Value Chains

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Purpose and Objective

Resource-Product Value chain analysis and visualization using I-O accounts as a data source.

A premise of this project is that double-entry I-O accounts contain a sufficient data set to derive directed acyclical graphs that can in turn be visualized. The graph (composed of nodes and edges) depicts the order and sequence of activities in a process that transforms a primary resource (e.g. wood from standing trees) into an end product (e.g., electricity generated by burning wood - biomass power generation). Graphs of this type are commonly referred to as "supply chains" or "resource value chains" or "resource-product value chains (RPVC)".

This project describes and illustrates methods used extract and visualize RPVCs from I-O accounts.

1. National Income and Product Accounts

Input-Output accounts are an artifact in a System of National Accounts or SNA (also referred to as National Income and Product Accounts or NIPA). Fundamental accounting identities hold for NIPA accounts and I-O accounts inherit the same identities.

National Income and Product identities (in scalar form):

$$v = c + i + g + (e - m) \qquad \text{(National Income = National Product)}$$
$$m + v = c + i + g + e \qquad \text{(GNI + imports = GNP)}$$
$$m + v = f \;\; \text{(imports + GNI = Total Final Demand = f)}$$

I-O accounts add an intermediate output and outlay variables to the identity to introduce the concept of total industry output and outlay:

$$x + m + v = x + f \;\; \text{(x=intermediate output/outlay; (x+f) and (m+v)= total output)}$$

2. Core Form I-O Accounts

2.1 Core Form I-O Accounts

The fundamental accounting identities (i.e., double-entry identities) for I-O are shown in equations (1) and (2) below. These matrix equations describe the *Core Form* (vs the *Leontief Form*) of I-O accounts

$$X + f = t \qquad \text{(Intermediate Demand + Final Demand = Total Output)} \; (1)$$
$$X + V + M = q \; \text{(Intermediate Outlay + Value Added + Imports = Total Outlay)} \; (2)$$

2,2 Direct input coefficients

Using the *Core Form* equations (1) and (2), and assuming a multi-sector set of accounts, we cannot associate the multi-sector pattern of inputs (V and M) with the multi-sector pattern of final products f. In order to do this, we need algebraically transform the *Core Form* accounts into the *Leontief Form* of the same I-O accounts. To accomplish the algebraic transformation, start with the assumption that the pattern of inputs for an activity's product can be given by a vector of *direct coefficients* as shown in equations (3) - (5).

$$\hat{q}^{-1}X \underset{i}{=} A \; \text{(Inter-industry Direct Input Coefficients matrix iA)} \; (3)$$
$$\hat{q}^{-1}V \underset{v}{=} A \; \text{(Value Added Direct Input Coefficients matrix vA)} \; (4)$$
$$\hat{q}^{-1}M \underset{m}{=} A \qquad \text{(Import Direct Input Coefficients matrix mA)} \; (5)$$

Total Industry Outlay Vector $[q]_{1\times 6}$:

$$[\mathbf{q}]_{\mathbf{1\times 6}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

Total Outlay Inverse (Reciprocals) Matrix $[\hat{q}^{-1}]_{6\times 6}$:

$$[\hat{\mathbf{q}}^{-1}]_{\mathbf{6\times 6}} = \begin{bmatrix} \left(\frac{1}{q_1}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{1}{q_2}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{1}{q_3}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{1}{q_4}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{1}{q_5}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{1}{q_6}\right) \end{bmatrix}$$

2.2.1 Inter-industry Direct Coefficients

$$\hat{q}^{-1}X \underset{i}{=} A \; \text{(Inter-industry Direct Input Coefficients matrix iA)} \; (3)$$

Inter-industry Transactions Matrix $[X]_{6\times 6}$:

$$[\mathbf{X}]_{\mathbf{6\times6}} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66} \end{bmatrix}$$

Inter-Industry Explicit Computation of Direct Coefficients Matrix $\left[\hat{q}^{-1}X\right]_{6\times6} = \left[{}_{i}A\right]_{6\times6}$:

