On Jacobian Matrix

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Abstract During the past decade the use of the word Jacobian in robotics has been extensively used and abused, in this document the topic will be covered together with some interesting insight. First the Jacobian matrix is defined, followed by its application to robotics of serial and parallel manipulators, and last I will remind the meaning of some commonly used formulas such as $J^T J$, $J^\# J$, and $I - J^T J$

1 Jacobian matrix

In vector calculus, the Jacobian is the matrix of all first-order partial derivatives of a vector-valued function, let's consider $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, such that $x \in \mathbb{R}^n$ is mapped to $y = f(x) \in \mathbb{R}^m$, the Jacobian matrix J of f is an $m \times n$ matrix, defined as follows:

$$\mathbf{J} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right] = \left[\frac{\partial \mathbf{f}}{\partial x_1}, \cdots, \frac{\partial \mathbf{f}}{\partial x_n}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \tag{1}$$

We can see that the Jacobian matrix defines a linear map $\mathbb{R}^n \mapsto \mathbb{R}^m$ which is the best linear approximation of the function $\mathbf{f}$ near the point x, thus it is just a generalization of the notion of derivative, intuitively the Jacobian can also be seen as the distortion that the transformation generates locally (i.e ”stretching”, ”rotating” etc.).

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2 Jacobian for robotic manipulators

The Jacobian in the field of robotics connects the velocities of the joints with the velocity of the end effector in cartesian space in two different ways depending on the architecture of the manipulator.

Serial Manipulators

The forward kinematics of a manipulator can be represented with:

$$\mathbf{x} = \mathbf{f}(\mathbf{q}),\ \mathbf{q} \in \mathbb{R}^n,\ \mathbf{x} \in \mathbb{R}^m \tag{2}$$

$$x_i = f_i(\mathbf{q}),\ i = 0..m \tag{3}$$

Then the time derivative of $\mathbf{x}$ can be written as:

$$\dot{\mathbf{x}} = \frac{d\mathbf{f}(\mathbf{q})}{dt} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial t} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{4}$$

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}} \tag{5}$$

$$\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{x}} \tag{6}$$

Thus we can think the Jacobian as how a differential change in joint variables influences a differential change in end effector pose, the forward kinematics is a non-linear function thus the Jacobian matches just locally its behavior.

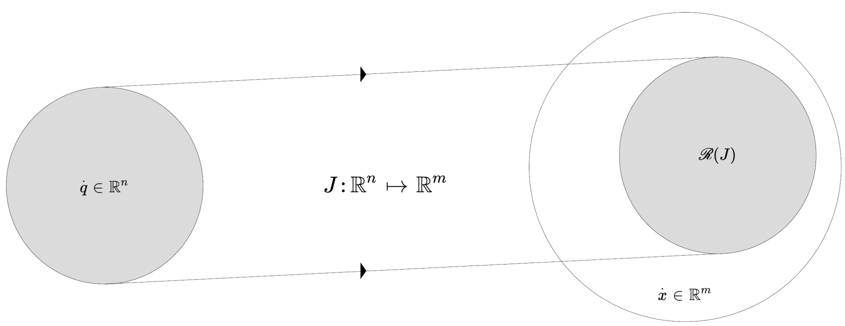

Fig. 1 Graphic representation of the Jacobian linear mapping for serial manipulators, from the set of joint velocities to end effector twist. $\mathscr{R}(J)$ is the Range of the linear mapping (i.e. all the possible $\dot{x}$ that can be achieved by the manipulator).

