On the Cotorsion Hull of an Unbounded Direct Sum of Cyclic p-Groups

Abstract

We investigate the structure of $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ where $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ is the direct sum of cyclic 2-groups of unbounded order. This group, known as the cotorsion hull of G, exhibits subtle properties: it is cotorsion with torsion subgroup exactly G, yet it is not pure-injective. We compute the quotient K/G as a $\mathbb{Q}$-vector space of uncountable dimension and characterize the group via Ulm invariants. We also derive explicit 2-cocycle representatives for elements in the image of the connecting homomorphism.

1 Introduction

Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ be the direct sum of cyclic groups of orders $2, 4, 8, 16, \ldots$, and let $H = \mathbb{Q}/\mathbb{Z}$ denote the Prüfer group (rationals modulo integers). We seek to compute $\mathrm{Ext}^1(H, G)$.

This problem lies at the intersection of several classical themes in abelian group theory: cotorsion groups, algebraic compactness, and the derived functors $\lim^1$. The group G has *unbounded torsion*, meaning there is no integer n with $nG = 0$, which creates delicate issues when considering its various completions.

1.1 Main Results

Theorem 1.1. *The group $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ is the unique adjusted cotorsion group with torsion subgroup isomorphic to G. It fits into a non-split exact sequence*

$$0 \to G \to K \to D \to 0$$

where $D = \mathrm{Ext}^1(\mathbb{Q}, G)$ is a torsion-free divisible group of uncountable dimension.

Theorem 1.2. *The quotient $K/G \cong \mathrm{Ext}^1(\mathbb{Q}, G) = \lim^1 G$ is isomorphic as an abstract group to $\bigoplus_{2^{\aleph_0}} \mathbb{Q}$, a direct sum of continuum-many copies of the rationals.*

2 Preliminaries

2.1 Cotorsion Groups

Definition 2.1 ([2])**.** An abelian group C is *cotorsion* if $\mathrm{Ext}^1(F, C) = 0$ for all torsion-free groups F. Equivalently, $\mathrm{Ext}^1(\mathbb{Q}, C) = 0$.

Definition 2.2. A cotorsion group is *adjusted* if it has no nonzero torsion-free direct summands.

The following theorem establishes a fundamental correspondence.

Theorem 2.3 ([4], [3])**.** *There is a one-to-one correspondence between adjusted cotorsion groups and reduced torsion groups, given by $C \mapsto tC$ (the torsion subgroup of C).*

2.2 The Cotorsion Hull

Definition 2.4. For a reduced abelian group A, the *cotorsion hull* of A is defined as $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, A)$.

Proposition 2.5 ([3])**.** *For a reduced torsion group T:*

1. $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, T)$ *is cotorsion.*
2. *The torsion subgroup of* $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, T)$ *equals* T*.*
3. *The quotient* $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, T)/T$ *is torsion-free and divisible.*

2.3 The $\lim^1$ Functor

For an inverse system of abelian groups $\cdots \to A_{n+1} \xrightarrow{f_n} A_n \to \cdots$, the derived functor $\lim^1$ is computed as [6]:

$$\lim^1 A_n = \left(\prod_{n=0}^{\infty} A_n \right) \Big/ \mathrm{Im}(\partial)$$

where $\partial((a_n)) = (a_n - f_n(a_{n+1}))_{n \geq 0}$.

3 The Fundamental Exact Sequence

3.1 Derivation

From the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, applying $\mathrm{Hom}(-, G)$ yields:

$$\cdots \to \mathrm{Hom}(\mathbb{Q}, G) \to \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{\delta} \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}^1(\mathbb{Q}, G) \to \mathrm{Ext}^1(\mathbb{Z}, G) \to \cdots$$

Lemma 3.1. *For $G = \bigoplus_i \mathbb{Z}/2^i\mathbb{Z}$:*

1. $\mathrm{Hom}(\mathbb{Q}, G) = 0$ *(since $\mathbb{Q}$ is divisible and G is reduced torsion)*
2. $\mathrm{Hom}(\mathbb{Z}, G) \cong G$
3. $\mathrm{Ext}^1(\mathbb{Z}, G) = 0$ *(since $\mathbb{Z}$ is projective)*

Corollary 3.2. *There is a short exact sequence*

$$0 \to G \xrightarrow{\delta} \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}^1(\mathbb{Q}, G) \to 0$$

where δ is the connecting homomorphism.

