Complete Lattice Analysis: Groups and Their Properties

1 The Lattice Structure

All groups embed into $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$. The containment lattice is:

$$\begin{array}{ccc} & \Pi = \prod_i \mathbb{Z}/2^i\mathbb{Z} & \\ & | & \\ & K + \bar{G} & \\ / & & \backslash \\ K & & \bar{G} \\ \backslash & & / \\ & G = \bigoplus_i \mathbb{Z}/2^i\mathbb{Z} & \\ & | & \\ & 0 & \end{array}$$

Notation:

- $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ (direct sum of cyclic 2-groups)
- $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ (cotorsion hull)
- $\bar{G} = \{(a_i) \in \Pi : \exists k,\ 2^k a_i = 0\ \forall i\}$ (torsion-complete group)
- $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ (full product)

2 Properties of Base Groups

2.1 Structural Properties

Group	Torsion	Torsion-Free	Mixed	Bounded	Countable
G	✓	×	×	×	✓
K	×	×	✓	×	×
$\bar{G}$	✓	×	×	per elt	×
$K + \bar{G}$	×	×	✓	×	×
Π	×	×	✓	×	×

2.2 Homological Properties

Group	Divisible	Reduced	Cotorsion	Pure-Inj	Injective	Projective
G	×	✓	×	×	×	×
K	×	✓	✓	×	×	×
$\bar{G}$	×	✓	✓	✓	×	×
$K + \bar{G}$	×	✓	?	×	×	×
Π	×	✓	✓	✓	×	×

2.3 Completeness Properties

Group	$\mathbb{Z}$-adic Complete	2-adic Complete	Torsion-Complete	Alg. Compact
G	$\times$	$\times$	$\times$	$\times$
K	$\times$	$\times$	$\times$	$\times$
$\bar{G}$	✓	✓	✓	✓
$K+\bar{G}$	$\times$	$\times$	$\times$	$\times$
Π	✓	✓	N/A	✓

2.4 Additional Invariants

Group	Torsion Subgroup	TF Rank	Adjusted Cotorsion	Cardinality
G	G	0	N/A	$\aleph_0$
K	G	$2^{\aleph_0}$	✓	$2^{\aleph_0}$
$\bar{G}$	$\bar{G}$	0	N/A	$2^{\aleph_0}$
$K+\bar{G}$	$\bar{G}$	$2^{\aleph_0}$	$\times$	$2^{\aleph_0}$
Π	$\bar{G}$	$2^{\aleph_0}$	$\times$	$2^{\aleph_0}$

3 Properties of All Quotients

3.1 Quotients by G

Quotient	Torsion	TF	Div	Red	Cotor	Pure-Inj	Inj
K/G	$\times$	✓	✓	$\times$	✓	$\times$	✓
$\bar{G}/G$	✓	$\times$	$\times$	✓	✓	✓	$\times$
$(K+\bar{G})/G$	mixed	$\times$	$\times$	✓	?	$\times$	$\times$
Π/G	mixed	$\times$	$\times$	✓	?	$\times$	$\times$

Key fact: $K/G \cong \mathrm{Ext}^1(\mathbb{Q}, G) \cong \lim^1 G$ is a $\mathbb{Q}$-vector space of dimension $2^{\aleph_0}$.

3.2 Quotients by K

Quotient	Torsion	TF	Div	Red	Cotor	Pure-Inj
$(K+\bar{G})/K \cong \bar{G}/G$	✓	$\times$	$\times$	✓	✓	✓
Π/K	mixed	$\times$	$\times$	✓	?	$\times$

3.3 Quotients by $\bar{G}$

Quotient	Torsion	TF	Div	Red	Cotor	Pure-Inj
$(K+\bar{G})/\bar{G} \cong K/G$	$\times$	✓	✓	$\times$	✓	$\times$
$\Pi/\bar{G}$	$\times$	✓	$\times$	✓	?	$\times$

3.4 Quotient by $K+\bar{G}$

Quotient	Torsion	TF	Div	Red	Cotor	Pure-Inj
$\Pi/(K+\bar{G})$	$\times$	✓	$\times$	✓	?	$\times$

4 Complete Quotient Matrix

Every quotient A/B where $B \subseteq A$:

$A \backslash B$	G	K	$\bar{G}$	$K+\bar{G}$	Π
G	0	—	—	—	—
K	K/G	0	—	—	—
$\bar{G}$	$\bar{G}/G$	—	0	—	—
$K+\bar{G}$	$(K+\bar{G})/G$	$\bar{G}/G$	K/G	0	—
Π	Π/G	Π/K	$\Pi/\bar{G}$	$\Pi/(K+\bar{G})$	0

5 Intersection and Join Tables

5.1 Intersections (Meets)

$A \cap B$	G	K	$\bar{G}$	$K+\bar{G}$	Π
G	G	G	G	G	G
K	G	K	G	K	K
$\bar{G}$	G	G	$\bar{G}$	$\bar{G}$	$\bar{G}$
$K+\bar{G}$	G	K	$\bar{G}$	$K+\bar{G}$	$K+\bar{G}$
Π	G	K	$\bar{G}$	$K+\bar{G}$	Π

Key fact: $K \cap \bar{G} = G$ (cotorsion hull and pure-injective envelope intersect at original group).

5.2 Sums (Joins)

$A+B$	G	K	$\bar{G}$	$K+\bar{G}$	Π
G	G	K	$\bar{G}$	$K+\bar{G}$	Π
K	K	K	$K+\bar{G}$	$K+\bar{G}$	Π
$\bar{G}$	$\bar{G}$	$K+\bar{G}$	$\bar{G}$	$K+\bar{G}$	Π
$K+\bar{G}$	$K+\bar{G}$	$K+\bar{G}$	$K+\bar{G}$	$K+\bar{G}$	Π
Π	Π	Π	Π	Π	Π

6 Ulm Invariants

For the torsion subgroups:

Group	u_0	u_1	u_2	$\cdots$	**Length**
$G = \text{torsion}(K)$	$\aleph_0$	0	0	0	1
$\bar{G} = \text{torsion}(\Pi)$	$2^{\aleph_0}$	$2^{\aleph_0}$	$2^{\aleph_0}$	$\cdots$	ω

7 Key Structural Results

1. $K \cap \bar{G} = G$ (incomparable in containment)
2. $(K+\bar{G})/G \cong K/G \oplus \bar{G}/G$ (direct sum since one is TF, other is torsion)
3. $K/G \hookrightarrow \Pi/\bar{G}$ is the maximal divisible subgroup
4. $\Pi/(K+\bar{G}) = (\Pi/\bar{G})/(K/G)$ is reduced torsion-free