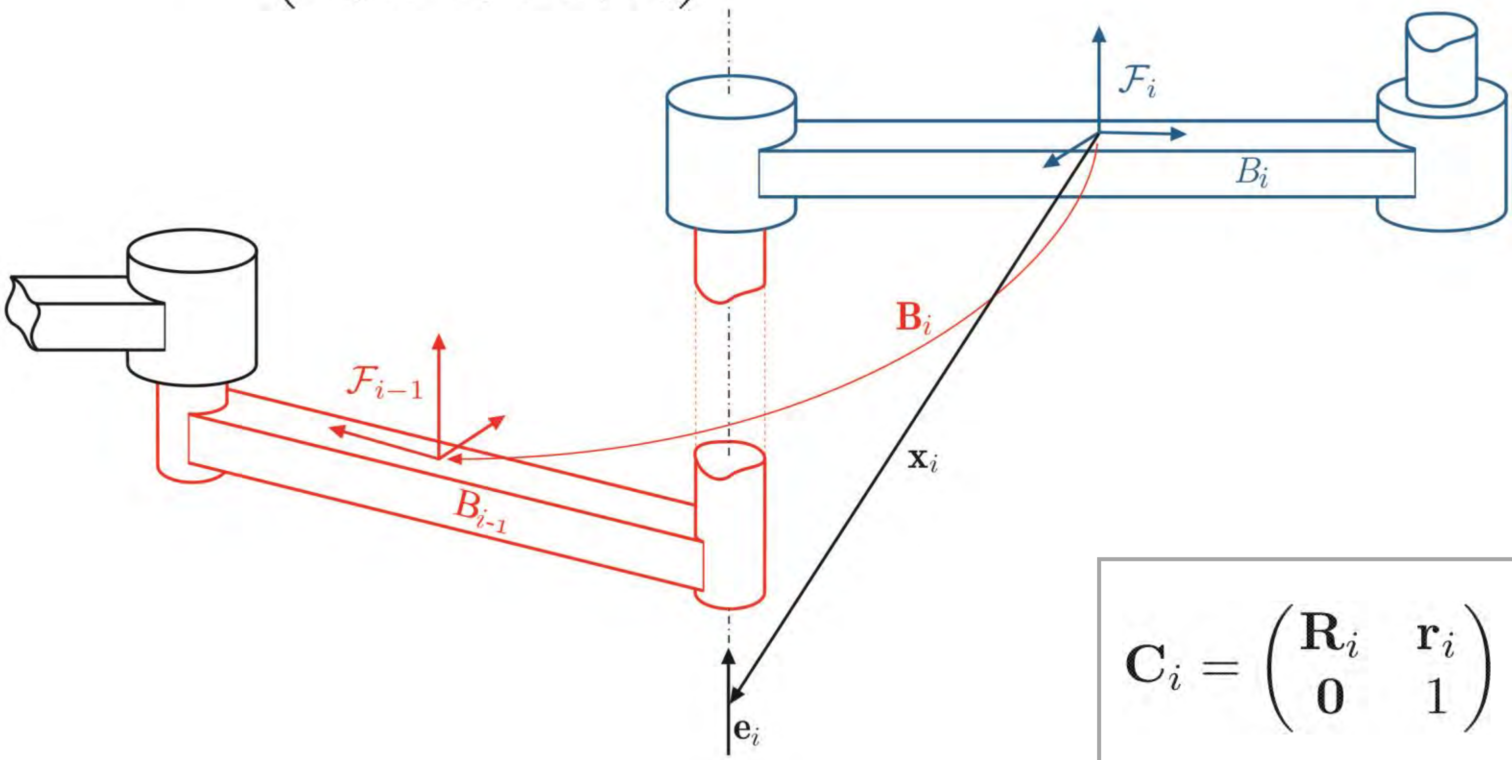


$$\mathbf{C}_i = \begin{pmatrix} \mathbf{R}_i & \mathbf{r}_i \\ \mathbf{0} & 1 \end{pmatrix}$$

$$\boxed{\mathbf{C}_{i-1,i} = \mathbf{B}_i \exp({}^i\mathbf{X}_i q_i)}$$

- Relative reference configuration for $q_i = 0$: $\mathbf{B}_i = \mathbf{C}_{i-1,i}(0)$
- Screw coordinate vector of joint i represented in $\mathcal{F}_i$:

$${}^i\mathbf{X}_i = \begin{pmatrix} {}^i\mathbf{e}_i \\ {}^i\mathbf{x}_i \times {}^i\mathbf{e}_i + h_i {}^i\mathbf{e}_i \end{pmatrix}$$

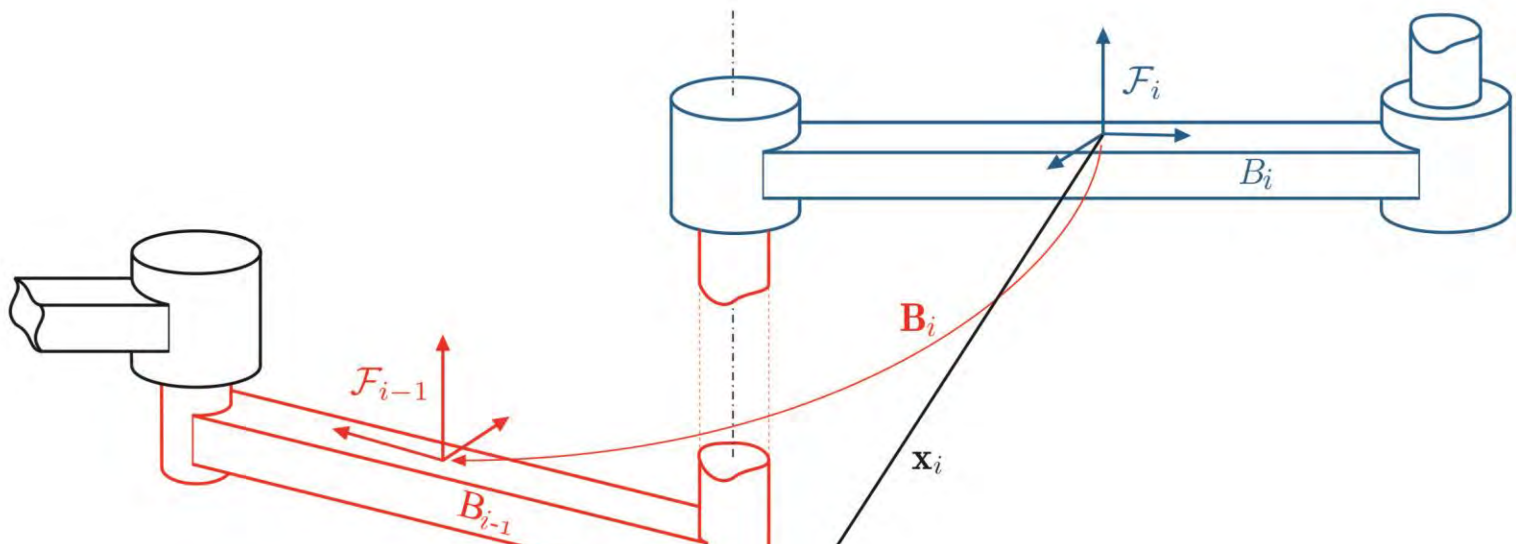


$$\mathbf{C}_i = \begin{pmatrix} \mathbf{R}_i & \mathbf{r}_i \\ \mathbf{0} & 1 \end{pmatrix}$$

Configuration/pose of links

- Configuration of body i

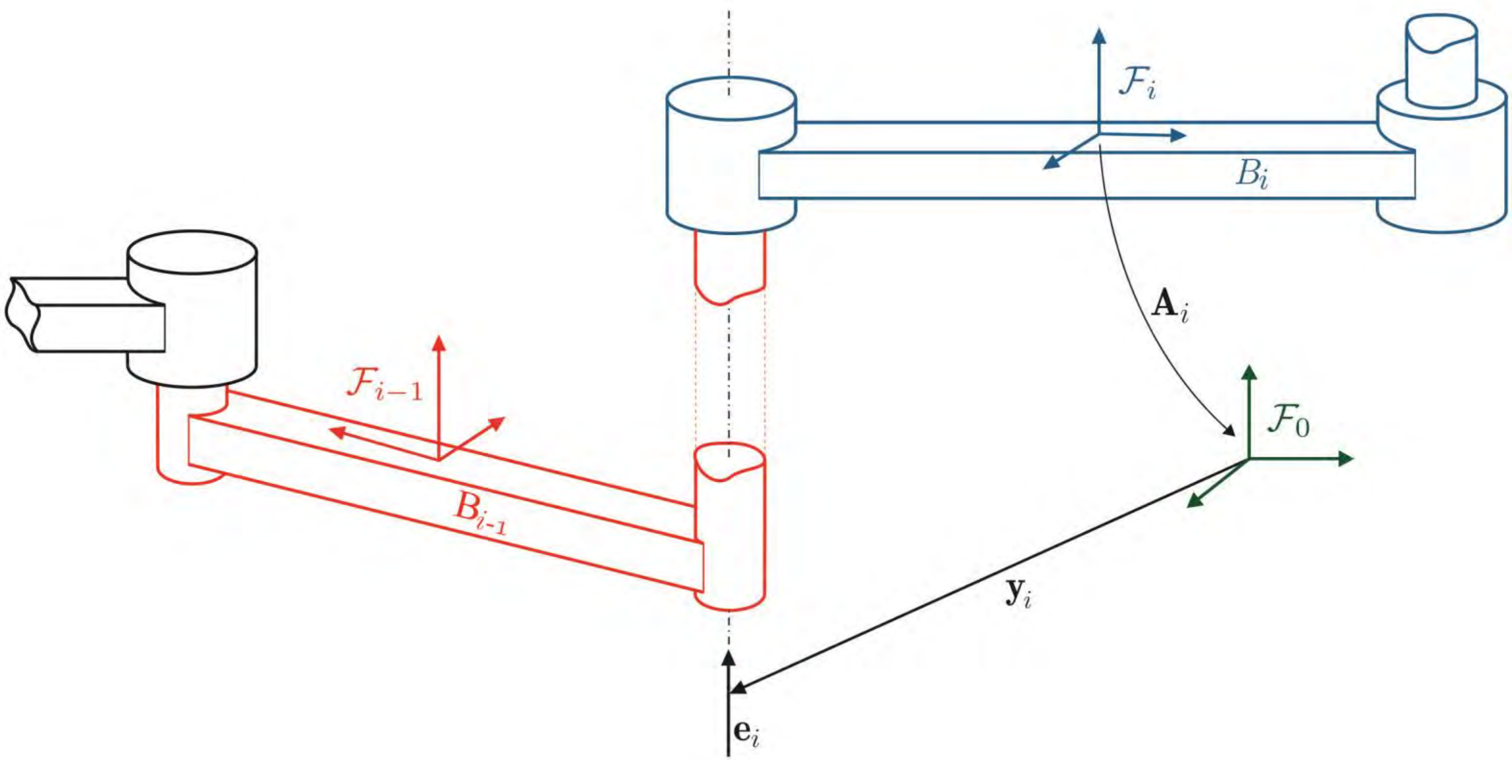
$$\mathbf{C}_i\left(\mathbf{q}\right) = \mathbf{B}_1 \exp({}^1\mathbf{X}_1 q_1) \cdot \mathbf{B}_2 \exp({}^2\mathbf{X}_2 q_2) \cdot \ldots \cdot \mathbf{B}_i \exp({}^i\mathbf{X}_i q_i)$$



$$\mathbf{C}_i = \begin{pmatrix} \mathbf{R}_i & \mathbf{r}_i \\ \mathbf{0} & 1 \end{pmatrix}$$

- Absolute reference configuration $\mathbf{A}_i = \mathbf{C}_i(\mathbf{0})$,
  i.e. the reference configuration of body i w.r.t. the IFR $\mathcal{F}_0$ for $\mathbf{q} = \mathbf{0}$
- Joint screw coordinates in spatial representations (measured + resolved in IFR)

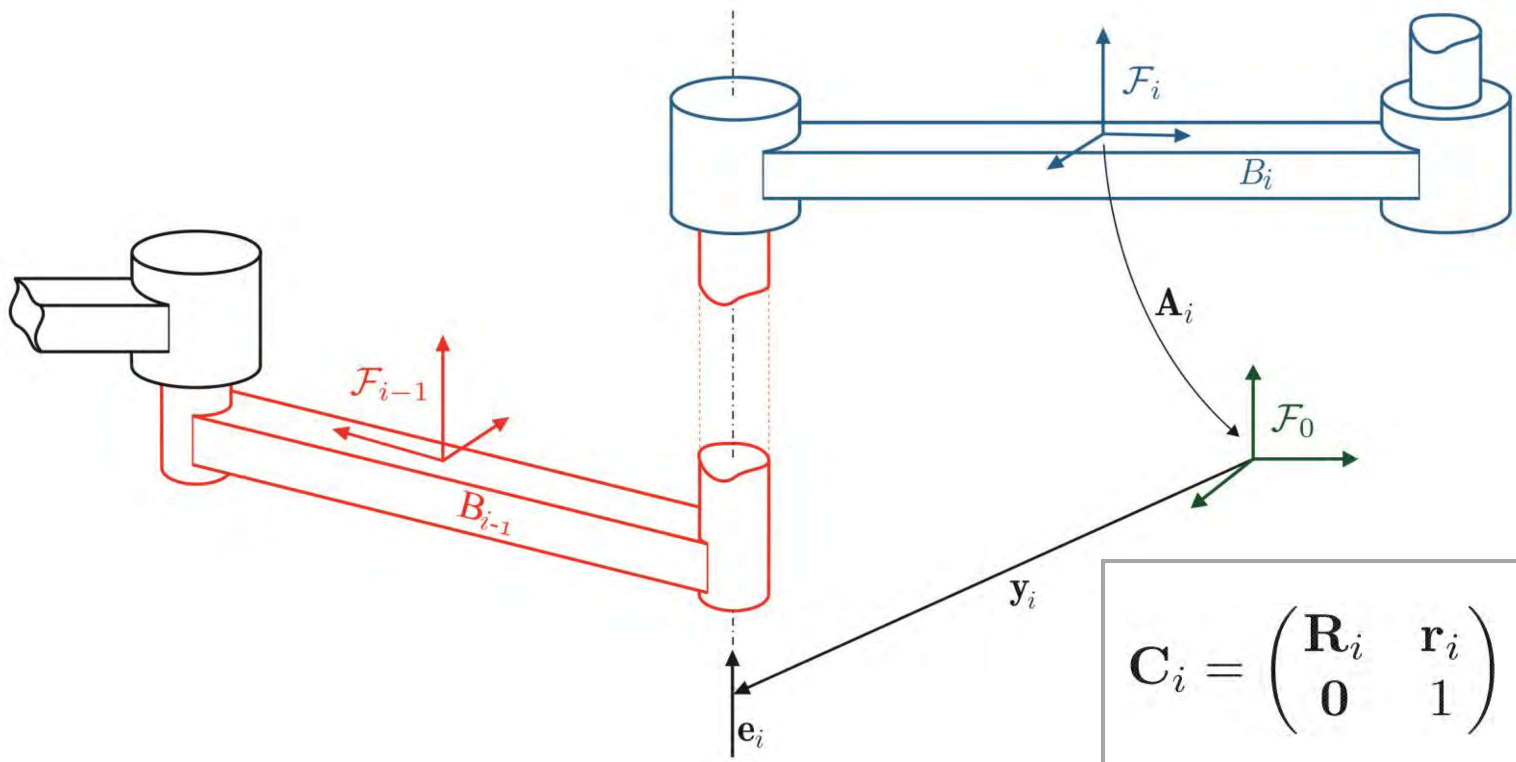
$$\mathbf{Y}_j = \begin{pmatrix} \mathbf{e}_j \\ \mathbf{y}_j \times \mathbf{e}_j + h_j \mathbf{e}_j \end{pmatrix} = \mathbf{Ad}_{\mathbf{A}_j} {}^j\mathbf{X}_j$$



MBS Kinematics: 3.) Without body-fixed JFR or BFR

- Configuration of body i

$$\mathbf{C}_i\left(\mathbf{q}\right) = \exp(\mathbf{Y}_1 q_1)\cdot\exp(\mathbf{Y}_2 q_2)\cdot\ldots\cdot\exp(\mathbf{Y}_i q_i)\mathbf{A}_i$$



$$\mathbf{C}_i = \begin{pmatrix} \mathbf{R}_i & \mathbf{r}_i \\ \mathbf{0} & 1 \end{pmatrix}$$

- Configuration of body i

$$\mathbf{C}_i\left(\mathbf{q}\right) = \mathbf{B}_1 \exp({}^1\mathbf{X}_1 q_1) \cdot \mathbf{B}_2 \exp({}^2\mathbf{X}_2 q_2) \cdot \ldots \cdot \mathbf{B}_i \exp({}^i\mathbf{X}_i q_i)$$
$$= \exp(\mathbf{Y}_1 q_1) \cdot \exp(\mathbf{Y}_2 q_2) \cdot \ldots \cdot \exp(\mathbf{Y}_i q_i)\mathbf{A}_i$$

- Motion of lower pair (Reuleaux pair) is a screw motion
- Configuration of kinematic chain expressed in terms of joint screw coordinates
- Frame invariance allows for various representations of screws

⇨ No need for body-fixed joint frames

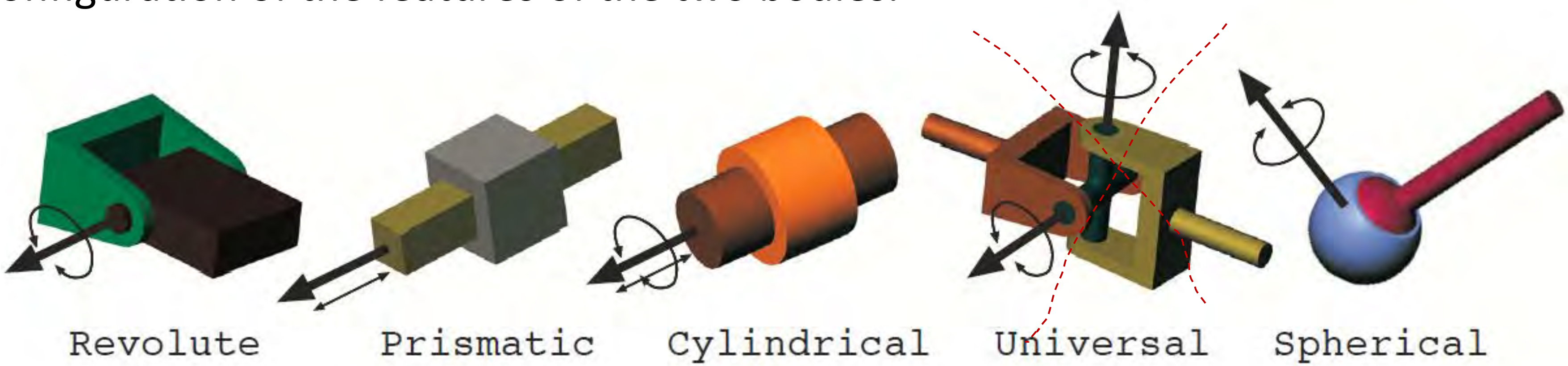
No modeling conventions, e.g. Denavit-Hartenberg

Definition:

A *kinematic joint* is a coincidence relation between features of two adjacent bodies.

Definition:

A *holonomic kinematic joint* imposes geometric constraints on the relative configuration of the features of the two bodies.

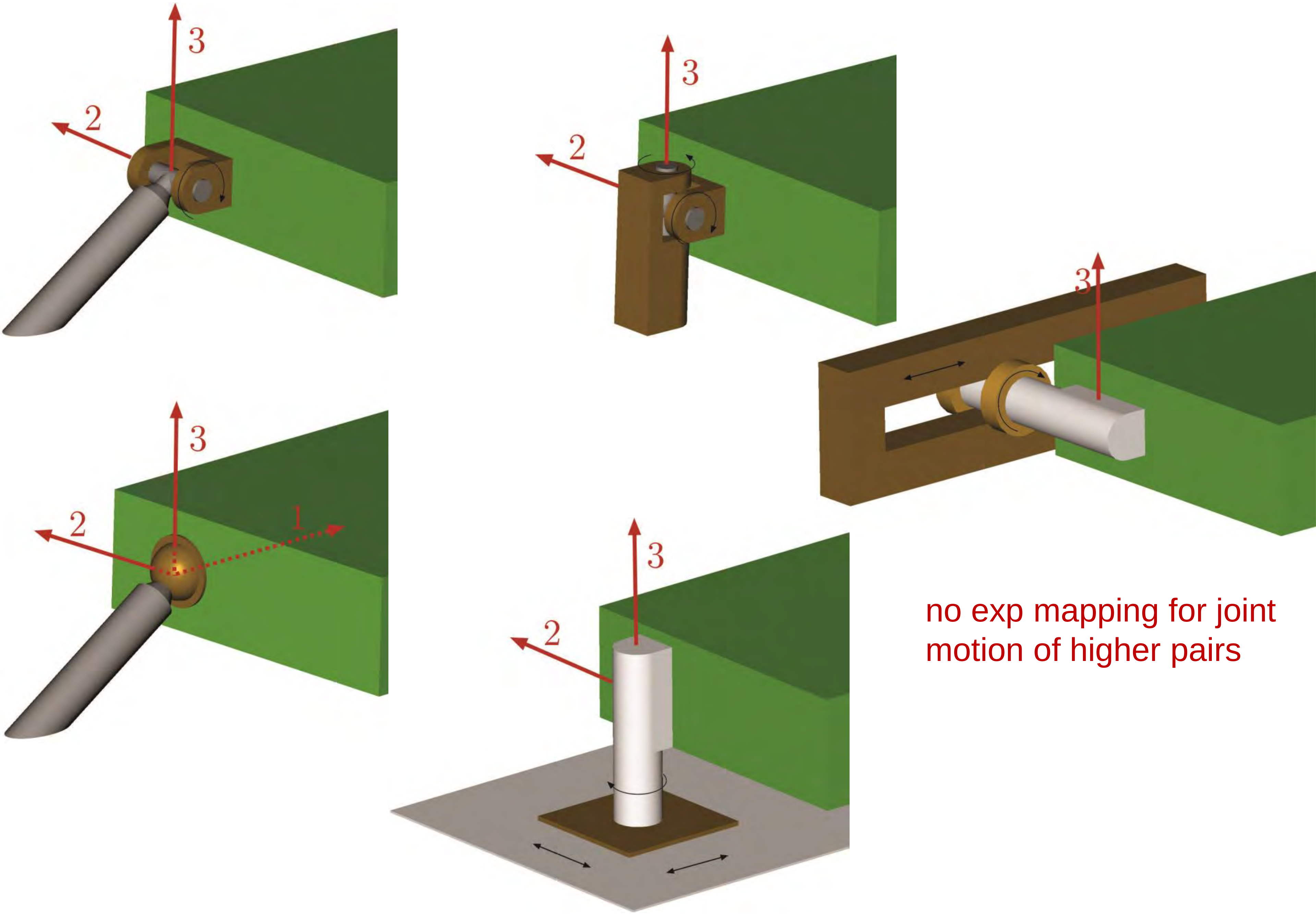


Definition:

A *lower pair (Reuleaux pair)* is a kinematic pair constraining two bodies so that they move relative to each other while remaining in surface contact.

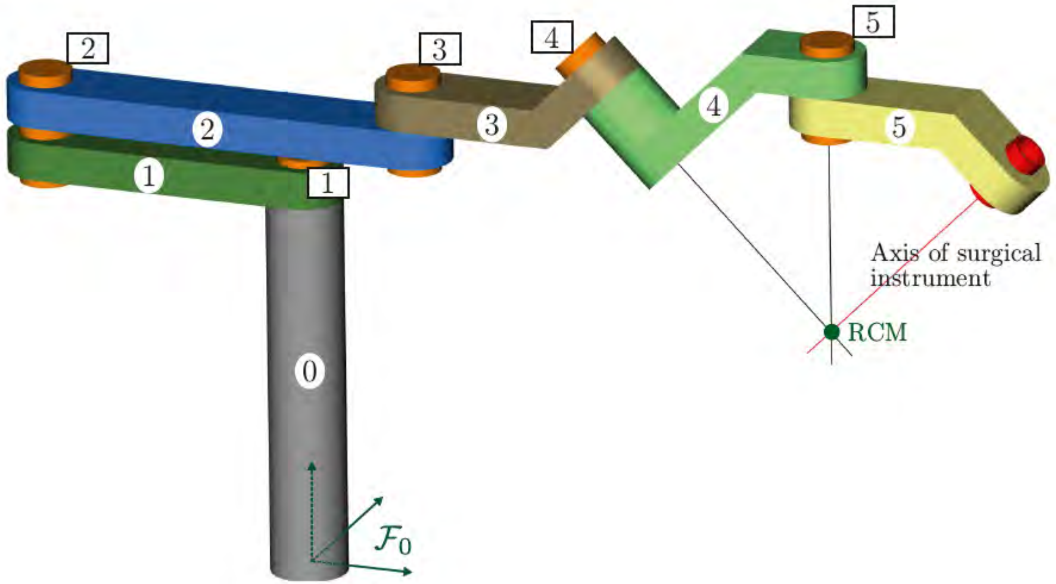
F. Reuleaux: Theoretische Kinematic: Grundzüge einer Theorie des Maschinenwesens, Vieweg, Braunschweig, 1875.
Translation: A.B.W. Kennedy as "The Kinematics of Machinery", Macmillan, London, 1876.
Reprinted, Dover, New York, 1963

Lower-Pairs vs. Higher Pairs



no exp mapping for joint motion of higher pairs

$$\mathbf{C}_i\left(\mathbf{q}\right) = \exp(\mathbf{Y}_1 q_1) \cdot \exp(\mathbf{Y}_2 q_2) \cdot \ldots \cdot \exp(\mathbf{Y}_i q_i)\mathbf{A}_i$$



$$\mathbf{Y}_j = \begin{pmatrix} \mathbf{e}_j \\ \mathbf{y}_j \times \mathbf{e}_j + h_j \mathbf{e}_j \end{pmatrix}$$

Readily available (e.g. CAD)!