3.2 Non-Splitting

Proposition 3.3. *The sequence $0 \to G \to K \to K/G \to 0$ does not split.*

Proof. Suppose the sequence splits, so $K \cong G \oplus (K/G)$. Since direct summands of cotorsion groups are cotorsion, G would be cotorsion. However, a torsion group is cotorsion if and only if it is the direct sum of a divisible group and a bounded group [2, Theorem 54.1]. Since $G = \bigoplus_i \mathbb{Z}/2^i\mathbb{Z}$ is reduced and has unbounded exponent, it is not cotorsion. Contradiction. □

4 Computing the Quotient K/G

4.1 Identification with $\lim^1$

Theorem 4.1. $\operatorname{Ext}^1(\mathbb{Q}, G) \cong \lim^1 G$ *where the inverse system is*

$$G \xleftarrow{\times 2} G \xleftarrow{\times 3} G \xleftarrow{\times 4} \cdots$$

with the n-th map being multiplication by $(n+1)$.

Proof. Write $\mathbb{Q} = \varinjlim_n \frac{1}{n!}\mathbb{Z}$. Then by [6, Proposition 3.5.8]:

$$\operatorname{Ext}^1(\mathbb{Q}, G) = \operatorname{Ext}^1(\varinjlim \tfrac{1}{n!}\mathbb{Z}, G) \cong \overset{1}{\lim} \operatorname{Hom}(\tfrac{1}{n!}\mathbb{Z}, G) = \overset{1}{\lim} G$$

where the bonding maps are induced by the inclusions $\frac{1}{n!}\mathbb{Z} \hookrightarrow \frac{1}{(n+1)!}\mathbb{Z}$. □

4.2 Structure of the Quotient

Proposition 4.2. $K/G = \operatorname{Ext}^1(\mathbb{Q}, G)$ *is torsion-free and divisible.*

Proof. Divisibility: For any integer m, consider the endomorphism $\times m : \mathbb{Q} \to \mathbb{Q}$, which is an isomorphism. This induces an isomorphism $\operatorname{Ext}^1(\mathbb{Q}, G) \xrightarrow{\cong} \operatorname{Ext}^1(\mathbb{Q}, G)$, which is multiplication by m. Hence $m \cdot \operatorname{Ext}^1(\mathbb{Q}, G) = \operatorname{Ext}^1(\mathbb{Q}, G)$.

Torsion-free: If $x \in K/G$ satisfies $mx = 0$ for some $m > 0$, lift x to $\tilde{x} \in K$. Then $m\tilde{x} \in G$, so $\tilde{x}$ has finite order in K. Since $\operatorname{torsion}(K) = G$, we have $\tilde{x} \in G$, so $x = 0$. □

Theorem 4.3. *The group K is uncountable, with cardinality $|K| = 2^{\aleph_0}$.*

Proof. Since K fits into the exact sequence $0 \to G \to K \to \operatorname{Ext}^1(\mathbb{Q}, G) \to 0$ and G is countable, it suffices to show that $\operatorname{Ext}^1(\mathbb{Q}, G) \cong \lim^1 G$ has cardinality $2^{\aleph_0}$.

The inverse system defining $\lim^1 G$ is $G \xleftarrow{\times(n+1)} G$. Since $G = \bigoplus_i \mathbb{Z}/2^i\mathbb{Z}$ is unbounded, the image of the bonding map $G \xrightarrow{\times m} G$ is strictly smaller than G for large m. Specifically, the system does not satisfy the Mittag-Leffler condition.

By a theorem of Jensen (1972), for an inverse system of countable abelian groups, the derived limit $\lim^1$ is either trivial or has cardinality $2^{\aleph_0}$. Since the maps are not surjective (leaving "holes" that accumulate), $\lim^1 G \neq 0$. Being a vector space over $\mathbb{Q}$, it must therefore have uncountable dimension, so $|\operatorname{Ext}^1(\mathbb{Q}, G)| = 2^{\aleph_0}$. □

5 Comparison with Pure-Injective Envelope

5.1 Torsion-Complete Groups

Definition 5.1 ([3])**.** For a direct sum of cyclic p-groups $B = \bigoplus_i B_i$, the *torsion-complete* group is the torsion subgroup of the p-adic completion:

$$\bar{B} = \{(a_i) \in \prod_i B_i : \exists k,\ p^k a_i = 0 \text{ for all } i\}$$