Parallel Manipulators

The meaning of the Jacobian matrix for parallel manipulators is quite different, the forward kinematics is complex due to the number of constraints arising from the multiple closed loops thus usually the Jacobian comes from a different analysis. Let's consider a p-dimensional function that takes into account all the kinematics constraints:

$$\mathbf{g}(\mathbf{x},\mathbf{q}) = \mathbf{0} \tag{7}$$

Taking the time derivative as we did for serial manipulators leads to:

$$\frac{d\mathbf{g}}{dt} = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial \mathbf{g}}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial t} = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \mathbf{g}}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{0} \tag{8}$$

and if we reorganize it:

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\dot{\mathbf{x}} = -\frac{\partial \mathbf{g}}{\partial \mathbf{q}}\dot{\mathbf{q}} \tag{9}$$

$$\mathbf{J}_x\dot{\mathbf{x}} = -\mathbf{J}_q\dot{\mathbf{q}} \tag{10}$$

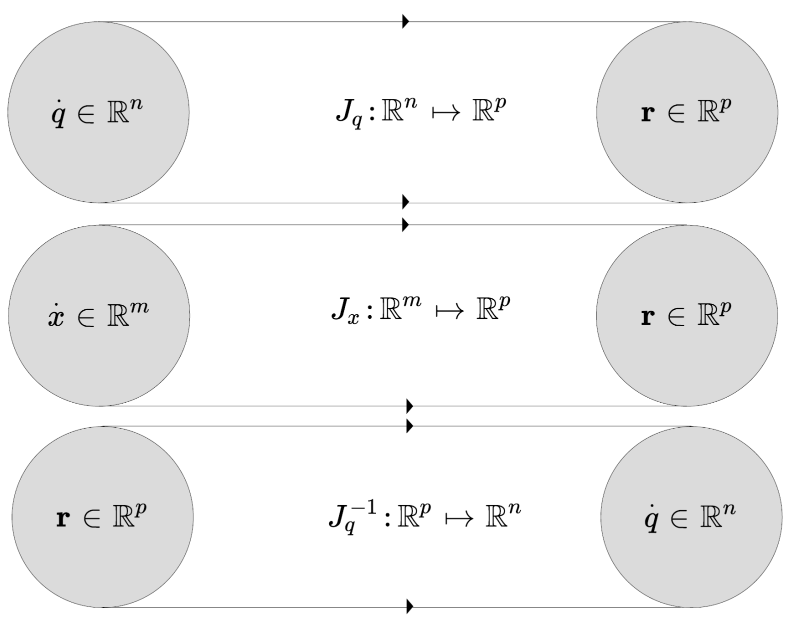

Fig. 2 Graphic representation of the intermediate linear mappings to get the parallel robot jacobian.

and finally:

$$\dot{\mathbf{q}} = \mathbf{J}\dot{\mathbf{x}},\ \mathbf{J} = \mathbf{J}_q^{-1}\mathbf{J}_x \tag{11}$$

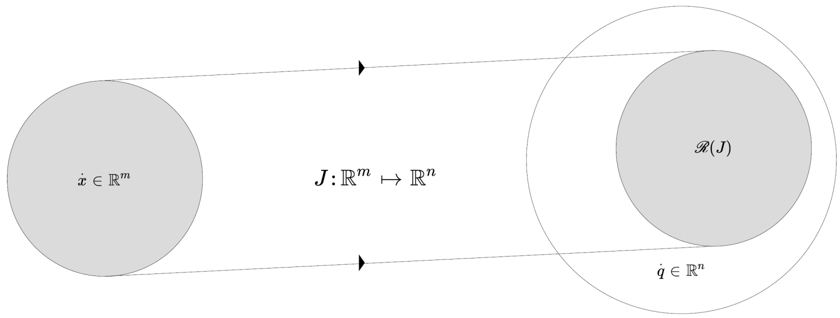

Fig. 3 Graphic representation of the Jacobian linear mapping for parallel manipulators, from the end effector twist to the set of joint velocities. $\mathscr{R}(J)$ is the Range of the linear mapping (i.e. all the possible $\dot{q}$ that satisfy the constraints of the closed loops).

Remark: It is important to understand that the Jacobian is just the matrix of the first order partial derivative of a vector valued function, there is nothing special about it, we could work with different Jacobians for every robots but in literature we are used to work with the particolar manipulator Jacobians described above, I personally find confusing that we have two complete different definition of Jacobians for serial and parallel manipulators but we call them with the same name.