Example: Remote Center of Motion (RCM) mechanism – spatial screws

$$\mathbf{C}_i\left(\mathbf{q}\right) = \exp(\mathbf{Y}_1 q_1) \cdot \exp(\mathbf{Y}_2 q_2) \cdot \ldots \cdot \exp(\mathbf{Y}_i q_i) \mathbf{A}_i$$

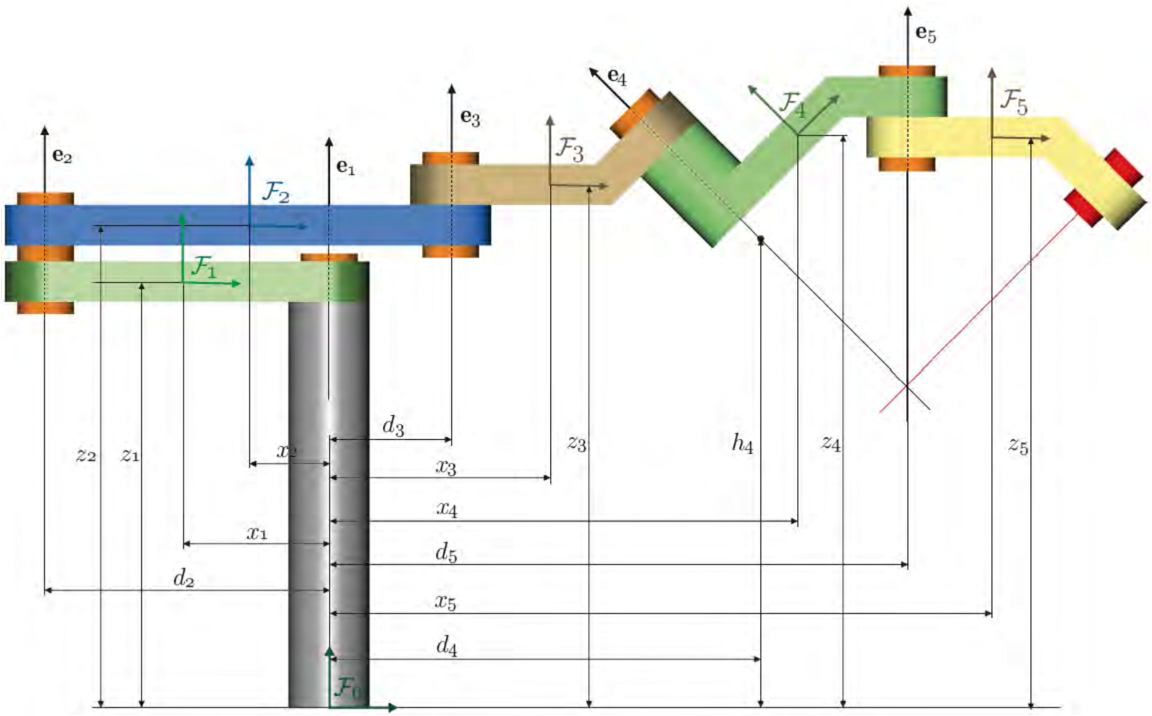
$$\mathbf{y}_1 = (0,0,0)^T, \mathbf{y}_2 = (-d_2, 0, 0)^T$$

$$\mathbf{y}_3 = (d_3, 0, 0)^T, \mathbf{y}_4 = (d_4, 0, h_4)^T$$

$$\mathbf{y}_5 = (d_5, 0, 0)^T$$

$$\mathbf{e}_1 = \mathbf{e}_2 = \mathbf{e}_3 = \mathbf{e}_5 = (0,0,1)^T$$

$$\mathbf{e}_4 = (-(1/\sqrt{2}, 0, 1/\sqrt{2})^T$$



$$\mathbf{Y}_1 = (0,0,1,0,0,0)^T, \mathbf{Y}_2 = (0,0,1,0,d_2,0)^T, \mathbf{Y}_3 = (0,0,1,0,-d_3,0)^T$$

$$\mathbf{Y}_4 = (-1\sqrt{2}, 0, 1/\sqrt{2}, 0, -d_4/\sqrt{2} - h_4/\sqrt{2}, 0)^T, \mathbf{Y}_5 = (0,0,1,0,-d_5,0)^T$$

$$\mathbf{C}_i(\mathbf{q}) = \mathbf{B}_1 \exp({}^1\mathbf{X}_1 q_1) \cdot \mathbf{B}_2 \exp({}^2\mathbf{X}_2 q_2) \cdot \ldots \cdot \mathbf{B}_i \exp({}^i\mathbf{X}_i q_i)$$

$${}^1\mathbf{e}_1 = {}^4\mathbf{e}_4 = {}^6\mathbf{e}_6 = (0,0,1)^T$$

$${}^2\mathbf{e}_2 = {}^3\mathbf{e}_3 = {}^5\mathbf{e}_5 = (0,1,0)^T$$

$${}^i\mathbf{x}_i = \mathbf{0}, i = 1, \ldots, 6$$

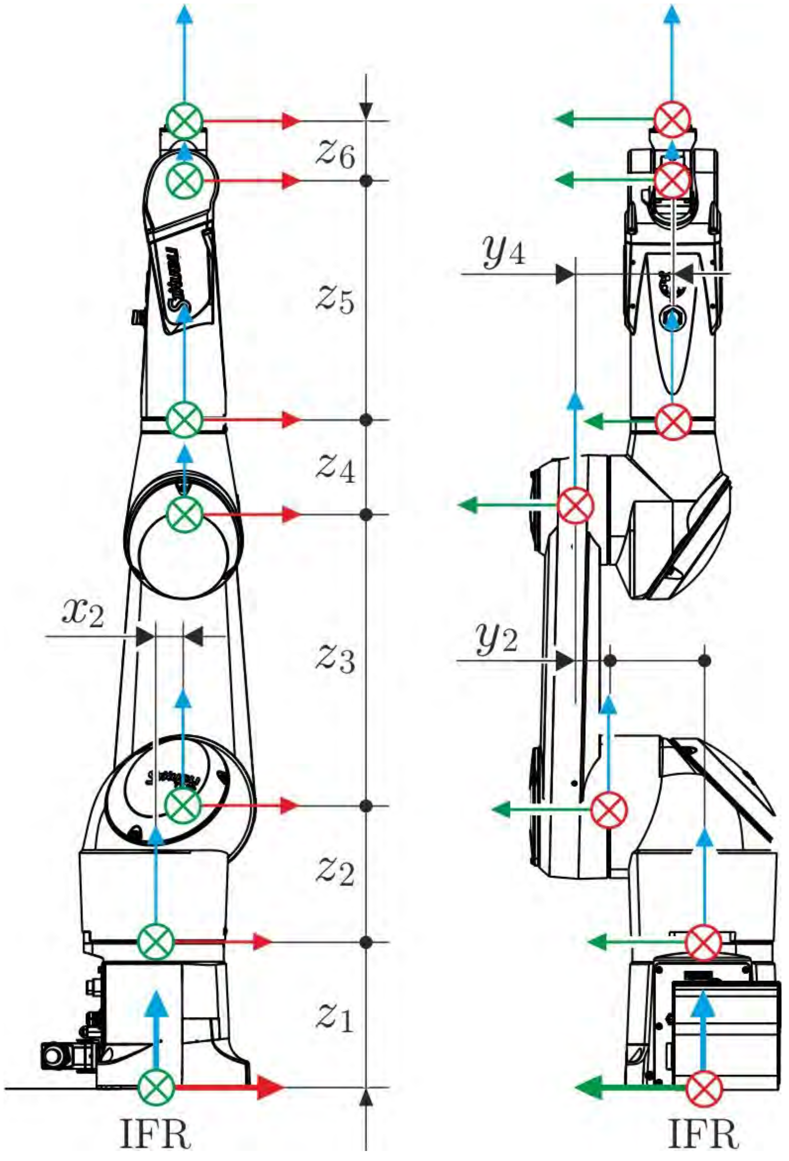
$$\mathbf{r}_1 = (0,0,z_1)^T, \mathbf{r}_2 = (x_2, y_2, z_2)^T$$

$$\mathbf{r}_3 = (0, y_3, z_3)^T, \mathbf{r}_4 = (0, -y_4, z_4)^T$$

$$\mathbf{r}_5 = (0,0,z_5)^T, \mathbf{r}_6 = (0,0,z_6)^T$$

$$\Rightarrow \quad {}^i\mathbf{X}_i = ({}^i\mathbf{e}_i, \mathbf{0})^T$$

$$\mathbf{B}_i = \begin{pmatrix} \mathbf{I} & \mathbf{r}_i \\ \mathbf{0} & 1 \end{pmatrix}$$



Example: Metamorphic Hand with Reconfigurable Palm – spatial screws

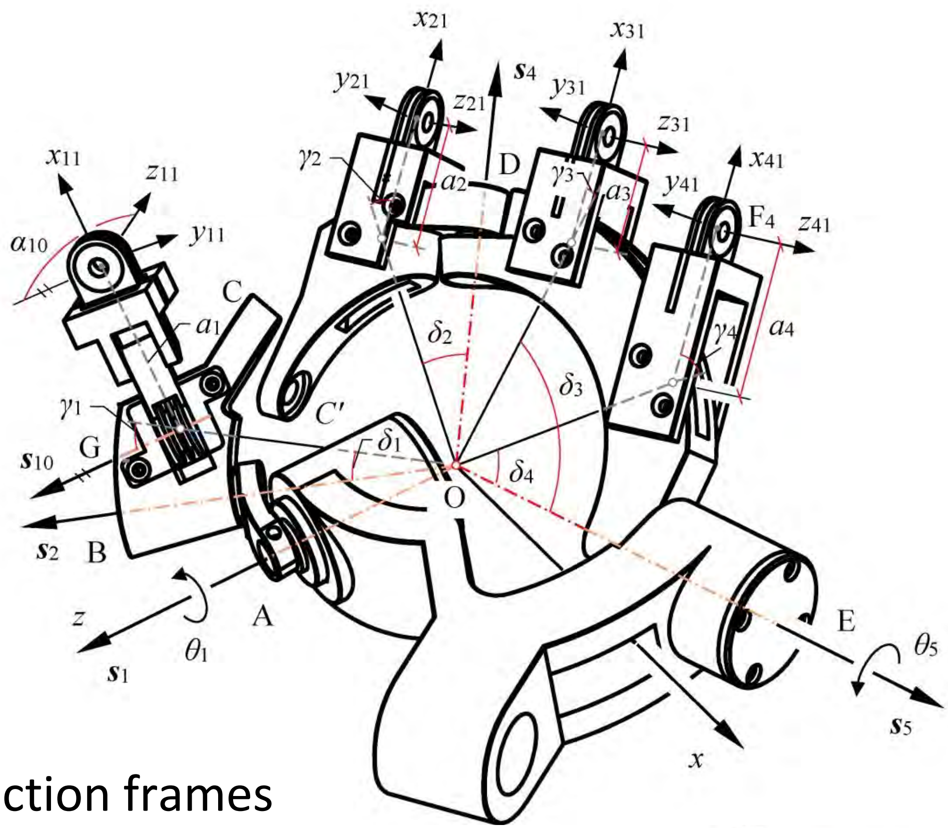
Palm is a spherical mechanism

- Joint screws in IFR at center of rotation:

$$\mathbf{Y}_1 = \begin{pmatrix} \mathbf{e}_A \\ \mathbf{0} \end{pmatrix} \qquad \mathbf{Y}_2 = \begin{pmatrix} \mathbf{e}_B \\ \mathbf{0} \end{pmatrix}$$

$$\mathbf{Y}_3 = \begin{pmatrix} \mathbf{e}_C \\ \mathbf{0} \end{pmatrix} \qquad \mathbf{Y}_4 = \begin{pmatrix} \mathbf{e}_D \\ \mathbf{0} \end{pmatrix}$$

$$\mathbf{Y}_5 = \begin{pmatrix} \mathbf{e}_E \\ \mathbf{0} \end{pmatrix}$$

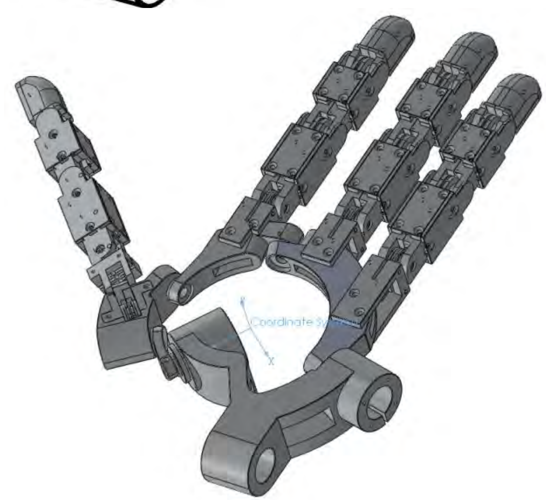


- It is most natural to locate the construction frames at center of curvature

$$\mathbf{C}_1(\mathbf{q}) = \exp(\mathbf{Y}_1 q^1) = \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$$

$$\mathbf{C}_2(\mathbf{q}) = \exp(\mathbf{Y}_1 q^1)\exp(\mathbf{Y}_2 q^2) = \begin{pmatrix} \mathbf{R}_2 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$$

…



Forward kinematic mapping of serial manipulators

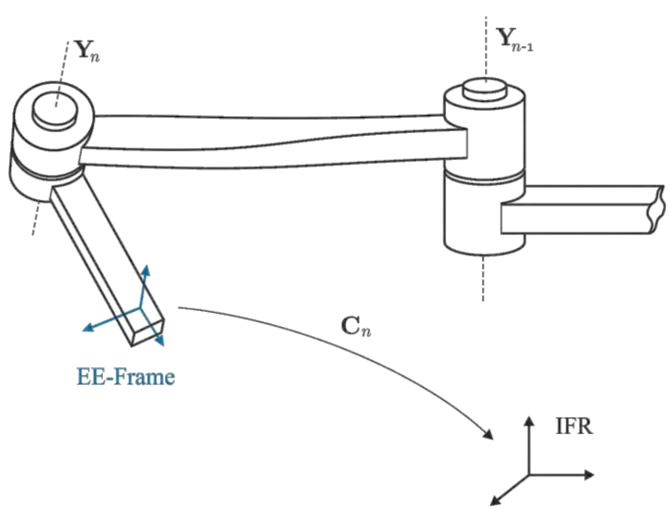
- Serial manipulator = Kinematic chain comprising n joints/link
- Forward kinematic mapping:

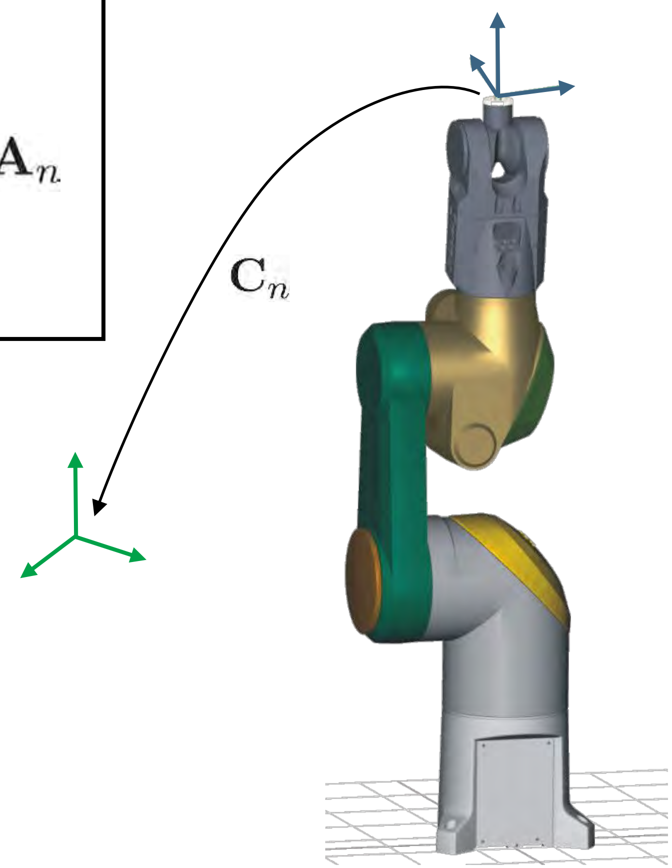
$$f_{\mathrm{FK}} : \mathbb{V}^n \rightarrow SE(3)$$

‘joint space’ workspace

$$\begin{aligned} f_{\mathrm{FK}}(\mathbf{q}) &= \exp(\mathbf{Y}_1 q_1) \cdot \exp(\mathbf{Y}_2 q_2) \cdot \ldots \cdot \exp(\mathbf{Y}_n q_n) \mathbf{A}_n \\ &= \mathbf{B}_1 \exp({}^1\mathbf{X}_1 q_1) \cdot \ldots \cdot \mathbf{B}_n \exp({}^n\mathbf{X}_n q_n) \end{aligned}$$

What happens for changing geometry?





- Kinematic loop comprising n 1-DOF lower-pair joints
- Geometric loop constraints:

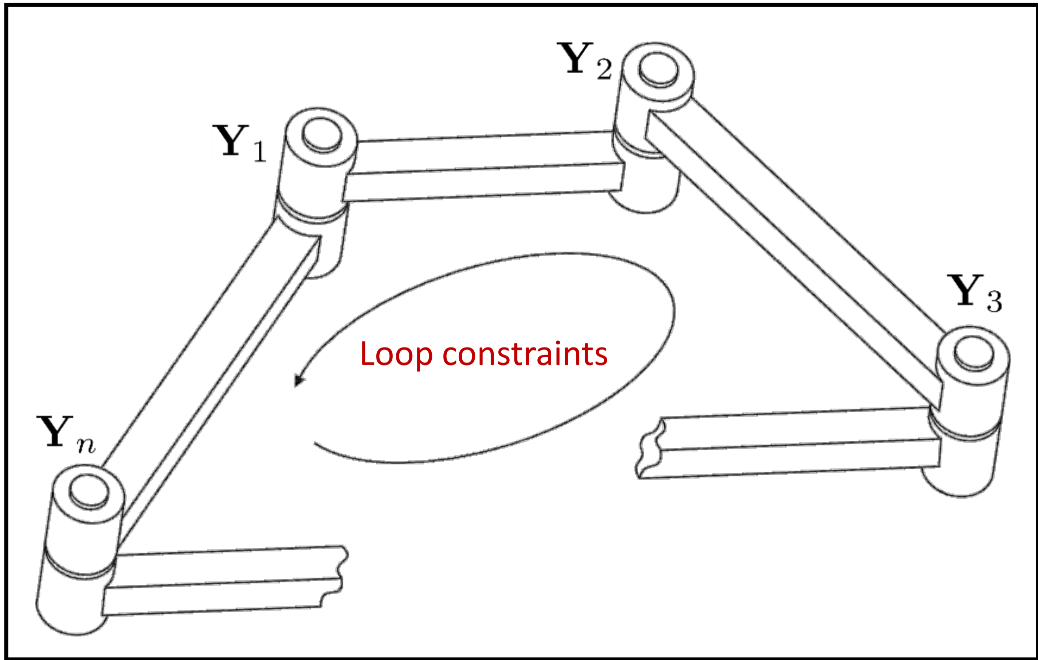
$$\boxed{f(\mathbf{q}) = \mathbf{I}}$$

with constraint mapping $f : \mathbb{V}^n \to SE(3)$

$$f\,(\mathbf{q}) = \exp(\mathbf{Y}_1 q_1)\exp(\mathbf{Y}_2 q_2) \cdot \ldots \cdot \exp(\mathbf{Y}_n q_n)$$

Screw coordinate vector of joint j in spatial representation:

$$\mathbf{Y}_j = \begin{pmatrix} \mathbf{e}_j \\ \mathbf{y}_j \times \mathbf{e}_j + h_j \mathbf{e}_j \end{pmatrix}$$



- Kinematic loop comprising n 1-DOF lower-pair joints
- Geometric loop constraints:

$$\boxed{f(\mathbf{q}) = \mathbf{I}}$$

with constraint mapping $f : \mathbb{V}^n \to SE(3)$

$$f(\mathbf{q}) = \exp(\mathbf{Y}_1 q_1)\exp(\mathbf{Y}_2 q_2)\cdot \ldots \cdot \exp(\mathbf{Y}_n q_n)$$

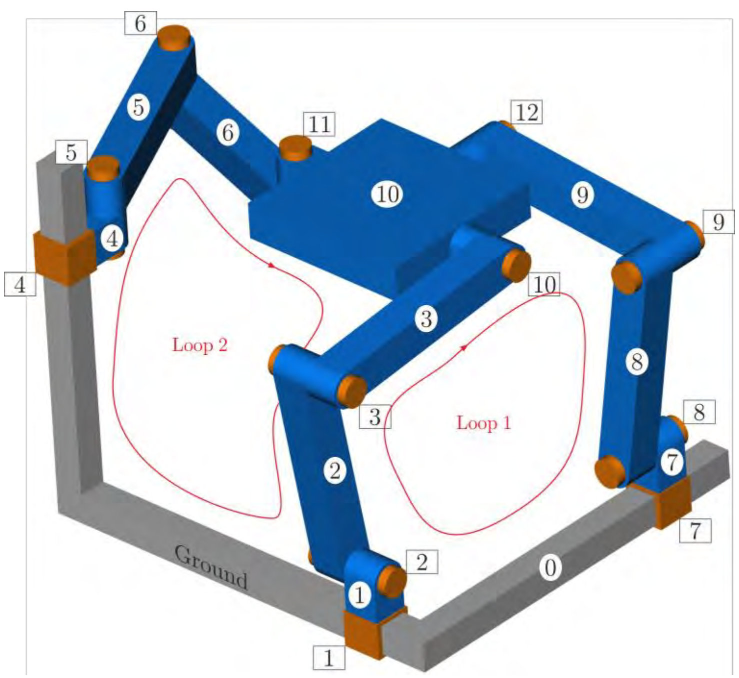
- Configuration space is the analytic variety:

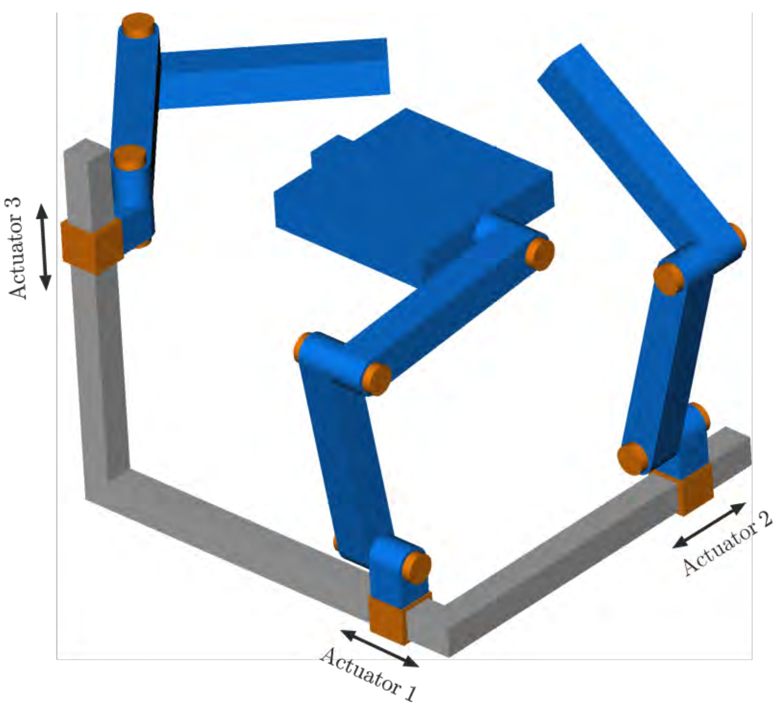
$$\boxed{V := \{\mathbf{q} \in \mathbb{V}^n | f(\mathbf{q}) = \mathbf{I}\}}$$

- The local DOF at $\mathbf{q} \in V$:

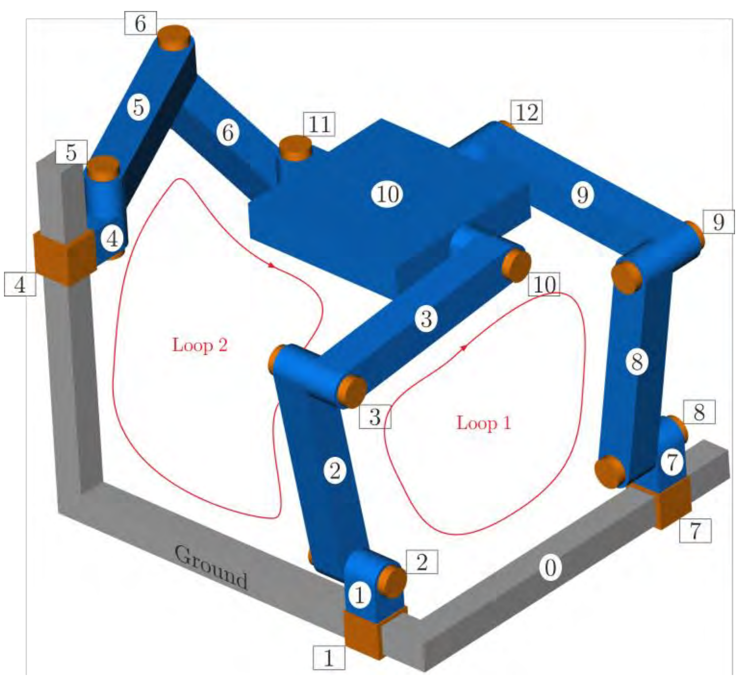
$$\boxed{\delta_{\mathrm{loc}}(\mathbf{q}) := \dim_{\mathbf{q}} V}$$