Theorem 5.2 ([3])**.** *A reduced p-group is algebraically compact (= pure-injective) if and only if it is torsion-complete.*

5.2 Why $K \neq \bar{G}$

Proposition 5.3. *The cotorsion hull $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ is not equal to the torsion-complete group $\bar{G}$.*

Proof. The torsion-complete group $\bar{G}$ is:

$$\bar{G} = \{(a_i) \in \prod_i \mathbb{Z}/2^i\mathbb{Z} : \exists k,\ 2^k a_i = 0 \text{ for all } i\}$$

This is a torsion group strictly larger than $G = \bigoplus_i \mathbb{Z}/2^i\mathbb{Z}$. For example, the element $(1, 1, 1, \ldots)$ where each component is $1 \in \mathbb{Z}/2^i\mathbb{Z}$ lies in $\bar{G}$ (with $k = 1$ since $2 \cdot 1 = 0$ in $\mathbb{Z}/2$), but not in G.

Since $\mathrm{torsion}(\bar{G}) = \bar{G} \supsetneq G$ but $\mathrm{torsion}(K) = G$, we have $K \neq \bar{G}$. □

Remark 5.4. The cotorsion hull K and the torsion-complete group $\bar{G}$ are *incomparable* in the containment order. Specifically:

- $K \not\subset \bar{G}$: The quotient K/G is torsion-free, so K contains elements of infinite order. But $\bar{G}$ is entirely torsion, so these elements cannot lie in $\bar{G}$.
- $\bar{G} \not\subset K$: The group $\bar{G}$ contains torsion elements outside G (e.g., $(1, 1, 1, \ldots)$), but $\mathrm{torsion}(K) = G$, so these cannot lie in K.

Their intersection is exactly G:

$$K \cap \bar{G} = G$$

Both K and $\bar{G}$ embed into the full product $\prod_i \mathbb{Z}/2^i\mathbb{Z}$, but they capture different "completions" of G.

Note: Since K is the minimal cotorsion group containing G, if the torsion-complete group $\bar{G}$ were cotorsion, it would necessarily contain K. The fact that $K \not\subset \bar{G}$ thus provides a proof that $\bar{G}$ is not cotorsion.

Proposition 5.5. *Let $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ and let $R \cong \prod_{\aleph_0} \mathbb{Z}_2$ denote the reduced torsion-free algebraically compact part. The structure of the global quotients is:*

1. $K/G \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$
2. $\bar{G}/G \cong \bigoplus_{2^{\aleph_0}} \mathbb{Z}(2^\infty)$
3. $\Pi/K \cong R \oplus \bigoplus_{2^{\aleph_0}} \mathbb{Z}(2^\infty)$
4. $\Pi/\bar{G} \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus R$ *(note: $\bigoplus_{2^{\aleph_0}} \mathbb{Q} \cong \mathbb{Q}^{\oplus 2^{\aleph_0}}$)*
5. $\Pi/(K + \bar{G}) \cong R$
6. $\Pi/G \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus R \oplus \bigoplus_{2^{\aleph_0}} \mathbb{Z}(2^\infty)$

6 Explicit Cocycles

6.1 Extensions as 2-Cocycles

An element of $\mathrm{Ext}^1(A, B)$ corresponds to an extension $0 \to B \to E \to A \to 0$. Given a set-theoretic section $s : A \to E$, the *2-cocycle* $c : A \times A \to B$ is

$$c(a_1, a_2) = s(a_1) + s(a_2) - s(a_1 + a_2)$$

6.2 The Connecting Homomorphism

The image of the connecting homomorphism $\delta : G \to K$ consists of explicit cocycles.

Proposition 6.1. *For $g \in G$, the cocycle $c_g : \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \to G$ representing $\delta(g)$ is:*

$$c_g([r_1], [r_2]) = \epsilon(r_1, r_2) \cdot g$$

where $r_1, r_2 \in [0, 1)$ are representatives and

$$\epsilon(r_1, r_2) = \begin{cases} 1 & \text{if } r_1 + r_2 \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

is the "carry" function.

