Explicit Description of the Cotorsion Hull

Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ and $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$. Define $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$, the cotorsion hull of G.

1 The Characterization

We characterize K by building up three increasingly refined conditions.

Definition 1. *For $x \in \Pi$ and $n \geq 1$, we say x is* n-divisible modulo G *if there exists $g \in G$ such that $n \cdot x - g \in G$. In this case, we call g an* n-remainder *of x.*

The first condition is simply that x is divisible by every positive integer modulo G:

Proposition 1 (Condition 1: Universal Divisibility)**.** *An element $x \in \Pi$ belongs to K only if x is n-divisible modulo G for every $n \geq 1$. That is, for each n, there exists $g_n \in G$ with $n \cdot x - g_n \in G$.*

However, this alone is not sufficient. We need the remainders to be *compatible*:

Proposition 2 (Condition 2: Compatibility of Remainders)**.** *If $x \in K$ with n-remainders g_n and m-remainders g_m, then these must satisfy:*

$$n \cdot g_m - m \cdot g_n \in G$$

for all $n, m \geq 1$. This ensures that "dividing by n then by m" gives the same result as "dividing by m then by n" (modulo G).

The final condition ensures these compatible remainders actually come from the $\varprojlim^1$ construction, which is the quotient K/G:

Proposition 3 (Condition 3: $\lim^1$ Compatibility)**.** *The remainders (g_n) must not only be compatible with each other, but must also arise from the specific structure of $K/G \cong \varprojlim^1 G$.*

To check this, we examine the factorial subsequence $(g_{1!}, g_{2!}, g_{3!}, \ldots)$. The requirement is that the "jumps" between consecutive terms have a specific form: for each $n \geq 1$, there must exist some $h_n \in G$ such that the difference $g_{n!} - (n+1) \cdot g_{(n+1)!}$ equals $h_n - (n+1) \cdot h_{n+1}$.

This condition ensures that $(g_{n!})$ represents a genuine element of $\varprojlim^1 G = (\prod_{n=0}^{\infty} G)/\mathrm{Im}(\partial)$, where the boundary map ∂ sends (a_n) to $(a_n - (n+1) \cdot a_{n+1})$. In other words, the sequence must be a valid "cocycle" in the lim^1 construction.

Theorem 1. *An element $x = (x_i)_{i \geq 1} \in \Pi$ belongs to K if and only if:*

1. *x is n-divisible modulo G for every $n \geq 1$ (with some choice of n-remainders $g_n \in G$).*
2. *The remainders (g_n) are compatible: $n \cdot g_m - m \cdot g_n \in G$ for all $n, m \geq 1$.*

3. *For each $n \geq 1$, there exists $h_n \in G$ such that:*

$$g_{n!} - (n+1) \cdot g_{(n+1)!} = h_n - (n+1) \cdot h_{n+1}$$

This ensures the factorial subsequence $(g_{n!})$ represents a well-defined element in $\varprojlim^1 G = (\prod_{n=0}^{\infty} G)/\mathrm{Im}(\partial)$, where $\partial((a_n)) = (a_n - (n+1) \cdot a_{n+1})$.

Proof. **Necessity:** Since $K/G \cong \varprojlim^1 G$ is divisible, for any $x \in K$ and $n \geq 1$, we can find $y \in K$ with $n \cdot y \equiv x \pmod{G}$. Taking $g_n = n \cdot y - x \in G$ gives condition (1). The compatibility (2) follows because both $n \cdot g_m$ and $m \cdot g_n$ represent ways of "dividing x by nm," which must agree. Condition (3) ensures the sequence $(g_{n!})$ actually represents the class of x mod G in $\varprojlim^1 G$.

Sufficiency: If $x \in \Pi$ satisfies (1)–(3), then the remainders (g_n) form a compatible system representing an element of $\varprojlim^1 G$. Since $K/G \cong \varprojlim^1 G$, this means x mod $G \in K/G$, so $x \in K$. $\square$