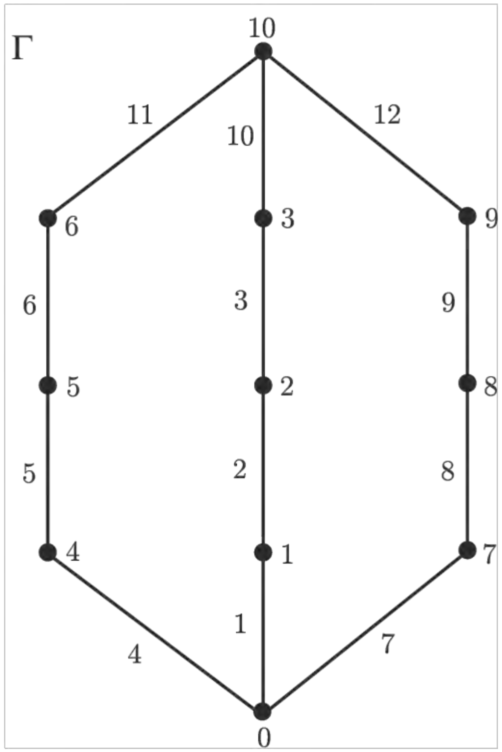
- Topologically independent loops: Fundamental Cycles (FC)



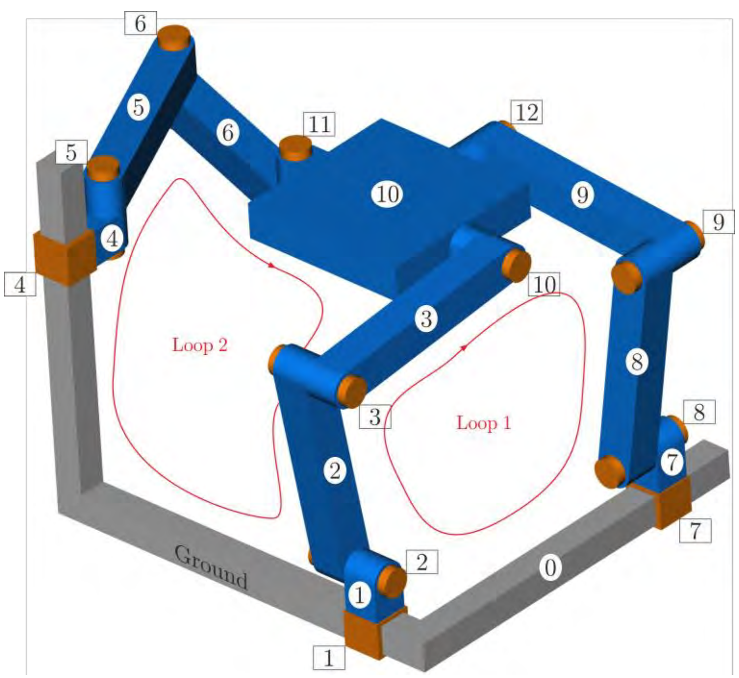


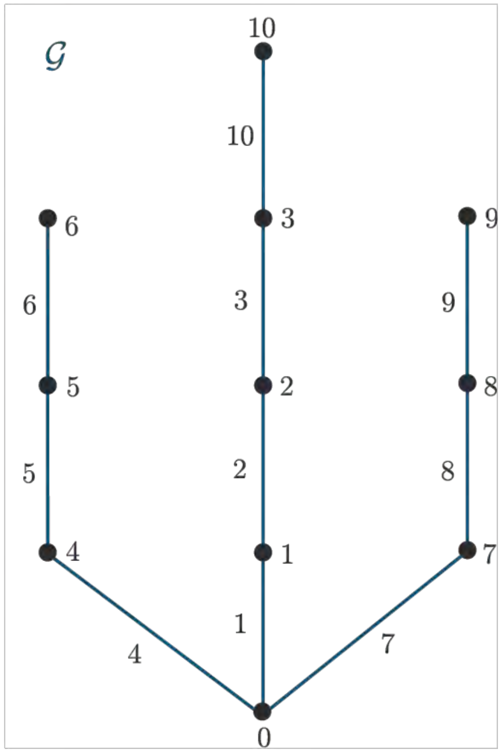
- Topologically independent loops: Fundamental Cycles (FC)





- Topologically independent loops: Fundamental Cycles (FC)



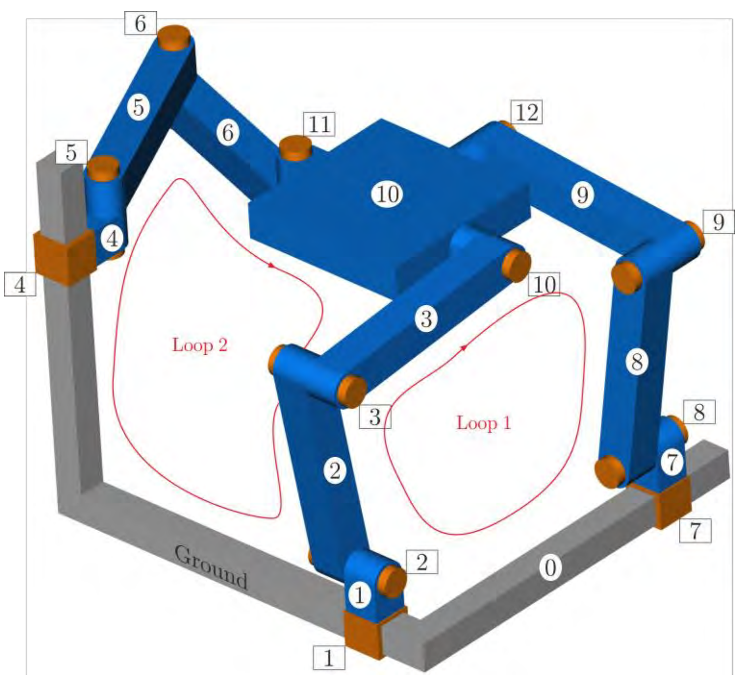


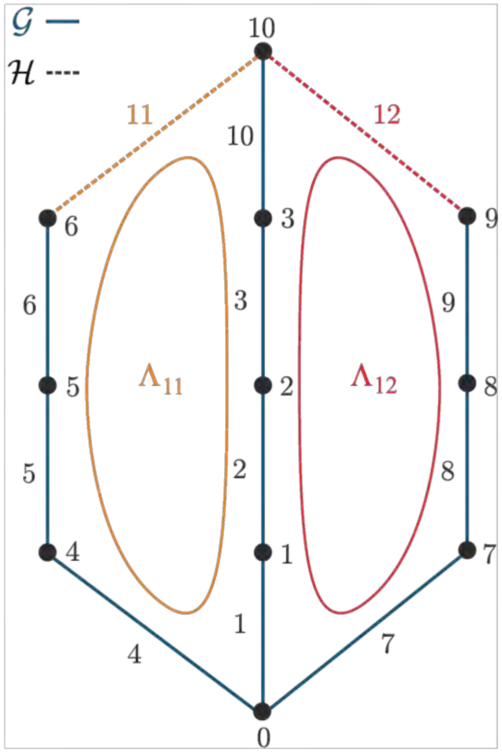
- Topologically independent loops: Fundamental Cycles (FC) $l = 1, \ldots, \gamma$
- Geometric loop constraints:

$$\boxed{f_l\left(\mathbf{q}\right) = \mathbf{I}, l = 1, \ldots, \gamma}$$

with constraint mapping of FC l:

$$f_l\left(\mathbf{q}\right) := \exp(\sigma_{l,\bar{l}} q_{\bar{l}} \mathbf{Y}_{\bar{l}}) \cdots \exp(\sigma_{l,l-1} q_{l-1} \mathbf{Y}_{l-1}) \exp(\mathbf{Y}_l q_l)$$





A rigid body motion is an *isometric and orientation preserving transformation*.
It can be considered as a frame transformation

Definition:

Special Euclidean Group (in 3 dimensions)

$$\boxed{SE(3) := \left\{ \begin{pmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix} \mid \mathbf{R} \in SO(3), \mathbf{r} \in \mathbb{R}^3 \right\}}$$

- Inverse of $\mathbf{C} \in SE(3)$: $\mathbf{C}^{-1} = \begin{pmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{r} \\ \mathbf{0} & 1 \end{pmatrix}$
- Identity: $\mathbf{I}_4$
- Group multiplication: $\mathbf{C}_1 \cdot \mathbf{C}_2$

Lie algebra of $SE(3)$:

$$\boxed{se(3) := \left\{ \begin{pmatrix} \widetilde{\boldsymbol{\xi}} & \boldsymbol{\eta} \\ \mathbf{0} & 0 \end{pmatrix} \mid \widetilde{\boldsymbol{\xi}} \in so(3), \boldsymbol{\eta} \in \mathbb{R}^3 \right\}}$$

- Velocity constraints at $\mathbf{q} \in V$:

$$\mathbf{0} = \frac{d}{dt} f(\mathbf{q}) = f(\mathbf{q})^{-1} \frac{d}{dt} f(\mathbf{q}) \qquad\qquad f(\mathbf{q}) = \mathbf{I}, \mathbf{q} \in V$$

$$\boxed{\mathbf{0} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{S}_1(\mathbf{q})\dot{q}_1 + \cdots + \mathbf{S}_n(\mathbf{q})\dot{q}_n}$$

Geometric Jacobian: $\mathbf{J}\left(\mathbf{q}\right) = \left(\mathbf{S}_1(\mathbf{q}), \mathbf{S}_2(\mathbf{q}), \cdots, \mathbf{S}_n(\mathbf{q})\right)$

Instantaneous joint screw coordinates: $\mathbf{S}_j(\mathbf{q}) := \mathbf{Ad}_{g_j(\mathbf{q})}\mathbf{Y}_j$

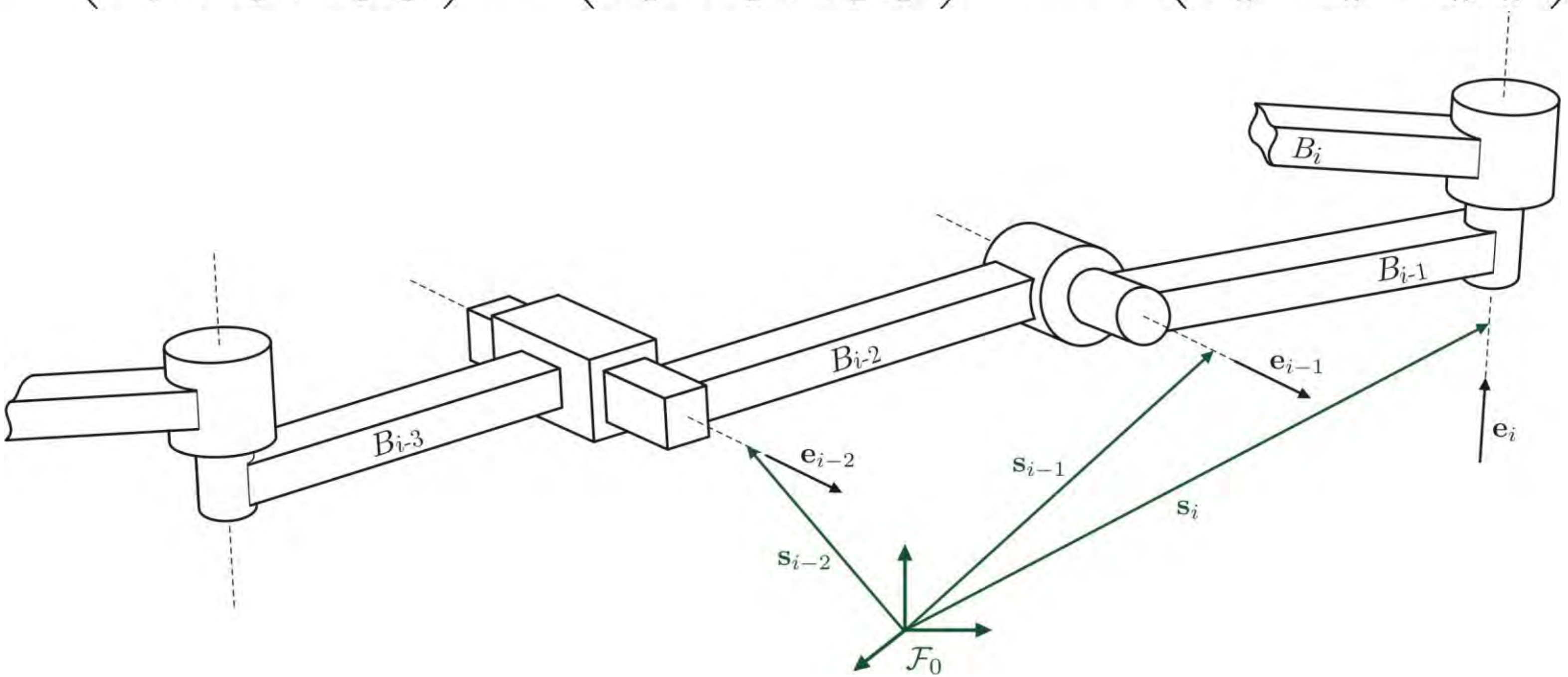
with $g_j(\mathbf{q}) := \exp(\mathbf{Y}_1 q_1) \cdot \ldots \cdot \exp(\mathbf{Y}_j q_j)$

- Velocity constraints at $\mathbf{q} \in V$:

$$\mathbf{0} = \frac{d}{dt} f(\mathbf{q}) = f(\mathbf{q})^{-1} \frac{d}{dt} f(\mathbf{q}) \qquad f(\mathbf{q}) = \mathbf{I}, \mathbf{q} \in V$$

$$\boxed{\mathbf{0} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{S}_1(\mathbf{q})\dot{q}_1 + \cdots + \mathbf{S}_n(\mathbf{q})\dot{q}_n}$$

$$\mathbf{0} = \dot{q}_1 \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{s}_1 \times \mathbf{e}_1 + h_1 \mathbf{e}_1 \end{pmatrix} + \dot{q}_2 \begin{pmatrix} \mathbf{e}_2 \\ \mathbf{s}_2 \times \mathbf{e}_2 + h_2 \mathbf{e}_2 \end{pmatrix} + \ldots + \dot{q}_n \begin{pmatrix} \mathbf{e}_n \\ \mathbf{s}_n \times \mathbf{e}_n + h_n \mathbf{e}_n \end{pmatrix}$$



- Velocity constraints at $\mathbf{q} \in V$:

$$\mathbf{0} = \frac{d}{dt} f(\mathbf{q}) = f(\mathbf{q})^{-1} \frac{d}{dt} f(\mathbf{q}) \qquad f(\mathbf{q}) = \mathbf{I}, \mathbf{q} \in V$$

$$\boxed{\mathbf{0} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{S}_1(\mathbf{q})\dot{q}_1 + \cdots + \mathbf{S}_n(\mathbf{q})\dot{q}_n}$$

- The differential DOF at $\mathbf{q} \in V$:

$$\boxed{\delta_{\mathrm{diff}}(\mathbf{q}) := n - \operatorname{rank} \mathbf{J}(\mathbf{q})}$$

(instantaneous mobility)

- If $\operatorname{rank} \mathbf{J}(\mathbf{q})$ is not maximal, then $\mathbf{q} \in \mathbb{V}^n$ is a critical point of $f : \mathbb{V}^n \to SE(3)$

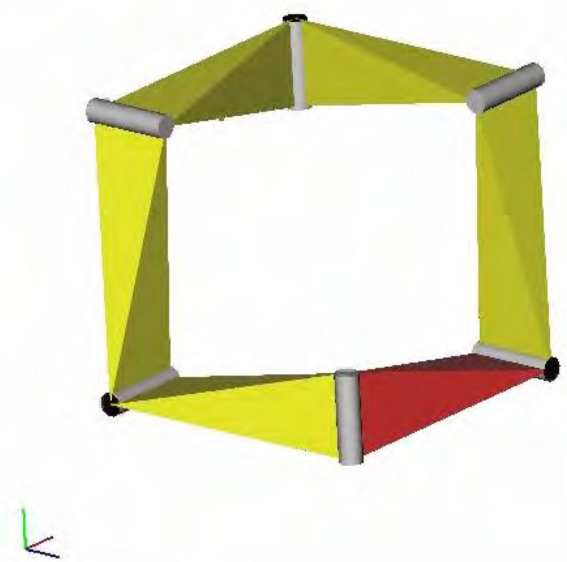
  That is, the constraints become dependent

Definition:

A constraint singularity of a kinematic chain is a configuration $\mathbf{q} \in V$ which is a critical point of the constraint mapping, i.e. $\mathbf{0} = \mathbf{J}(\mathbf{q})$ is not full rank

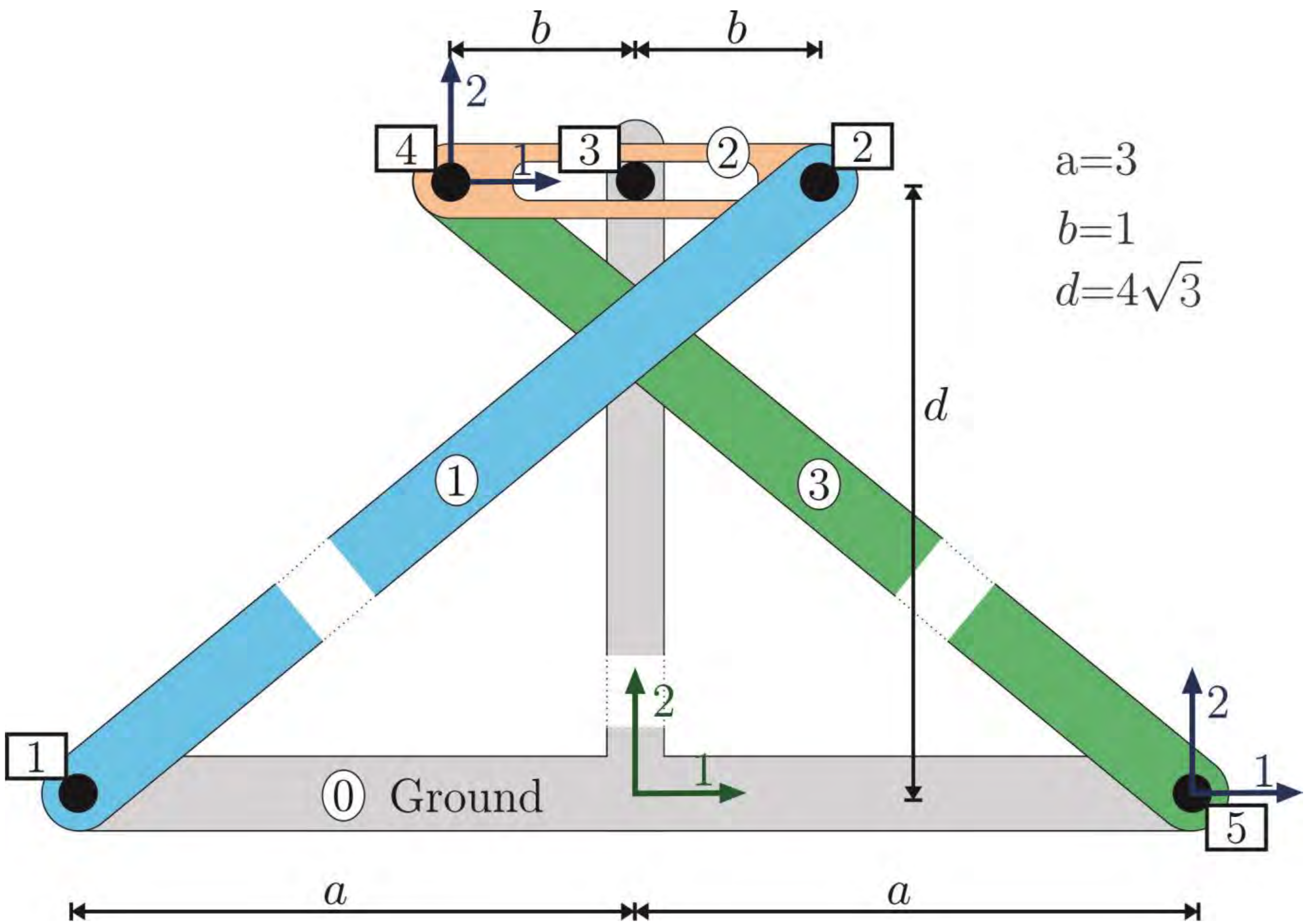
The c-space of overconstrained mechanisms consists of manifolds of constraint singularities

It depends on the constraints used to describe the mechanism



$\operatorname{rank} \mathbf{J}(\mathbf{q}) = 5$

What is the DOF of this two-loop mechanism?



1. Phenomenology
   - Mobility
   - Constraint-, C-Space-, Input-, Output-Singularities
   - A Model for the Mechanism Kinematics
2. Kinematics Modeling
   - Screws and the Product-of-Exponentials
   - Geometric Jacobian
   - Higher-Order Kinematics
3. Local analysis of C-Space and ‘singular set’
   - Tangent space, Tangent cone
   - ‘Kinematic tangent cone’
4. Further Topics
   - Singularities of Non-Holonomic Systems
   - Escapement from singularities
   - Combinatorial algorithm: Pebble Game

Definition 1:

A kinematic singularity is a configuration $\mathbf{q} \in V$ where the differential (instantaneous) DOF is not constant in any neighborhood $U(\mathbf{q})$ of $\mathbf{q}$.

- Property if the functions defining V – the generating ideal

⇨ Investigate set of points of certain rank

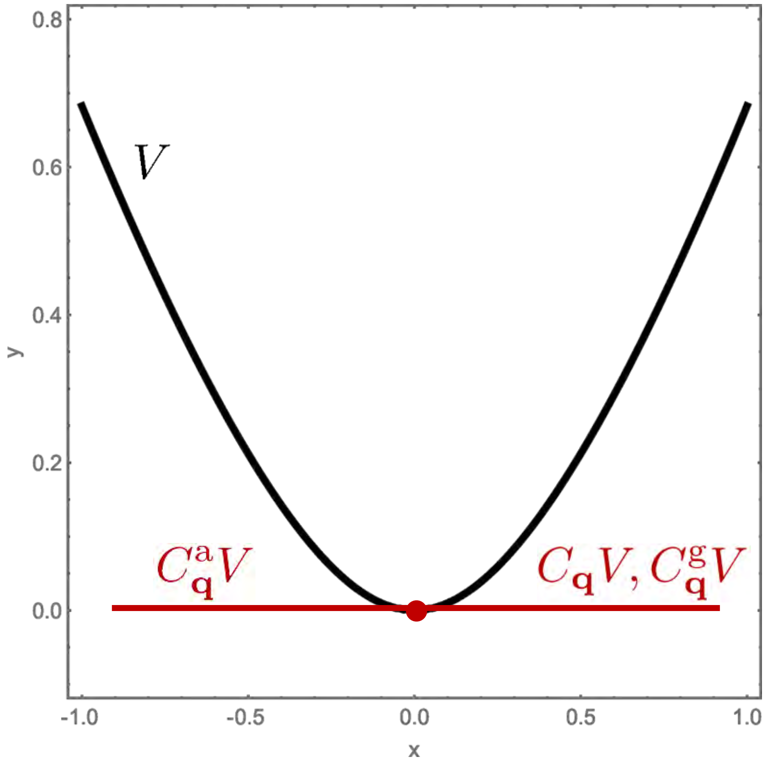
Definition 2:

A C-space singularity is a configuration $\mathbf{q} \in V$ where V is not a smooth manifold.