$$]_{\mathbf{6\times6}} = [{}_{\mathbf{i}}\mathbf{A}]_{\mathbf{6\times6}} = \begin{bmatrix} \left(\left(\frac{1}{q_1}\right)*x_{11}\right) \underset{i}{=} a_{11} & \left(\left(\frac{1}{q_1}\right)*x_{12}\right) \underset{i}{=} a_{12} & \left(\left(\frac{1}{q_1}\right)*x_{13}\right) \underset{i}{=} a_{13} & \left(\left(\frac{1}{q_1}\right)*x_{13}\right) \underset{i}{=} a_{14} & \left(\left(\frac{1}{q_1}\right)*x_{15}\right) \underset{i}{=} a_{15} & \left(\left(\frac{1}{q_1}\right)*x_{16}\right) = \\ \left(\left(\frac{1}{q_2}\right)*x_{21}\right) \underset{i}{=} a_{21} & \left(\left(\frac{1}{q_2}\right)*x_{22}\right) \underset{i}{=} a_{22} & \left(\left(\frac{1}{q_2}\right)*x_{23}\right) \underset{i}{=} a_{23} & \left(\left(\frac{1}{q_2}\right)*x_{23}\right) \underset{i}{=} a_{24} & \left(\left(\frac{1}{q_2}\right)*x_{25}\right) \underset{i}{=} a_{25} & \left(\left(\frac{1}{q_2}\right)*x_{26}\right) = \\ \left(\left(\frac{1}{q_3}\right)*x_{31}\right) \underset{i}{=} a_{31} & \left(\left(\frac{1}{q_3}\right)*x_{32}\right) \underset{i}{=} a_{32} & \left(\left(\frac{1}{q_3}\right)*x_{33}\right) \underset{i}{=} a_{33} & \left(\left(\frac{1}{q_3}\right)*x_{33}\right) \underset{i}{=} a_{34} & \left(\left(\frac{1}{q_3}\right)*x_{35}\right) \underset{i}{=} a_{35} & \left(\left(\frac{1}{q_3}\right)*x_{36}\right) = \\ \left(\left(\frac{1}{q_4}\right)*x_{41}\right) \underset{i}{=} a_{41} & \left(\left(\frac{1}{q_4}\right)*x_{42}\right) \underset{i}{=} a_{42} & \left(\left(\frac{1}{q_4}\right)*x_{43}\right) \underset{i}{=} a_{43} & \left(\left(\frac{1}{q_4}\right)*x_{43}\right) \underset{i}{=} a_{44} & \left(\left(\frac{1}{q_4}\right)*x_{45}\right) \underset{i}{=} a_{45} & \left(\left(\frac{1}{q_4}\right)*x_{46}\right) = \\ \left(\left(\frac{1}{q_5}\right)*x_{51}\right) \underset{i}{=} a_{51} & \left(\left(\frac{1}{q_5}\right)*x_{52}\right) \underset{i}{=} a_{52} & \left(\left(\frac{1}{q_5}\right)*x_{53}\right) \underset{i}{=} a_{53} & \left(\left(\frac{1}{q_5}\right)*x_{53}\right) \underset{i}{=} a_{54} & \left(\left(\frac{1}{q_5}\right)*x_{55}\right) \underset{i}{=} a_{55} & \left(\left(\frac{1}{q_5}\right)*x_{56}\right) = \\ \left(\left(\frac{1}{q_6}\right)*x_{61}\right) \underset{i}{=} a_{61} & \left(\left(\frac{1}{q_6}\right)*x_{62}\right) \underset{i}{=} a_{62} & \left(\left(\frac{1}{q_6}\right)*x_{63}\right) \underset{i}{=} a_{63} & \left(\left(\frac{1}{q_6}\right)*x_{63}\right) \underset{i}{=} a_{64} & \left(\left(\frac{1}{q_6}\right)*x_{65}\right) \underset{i}{=} a_{65} & \left(\left(\frac{1}{q_6}\right)*x_{66}\right) = \end{bmatrix}$$

Inter-Industry Direct Coefficients Matrix $\left[{}_{i}A\right]_{6\times6}$:

$$[{}_{\mathbf{i}}\mathbf{A}]_{\mathbf{6\times6}} = \begin{bmatrix} {}_{i}a_{11} & {}_{i}a_{12} & {}_{i}a_{13} & {}_{i}a_{14} & {}_{i}a_{15} & a_{16} \\ {}_{i}a_{21} & {}_{i}a_{22} & {}_{i}a_{23} & {}_{i}a_{24} & {}_{i}a_{25} & a_{26} \\ {}_{i}a_{31} & {}_{i}a_{32} & {}_{i}a_{33} & {}_{i}a_{34} & {}_{i}a_{35} & a_{36} \\ {}_{i}a_{41} & {}_{i}a_{42} & {}_{i}a_{43} & {}_{i}a_{44} & {}_{i}a_{45} & a_{46} \\ {}_{i}a_{51} & {}_{i}a_{52} & {}_{i}a_{53} & {}_{i}a_{54} & {}_{i}a_{55} & a_{56} \\ {}_{i}a_{61} & {}_{i}a_{62} & {}_{i}a_{63} & {}_{i}a_{64} & {}_{i}a_{65} & a_{66} \end{bmatrix}$$