3 Dual spaces and J^T

In order to understand the meaning of the transpose of a matrix is useful to introduce some basic concepts:

Vector Space over a Field: set of objects called vectors, two operations are defined in a vector space: *i)* addition of element of the vector space (parallelogram rule), *ii)* multiplication of an element of the vector space with an element of the field (in robotics the field is most likely $\mathbb{R}$).

Functional: linear map from a vector space to its field, if we consider the case of column vectors (i.e. joint velocities), then the functionals are row vectors (here a first exposure to the concept of **transpose**), their action on vectors is given by the dot product. The functional can also be called **covector**.

Dual Space: in any vector space $\mathbf{V}$ we can introduce a dual space $\mathbf{V}^*$ which consists of all linear functionals on $\mathbf{V}$, together with the vector space structure (i.e. addition of elements of **V** and multiplication by elements of the field). $v \in V : w^T v = f(v), w \in V^*$.

Transpose: the transpose of a linear map between two vector spaces, defined over the same field, is an induced map between the dual spaces of the two vector spaces.

In our case limited to robotics the linear map is the Jacobian matrix and the two vector spaces are the joint velocities and end effector twist. The dual of joint velocities is the space of joint torques and the dual of the cartesian twist is the space of cartesian wrench.

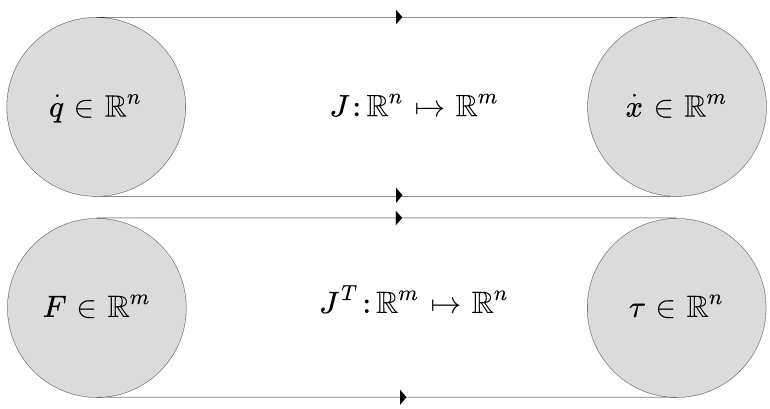

Fig. 4 Transpose of the Jacobian for serial manipulators, mapping from the end effector wrench (dual of the end effector twist) to the set of joint torques (dual of the joint velocities).

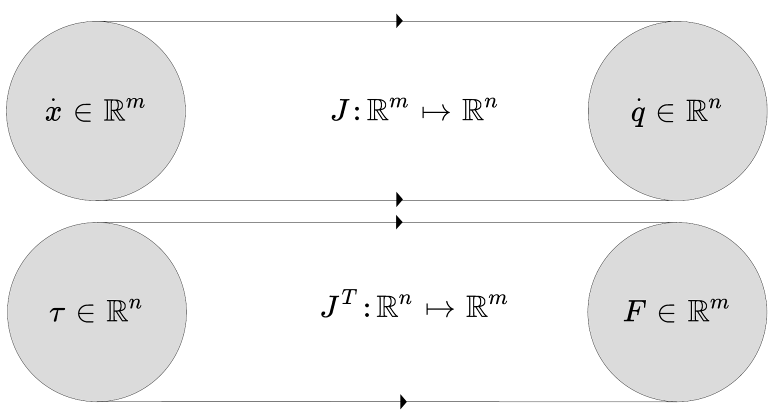

Fig. 5 Transpose of the Jacobian for parallel manipulators, mapping from the set of joint torques to the end effector wrench.