- Topological property of the point set V

⇨ Investigate local geometry of V

- Geometry vs. Kinematics:



$$V = \mathbf{V}(y^3 + yx^2 - x^4) \subset \mathbb{R}^2$$

$$\mathbf{J}(\mathbf{q}) = \left(-4x^3 + 2xy \quad x^2 + 3y^2 \right)$$

V is a smooth manifold

but $\operatorname{rank} \mathbf{J}(\mathbf{q}) = \begin{cases} 0, & \mathbf{q} = \mathbf{0} \\ 1, & \mathbf{q} \neq \mathbf{0} \end{cases}$

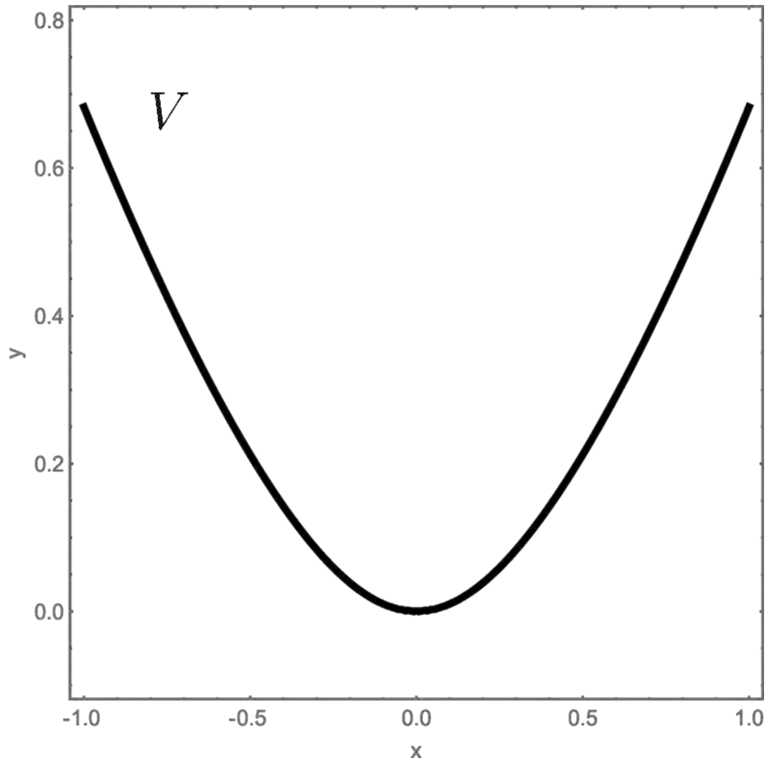
$\mathbf{q} = \mathbf{0}$ is a kinematic singularity

$T_{\mathbf{0}}V = \ker \mathbf{J}(\mathbf{0}) = \mathbb{R}^2$?

$$\begin{aligned} C^{\mathrm{a}}_{\mathbf{q}}V &= \mathbf{V}(y^3 + yx^2) \\ &= \{(x,y) | y = 0\} \end{aligned}$$

- Let V be a real point set in $\mathbb{V}^n$. V is an analytic variety if and only if it is closed and for each $\mathbf{q} \in \mathbb{V}^n$ there is a neighborhood $U(\mathbf{q})$ in $\mathbb{V}^n$ and analytic functions $f_1, \ldots, f_m$ such that $V \cap U(\mathbf{q})$ is the set of common zeros of f_i in $U(\mathbf{q})$.

Point set in $\mathbb{R}^2$:



Function:

$$f(x, y) = y^3 + 2x^2y - x^4)$$

Variety:

$$V = \mathbf{V}(y^3 + 2x^2y - x^4) \subset \mathbb{R}^2$$

- Let V be a real point set in $\mathbb{V}^n$. V is an analytic variety if and only if it is closed and for each $\mathbf{q} \in \mathbb{V}^n$ there is a neighborhood $U(\mathbf{q})$ in $\mathbb{V}^n$ and analytic functions $f_1, \ldots, f_m$ such that $V \cap U(\mathbf{q})$ is the set of common zeros of f_i in $U(\mathbf{q})$.
- The tangent space $T_{\mathbf{q}}V$ to the variety V at $\mathbf{q}$ is the vector space

$$T_{\mathbf{q}}V = \{\mathbf{x} \in \mathbb{R}^n | df(\mathbf{q})\,\mathbf{x} = 0, \forall f \in I(V, \mathbf{q})\}$$

where $I(V, \mathbf{q})$ is the ideal of analytic germs vanishing over V at $\mathbf{q}$
(Think of all function that are zero at $\mathbf{q}$ and locally look identical)

- Let V be a real point set in $\mathbb{V}^n$. V is an analytic variety if and only if it is closed and for each $\mathbf{q} \in \mathbb{V}^n$ there is a neighborhood $U(\mathbf{q})$ in $\mathbb{V}^n$ and analytic functions $f_1, \ldots, f_m$ such that $V \cap U(\mathbf{q})$ is the set of common zeros of f_i in $U(\mathbf{q})$.
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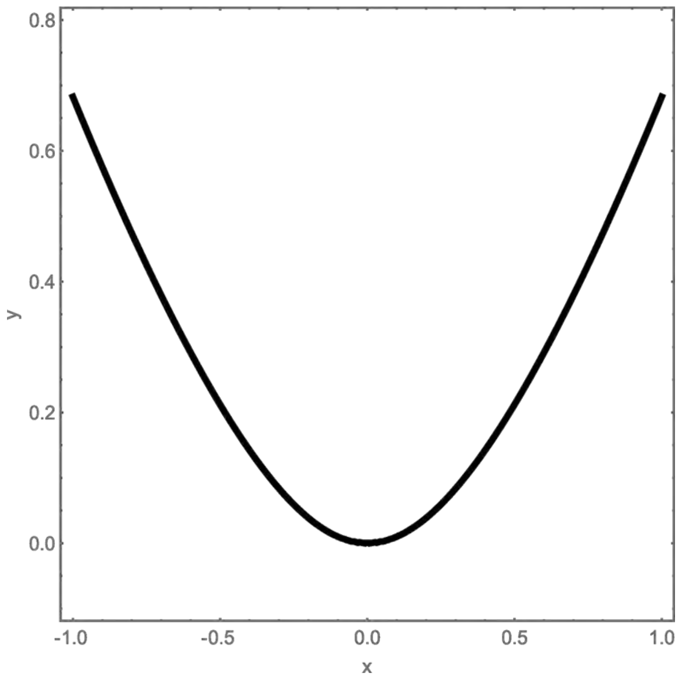


$$f(x, y) = y^3 + 2x^2y - x^4)$$

$$\mathbf{J}(\mathbf{q}) = \begin{pmatrix} -4x^3 + 2xy & x^2 + 3y^2 \end{pmatrix}$$

$$\operatorname{rank} \mathbf{J}(\mathbf{q}) = \begin{cases} 0, & \mathbf{q} = \mathbf{0} \\ 1, & \mathbf{q} \neq \mathbf{0} \end{cases}$$

$$T_{\mathbf{0}}V = \ker \mathbf{J}(\mathbf{0}) = \mathbb{R}^2$$

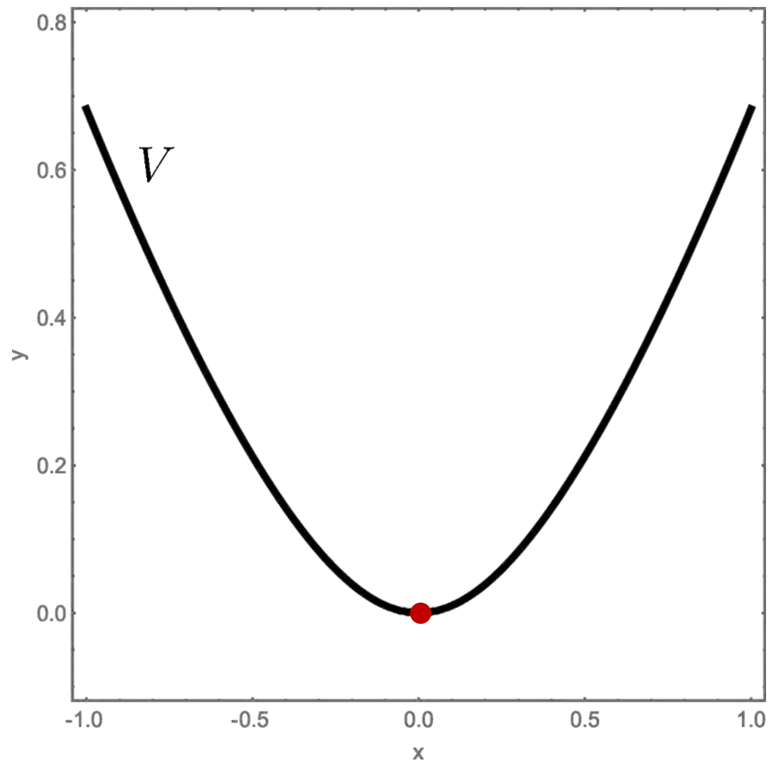
- Let V be a real point set in $\mathbb{V}^n$. V is an analytic variety if and only if it is closed and for each $\mathbf{q} \in \mathbb{V}^n$ there is a neighborhood $U(\mathbf{q})$ in $\mathbb{V}^n$ and analytic functions $f_1, \ldots, f_m$ such that $V \cap U(\mathbf{q})$ is the set of common zeros of f_i in $U(\mathbf{q})$.
- The tangent space $T_{\mathbf{q}}V$ to the variety V at $\mathbf{q}$ is the vector space

$$T_{\mathbf{q}}V = \{\mathbf{x} \in \mathbb{R}^n | df(\mathbf{q})\mathbf{x} = 0, \forall f \in I(V, \mathbf{q})\}$$

where $I(V, \mathbf{q})$ is the ideal of analytic germs vanishing over V at $\mathbf{q}$ (Think of all function that are zero at $\mathbf{q}$ and locally look identical)

- $\mathbf{q}$ is a regular point of the variety V iff there is a neighborhood $U(\mathbf{q})$ and m analytic functions such that the differentials $df_1, \ldots, df_m$ are linearly independent and $V \cap U(\mathbf{q})$ is the set of common zeros of f_i in $U(\mathbf{q})$.

$\Rightarrow$ $\mathbf{q}$ is a regular point of the variety iff the dimension of the tangent space is constant in some neighborhood



$\mathbf{q} = \mathbf{0}$ is a singularity of the variety

$$\operatorname{rank} \mathbf{J}(\mathbf{q}) = \begin{cases} 0, & \mathbf{q} = \mathbf{0} \\ 1, & \mathbf{q} \neq \mathbf{0} \end{cases}$$

- $\mathbf{q}$ is a regular point of the variety V iff there is a neighborhood $U(\mathbf{q})$ and m analytic functions such that the differentials $df_1, \ldots, df_m$ are linearly independent and $V \cap U(\mathbf{q})$ is the set of common zeros of f_i in $U(\mathbf{q})$.

⇨ $\mathbf{q}$ is a regular point of the variety iff the dimension of the tangent space is constant in some neighborhood

- The set V is a manifold at $\mathbf{q} \in \mathbb{V}^n$ if $\mathbf{q}$ is a regular point of the variety V.

- It is important to distinguish between the topological set V and the variety.

- From a kinematic perspective the set V (i.e. the c-space) seems to be of primary interest.
  This is not always sufficient for capturing kinematic singularities, however.

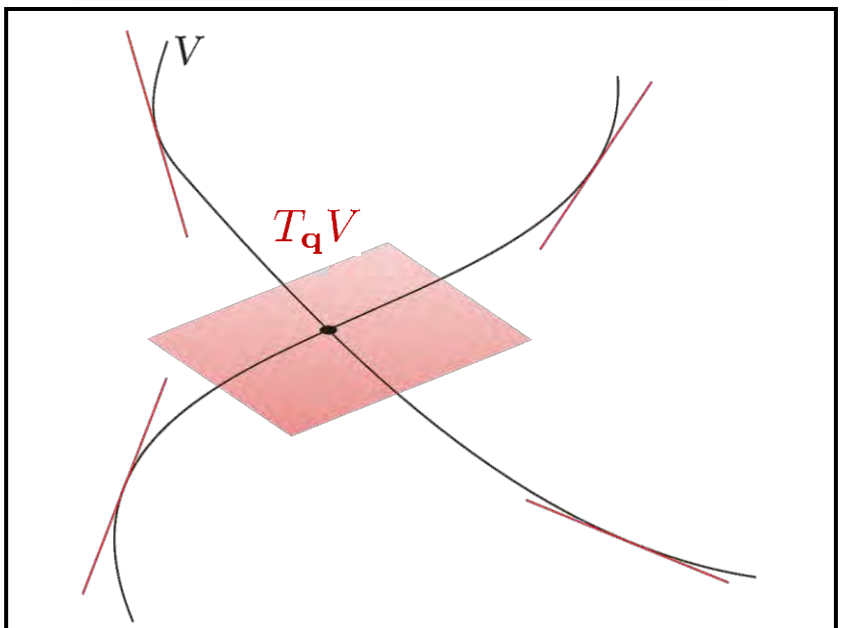
- The tangent space to the c-space V can in general not be defined as the kernel of $\mathbf{J}$

- Joint variables: $\mathbf{q} \in \mathbb{V}^n$
- Constraints: $h(\mathbf{q}) = \mathbf{0}$

$$\mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$$

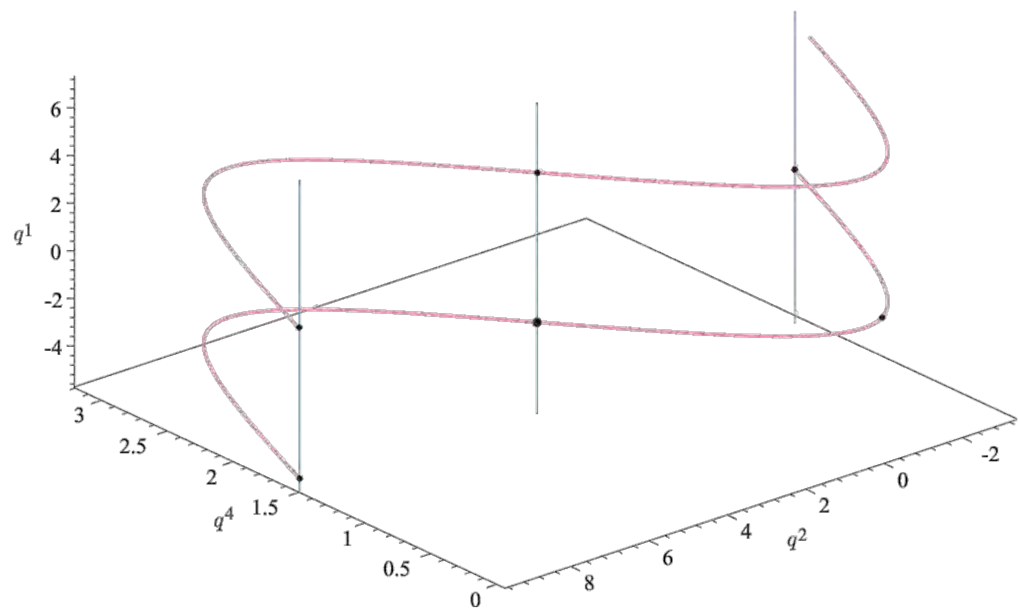
- Differential DOF: $\delta_{\text{diff}}(\mathbf{q}) := n - \operatorname{rank} \mathbf{J}(\mathbf{q})$
- Configuration space: $V := \{\mathbf{q} \in \mathbb{V}^n | h(\mathbf{q}) = \mathbf{0}\}$
- Local (finite) DOF: $\delta_{\text{loc}}(\mathbf{q}) := \dim_{\mathbf{q}} V$

Rank drops at bifurcation point



$\delta_{\text{diff}} = 1, 2$ $\delta_{\text{loc}} = 1$

C-space is non-smooth at bifurcation point



- Joint variables: $\mathbf{q} \in \mathbb{V}^n$
- Constraints: $h(\mathbf{q}) = \mathbf{0}$

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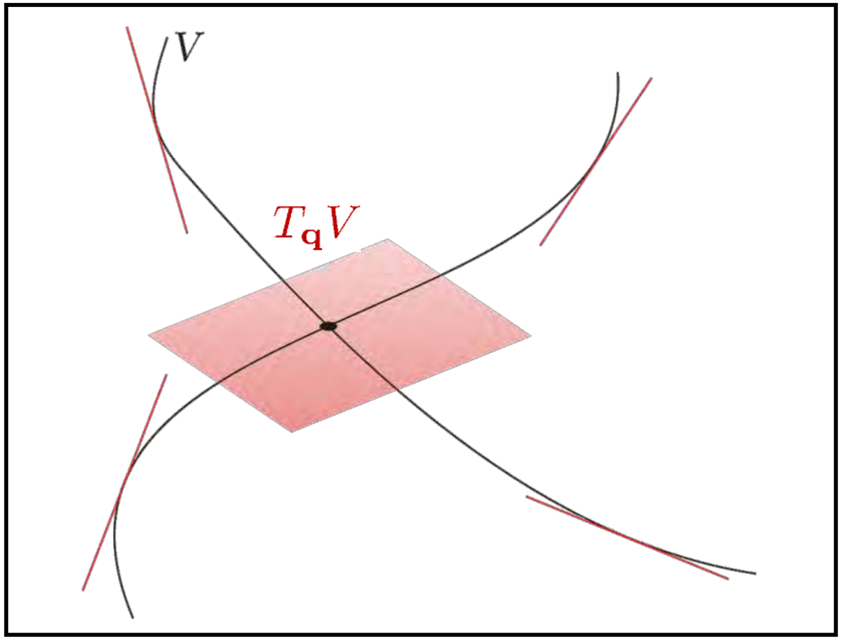

$\delta_{\text{diff}} = 1, 2$ $\delta_{\text{loc}} = 1$

C-space is non-smooth at bifurcation point

- Tangent space at $\mathbf{q} \in V$:

  $T_{\mathbf{q}}V =$ Vector space of tangents to V at $\mathbf{q}$

  $= \ker \mathbf{J}(\mathbf{q})$ (for this linkage)

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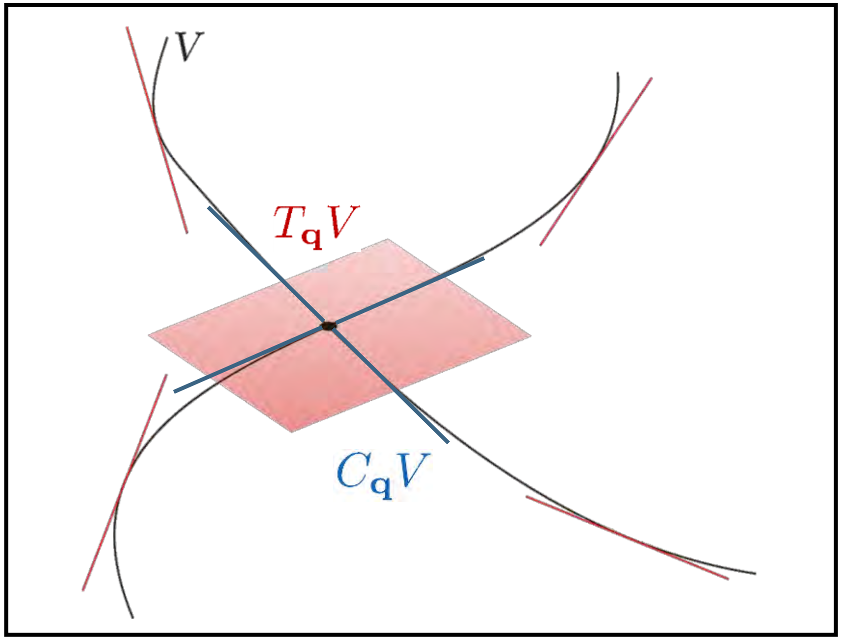
Rank drops at bifurcation point



$\delta_{\text{diff}} = 1, 2$ $\delta_{\text{loc}} = 1$

C-space is non-smooth at bifurcation point

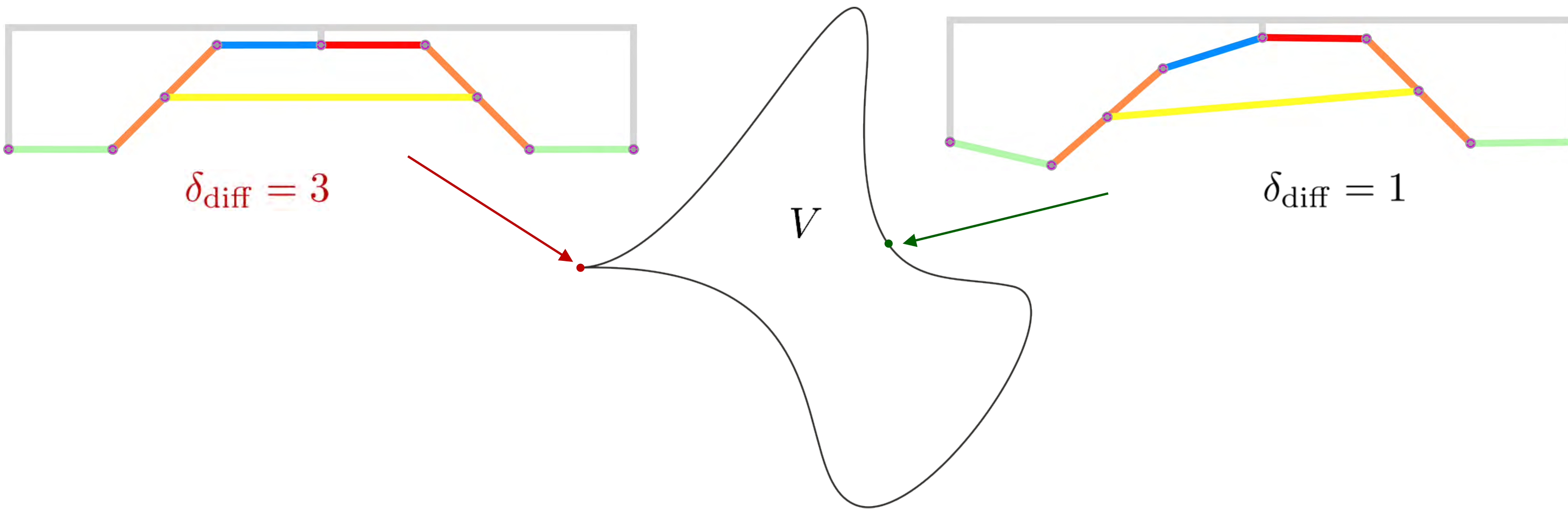
- Tangent space at $\mathbf{q} \in V$:

$$T_{\mathbf{q}}V = \text{Vector space of tangents to } V \text{ at } \mathbf{q}$$
$$= \ker \mathbf{J}(\mathbf{q}) \text{ (for this linkage)}$$

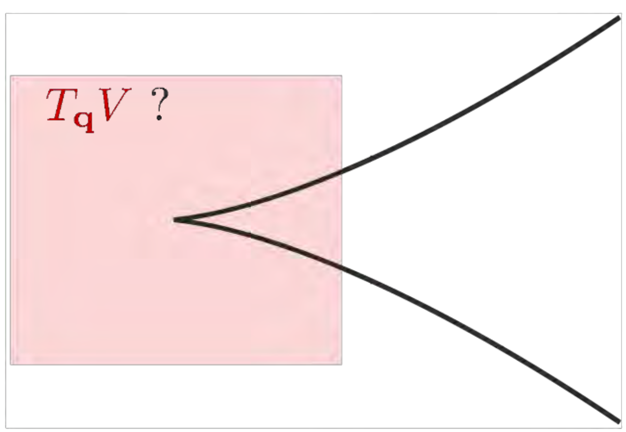
- 'Tangent cone' at $\mathbf{q} \in V$: [Whitney, 1965]

$$C_{\mathbf{q}}V = \text{Limit of tangent spaces near } \mathbf{q}$$

Some examples: ‚Double Watt‘ Linkage



Tangent space?

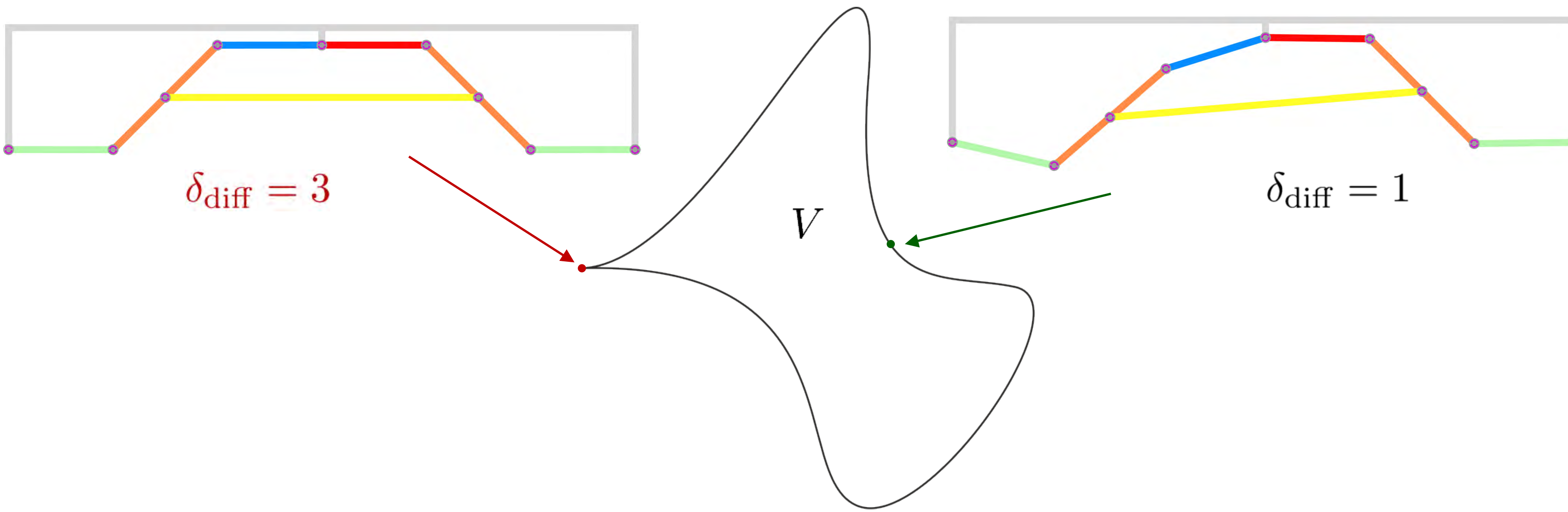


- Definition (simple version):

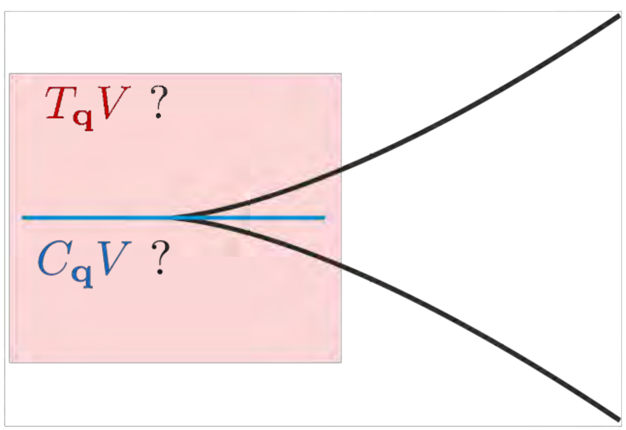
  Tangent space is the set of tangents to V at $\mathbf{q}$

- There are no tangents at cusps!