2.2.2 Total Value Added Direct Coefficients

$$\hat{q}^{-1}V \underset{v}{=} A \text{ (Value Added Direct Input Coefficients matrix vA) } (4)$$

Total Value Added Matrix $[V]_{6\times6}$:

$$[\mathbf{V}]_{\mathbf{6\times6}} = \begin{bmatrix} v_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & v_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & v_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & v_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & v_{66} \end{bmatrix}$$

Explicit Computation of Total Value Added Direct Coefficients Matrix $[\hat{q}^{-1}v]_{6\times6} = [{}_vA]_{6\times6}$:

$$V]_{\mathbf{6\times6}} = [{}_{\mathbf{v}}\mathbf{A}]_{\mathbf{6\times6}} = \begin{bmatrix} \left(\left(\frac{1}{q_1}\right)*v_{11}\right) \underset{v}{=} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\left(\frac{1}{q_2}\right)*v_{22}\right) \underset{v}{=} a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\left(\frac{1}{q_3}\right)*v_{33}\right) \underset{v}{=} a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\left(\frac{1}{q_4}\right)*v_{43}\right) \underset{v}{=} a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\left(\frac{1}{q_5}\right)*v_{55}\right) \underset{v}{=} a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\left(\frac{1}{q_6}\right)*v_{66}\right) = \end{bmatrix}$$

Total Value Added Direct Coefficients Matrix $[{}_vA]_{6\times6}$:

$$[{}_{\mathbf{v}}\mathbf{A}]_{\mathbf{6\times6}} = \begin{bmatrix} {}_va_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_va_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & {}_va_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & {}_va_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & {}_va_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & {}_va_{66} \end{bmatrix}$$

2.2.3 Total Imports Direct Coefficients

$$\hat{q}^{-1}M \underset{m}{=} A \text{ (Import Direct Input Coefficients matrix mA) } (5)$$

Total Imports Matrix $[M]_{6\times6}$:

$$[\mathbf{M}]_{\mathbf{6\times6}} = \begin{bmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{66} \end{bmatrix}$$

Explicit Computation of Total Imports Direct Coefficients Matrix $[\hat{q}^{-1}M]_{6\times6} = [{}_mA]_{6\times6}$:

$$\mathbf{{}_{\times 6}} = [{}_{\mathbf{m}}\mathbf{A}]_{\mathbf{6\times6}} = \begin{bmatrix} \left(\left(\frac{1}{q_1}\right)^{*}m_{11}\right)\underset{m}{=}a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\left(\frac{1}{q_2}\right)^{*}m_{22}\right)\underset{m}{=}a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\left(\frac{1}{q_3}\right)^{*}m_{33}\right)\underset{m}{=}a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\left(\frac{1}{q_4}\right)^{*}m_{43}\right)\underset{m}{=}a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(\left(\frac{1}{q_5}\right)^{*}m_{55}\right)\underset{m}{=}a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\left(\frac{1}{q_6}\right)^{*}m_{66} \end{bmatrix}$$

Total Value Added Direct Coefficients Matrix $[{}_{m}A]_{6\times6}$:

$$[{}_{\mathbf{m}}\mathbf{A}]_{\mathbf{6\times6}} = \begin{bmatrix} {}_{m}a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & {}_{m}a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & {}_{m}a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & {}_{m}a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & {}_{m}a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & {}_{m}a_{66} \end{bmatrix}$$

Direct Coefficients Matrix $[\hat{q}^{-1}X]_{6\times6} = [{}_{i}A]_{6\times6}$:

$$]_{\mathbf{6\times6}} = [{}_{\mathbf{i}}\mathbf{A}]_{\mathbf{6\times6}} = \begin{bmatrix} \left(\left(\frac{1}{q_1}\right)^{*}x_{11}\right)\underset{i}{=}a_{11} & \left(\left(\frac{1}{q_1}\right)^{*}x_{12}\right)\underset{i}{=}a_{12} & \left(\left(\frac{1}{q_1}\right)^{*}x_{13}\right)\underset{i}{=}a_{13} & \left(\left(\frac{1}{q_1}\right)^{*}x_{13}\right)\underset{i}{=}a_{14} & \left(\left(\frac{1}{q_1}\right)^{*}x_{15}\right)\underset{i}{=}a_{15} & \left(\left(\frac{1}{q_1}\right)^{*}x_{16}\right) = \\ \left(\left(\frac{1}{q_2}\right)^{*}x_{21}\right)\underset{i}{=}a_{21} & \left(\left(\frac{1}{q_2}\right)^{*}x_{22}\right)\underset{i}{=}a_{22} & \left(\left(\frac{1}{q_2}\right)^{*}x_{23}\right)\underset{i}{=}a_{23} & \left(\left(\frac{1}{q_2}\right)^{*}x_{23}\right)\underset{i}{=}a_{24} & \left(\left(\frac{1}{q_2}\right)^{*}x_{25}\right)\underset{i}{=}a_{25} & \left(\left(\frac{1}{q_2}\right)^{*}x_{26}\right) = \\ \left(\left(\frac{1}{q_3}\right)^{*}x_{31}\right)\underset{i}{=}a_{31} & \left(\left(\frac{1}{q_3}\right)^{*}x_{32}\right)\underset{i}{=}a_{32} & \left(\left(\frac{1}{q_3}\right)^{*}x_{33}\right)\underset{i}{=}a_{33} & \left(\left(\frac{1}{q_3}\right)^{*}x_{33}\right)\underset{i}{=}a_{34} & \left(\left(\frac{1}{q_3}\right)^{*}x_{35}\right)\underset{i}{=}a_{35} & \left(\left(\frac{1}{q_3}\right)^{*}x_{36}\right) = \\ \left(\left(\frac{1}{q_4}\right)^{*}x_{41}\right)\underset{i}{=}a_{41} & \left(\left(\frac{1}{q_4}\right)^{*}x_{42}\right)\underset{i}{=}a_{42} & \left(\left(\frac{1}{q_4}\right)^{*}x_{43}\right)\underset{i}{=}a_{43} & \left(\left(\frac{1}{q_4}\right)^{*}x_{43}\right)\underset{i}{=}a_{44} & \left(\left(\frac{1}{q_4}\right)^{*}x_{45}\right)\underset{i}{=}a_{45} & \left(\left(\frac{1}{q_4}\right)^{*}x_{46}\right) = \\ \left(\left(\frac{1}{q_5}\right)^{*}x_{51}\right)\underset{i}{=}a_{51} & \left(\left(\frac{1}{q_5}\right)^{*}x_{52}\right)\underset{i}{=}a_{52} & \left(\left(\frac{1}{q_5}\right)^{*}x_{53}\right)\underset{i}{=}a_{53} & \left(\left(\frac{1}{q_5}\right)^{*}x_{53}\right)\underset{i}{=}a_{54} & \left(\left(\frac{1}{q_5}\right)^{*}x_{55}\right)\underset{i}{=}a_{55} & \left(\left(\frac{1}{q_5}\right)^{*}x_{56}\right) = \\ \left(\left(\frac{1}{q_6}\right)^{*}x_{61}\right)\underset{i}{=}a_{61} & \left(\left(\frac{1}{q_6}\right)^{*}x_{62}\right)\underset{i}{=}a_{62} & \left(\left(\frac{1}{q_6}\right)^{*}x_{63}\right)\underset{i}{=}a_{63} & \left(\left(\frac{1}{q_6}\right)^{*}x_{63}\right)\underset{i}{=}a_{64} & \left(\left(\frac{1}{q_6}\right)^{*}x_{65}\right)\underset{i}{=}a_{65} & \left(\left(\frac{1}{q_6}\right)^{*}x_{66}\right) = \end{bmatrix}$$

3. Inter-Industry Adjacency Matrix

3.1 Identifying an RPVC Path in an Adjacency Matrix

The $[{}_{i}A]$ matrix is commonly referred to as the *Direct Coefficients* matrix. The term *Direct* refers to the *one-step* relationships between an industry activity and its input suppliers. That is, each industry activity is one transaction (i.e., *one step*) removed from each of its suppliers. In aggregate, each industry activity and its suppliers have a *one-to-many* relationship (i.e., one activity has many suppliers). Each activity's column vector in the $[_iA]$ matrix describes that activity's *one-to-many* relationships with its input suppliers.

An *Adjacency Matrix* derived from the $[_iA]$ matrix can be used to determine the number of one-step relationships (i.e., transactions) between any pair of purchasing and suppling activities. In other words, an *Adjacency Matrix* can reveal the path and distance (number of steps) between an origin activity and a destination activity (i.e., an RPVC).

Derive adjacency matrix $[B]$ from the $[_iA]$ matrix. Successive powers of $[B^n]$ will identify successive paths of step-length ***n***.