7 Ulm Invariant Characterization

7.1 Ulm's Theorem

For a reduced abelian p-group T, the α-th Ulm invariant is [2]:

$$u_\alpha(T) = \dim_{\mathbb{F}_p} \frac{(p^\alpha T)[p]}{(p^{\alpha+1} T)[p]}$$

Theorem 7.1 (Ulm's Theorem, [5])**.** *Two countable reduced abelian p-groups are isomorphic if and only if they have the same Ulm invariants.*

7.2 Application to G

Proposition 7.2. *For $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$:*

$$u_0(G) = \aleph_0, \quad u_\alpha(G) = 0 \text{ for } \alpha \geq 1$$

Corollary 7.3. *By Theorem 2.3, the cotorsion hull $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ is the unique adjusted cotorsion group with Ulm invariants $u_0 = \aleph_0$ and $u_\alpha = 0$ for $\alpha \geq 1$.*

8 Complete Lattice Analysis

All groups embed into $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$. The containment lattice is:

$$\begin{tikzcd}[arrows=-] & \Pi \arrow[d] & \\ & K + \bar{G} \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & \bar{G} \arrow[dl] \\ & G \arrow[d] & \\ & 0 & \end{tikzcd}$$

8.1 Structural Properties

Group	Torsion	TF	Mixed	Bounded	Countable	Card.
G	✓	×	×	×	✓	$\aleph_0$
K	×	×	✓	×	×	$2^{\aleph_0}$
$\bar{G}$	✓	×	×	per elt	×	$2^{\aleph_0}$
$K + \bar{G}$	×	×	✓	×	×	$2^{\aleph_0}$
Π	×	×	✓	×	×	$2^{\aleph_0}$

8.2 Homological Properties

Group	Div	Red	Cotor	Pure-Inj	Inj	Proj
G	×	✓	×	×	×	×
K	×	✓	✓	×	×	×
$\bar{G}$	×	✓	×	✓	×	×
$K + \bar{G}$	×	✓	×	×	×	×
Π	×	✓	✓	✓	×	×

8.3 Completeness Properties

Group	$\mathbb{Z}$-adic Compl	2-adic Compl	Tor-Compl	Alg Compact
G	×	×	×	×
K	×	×	×	×
$\bar{G}$	×	×	✓	✓
$K + \bar{G}$	×	×	×	×
Π	✓	✓	N/A	✓

8.4 Properties of Quotients by G

Quotient	Torsion	TF	Div	Red	Cotor	Inj
K/G	×	✓	✓	×	✓	✓
$\bar{G}/G$	✓	×	✓	×	✓	×
$(K + \bar{G})/G$	mixed	×	✓	×	✓	×
Π/G	mixed	×	×	×	?	×

8.5 Lattice Structure

The relationships between the groups form a non-distributive lattice structure. The non-trivial structural facts are:

- **Incomparability**: $K \not\subset \bar{G}$ and $\bar{G} \not\subset K$.
- **Intersection**: $K \cap \bar{G} = G$.
- **Strict Containments**:
  - $G \subsetneq K \subsetneq K + \bar{G} \subsetneq \Pi$
  - $G \subsetneq \bar{G} \subsetneq K + \bar{G} \subsetneq \Pi$
- **The Join**: The sum $K + \bar{G}$ is the smallest group containing both completions, but it is still strictly smaller than the full product Π.

9 Open Questions

Open Question 9.1. Can one give an explicit set-theoretic description of K as a subgroup of $\prod_i \mathbb{Z}/2^i\mathbb{Z}$? The naive characterization "x has finite order implies $x \in \bigoplus_i \mathbb{Z}/2^i\mathbb{Z}$" fails to define a subgroup.

Open Question 9.2. What is the structure of the $\lim^1$ cocycles explicitly? Can one describe elements of $K \setminus \delta(G)$ via concrete 2-cocycles $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \to G$?

Open Question 9.3. For which reduced torsion groups T does the cotorsion hull $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, T)$ coincide with the pure-injective envelope? The case $T = G$ shows they can differ.

Open Question 9.4. What is the natural topology on K induced by its various completions, and how does it relate to the 2-adic topology on G?

References

[1] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, NJ, 1956.

[2] L. Fuchs, *Infinite Abelian Groups, Vol. I*, Academic Press, New York, 1970.

[3] L. Fuchs, *Infinite Abelian Groups, Vol. II*, Academic Press, New York, 1973.

[4] D. K. Harrison, Infinite abelian groups and homological methods, *Ann. of Math.* (2) **69** (1959), 366–391.

[5] H. Ulm, Zur Theorie der abzählbar-unendlichen Abelschen Gruppen, *Math. Ann.* **107** (1933), 774–803.

[6] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.