Some examples: ‚Double Watt‘ Linkage



Tangent cone?

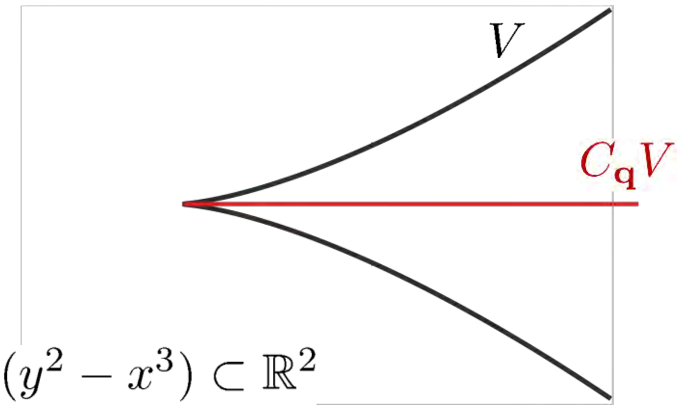


- Definition (simple version):

  Tangent space is the set of tangents to V at $\mathbf{q}$

- There are no tangents at cusps!

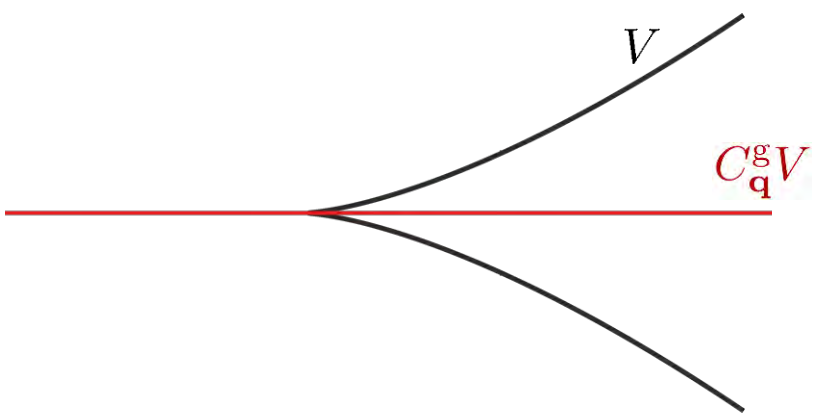
- Definition 1:
  Tangent semi-cone $C_{\mathbf{q}}V$ consists of the limits to secant rays originating from $\mathbf{q}$



$V = \mathbf{V}(y^2 - x^3) \subset \mathbb{R}^2$

$C_{\mathbf{q}}V$ may not be symmetric

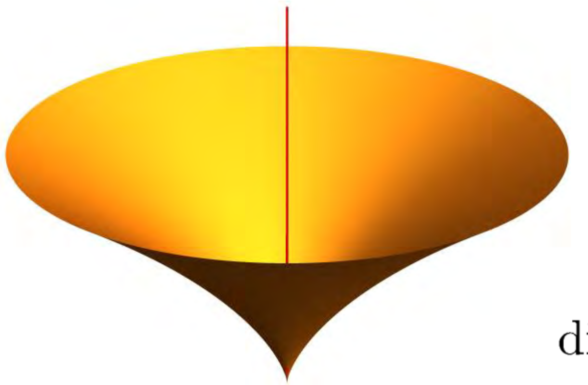
- Definition 2:
  Geometric tangent cone $C^{\mathrm{g}}_{\mathbf{q}}V$ consists of the limits to secant lines originating from $\mathbf{q}$



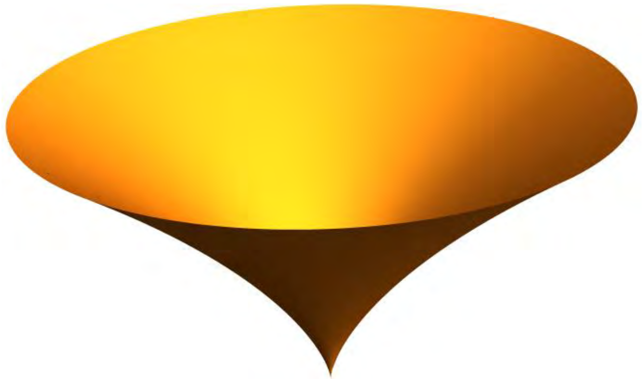
$C^{\mathrm{g}}_{\mathbf{q}}V$ is symmetric:

$\mathbf{x} \in C^{\mathrm{s}}_{\mathbf{q}}V \leftrightarrow -\mathbf{x} \in C^{\mathrm{s}}_{\mathbf{q}}V$

- Definition 1:
  Tangent semi-cone $C^{\mathrm{s}}_{\mathbf{q}}V$ consists of the limits to secant rays originating from $\mathbf{q}$

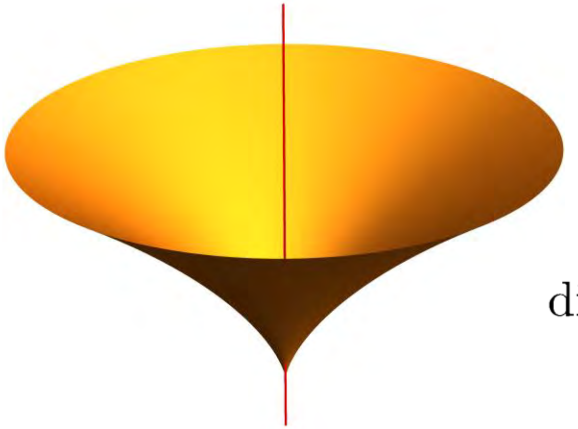


$$\dim C^{\mathrm{s}}_{\mathbf{q}}V \neq \dim V$$



$$V = \mathbf{V}(x^2 + y^2 - z^3) \subset \mathbb{R}^3$$

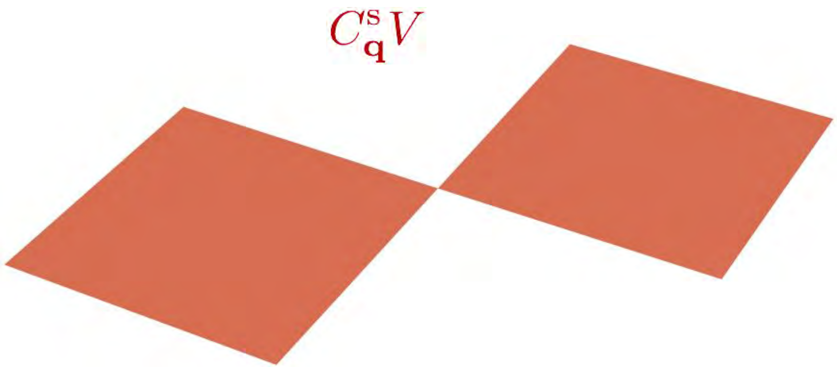
- Definition 2:
  Geometric tangent cone $C^{\mathrm{g}}_{\mathbf{q}}V$ consists of the limits to secant lines originating from $\mathbf{q}$

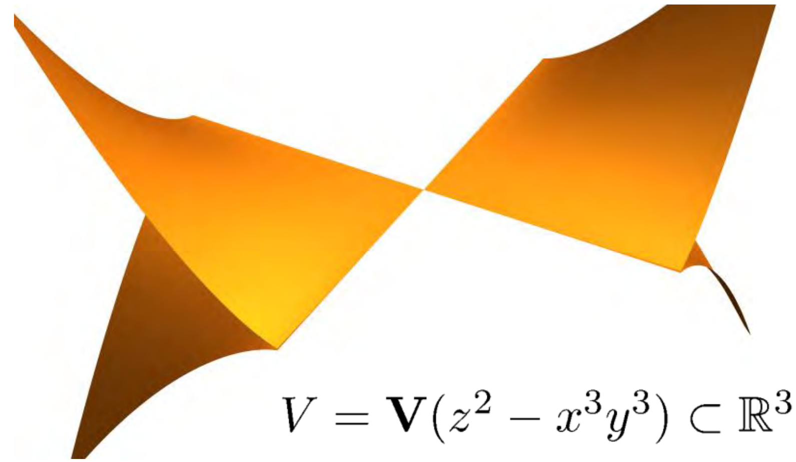


$$\dim C^{\mathrm{g}}_{\mathbf{q}}V \neq \dim V$$

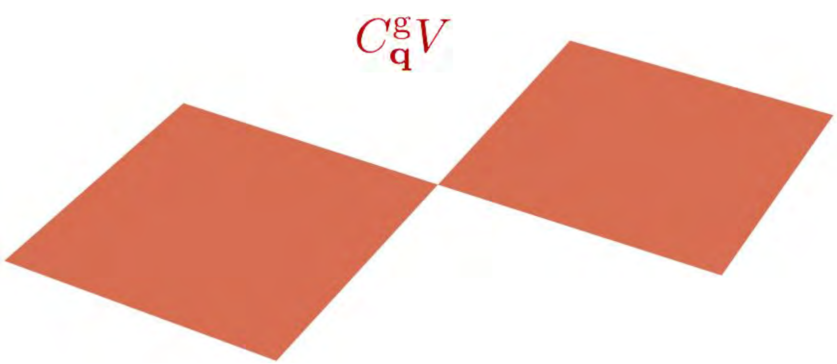
Tangent Cone(s)

- Definition 1:
  Tangent semi-cone $C^{\mathrm{s}}_{\mathbf{q}}V$ consists of the limits to secant rays originating from $\mathbf{q}$

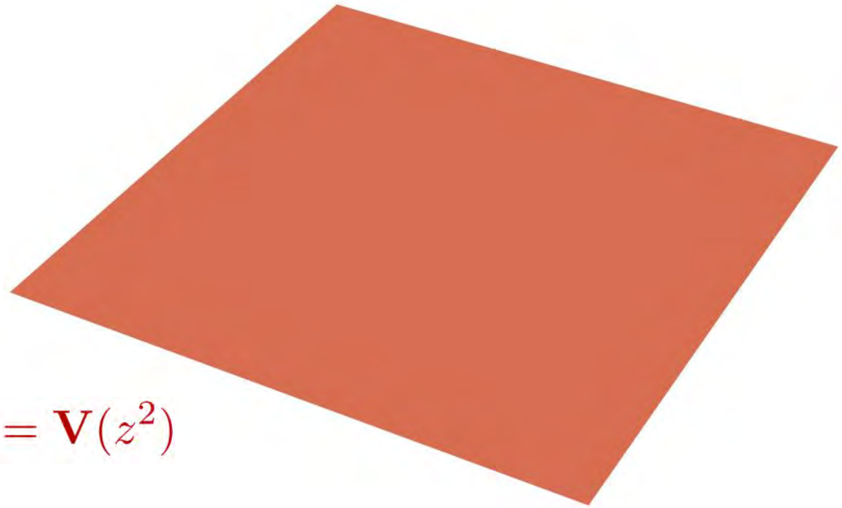




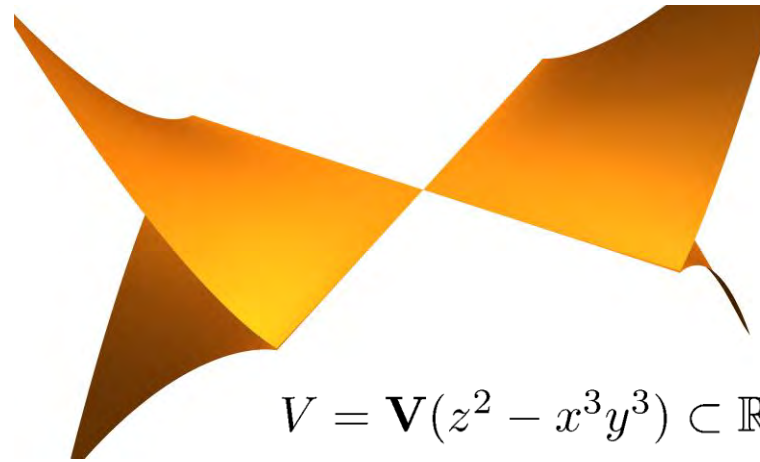
- Definition 2:
  Geometric tangent cone $C^{\mathrm{g}}_{\mathbf{q}}V$ consists of the limits to secant lines originating from $\mathbf{q}$



- Definition 3:
  Algebraic tangent cone $C^{\mathrm{a}}_{\mathbf{q}}V$ is defined by the vanishing of initial polynomials

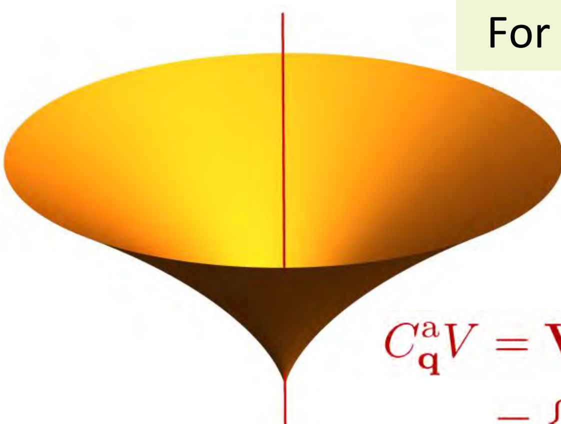


$$C^{\mathrm{a}}_{\mathbf{q}}V = \mathbf{V}(z^2)$$
$$= \{(x, y, z) | z = 0\}$$

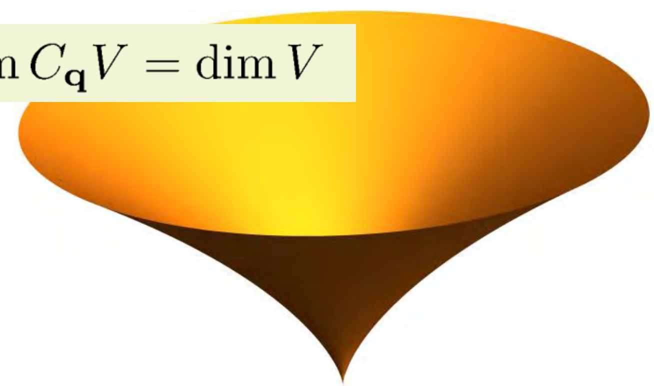


$$V = \mathbf{V}(z^2 - x^3y^3) \subset \mathbb{R}^3$$

For complex varieties $\dim C_{\mathbf{q}}V = \dim V$

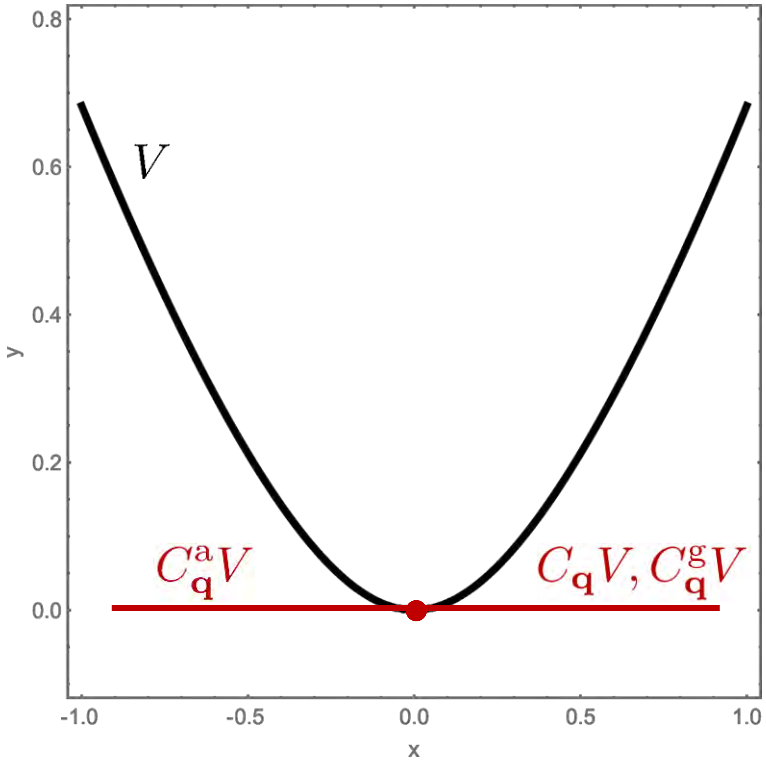


$$C^{\mathrm{a}}_{\mathbf{q}}V = \mathbf{V}(x^2 + y^2)$$
$$= \{(x, y, z) | x = y = 0\}$$



$$V = \mathbf{V}(x^2 + y^2 - z^3) \boxed{\subset \mathbb{R}^3}$$

- Geometry vs. Kinematics:



$$V = \mathbf{V}(y^3 + yx^2 - x^4) \subset \mathbb{R}^2$$

$$\mathbf{J}(\mathbf{q}) = \big(-4x^3 + 2xy \quad x^2 + 3y^2\big)$$

V is a smooth manifold

but $\operatorname{rank} \mathbf{J}(\mathbf{q}) = \begin{cases} 0, & \mathbf{q} = \mathbf{0} \\ 1, & \mathbf{q} \neq \mathbf{0} \end{cases}$

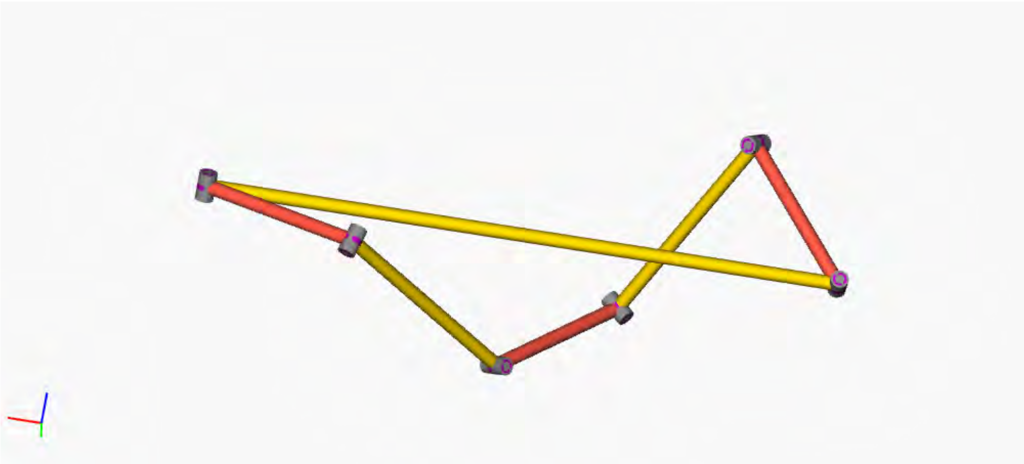
$\mathbf{q} = \mathbf{0}$ is a kinematic singularity

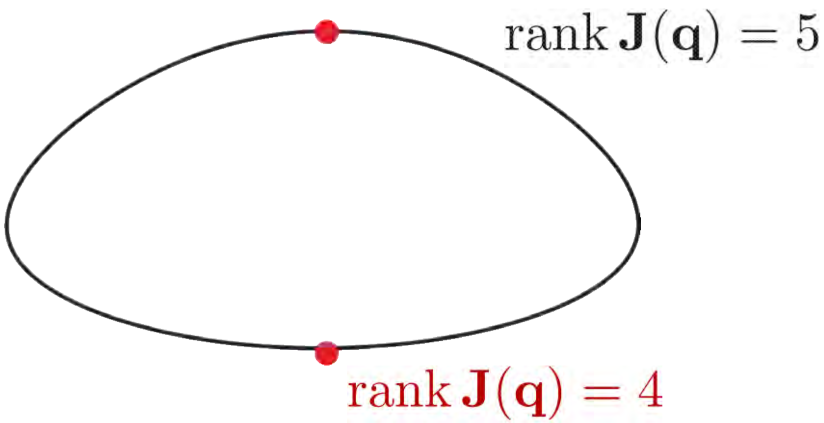
$T_{\mathbf{0}}V = \ker \mathbf{J}(\mathbf{0}) = \mathbb{R}^2$?

$$C^{\mathrm{a}}_{\mathbf{q}}V = \mathbf{V}(y^3 + yx^2) = \mathbf{V}(y(y^2 + x^2))$$

$$= \{(x,y)|y = 0\} \cup \{(x,y)|x = y = 0\}$$

- Geometry vs. Kinematics:





- Velocity constraints at $\mathbf{q} \in V$:

$$\boxed{\mathbf{0} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} = \sum_{i \leq n} \mathbf{S}_i(\mathbf{q})\dot{q}_i}$$

- Acceleration constraints at $\mathbf{q} \in V$:

$$\mathbf{0} = \mathbf{J}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{J}}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$$
$$= \sum_{i \leq n} \mathbf{S}_i \ddot{q}_i + \sum_{i \leq n} \sum_{j < i} [\mathbf{S}_j, \mathbf{S}_i]\dot{q}_j \dot{q}_i$$

$$\boxed{\frac{\partial \mathbf{S}_i}{\partial q_j} = [\mathbf{S}_j, \mathbf{S}_i], j < i}$$

- Jerk constraints…

Remember: for $\mathbf{q} = \mathbf{0}$ we have $\mathbf{S}_i(\mathbf{0}) = \mathbf{Y}_i$

Key Relations

- Partial derivative: $$\frac{\partial \mathbf{S}_i}{\partial q_j} = [\mathbf{S}_j, \mathbf{S}_i], j < i$$

  with screw product $$[\mathbf{X}, \mathbf{Y}] = (\boldsymbol{\xi}_X \times \boldsymbol{\xi}_Y, \boldsymbol{\xi}_X \times \boldsymbol{\eta}_Y - \boldsymbol{\xi}_Y \times \boldsymbol{\eta}_X)$$
  $$= \mathbf{ad}_{\mathbf{X}}\mathbf{Y}$$

  $$\boxed{\mathbf{ad}_{\mathbf{Y}} = \begin{pmatrix} \widetilde{\xi}_Y & 0 \\ \widetilde{\eta}_Y & \widetilde{\xi}_Y \end{pmatrix}}$$

- Bilinearity of Lie bracket:

$$[a_1\mathbf{X}_1 + a_2\mathbf{X}_2, b_1\mathbf{Y}_1 + b_2\mathbf{Y}_2] = a_1b_1[\mathbf{X}_1, \mathbf{Y}_1] + a_1b_2[\mathbf{X}_1, \mathbf{Y}_2] + a_2b_1[\mathbf{X}_2, \mathbf{Y}_1] + a_2b_2[\mathbf{X}_2, \mathbf{Y}_2]$$

- Skew symmetry of Lie bracket:

$$[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}]$$

- Jacobi identity: $$[\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] + [\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] = 0$$

- *Higher-order* derivatives

$$\frac{\partial \mathbf{S}_i}{\partial q_j} = [\mathbf{S}_j, \mathbf{S}_i], j < i \quad \Longrightarrow \quad \frac{\partial^2 \mathbf{S}_i}{\partial q_l \partial q_j} = [\frac{\partial \mathbf{S}_j}{\partial q_l}, \mathbf{S}_i] + [\mathbf{S}_j, \frac{\partial S_i}{\partial q_l}], \quad j < i$$

$$\frac{\partial^2 \mathbf{S}_i}{\partial q_l \partial q_j} = \begin{cases} [\mathbf{S}_l, [\mathbf{S}_j, \mathbf{S}_i]] & , \; l < j < i \\ [\mathbf{S}_j, [\mathbf{S}_l, \mathbf{S}_i]] & , \; j \leq l < i \end{cases}$$

$$\frac{\partial^3 \mathbf{S}_i}{\partial q_m \partial q_l \partial q_j} = \begin{cases} [\mathbf{S}_l, [\mathbf{S}_m, [\mathbf{S}_j, \mathbf{S}_i]]] \,, & l < m < j < i \\ [\mathbf{S}_m, [\mathbf{S}_l, [\mathbf{S}_j, \mathbf{S}_i]]] \,, & m \leq l < j < i \\ [\mathbf{S}_l, [\mathbf{S}_j, [\mathbf{S}_m, \mathbf{S}_i]]] \,, & l < j \leq m < i \\ [\mathbf{S}_j, [\mathbf{S}_l, [\mathbf{S}_m, \mathbf{S}_i]]] \,, & j \leq l < m < i \\ [\mathbf{S}_m, [\mathbf{S}_j, [\mathbf{S}_l, \mathbf{S}_i]]] \,, & m < j \leq l < i \\ [\mathbf{S}_j, [\mathbf{S}_m, [\mathbf{S}_l, \mathbf{S}_i]]] \,, & j \leq m \leq l < i \end{cases}$$

The non-zero terms of the ν-th derivative of $\mathbf{S}_i$ are

$$\frac{\partial^\nu \mathbf{S}_i}{\partial q_{\alpha_1} \partial q_{\alpha_2} \cdots \partial q_{\alpha_\nu}} = \left[\mathbf{S}_{\beta_\nu}, \left[\mathbf{S}_{\beta_{\nu-1}}, \left[\mathbf{S}_{\beta_{\nu-2}}, \ldots \left[\mathbf{S}_{\beta_1}, \mathbf{S}_i\right] \ldots\right]\right]\right]$$

$$= \mathbf{ad}_{\mathbf{S}_{\beta_\nu}} \mathbf{ad}_{\mathbf{S}_{\beta_{\nu-1}}} \mathbf{ad}_{\mathbf{S}_{\beta_{\nu-2}}} \cdots \mathbf{ad}_{\mathbf{S}_{\beta_1}} \mathbf{S}_i$$

where $\beta_\nu \leq \beta_{\nu-1} \leq \cdots \leq \beta_1 < i$ is the ordered sequence of the indices $\alpha_1, \ldots, \alpha_\nu$

There are ν! different non-vanishing terms!