To illustrate an adjacency matrix, consider a portion of an $[_iA]$ matrix with only wood-related activities. Assume industry ***1*** is *Grow*, industry ***2*** is *Harvest*, industry ***3*** is *Saw Mill*, industry ***4*** is *Residuals*, industry ***5*** is *Bio-Power*, and industry ***6*** is *Other*. Assume the commodity produced by industry ***1*** is *Stumpage* (*m³* of solid wood), the commodity produced by industry ***2*** is *Logs* (containing *m³* of solid wood), the commodity produced by industry ***3*** is *Dimension Lumber* (containing *m³* of solid wood), the commodity produced by industry ***4*** is *Sawdust Residuals* (containing *m³* of solid wood), the commodity produced by industry ***5*** is *Electricity* (generated by using *m³* of solid wood *Sawdust Residuals* as fuel), and commodity produced by industry ***6*** is *Other*.

The Inter-industry Direct Coefficients Matrix $[_iA]_{6\times 6}$:

$$[_\mathbf{i}\mathbf{A}]_{\mathbf{6\times 6}} = \begin{bmatrix} _ia_{11} & _ia_{12} & _ia_{13} & _ia_{14} & _ia_{15} & _ia_{16} \\ _ia_{21} & _ia_{22} & _ia_{23} & _ia_{24} & _ia_{25} & _ia_{26} \\ _ia_{31} & _ia_{32} & _ia_{33} & _ia_{34} & _ia_{35} & _ia_{36} \\ _ia_{41} & _ia_{42} & _ia_{43} & _ia_{44} & _ia_{45} & _ia_{46} \\ _ia_{51} & _ia_{52} & _ia_{53} & _ia_{54} & _ia_{55} & _ia_{56} \\ _ia_{61} & _ia_{62} & _ia_{63} & _ia_{64} & _ia_{65} & _ia_{66} \end{bmatrix}$$

Inter-industry Direct Coefficients Matrix $[_iA]_{6\times 6}$ where $_ia_{nn} \neq 0$:

$$[_\mathbf{i}\mathbf{A}]_{\mathbf{6\times 6}} = \begin{bmatrix} _ia_{11} & _ia_{12} & 0 & 0 & 0 & 0 \\ 0 & _ia_{22} & _ia_{23} & 0 & 0 & 0 \\ 0 & 0 & _ia_{33} & _ia_{34} & 0 & 0 \\ 0 & 0 & 0 & _ia_{44} & _ia_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency matrix $[B^1_{6\times 6}]$ (no 1-step path between Activity 5 and Activity 1 indicated by $b^1_{15} = 0$):

$$[\mathbf{B^1}]_{\mathbf{6\times 6}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrix $\left[B^2_{6\times 6}\right]$ (no 2-step path between Activity 5 and Activity 1 indicated by $b^2_{15} = 0$):

$$[\mathbf{B^2}]_{\mathbf{6\times 6}} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrix $\left[B^3_{6\times 6}\right]$ (no 3-step path between Activity 5 and Activity 1 indicated by $b^3_{15} = 0$):

$$[\mathbf{B^3}]_{\mathbf{6\times 6}} = \begin{bmatrix} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrix $\left[B^4_{6\times 6}\right]$ (one 4-step path between Activity 5 and Activity 1 indicated by $b^4_{15} = 1$):

$$[\mathbf{B^4}]_{\mathbf{6\times 6}} = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 & 0 \\ 0 & 1 & 4 & 5 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The *trace* of the *path* of *length* 4 from Activity 5 to Activity 1 in $\left[B^4\right]$:

step 1: (b_{55} > b_{45}), step 2: (b_{43} > b_{33}), step 3: (b_{32} > b_{22}), step 4: (b_{21} > b_{11})

A *path* like this can be used below to determine the order and number of steps (column-space expansions) necessary to decompose a specific RPVC from a *Leontief Form* product vector.

