The cotorsion hull of the sum of cyclic groups of order powers of 2 – Gemini 3 version

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Abstract

In this note, we compute the group $\mathrm{Ext}(H, G)$ where $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ and $H = \mathbb{Q}/\mathbb{Z}$. We verify that $\mathrm{Ext}(H, G)$ is the cotorsion hull of G. We distinguish this group from the 2-adic completion of G using Ulm invariants, showing that $\mathrm{Ext}(H, G)$ is the unique reduced algebraically compact 2-primary group with Ulm invariants $f_n = 1$ for all n.

1 Problem Statement

Let G be the direct sum of cyclic groups of order 2^i:

$$G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$$

Let H be the quotient of the rationals by the integers:

$$H = \mathbb{Q}/\mathbb{Z}$$

We wish to compute the group $\mathrm{Ext}(H, G)$. Throughout this document, Ext denotes the first derived functor of Hom, i.e., $\mathrm{Ext}^1_{\mathbb{Z}}$.

2 Preliminaries

We recall several standard results from the theory of abelian groups.

Definition 1 (Cotorsion Group)**.** *An abelian group C is called cotorsion if $Ext(F, C) = 0$ for all torsion-free groups F. By a result of Harrison [1], it suffices to check this condition for $F = \mathbb{Q}$.*

Definition 2 (Cotorsion Hull)**.** *The cotorsion hull (or cotorsion envelope) of an abelian group A is a cotorsion group $A^\bullet$ containing A such that $A^\bullet/A$ is torsion-free and divisible. It is unique up to isomorphism over A. Standard homological algebra identifies the cotorsion hull as:*

$$A^\bullet \cong Ext(\mathbb{Q}/\mathbb{Z}, A)$$

3 Derivation

3.1 Identification as the Cotorsion Hull

We compute $\mathrm{Ext}(H, G) = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ using the standard short exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

Applying the functor $\mathrm{Hom}(-, G)$, we obtain the long exact sequence:

$$\cdots \to \mathrm{Hom}(\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}, G) \to \mathrm{Ext}(\mathbb{Z}, G) \to 0$$

Since $\mathbb{Z}$ is projective, $\mathrm{Ext}(\mathbb{Z}, G) = 0$. Also, $\mathrm{Hom}(\mathbb{Z}, G) \cong G$. Thus, we have the short exact sequence:

$$0 \to G \to \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) \to \mathrm{Ext}(\mathbb{Q}, G) \to 0$$

This sequence precisely characterizes $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ as an essential extension of G by the torsion-free divisible group $\mathrm{Ext}(\mathbb{Q}, G)$. This matches the definition of the cotorsion hull of G [2].

3.2 Torsion and Structure

Let $E = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$. From the exact sequence, the quotient $E/G \cong \mathrm{Ext}(\mathbb{Q}, G)$ is torsion-free (as it is a vector space over $\mathbb{Q}$). Therefore, the torsion subgroup of E, denoted $t(E)$, must be contained in G:

$$t(E) = G$$

This distinguishes E from the 2-adic completion (the direct product), which contains torsion elements not in G.

3.3 Ulm Invariants

To uniquely identify the isomorphism class of E, we use Ulm invariants. Since E is algebraically compact (being a cotorsion hull), it is determined by its Ulm invariants and its maximal divisible subgroup [3].

Since G is reduced, E is also reduced. The Ulm invariants $f_n(A)$ of a p-group A are defined as the dimension of the vector space $(p^n A)[p]/(p^{n+1}A)[p]$ over $\mathbb{Z}/p\mathbb{Z}$. For $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$, the invariants are:

$$f_n(G) = 1 \quad \text{for all } n \geq 0$$

Since G is pure in its cotorsion hull E (actually G is the torsion part of E), they share the same Ulm invariants:

$$f_n(E) = 1 \quad \text{for all } n \geq 0$$

In contrast, the direct product $P = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ has invariants of cardinality $2^{\aleph_0}$.

4 Conclusion

The group $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z})$ is the unique reduced algebraically compact 2-primary group with Ulm invariants $f_n = 1$ for all n. It is the cotorsion hull of G.

References

[1] Harrison, D. K. (1959). Infinite Abelian Groups and Homological Methods. *Annals of Mathematics*, 69(2), 366-391.

[2] Fuchs, L. (2015). *Abelian Groups*. Springer International Publishing.

[3] Kaplansky, I. (1954). *Infinite Abelian Groups*. University of Michigan Press.