Introduce
$$\mathsf{S}_i^{\nu}(\mathbf{q},\mathbf{q}^{(\nu)}) := \sum_{j\leq i} \mathbf{S}_j\left(\mathbf{q}\right) \frac{d^{\nu}}{dt^{\nu}} q_j$$

$$\Rightarrow \quad \dot{\mathsf{S}}_i^{\nu} = \sum_{j<i} \left[\mathsf{S}_j^1, \mathbf{S}_j\right] \frac{d^{\nu}}{dt^{\nu}} q^j + \mathsf{S}_i^{\nu+1}, \quad \dot{\mathbf{S}}_i = \left[\mathsf{S}_i^1, \mathbf{S}_i\right]$$

- Arbitrary k-th order constraints:

Velocity constraints:

$$\mathbf{0} = \mathsf{S}_n^1(\mathbf{q},\dot{\mathbf{q}})$$

Acceleration constraints:

$$\begin{aligned}\mathbf{0} &= \dot{\mathsf{S}}_n^1(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}) \\ &= \sum_{j<i} \left[\mathsf{S}_j^1, \mathbf{S}_j\right] \dot{q}_j + \mathsf{S}_n^2\end{aligned}$$

Jerk constraints:

$$\begin{aligned}\mathbf{0} &= \ddot{\mathsf{S}}_n^1(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\dddot{\mathbf{q}}) \\ &= \sum_{j<n} ([\dot{\mathsf{S}}_j^1, \mathbf{S}_j]\dot{q}_j + [\mathsf{S}_j^1, \dot{\mathbf{S}}_j]\dot{q}_j + [\mathsf{S}_j^1, \mathbf{S}_j]\ddot{q}_j) + \dot{\mathsf{S}}_n^2 \\ &= \sum_{j<n} ([[\mathsf{S}_j^1, \mathbf{S}_j], \mathsf{S}_n^1]\dot{q}_j + 2[\mathsf{S}_j^1, [\mathsf{S}_j^1, \mathbf{S}_n]]\dot{q}_j + [\mathsf{S}_j^2, \mathbf{S}_j]\dot{q}_j + 2[\mathsf{S}_j^1, \mathbf{S}_j]\ddot{q}_j) + \mathsf{S}_n^3\end{aligned}$$

Introduce
$$\mathsf{S}_i^{\nu}(\mathbf{q},\mathbf{q}^{(\nu)}) := \sum_{j\le i} \mathbf{S}_j\left(\mathbf{q}\right)\frac{d^{\nu}}{dt^{\nu}}q_j$$

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- Arbitrary k-th order constraints:

$$D^{(k)}\mathbf{S}_i = \sum_{r=0}^{k-1}\binom{k-1}{r}\left[D^{(r)}\mathsf{S}_i^1, D^{(k-r-1)}\mathbf{S}_i\right]$$

$$D^{(k)}\mathsf{S}_i^{\nu} = \sum_{j\le i}\sum_{r=0}^{k-1}\binom{k-1}{r}D^{(k-r)}\mathbf{S}_j q_j^{(k+r)} + D^{(k-1)}\mathsf{S}_i^{\nu+1}$$

$$D^{(k)} := \frac{d^k}{dt^k}$$

- A finite trajectory $\mathbf{q}(t)$ of a mechanism satisfies constraints of any order:

$$\mathbf{0} = \mathsf{S}_n^1(\mathbf{q}, \dot{\mathbf{q}})$$

$$\mathbf{0} = \dot{\mathsf{S}}_n^1(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$$

$$\mathbf{0} = \ddot{\mathsf{S}}_n^1(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \dddot{\mathbf{q}})$$

$$\vdots$$

$$\mathbf{0} = \frac{d^\nu}{dt^\nu}\mathsf{S}_n^1(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \mathbf{q}^{(\nu+1)})$$

$$\vdots$$

- i-th order Cone: $K_{\mathbf{q}}^{i} := \{\mathbf{x} | \exists \mathbf{y}, \mathbf{z}, \ldots \in \mathbb{R}^{n} : \quad \begin{array}{l} H^{(1)}(\mathbf{q}, \mathbf{x}) = \mathbf{0}, \\ H^{(2)}(\mathbf{q}, \mathbf{x}, \mathbf{y}) = \mathbf{0}, \\ H^{(3)}(\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}, \\ \quad \cdots \\ H^{(i)}(\mathbf{q}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots) = \mathbf{0} \quad \} \end{array}$

- with i-th order constraint Mappings:

$$\begin{aligned} H^{(1)}(\mathbf{q}, \dot{\mathbf{q}}) &:= \mathsf{S}_{n}^{1}(\mathbf{q}, \dot{\mathbf{q}}) \\ H^{(2)}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) &:= \frac{d}{dt} \mathsf{S}_{n}^{1}(\mathbf{q}, \dot{\mathbf{q}}) \\ &\cdots \\ H^{(i)}(\mathbf{q}, \dot{\mathbf{q}}, \ldots, \mathbf{q}^{(i)}) &:= D^{(i-1)} \mathsf{S}_{n}^{1}(\mathbf{q}, \dot{\mathbf{q}}) . \end{aligned}$$

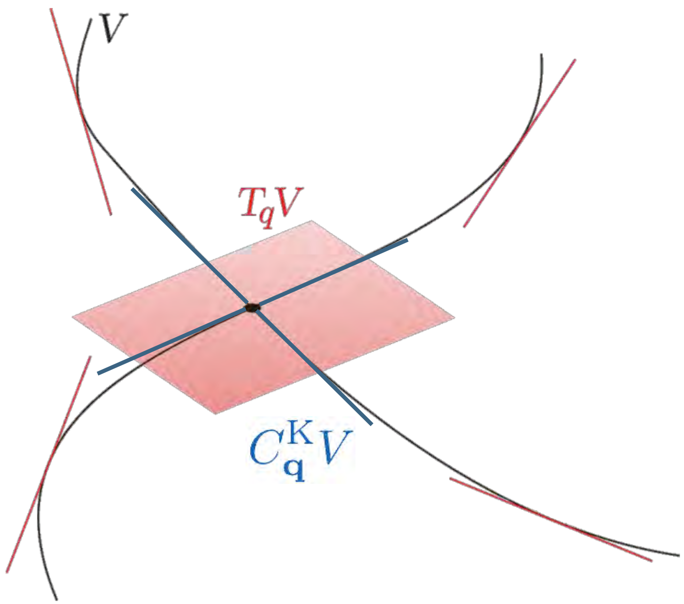
➡ ‘Kinematic’ Tangent Cone:

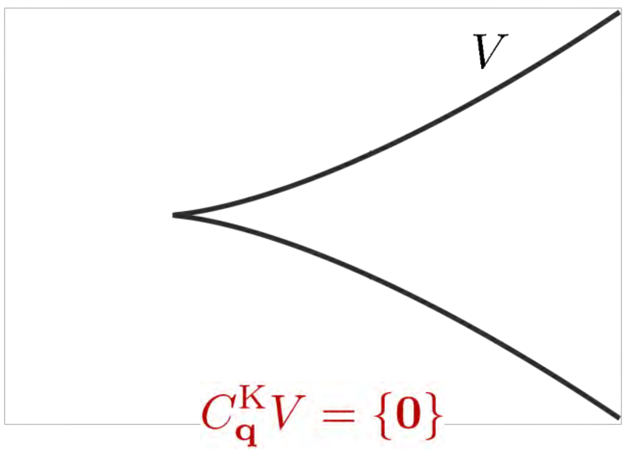
$$C_{\mathbf{q}}^{\mathrm{K}} V = K_{\mathbf{q}}^{\kappa} \subset \ldots \subset K_{\mathbf{q}}^{3} \subset K_{\mathbf{q}}^{2} \subset K_{\mathbf{q}}^{1}$$

Definition:

The kinematic tangent cone to V at $\mathbf{q}$ is the set $C^{\mathrm{K}}_{\mathbf{q}}V$ of tangents to smooth curves in V *through* $\mathbf{q}$

- $\mathbf{x} \in C^{\mathrm{K}}_{\mathbf{q}}V$ can be thought of as velocities
- $C^{\mathrm{K}}_{\mathbf{q}}V$ reveals the finite motions *through* $\mathbf{q}$
- May not describe the local geometry of V





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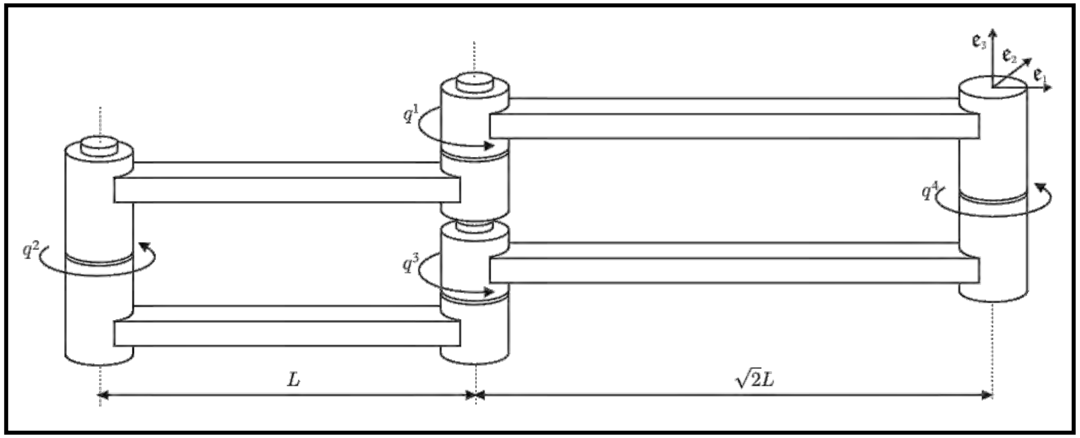
- May not describe the local geometry of V

$$\delta_{\mathrm{loc}}(\mathbf{q}) = \mathrm{dim}_{\mathbf{q}}V \geq \mathrm{dim}C_{\mathbf{q}}V$$



- Screw coordinates at $\mathbf{q}_0 = \mathbf{0}$

$$\mathbf{Y}_1 = \mathbf{Y}_3 = (0,0,1,0,-L,0)^T$$
$$\mathbf{Y}_2 = (0,0,1,0,-2L,0)^T$$
$$\mathbf{Y}_4 = (0,0,1,0,0,0)^T$$

- Tangent cone: $K^1_{\mathbf{0}} = \mathbf{V}(x_1+x_3+2x_4, x_2-x_4) \subset \mathbb{R}^4$

$$K^2_{\mathbf{0}} = K^{2,1}_{\mathbf{0}} \cup K^{2,2}_{\mathbf{0}} \quad \text{with} \quad K^{2,1}_{\mathbf{0}} = \mathbf{V}(x_1+x_3, x_2, x_4)$$
$$K^{2,2}_{\mathbf{0}} = \mathbf{V}(x_1+x_2, x_3+x_2, x_4-x_2)$$
$$K^3_{\mathbf{0}} = K^2_{\mathbf{0}} = C_{\mathbf{0}}V.$$

- Mobility: $\delta_{\text{diff}}(\mathbf{0}) = \dim K^1_{\mathbf{0}} = 2$

$$\delta_{\text{loc}}(\mathbf{0}) = \dim C_{\mathbf{0}}V = 1$$

- Singularity?:

$$K^1_{\mathbf{0}} = T_{\mathbf{0}}V \equiv \operatorname{span} C_{\mathbf{q}}V \neq C_{\mathbf{q}}V$$

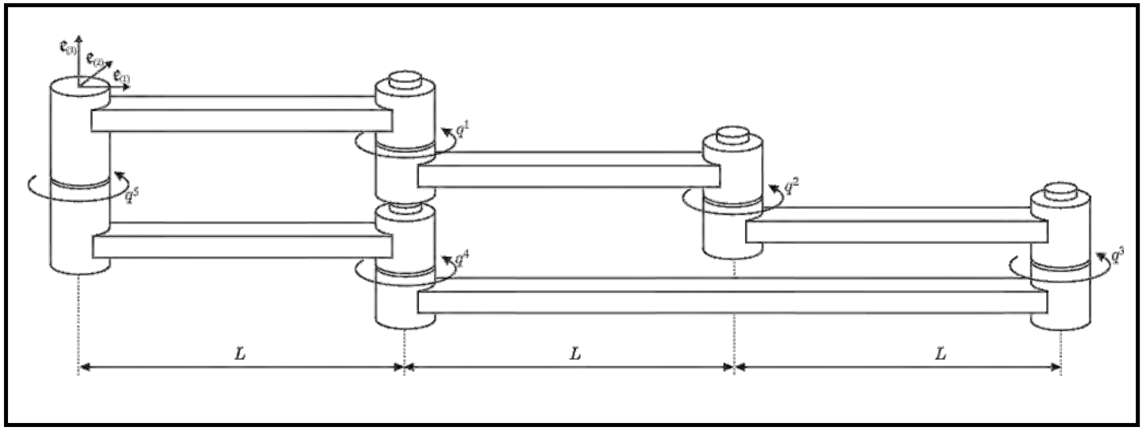
➡ $\mathbf{q}_0 = \mathbf{0}$ is a c-space singularity

➡ $\mathbf{q}_0 = \mathbf{0}$ is a kinematic singularity

Example: planar 5-bar linkage

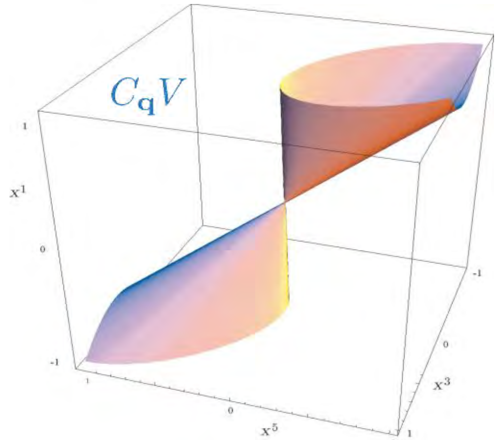
- Screw coordinates at $\mathbf{q}_0 = \mathbf{0}$

$$\mathbf{Y}_1 = (0,0,1,0,0,0)^T$$
$$\mathbf{Y}_2 = \mathbf{Y}_5 = (0,0,1,0,-1,0)^T$$
$$\mathbf{Y}_3 = (0,0,1,0,-2,0)^T$$
$$\mathbf{Y}_4 = (0,0,1,0,-3,0)^T$$



- Tangent cone: $K_{\mathbf{0}}^1 = \mathbf{V}(x_1 - x_3 - 2x_4, x_2 + 2x_3 + 3x_4 + x_5) \subset \mathbb{R}^5$

$$K_{\mathbf{0}}^2 = \mathbf{V}\big(x_2(x_3+2x_4) + (x_3+x_4)(x_3+3x_4), x_1 - x_3 - 2x_4$$
$$x_4^2 + (x_3+2x_4)(x_3+x_4+x_5)\big)$$
$$K_{\mathbf{0}}^3 = K_{\mathbf{0}}^2 = C_{\mathbf{0}}V.$$



- Mobility: $\delta_{\text{diff}}(\mathbf{0}) = \dim K_{\mathbf{0}}^1 = 3$

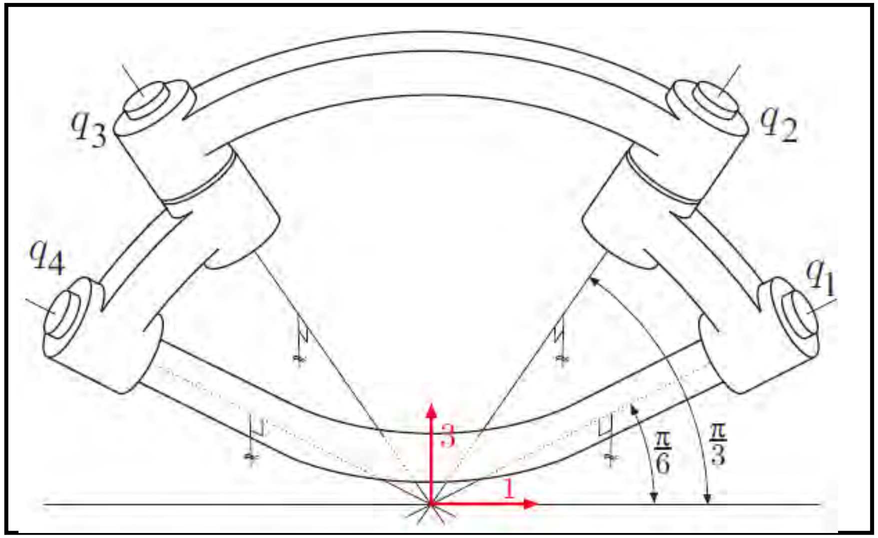
$$\delta_{\text{loc}}(\mathbf{0}) = \dim C_{\mathbf{0}}V = 2.$$

- Singularity?:

$$K_{\mathbf{0}}^1 = T_{\mathbf{0}}V \equiv \operatorname{span} C_{\mathbf{q}}V \neq C_{\mathbf{q}}V$$

➡ $\mathbf{q}_0 = \mathbf{0}$ is a c-space singularity

➡ $\mathbf{q}_0 = \mathbf{0}$ is a kinematic singularity



- Screw coordinates at $\mathbf{q}_0 = \mathbf{0}$

$$\mathbf{Y}_1 = (\sqrt{3}/2, 0, 1/2, 0, 0, 0)^T$$
$$\mathbf{Y}_2 = (1/2, 0, \sqrt{3}/2, 0, 0, 0)^T$$
$$\mathbf{Y}_3 = (-1/2, 0, \sqrt{3}/2, 0, 0, 0)^T$$
$$\mathbf{Y}_4 = (-\sqrt{3}/2, 0, 1/2, 0, 0, 0)^T$$

- Tangent cone:

$$K_{\mathbf{0}}^1 = \mathbf{V}(x_1 - \sqrt{3}x_3 - 2x_4, x_2 - 2x_3 + \sqrt{3}x_4) \subset \mathbb{R}^4$$
$$K_{\mathbf{0}}^2 = \{\mathbf{0}\},\ K_{\mathbf{0}}^3 = K_{\mathbf{0}}^2 = C_{\mathbf{0}}V.$$

- Mobility: $\delta_{\text{diff}}(\mathbf{0}) = \dim K_{\mathbf{0}}^1 = 2$

$$\delta_{\text{loc}}(\mathbf{0}) = \dim C_{\mathbf{0}}V = (\delta_{\text{loc}}(\mathbf{0}) = \dim C_{\mathbf{0}}V = 0$$

- Singularity?: $T_{\mathbf{0}}V \equiv \operatorname{span} C_{\mathbf{q}}V = C_{\mathbf{q}}V$ ➡ $\mathbf{q}_0 = \mathbf{0}$ is a regular point of V

$\delta_{\text{loc}}(\mathbf{q}) = 0$ ➡ $\mathbf{q}_0 = \mathbf{0}$ is not a kinematic singularity

$\delta_{\text{diff}}(\mathbf{0}) \neq \delta_{\text{loc}}(\mathbf{0})$ ➡ Linkage is underconstrained

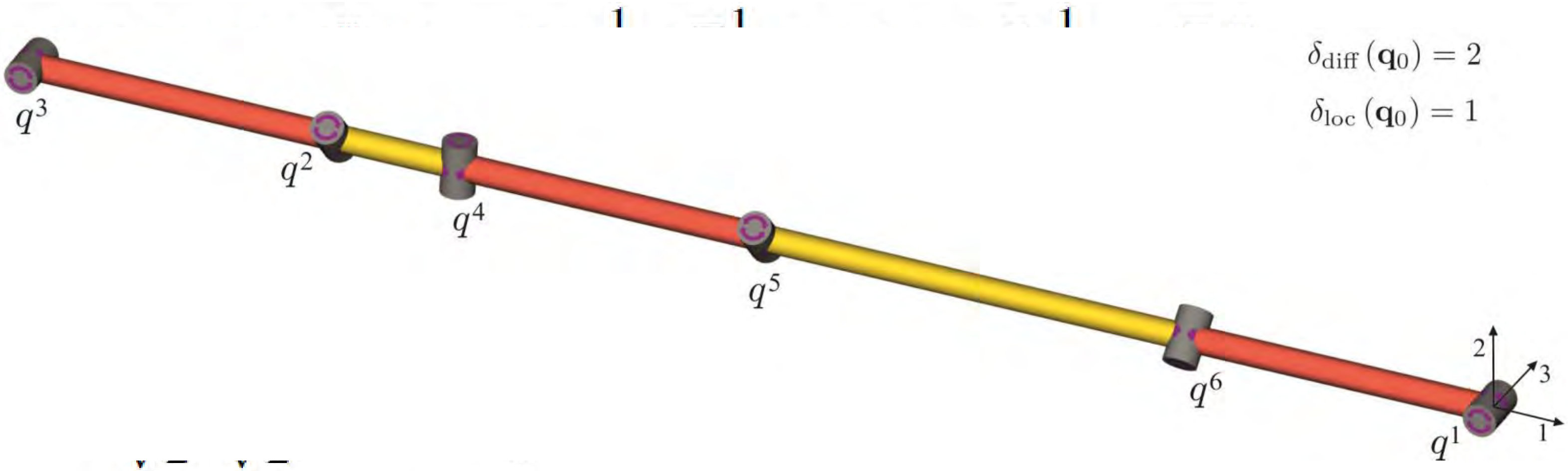
- Screw coordinates at $\mathbf{q}_0 = \mathbf{0}$

$\mathbf{Y}_1 =$

$\mathbf{Y}_3 =$

$\mathbf{Y}_5 =$

$\mathbf{Y}_6 =$



- Tangent cone:

$$K_{\mathbf{0}}^1 = \mathbf{V}\big(2x_1 + x_5 + (\sqrt{2}-1)x_6, 2x_3 - x_5 + (\sqrt{2}-1)x_6, x_2 + (1+\sqrt{2})x_5 - x_6, x_4 - x_5 + \sqrt{2}x_6\big) \subset \mathbb{R}^6$$

$$K_{\mathbf{0}}^2 = \mathbf{V}\big(x_1 + x_5, x_2, x_3, x_4 + (1+\sqrt{2})x_5, x_6 - (1+\sqrt{2})x_5\big)$$

$$K_{\mathbf{0}}^3 = K_{\mathbf{0}}^2 = C_{\mathbf{0}}V.$$

- Mobility:

$$\delta_{\text{diff}}(\mathbf{0}) = \dim K_{\mathbf{0}}^1 = 2$$

$$\delta_{\text{loc}}(\mathbf{0}) = \dim C_{\mathbf{0}}V = 1$$

- Singularity?: $T_{\mathbf{0}}V \equiv \operatorname{span} C_{\mathbf{q}}V = C_{\mathbf{q}}V$ ➡ $\mathbf{q}_0 = \mathbf{0}$ is a regular point of V

But $\mathbf{q}_0 = \mathbf{0}$ is a kinematic singularity! But how do we know that?