3.2 Visual Analysis of a Path in Core Form I-O Accounts

3.2.1 Build Sets of Nodes and Edges from I-O Matrices

Build relationship matrices (node-by-node matrix for a given relationship)

a) Nodes are (1) Industries, (2) Transactions Nodes have properties (1) Subject or (2) Object

b) Edges (relationships or Predicates) consist of (1) "Interacts With"

c) Semantic Triple EXAMPLE 1: Subject(Industry 1) > Predicate("Interacts with") > Object(Transaction 1)

d) Semantic Triple EXAMPLE 2: Subject(Transaction 1) > Predicate("Interacts with") > Object(Industry 2)

e) Semantic Chain EXAMPLE 3 where Subject ***MATCHES*** Object: Subject(Industry 1) > Predicate("Interacts with") > Object(Transaction 1) Subject(Transaction 1) > Predicate("Interacts with") > Object(Industry 2)

f) SEMANTIC TRANSACTION CHAIN: (Industry 1) > ("Interacts with") > (Transaction 1) > ("Interacts with") > (Industry 2)

3.2.2 Visualization of a Path in a Graph

TBD

4. Leontief Form I-O Accounts:

Leontief-Form I-O accounts map multi-sector inputs to multi-sector output. This is achieved by deriving a multi-sector transformation matrix to transform the *Core Form* I-O accounts into the *Leontief-Form* I-O accounts. The transformation from *Core Form* to *Netput Form* to *Leontief Form* I-O accounts (including the derivation of a "multiplier matrix") is derived algebraically as follows:

$$
\begin{aligned}
X + f &= x && \text{(Core Form I-O Accounts)} \quad (6)\\
X &\underset{i}{=} Ax && \text{(Substitute)} \quad (7)\\
{}_iAx + f &= x && \text{(Core Form I-O Accounts)} \quad (8)\\
f &= (I -_i A)x && \text{(Rearrange; Netput Form of I-O Accounts)} \quad (9)\\
N &= (I -_i A)^{-1} && \text{(Substitute)} \quad (10)\\
f &= Nx && \text{(Netput Form of I-O Accounts)} \quad (11)\\
x &= N^{-1}f && \text{(Rearrange; Leontief Form I-O Accounts)} \quad (12)\\
Z &= N^{-1} && \text{(Substitute)} \quad (13)\\
x &= Zf && \text{(Leontief Form I-O Accounts)} \quad (14)
\end{aligned}
$$

4.1 Leontief Form, Output Space (Primal)

Leontief Form Output Space I-O Accounting identites:

$$x = Zf \text{ (Leontief Form for Total Output)} \quad (15)$$

Total Output vector $[x]_{1\times 6}$:

$$[\mathbf{x}]_{\mathbf{1}\times\mathbf{6}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Total Final Demand Vector $[f]_{1\times 6}$:

$$[\mathbf{f}]_{\mathbf{1}\times\mathbf{6}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

Leontief Multiplier Matrix $[Z]_{6\times 6}$:

$$[\mathbf{Z}]_{\mathbf{6}\times\mathbf{6}} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} \end{bmatrix}$$

4.2 Leontief Form, Input Space (Dual)

Leontief Form I-O Input Space Accounting Identites:

$$v = \widehat{{}_vA}Zf \qquad \text{(Leontief Form for Total Value Added)} \quad (16)$$

$$m = \widehat{{}_mA}Zf \qquad \text{(Leontief Form for Total Imports)} \quad (17)$$

$$(v+m) = \left(\widehat{{}_vA}Zf + \widehat{{}_mA}Zf\right) \quad \text{(Leontief Form for Total Primary Inputs (v+m))} \quad (18)$$

Total Value Added vector $[v]_{1\times 6}$:

$$[\mathbf{v}]_{\mathbf{1}\times\mathbf{6}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

Total Value Added Direct Coefficients matrix $\left[\widehat{{}_vA}\right]_{6\times 6}$:

$$\left[\widehat{_\mathbf{v}\mathbf{A}}\right]_{\mathbf{6\times 6}} = \begin{bmatrix} {_v a_{11}} & 0 & 0 & 0 & 0 & 0 \\ 0 & {_v a_{22}} & 0 & 0 & 0 & 0 \\ 0 & 0 & {_v a_{33}} & 0 & 0 & 0 \\ 0 & 0 & 0 & {_v a_{44}} & 0 & 0 \\ 0 & 0 & 0 & 0 & {_v a_{55}} & 0 \\ 0 & 0 & 0 & 0 & 0 & {_v a_{66}} \end{bmatrix}$$

Total Final Demand Vector $[f]_{1\times 6}$:

$$[\mathbf{f}]_{\mathbf{1\times 6}} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}$$

Leontief Form Multiplier Matrix $[Z]_{6\times 6}$:

$$[\mathbf{Z}]_{\mathbf{6\times 6}} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} \end{bmatrix}$$