Where does this duality come from? It comes from the physical constraint imposed by the fact that the work done at the joint level is the same as the one at the end effector. Let's consider the instantaneous work of a generic serial manipulator (remark: $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$, and $(\mathbf{ab})^T = \mathbf{b}^T \mathbf{a}^T$):

$$\delta W = \dot{\mathbf{x}} \cdot F = v \cdot f + \omega \cdot \tau \tag{12}$$

$$W = \int_{t_1}^{t_2} \dot{\mathbf{q}} \cdot \tau dt = \int_{t_1}^{t_2} \dot{\mathbf{x}} \cdot \mathbf{F} dt \tag{13}$$

$$\dot{\mathbf{q}}^T \tau = \dot{\mathbf{x}}^T \mathbf{F} \tag{14}$$

$$\dot{\mathbf{q}}^T \tau = (\mathbf{J}\dot{\mathbf{q}})^T \mathbf{F} \tag{15}$$

$$\dot{\mathbf{q}}^T \tau = \dot{\mathbf{q}}^T \mathbf{J}^T \mathbf{F} \tag{16}$$

$$\tau = \mathbf{J}^T \mathbf{F} \tag{17}$$

4 Meaning of $J^T J$, JJ^T, Pseudoinverse, $J^\# J$, and $I - J^\# J$

In this section different operations on the Jacobian matrix found in literature are briefly analyzed (for serial manipulators):

$J^T J$: even though the two matrices are dimensionally coherent, their matrix product does not make much sense since $\mathbf{J}$ and $\mathbf{J}^T$ work on different spaces as we have seen in the previous section. In order to have a mathematically correct multiplication we need to have an intermediate mapping in between, such a matrix has to be symmetric, positive definite, and has to fix the units. A possible choice is the generalized inertia matrix. A correct representation would be $J^T MJ$. The mapping from twists to wrenches is usually called **Mechanical Impedance** matrix, its inverse **Mobility** matrix.

JJ^T: same as above but with M^{-1}.

$J^\#$: Penrose pseudoinverse, for $A \in \mathbb{R}^{n \times m}$ is a matrix which satisfies i) $AXA = A$, ii) $XAX = X$, iii) $(AX)^T = AX$, and iv) $(XA)^T = XA$, with $X \in \mathbb{R}^{m \times n}$. The left pseudoinverse is used for kinematically redundant manipulators and is formulated as $J^\# = J^T (JJ^T)^{-1}$.

$I - J^\# J$: Redundant Space (RS), maps the space of joint velocities $\dot{q} \in \mathbb{R}^n$ to the subset of $\dot{q} \in \mathbb{R}^n$ that does not cause end effector velocities.

$J^\# J$: Orthogonal Complement of the Redundant Space (OCRS), maps $\dot{q} \in \mathbb{R}^n$ to the subspace of $\dot{q} \in \mathbb{R}^n$ of the joint velocities achievable by the manipulator to the end effector subspace.

$JJ^\#$: Manipulable Space (MS), maps the end effector velocities $\dot{x} \in \mathbb{R}^m$ to the subspace of $\dot{x} \in \mathbb{R}^m$ of the kinematically realizable end effector velocities.

$I - JJ^\#$: Orthogonal Complement of the Manipulable Space (OCMS), maps the end effector velocities $\dot{x} \in \mathbb{R}^m$ to the subspace of $\dot{x} \in \mathbb{R}^m$ of the velocities that cannot be achieved by the manipulator.