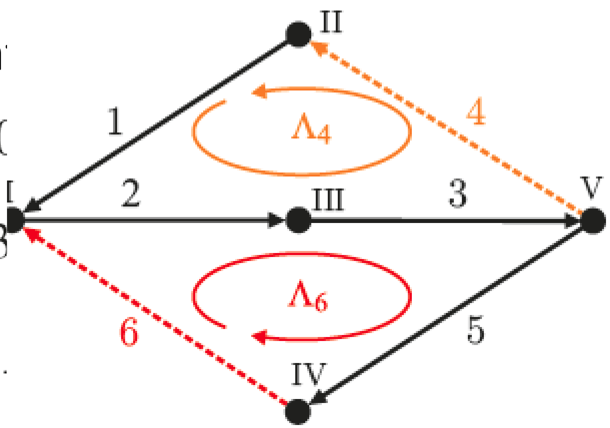
Example: Shaky 2-Loop Linkage (Assur graph)

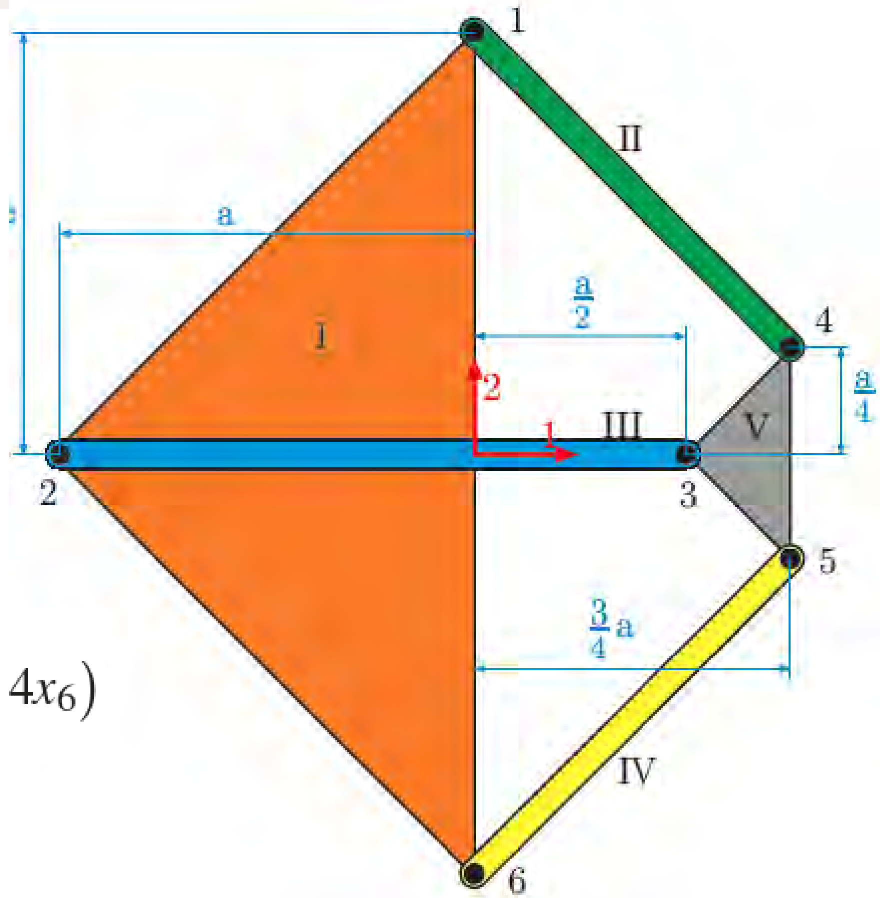
- Screw coordina

$\mathbf{Y}_1 = (0,0,1,6,0,($

$\mathbf{Y}_3 = (0,0,1,0,-3$

$\mathbf{Y}_5 = (0,0,1,-\frac{3}{2},$





- Tangent cone:

$$\boxed{K_{\mathbf{0}}^1 = K_{\mathbf{0}}^2 = K_{\mathbf{0}}^3}$$
$$= \mathbf{V}\left(x_1 - x_6, x_2 + x_6, x_3 - 4x_6, x_4 + 4x_6, x_5 + 4x_6\right)$$
$$K_{\mathbf{0}}^4 = C_{\mathbf{0}}V = \{\mathbf{0}\}$$

- Mobility: $\delta_{\text{diff}}(\mathbf{0}) = \dim K_{\mathbf{0}}^1 = 1$

$\delta_{\text{loc}}(\mathbf{0}) = \dim C_{\mathbf{0}}V = 0.$

- Singularity?: $T_{\mathbf{0}}V \equiv \operatorname{span} C_{\mathbf{q}}V = C_{\mathbf{q}}V$ ➡ $\mathbf{q}_0 = \mathbf{0}$ is a regular point of V

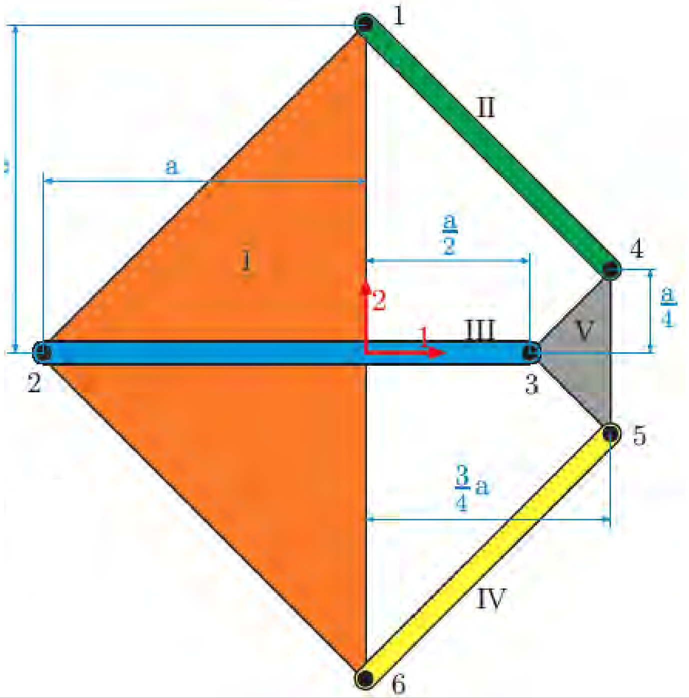
$\delta_{\text{loc}}(\mathbf{q}) = 0$ ➡ $\mathbf{q}_0 = \mathbf{0}$ is not a kinematic singularity

$\delta_{\text{diff}}(\mathbf{0}) \neq \delta_{\text{loc}}(\mathbf{0})$ ➡ Linkage is underconstrained

- Definition:

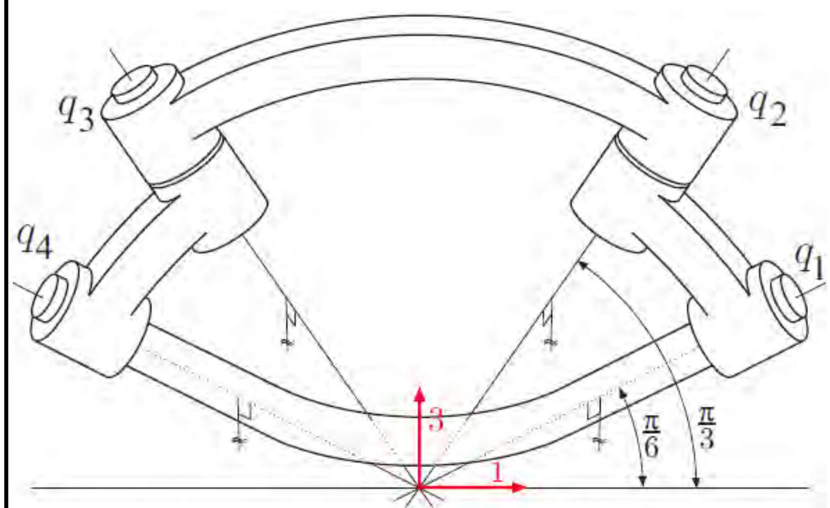
If $C_{\mathbf{q}}V = K_{\mathbf{q}}^{\kappa} \subset K_{\mathbf{q}}^{\kappa-1}$ for $\kappa > 1$, the linkage is shaky of order $i = \kappa - 1$ and the configuration $\mathbf{q}$ is called shaky. The dimension $\dim K_{\mathbf{q}}^{i}$ is referred to as the ith-order DOF at $\mathbf{q}$. If there is a regular shaky configuration, the linkage is underconstrained.

Shaky of order $i = 1, 2, 3$



Shaky of order $i = 1, 2$

Underconstrained



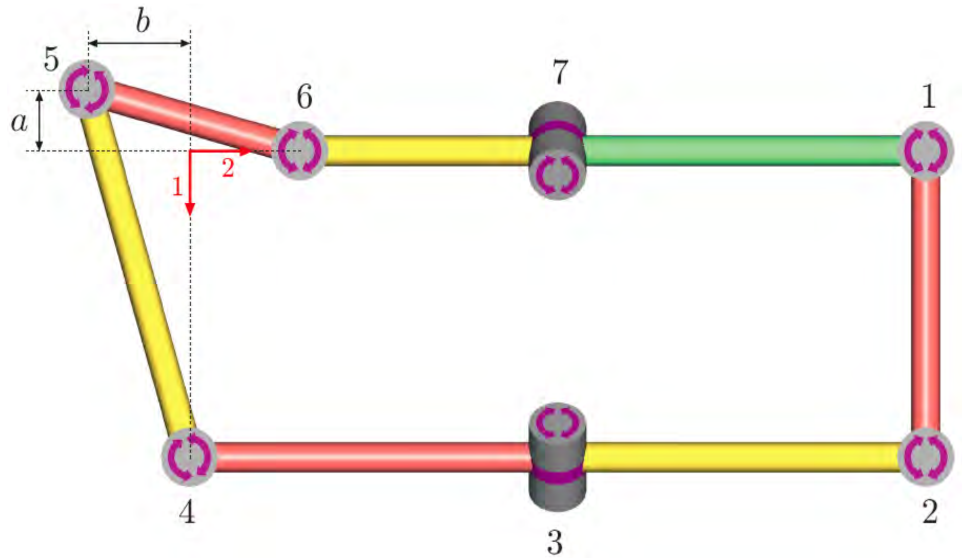
- Screw coordinates:

$\mathbf{Y}_1 = (0,0,1,110,0,0)^T, \mathbf{Y}_2 = (0,0,1,110,-40,0)^T$

$\mathbf{Y}_3 = (-(1/2),0,\frac{1}{2}\sqrt{3},30\sqrt{3},-20\sqrt{3},30)^T$

$\mathbf{Y}_4 = (0,0,1,0,-40,0)^T, \mathbf{Y}_5 = (0,0,1,-b,a,0)^T$

- $\mathbf{Y}_6 = (0,0,1,25,0,0)^T, \mathbf{Y}_7 = (-\frac{1}{2},0,\frac{1}{2}\sqrt{3},30\sqrt{3},0,30)^T$



X. Kong, M. Pfurner,
Mech. Mach. Teory, Vol. 85, 2015

$$K_0^1 = \mathbf{V}\big(x_3+x_7, 40x_2+20\sqrt{3}x_3+40x_4-ax_5, \\ x_1+x_2+\frac{1}{2}\sqrt{3}x_3+x_4+x_5+x_6+\frac{1}{2}\sqrt{3}x_7, \\ 110(x_1+x_2)+30\sqrt{3}(x_3+x_7)-bx_5+25x_6\big) \in \mathbb{R}^7$$

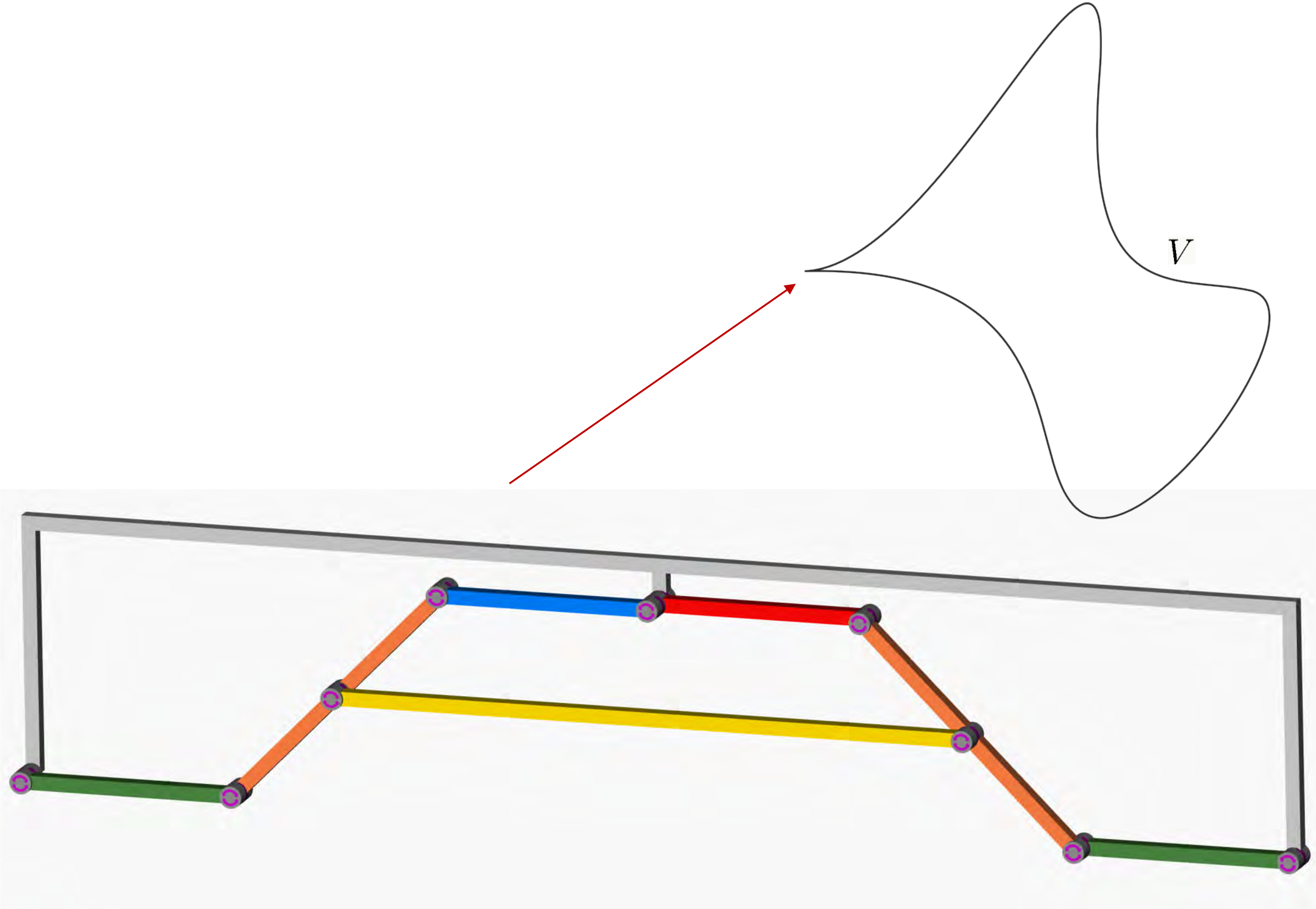
$$K_0^2 = K_0^{2,1} \cup K_0^{2,2} \qquad K_0^3 = K_0^2 = C_0V$$

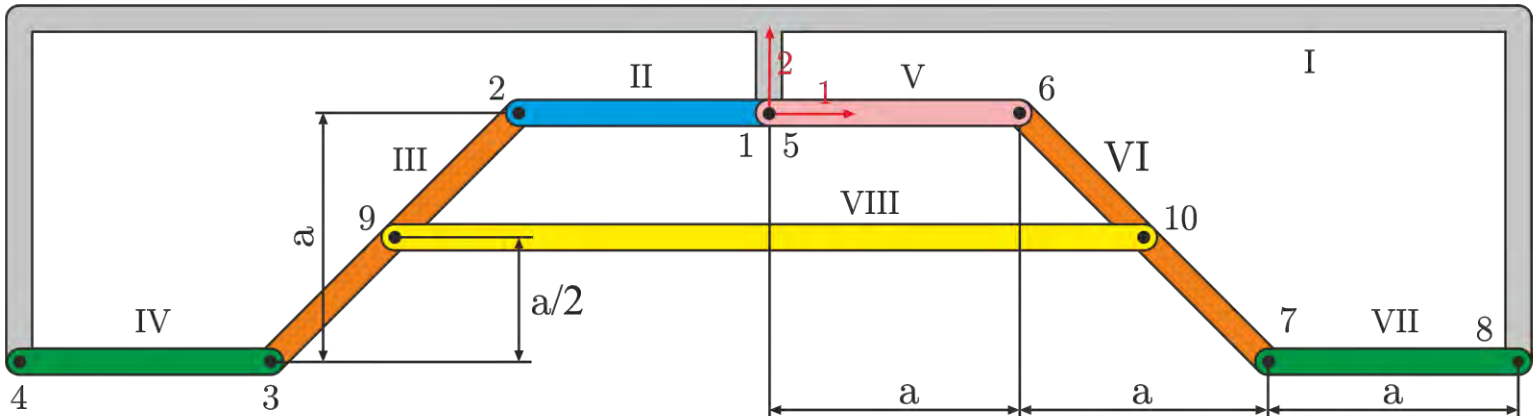
with

$$K_0^{2,1} = \mathbf{V}\big(200x_1-(200+5a+8b)x_5, 200x_2+(200+5a+8b)x_5, \\ 25x_4+25x_5+bx_5, 150x_7-\sqrt{3}(200+5a+8b)x_5x_7, \\ x_6-b/25x_5, \sqrt{3}(200+5a+8b)x_5-150x_3x_7\big).$$

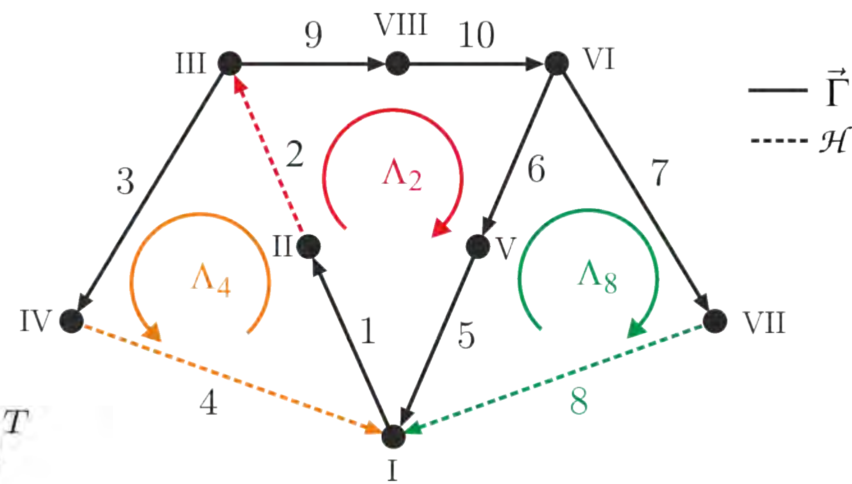
$$K_0^{2,2} = \mathbf{V}\big(680x_1+880x_2-(200+5a+8b)x_5, 40x_4+40x_2-ax_5, \\ 340x_6+(440+11a+4b)x_5-440x_2, x_3, x_7\big)$$

$$\dim K_0^1 = 3 \quad \boxed{\dim K_0^{2,1} = 1, \dim K_0^{2,2} = 2}$$


V



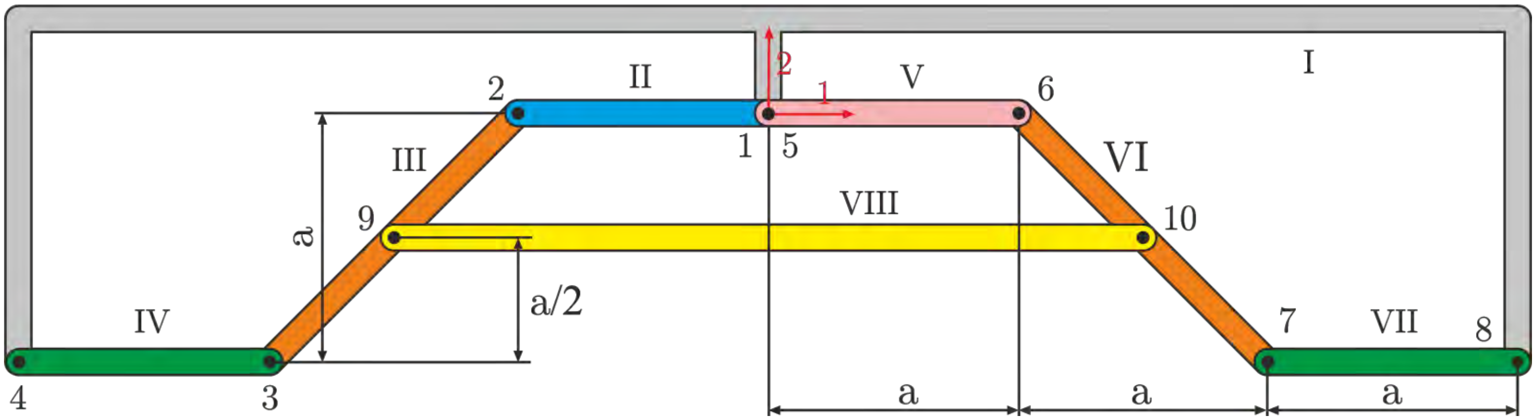
- Fundamental cycles: $\gamma = 3$



- Joint screw coordinates:

$$\mathbf{Y}_1 = \mathbf{Y}_5 = (0,0,1,0,0,0)^T, \mathbf{Y}_2 = (0,0,1,0,1,0)^T$$
$$\mathbf{Y}_3 = (0,0,1,-1,2,0)^T, \mathbf{Y}_4 = (0,0,1,-1,3,0)^T$$
$$\mathbf{Y}_6 = (0,0,1,0,-1,0)^T, \mathbf{Y}_7 = (0,0,1,-1,-2,0)^T$$
$$\mathbf{Y}_8 = (0,0,1,-1,-3,0)^T, \mathbf{Y}_9 = (0,0,1,-1/2,3/2,0)^T$$
$$\mathbf{Y}_{10} = (0,0,1,-1/2,-3/2,0)^T$$

Examples: Double-Watt linkage



- Cones:

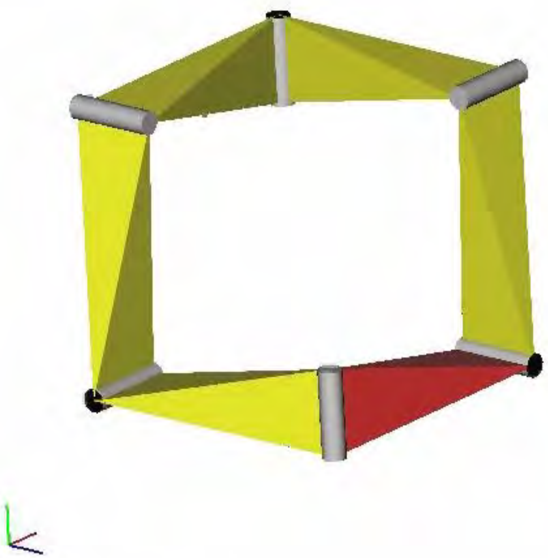
$$K^1_{\mathbf{q}_0} = \mathbf{V}\big(x_1 + x_2, x_1 + x_3, x_1 - x_4, x_5 + x_6, x_5 - x_7,$$
$$x_5 + x_8, x_1 - x_5 - 3x_9, x_1 + 3x_{10} - x_5\big)$$

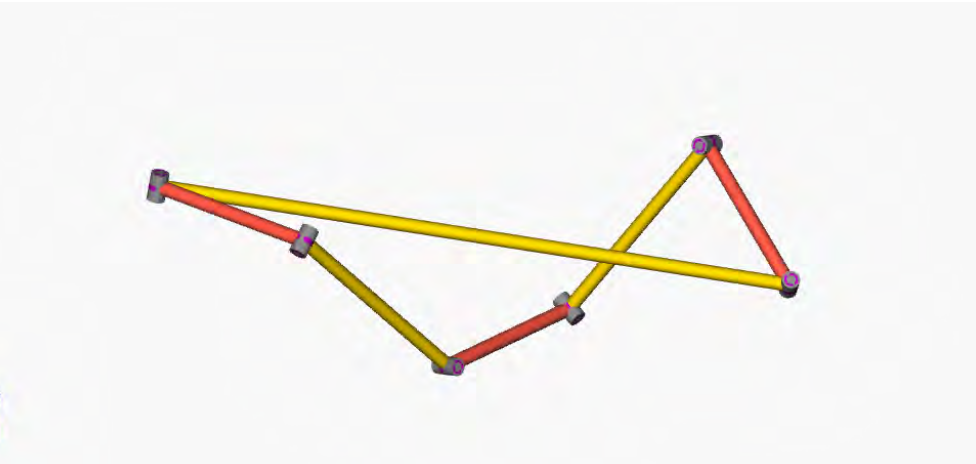
$$K^2_{\mathbf{q}_0} = \mathbf{V}\big(x_1 + x_2, x_1 + x_3, x_4 - x_1, x_5 - x_1, x_1 + x_6,$$
$$x_7 - x_1, x_1 + x_8, x_9, x_{10}\big)$$

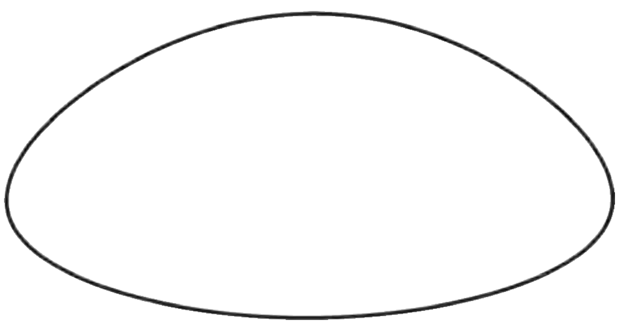
$$K^3_{\mathbf{q}_0} = \{\mathbf{0}\}$$

$$\Longrightarrow \quad C^{\mathrm{K}}_{\mathbf{q}_0} V = \{\mathbf{0}\}$$

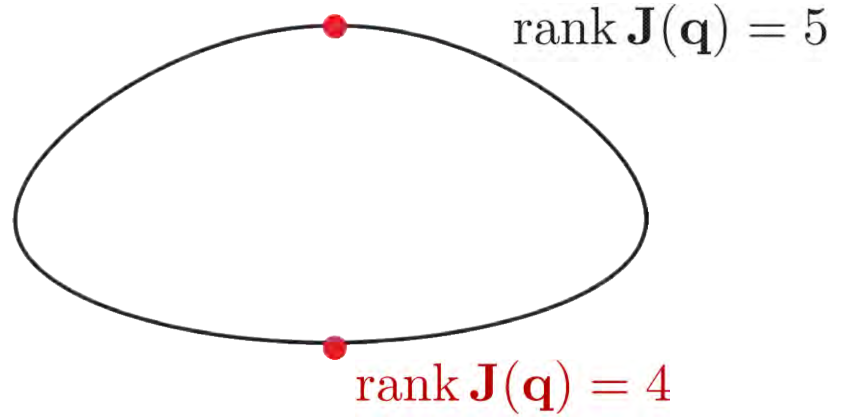
- Identifying kinematic singularities boils down to determine the varieties of points with certain rank