Intermediate product matrix $\left[\widehat{_v A}Z\right]_{6\times 6}$:

$$\left[\widehat{_\mathbf{v}\mathbf{A}}\mathbf{Z}\right]_{\mathbf{6\times 6}} = \begin{bmatrix} ({_v a_{11}}{*}z_{11}) & ({_v a_{11}}{*}z_{12}) & ({_v a_{11}}{*}z_{13}) & ({_v a_{11}}{*}z_{14}) & ({_v a_{11}}{*}z_{15}) & ({_v a_{11}}{*}z_{16}) \\ ({_v a_{21}}{*}z_{21}) & ({_v a_{21}}{*}z_{22}) & ({_v a_{21}}{*}z_{23}) & ({_v a_{21}}{*}z_{24}) & ({_v a_{21}}{*}z_{25}) & ({_v a_{21}}{*}z_{26}) \\ ({_v a_{31}}{*}z_{31}) & ({_v a_{31}}{*}z_{32}) & ({_v a_{31}}{*}z_{33}) & ({_v a_{31}}{*}z_{34}) & ({_v a_{31}}{*}z_{35}) & ({_v a_{31}}{*}z_{36}) \\ ({_v a_{41}}{*}z_{41}) & ({_v a_{41}}{*}z_{42}) & ({_v a_{41}}{*}z_{43}) & ({_v a_{41}}{*}z_{44}) & ({_v a_{41}}{*}z_{45}) & ({_v a_{41}}{*}z_{46}) \\ ({_v a_{51}}{*}z_{51}) & ({_v a_{51}}{*}z_{52}) & ({_v a_{51}}{*}z_{53}) & ({_v a_{51}}{*}z_{54}) & ({_v a_{51}}{*}z_{55}) & ({_v a_{51}}{*}z_{56}) \\ ({_v a_{51}}{*}z_{61}) & ({_v a_{51}}{*}z_{62}) & ({_v a_{51}}{*}z_{63}) & ({_v a_{51}}{*}z_{64}) & ({_v a_{51}}{*}z_{65}) & ({_v a_{51}}{*}z_{66}) \end{bmatrix}$$

Total Value Added vector v in explicit computational form $[v]_{1\times 6} = \left[\left[\widehat{_v A}Z\right][f]\right]_{1\times 6}$:

$$_{\mathbf{1\times6}} = \left[\left[\widehat{_{\mathbf{v}}\mathbf{A}\mathbf{Z}}\right][\mathbf{f}]\right]_{\mathbf{1\times6}} = \begin{bmatrix} ((_va_{11}{}^*z_{11})^*f_1) & +((_va_{11}{}^*z_{12})^*f_2) & +((_va_{11}{}^*z_{13})^*f_3) & +((_va_{11}{}^*z_{14})^*f_4) & +((_va_{11}{}^*z_{15})^*f_5) & +((_va_{11}{}^*z_{16})^*f_6 \\ ((_va_{21}{}^*z_{21})^*f_1) & +((_va_{21}{}^*z_{22})^*f_2) & +((_va_{21}{}^*z_{23})^*f_3) & +((_va_{21}{}^*z_{24})^*f_4) & +((_va_{21}{}^*z_{25})^*f_5) & +((_va_{21}{}^*z_{26})^*f_6 \\ ((_va_{31}{}^*z_{31})^*f_1) & +((_va_{31}{}^*z_{32})^*f_2) & +((_va_{31}{}^*z_{33})^*f_3) & +((_va_{31}{}^*z_{34})^*f_4) & +((_va_{31}{}^*z_{35})^*f_5) & +((_va_{31}{}^*z_{36})^*f_6 \\ ((_va_{41}{}^*z_{41})^*f_1) & +((_va_{41}{}^*z_{42})^*f_2) & +((_va_{41}{}^*z_{43})^*f_3) & +((_va_{41}{}^*z_{44})^*f_4) & +((_va_{41}{}^*z_{45})^*f_5) & +((_va_{41}{}^*z_{46})^*f_6 \\ ((_va_{51}{}^*z_{51})^*f_1) & +((_va_{51}{}^*z_{52})^*f_2) & +((_va_{51}{}^*z_{53})^*f_3) & +((_va_{51}{}^*z_{54})^*f_4) & +((_va_{51}{}^*z_{55})^*f_5) & +((_va_{51}{}^*z_{56})^*f_6 \\ ((_va_{51}{}^*z_{61})^*f_1) & +((_va_{51}{}^*z_{62})^*f_2) & +((_va_{51}{}^*z_{63})^*f_3) & +((_va_{51}{}^*z_{64})^*f_4) & +((_va_{51}{}^*z_{65})^*f_5) & +((_va_{51}{}^*z_{66})^*f_6 \end{bmatrix}$$