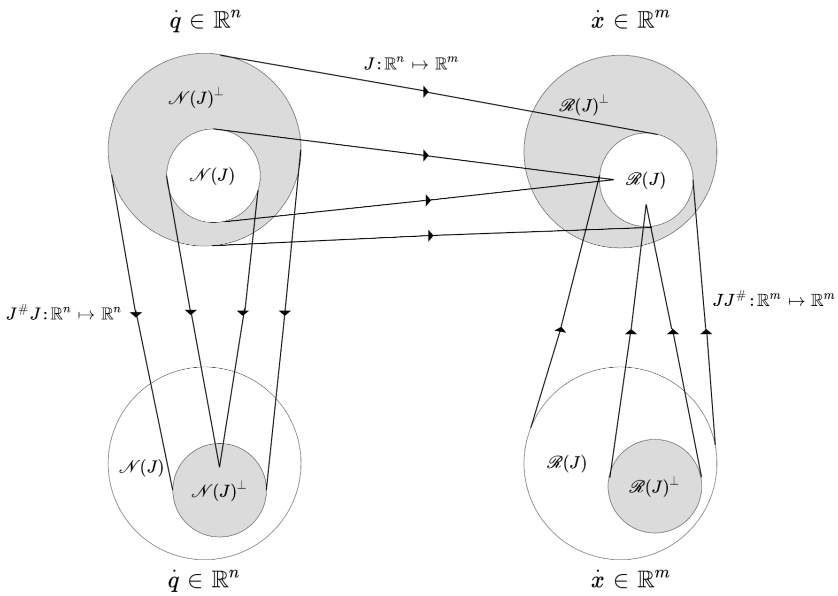

Fig. 6 Drawing of the projection made by J,$J^\# J$, and $JJ^\#$. Notice that the following equalities arise: i) $\mathcal{R}(J) = \mathcal{N}(J^\#)^\perp = \mathcal{R}(JJ^\#)$, ii) $\mathcal{R}(J)^\perp = \mathcal{N}(J^\#) = \mathcal{N}(JJ^\#)$, iii) $\mathcal{N}(J) = \mathcal{R}(J^\#)^\perp = \mathcal{N}(J^\# J)$, iv) $\mathcal{N}(J)^\perp = \mathcal{R}(J^\#) = \mathcal{R}(J^\# J)$.

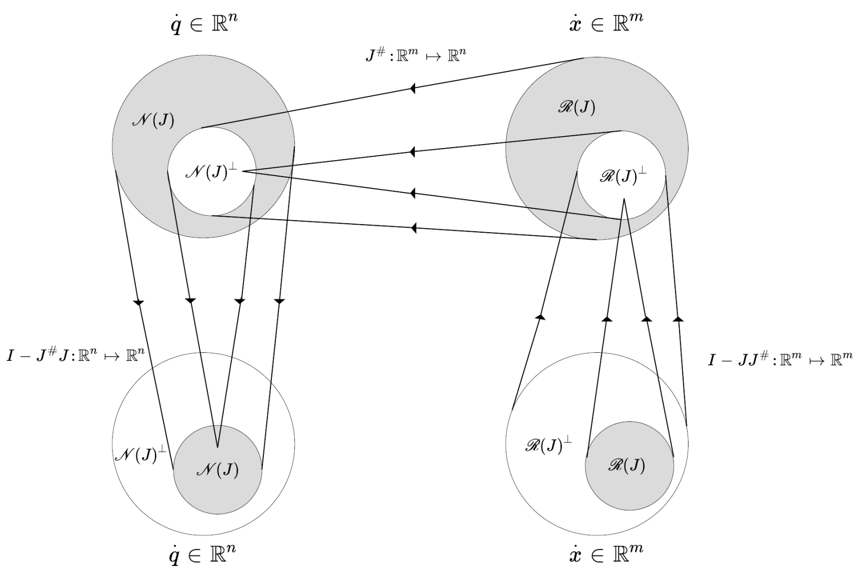

Fig. 7 Drawing of the projection made by $J^\#$,$I - J^\# J$, and $I - JJ^\#$. Notice that the following equalities arise: i) $\mathcal{R}(J) = \mathcal{N}(J^\#)^\perp = \mathcal{N}(I - JJ^\#)$, ii) $\mathcal{R}(J)^\perp = \mathcal{N}(J^\#) = \mathcal{R}(I - JJ^\#)$, iii) $\mathcal{N}(J) = \mathcal{R}(J^\#)^\perp = \mathcal{R}(I - J^\# J)$, iv) $\mathcal{N}(J)^\perp = \mathcal{R}(J^\#) = \mathcal{N}(I - J^\# J)$.