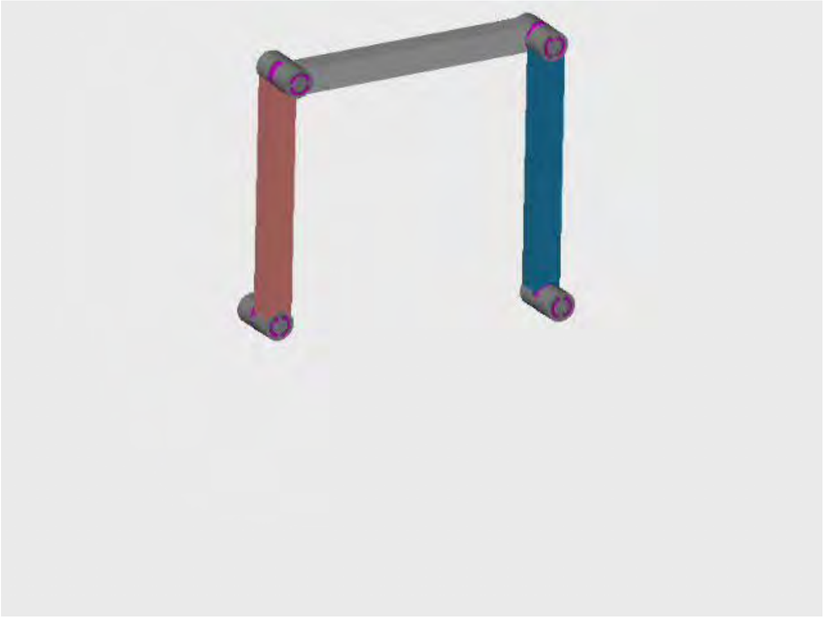


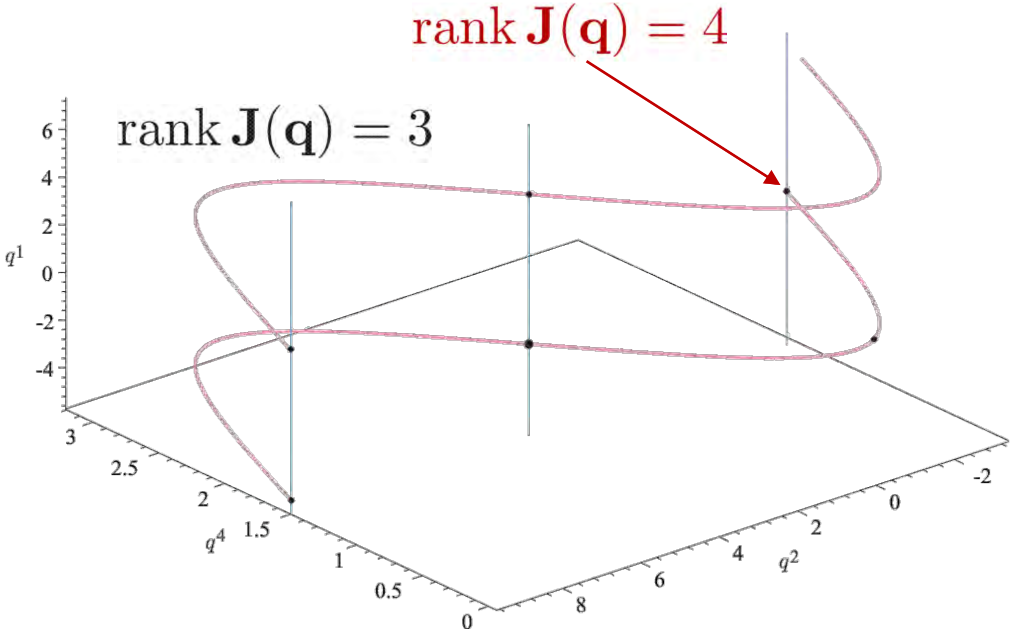


$\operatorname{rank} \mathbf{J}(\mathbf{q}) = 5 \quad \forall \mathbf{q} \in V$



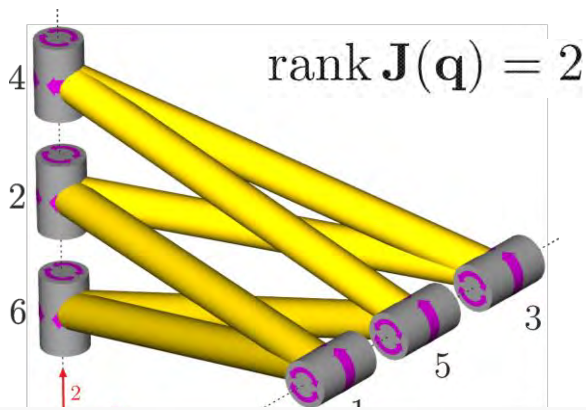
- Identifying kinematic singularities boils down to determine the varieties of points with certain rank

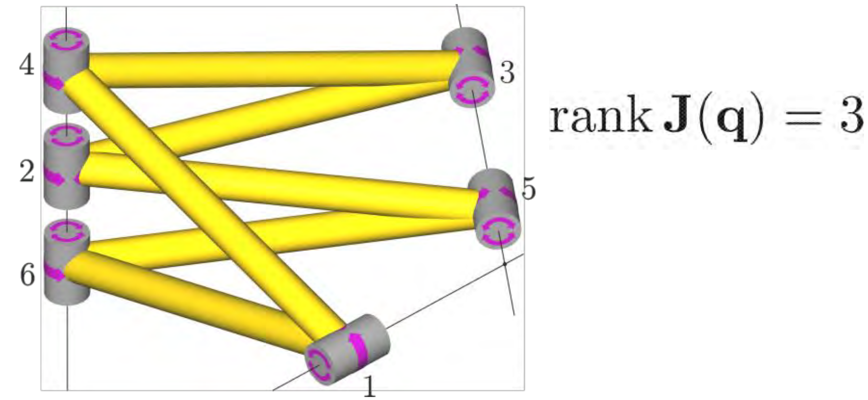


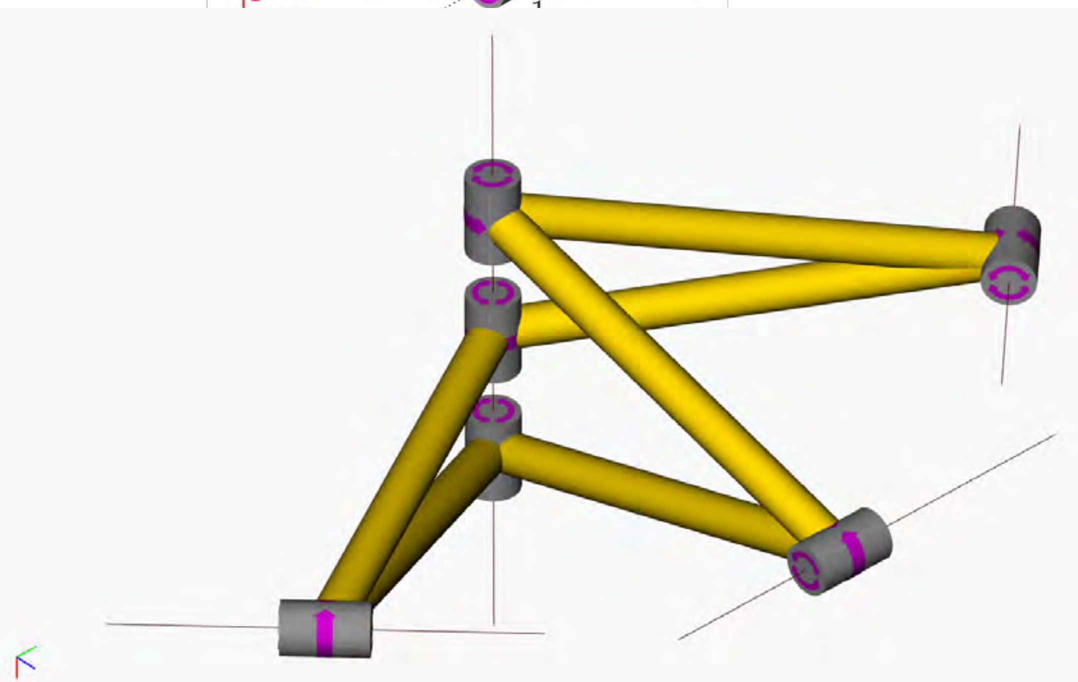


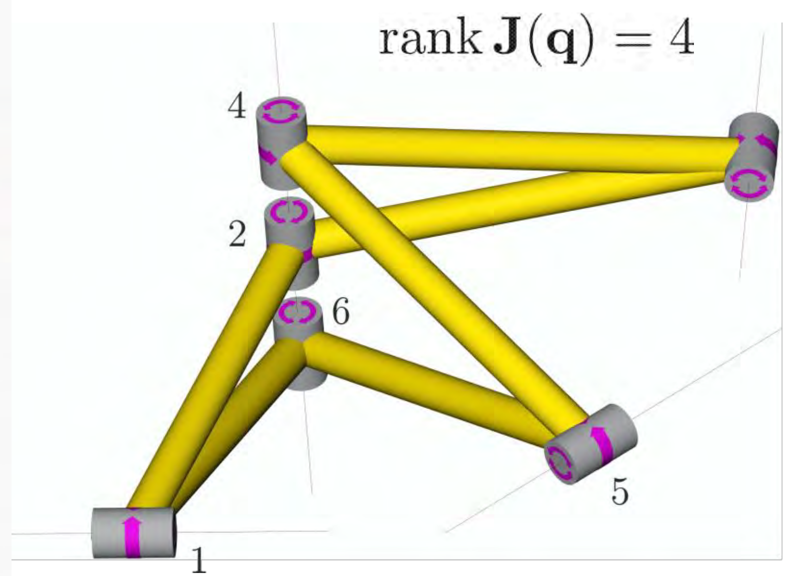
C-space stratification by (co)rank

- Identifying kinematic singularities boils down to determine the varieties of points with certain rank









- Set of critical points with rank less than k:

$$L_k := \{\mathbf{q} \in V | \mathrm{rank}\ \mathbf{J}(\mathbf{q}) < k\}$$
$$= \{\mathbf{q} \in \mathbb{V}^n | f(\mathbf{q}) = \mathbf{I}, m_{\mathbf{ab}}(\mathbf{q}) = 0$$
$$\forall \mathbf{a} \subseteq \{1, \ldots, 6\}, \mathbf{b} \subseteq \{1, \ldots, n\}, |\mathbf{a}| = |\mathbf{b}| = k\}$$

where $m_{\mathbf{ab}}(\mathbf{q}) := \det \mathbf{J}_{\mathbf{ab}}(\mathbf{q})$

$\mathbf{J}_{\mathbf{ab}}$ is the $\mathbf{ab}$-minor of $\mathbf{J}$ of order k

- $C^{\mathrm{K}}_{\mathbf{q}} L_k$ consists of tangents to finite curves (motions) through $\mathbf{q}$ with $\mathrm{rank}\ \mathbf{J} < k$

- Computation…

$\mathbf{J} = (\mathbf{S}_1 \cdots \mathbf{S}_k)$ $\qquad$ $\mathbf{S}_i(\mathbf{q})$ – screw coordinates of joint i

We need $\dfrac{d^i}{dt^i} m_{\mathbf{ab}}(\mathbf{q}(t)) = \dfrac{d^i}{dt^i} \det \mathbf{J}_{\mathbf{ab}}(\mathbf{q}(t))$

- Geometric loop constraints:

$$f\left(\mathbf{q}\right) = \mathbf{I}$$

with $\quad f\left(\mathbf{q}\right) := \exp\left(\mathbf{Y}_1 q_1\right)\exp\left(\mathbf{Y}_2 q_2\right)\cdot\ldots\cdot\exp\left(\mathbf{Y}_n q_n\right)$

- Velocity constraints

$$\mathbf{J}\left(\mathbf{q}\right)\dot{\mathbf{q}} = \mathbf{0}$$

- Derivative of determinant

$$\boxed{\begin{aligned}\frac{d^i}{dt^i} m_{\mathbf{ab}}(\mathbf{q}(t)) &= \frac{d^i}{dt^i}\det \mathbf{J}_{\mathbf{ab}}(\mathbf{q}(t)) \\ &= \sum_{\substack{\mu_1 \neq \ldots \neq \mu_p \in \mathbf{b} \\ \pi \in P(i)}} \left|\mathbf{S}_{\mathbf{a}b_1} \cdots \mathbf{S}^{(\pi_1)}_{\mathbf{a}\mu_1} \cdots \mathbf{S}^{(\pi_p)}_{\mathbf{a}\mu_p} \cdots \mathbf{S}_{\mathbf{a}b_k}\right| \frac{i!}{\pi_1! \cdots \pi_p! n_1! \cdots n_i!}\end{aligned}}$$

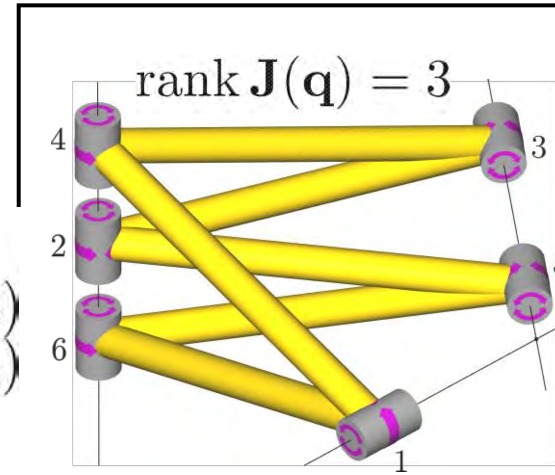
$$\mathbf{S}^{(k)}_{\mathbf{a}j} := D^{(k)}\mathbf{S}_{\mathbf{a}j} \qquad D^{(k)}\mathbf{S}_i = \sum_{r=0}^{k-1}\binom{k-1}{r}\left[D^{(r)}\mathsf{S}^1_i, D^{(k-r-1)}\mathbf{S}_i\right] \qquad D^{(k)} := \frac{d^k}{dt^k}$$

- $$C_{\mathbf{q}}^{\mathrm{K}}V = \mathbf{V}\left(x_1+x_3+x_5, x_2, x_4, x_6\right) \cup \mathbf{V}\left(x_1+x_3, x_4+x_6, x_2, x_5\right) \cup \mathbf{V}\left(x_2+x_6, x_3+x_5, x_1, x_4\right) \cup \mathbf{V}\left(x_2+x_4+x_6, x_1, x_3, x_5\right) \cup \mathbf{V}\left(x_1+x_5, x_2+x_4, x_3, x_6\right)$$

  $$\delta_{\mathrm{loc}}\left(\mathbf{q}\right) = \dim C_{\mathbf{q}}^{\mathrm{K}}V = 2$$

- $$C_{\mathbf{q}}^{\mathrm{K}}L_6 = C_{\mathbf{q}}^{\mathrm{K}}L_5 = C_{\mathbf{q}}^{\mathrm{K}}V$$

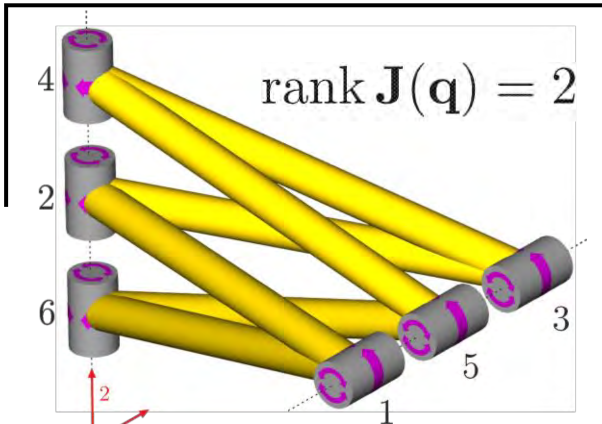
  ➡ all points have (locally) maximal rank 4



- $$C_{\mathbf{q}_0}^{\mathrm{K}}L_4 = \mathbf{V}\left(x_1, x_2, x_3, x_4+x_6, x_5\right) \cup \mathbf{V}\left(x_1, x_2, x_3+x_5, x_4, x_6\right) \cup \mathbf{V}\left(x_1, x_2+x_6, x_3, x_4, x_5\right) \cup \mathbf{V}\left(x_1, x_2+x_4, x_3, x_5, x_6\right) \cup \mathbf{V}\left(x_1+x_3, x_2, x_4, x_5, x_6\right) \cup \mathbf{V}\left(x_1+x_5, x_2, x_3, x_4, x_6\right)$$

  $$\dim C_{\mathbf{q}}^{\mathrm{K}}L_4 = 1$$

  ➡ points with $\operatorname{rank}\mathbf{J}(\mathbf{q}) \leq 4$ (locally) 1-dim manifold



- $$C_{\mathbf{q}}^{\mathrm{K}}L_3 = \{\mathbf{0}\}$$

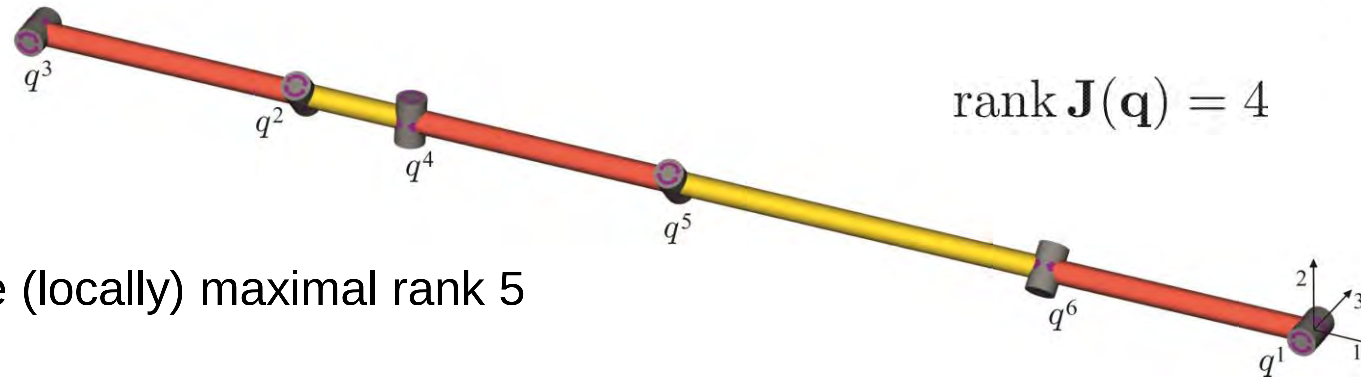
  $$\dim C_{\mathbf{q}}^{\mathrm{K}}L_3 = 0$$

Examples

- $C_{\mathbf{q}}^{\mathrm{K}} V = \mathbf{V}(x_1 + x_5, x_2, x_3, x_4 + (1+\sqrt{2})x_5, x_6 - (1+\sqrt{2})x_5)$

  $\delta_{\mathrm{loc}}(\mathbf{q}) = \dim C_{\mathbf{q}}^{\mathrm{K}} V = 1$

$\operatorname{rank} \mathbf{J}(\mathbf{q}) = 4$

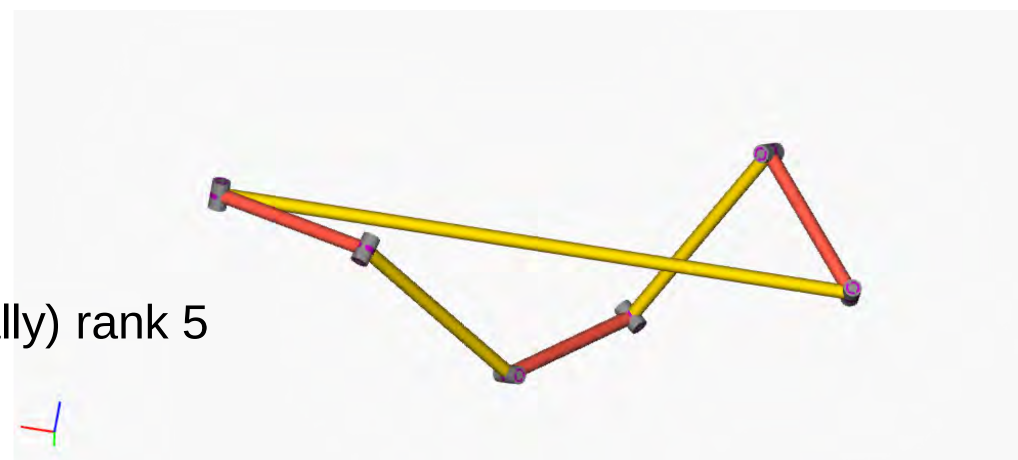


- $C_{\mathbf{q}}^{\mathrm{K}} L_6 = C_{\mathbf{q}}^{\mathrm{K}} V$

  ➡ all points have (locally) maximal rank 5

- $C_{\mathbf{q}}^{\mathrm{K}} L_5 = \{\mathbf{0}\}$

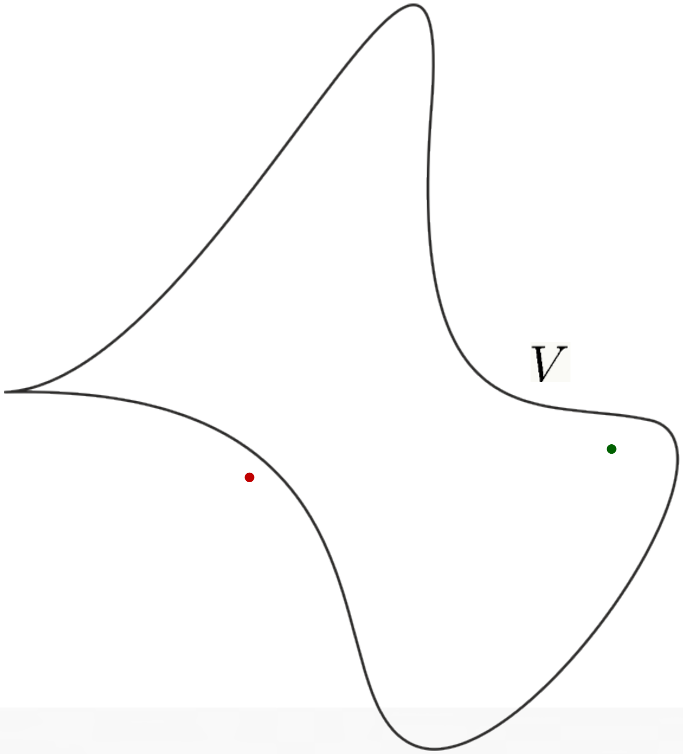
  $\dim C_{\mathbf{q}}^{\mathrm{K}} L_5 = 0$

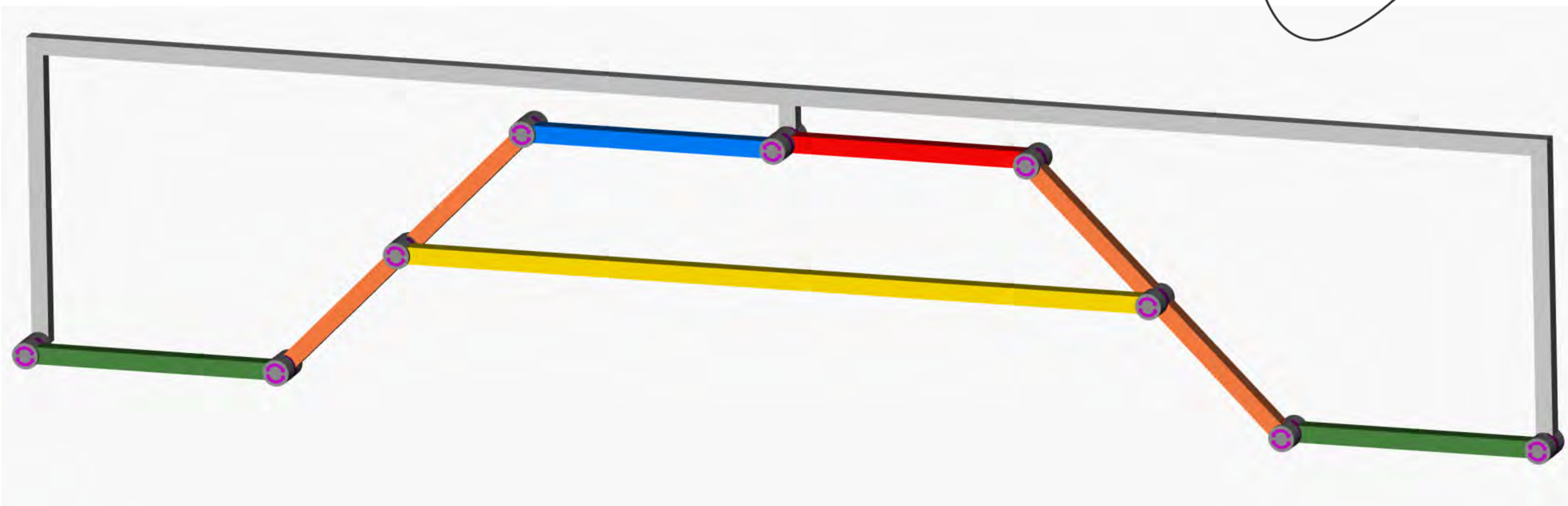


  ➡ all points, except $\mathbf{q}$ have (locally) rank 5

- $C_{\mathbf{q}}^{\mathrm{K}} V = C_{\mathbf{q}}^{\mathrm{K}} L_5 = \{\mathbf{0}\}$

$$\dim C_{\mathbf{q}}^{\mathrm{K}} V = 0 \neq \delta_{\mathrm{loc}}\left(\mathbf{q}\right) = 1$$





Conclusion

- Understanding of kinematic singularities of general linkages is still open
- No generally applicable math framework/algorithm for (local) analysis
- We have a method to investigate finite motions with certain rank

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