4.3 Column space expansion of Leontief Form I-O Output Space:

Equation:

$$x = Zf \qquad \text{(Leontief Form for Total Output)} \quad (15)$$

$$O = Z\hat{f}(\hat{f} \text{ is a diagonal matrix of final demands; O is the column space of x})$$

Where: Vector $\left[\hat{f}\right]_{1\times6}$:

$$\left[\hat{\mathbf{f}}\right]_{\mathbf{1\times6}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Vector f:

$$\mathbf{f} = \begin{bmatrix} f_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & f_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & f_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & f_{66} \end{bmatrix}$$

Matrix Z:

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} & z_{16} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} & z_{26} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} & z_{36} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} & z_{46} \\ z_{51} & z_{52} & z_{53} & z_{54} & z_{55} & z_{56} \\ z_{61} & z_{62} & z_{63} & z_{64} & z_{65} & z_{66} \end{bmatrix}$$

Column-space expansion Matrix O:

$$\mathbf{O} = \begin{bmatrix} o_{11} & o_{12} & o_{13} & o_{14} & o_{15} & o_{16} \\ o_{21} & o_{22} & o_{23} & o_{24} & o_{25} & o_{26} \\ o_{31} & o_{32} & o_{33} & o_{34} & o_{35} & o_{36} \\ o_{41} & o_{42} & o_{43} & o_{44} & o_{45} & o_{46} \\ o_{51} & o_{52} & o_{53} & o_{54} & o_{55} & o_{56} \\ o_{61} & o_{62} & o_{63} & o_{64} & o_{65} & o_{66} \end{bmatrix}$$

4.5 Visualization of Tier 1 Column Space Expansion

TBD

4.6 Column space expansion of Leontief Form I-O Input Space:

TBD

4.7 Visualization of Tier 1 Column Space Expansion

5. Column Space Expansion of RPVCs

5.1 Supply Chain Tier 1 Column Space Expansion of Total Output of Vector $\mathbf{o}_5$

SECTION 3.3 TBD!!!!

Column space expansion of output vector o_5:

$o = Zf$ (Leontief transformation function)

$O = Z\hat{f}$ (f-hat is a diagonal matrix of intermediate demands by Biopwr; O is the column space of o)

(Product Electricity from Biomass Power Generation) Vector o_5:

$$\mathbf{o_5} = \begin{bmatrix} o_{15} \\ o_{25} \\ o_{35} \\ o_{45} \\ o_{55} \\ o_{65} \end{bmatrix}$$

Column space expansion of output vector o_5:

$o = Zf$ (Leontief transformation function)

$O = Z\hat{f}$ (f-hat is a diagonal matrix of intermediate demands by Biopwr; O is the column space of o)

Matrix O:

$$\mathbf{O} = \begin{bmatrix} o_{11} & o_{12} & o_{13} & o_{14} & o_{15} & o_{16} \\ o_{21} & o_{22} & o_{23} & o_{24} & o_{25} & o_{26} \\ o_{31} & o_{32} & o_{33} & o_{34} & o_{35} & o_{36} \\ o_{41} & o_{42} & o_{43} & o_{44} & o_{45} & o_{46} \\ o_{51} & o_{52} & o_{53} & o_{54} & o_{55} & o_{56} \\ o_{61} & o_{62} & o_{63} & o_{64} & o_{65} & o_{66} \end{bmatrix}$$

5.2 Supply Chain Tier 2 Column Space Expansion of Total Output of Vector $\mathbf{o}_4$

5.3 Supply Chain Tier 3 Column Space Expansion of Total Output of Vector $\mathbf{o}_3$

5.4 Supply Chain Tier 4 Column Space Expansion of Total Output of Vector $\mathbf{o}_2$

6. Supply Chain Visualization

6.1 Assemble a Supply Chain Knowlegebase

6.2 Query Knowlegebase for Supply Chain Pattern

6.3 Visualization of Supply Chain

$$\begin{bmatrix} W_{11} \\ W_{12} \\ \vdots \\ W_{1n} \end{bmatrix} + \begin{bmatrix} W_{21} \\ W_{22} \\ \vdots \\ W_{2n} \end{bmatrix} + \ldots + \begin{bmatrix} W_{n1} \\ W_{n2} \\ \vdots \\ W_{nn} \end{bmatrix} = \begin{bmatrix} \frac{W_{11}+W_{21}+\ldots+W_{n1}}{n} \\ \frac{W_{12}+W_{22}+\ldots+W_{n2}}{n} \\ \vdots \\ \frac{W_{1n}+W_{2n}+\ldots+W_{nn}}{n} \end{bmatrix}$$

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