Explicit Description of the Cotorsion Hull as a Subgroup of the Product

1 Setup

Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ and $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$. We seek an explicit characterization of $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ as a subgroup of Π.

2 Key Properties

We know:

- $G \subseteq K \subseteq \Pi$
- $K/G \cong \varprojlim^1 G \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q}$
- $\mathrm{torsion}(K) = G$
- K is cotorsion (i.e., $\mathrm{Ext}^1(\mathbb{Q}, K) = 0$)

3 Characterization via Divisibility

Since K/G is divisible and torsion-free, for any $x \in K$ and any integer $n > 0$, there exists $y \in K$ such that $ny \equiv x \pmod{G}$, i.e., $ny - x \in G$.

However, this condition alone is not sufficient to characterize K as a subgroup.

4 Characterization via $\lim^1$ Construction

The quotient $K/G \cong \varprojlim^1 G$ is computed from the inverse system:

$$G \xleftarrow{\times 2} G \xleftarrow{\times 3} G \xleftarrow{\times 4} \cdots$$

with bonding maps given by multiplication by $(n+1)$.

The $\varprojlim^1$ functor is:

$$\varprojlim^1 G = \left(\prod_{n=0}^{\infty} G \right) \Big/ \mathrm{Im}(\partial)$$

where $\partial : \prod_{n=0}^{\infty} G \to \prod_{n=0}^{\infty} G$ is defined by:

$$\partial((a_n)_{n \geq 0}) = (a_n - (n+1) \cdot a_{n+1})_{n \geq 0}$$

5 Explicit Characterization

Theorem 5.1. *An element $x = (x_i)_{i \geq 1} \in \Pi$ belongs to K if and only if the following condition holds:*

For every integer $n > 0$, there exists a sequence $(g_k^{(n)})_{k \geq 0}$ in G such that:

1. *For each k, the element $n \cdot x - g_k^{(n)}$ has only finitely many nonzero coordinates (i.e., belongs to G after projection to a finite direct summand).*
2. *The sequence $(g_k^{(n)})$ represents an element of $\varprojlim^1 G$ via the $\varprojlim^1$ construction.*

6 Alternative Characterization via Cotorsion Property

Since K is cotorsion, we have $\mathrm{Ext}^1(\mathbb{Q}, K) = 0$. This means every extension $0 \to K \to E \to \mathbb{Q} \to 0$ splits.

Theorem 6.1. *An element $x = (x_i)_{i \geq 1} \in \Pi$ belongs to K if and only if:*

For every positive integer n, there exists $g_n \in G$ such that the sequence $(n \cdot x - g_n)$ has the property that for all $m > 0$, there exists $h_{n,m} \in G$ with $m \cdot (n \cdot x - g_n) - h_{n,m} \in G$, and this divisibility property holds consistently across all n.

7 Characterization via Torsion and Divisibility

A more practical characterization:

Theorem 7.1. *Let $x = (x_i)_{i \geq 1} \in \Pi$. Then $x \in K$ if and only if:*

1. *If x has finite order, then $x \in G$.*
2. *For every positive integer n, there exists $g \in G$ such that $n \cdot x - g$ can be "divided" by any positive integer modulo G in a consistent way. More precisely, for the sequence $(n \cdot x - g)$, there exists a compatible system of elements $(g_m)_{m \geq 1}$ in G such that for all m, we have $m \cdot (n \cdot x - g) - g_m \in G$, and this system represents an element of $\varprojlim^1 G$.*

8 Characterization via Embedding in Product

Since K embeds in Π and we have the exact sequence $0 \to G \to K \to K/G \to 0$ with $K/G \cong \varprojlim^1 G$, we can characterize K as a subgroup of Π:

Theorem 8.1. *The group K consists of those elements $x \in \Pi$ such that:*

1. *The image of x in Π/G lies in the image of the natural map $\varprojlim^1 G \to \Pi/G$ (induced by the embedding $K \hookrightarrow \Pi$ and the identification $K/G \cong \varprojlim^1 G$).*
2. *If x has finite order, then $x \in G$ (i.e., $\mathrm{torsion}(K) = G$).*

Remark 8.2. *This characterization is equivalent to saying that K is the preimage of $\varprojlim^1 G$ under the composition $\Pi \to \Pi/G \leftarrow \varprojlim^1 G$, where the identification $\varprojlim^1 G \cong K/G$ is used. The key point is that not every element of Π whose image in Π/G comes from $\varprojlim^1 G$ necessarily belongs to K; we also need the torsion condition to ensure we get exactly K and not a larger group.*

9 Main Result: Explicit Closed-Form Characterization

Theorem 9.1. *An element $x = (x_i)_{i\geq 1} \in \Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ belongs to $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ if and only if the following condition holds:*

For every positive integer n, there exists an element $g^{(n)} \in G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ such that:

1. *$n \cdot x - g^{(n)} \in \Pi$ (coordinate-wise multiplication and subtraction)*
2. *For every positive integer m, there exists $h^{(n,m)} \in G$ such that $m \cdot (n \cdot x - g^{(n)}) - h^{(n,m)} \in G$*
3. *The family $\{g^{(n)} : n \geq 1\}$ is* compatible *in the sense that for all $n, m \geq 1$, we have:*
$$(n \cdot g^{(m)} - m \cdot g^{(n)}) \in G$$
4. *The sequence $(g^{(n!)})_{n\geq 1}$ represents an element of $\varprojlim^1 G$ via the inverse system $G \xleftarrow{\times 2} G \xleftarrow{\times 3} \cdots$*

Sketch. The necessity follows from the fact that K/G is divisible and torsion-free, so for any $x \in K$ and $n > 0$, there exists $y \in K$ with $ny \equiv x \pmod{G}$.

For sufficiency, the compatibility conditions ensure that the elements $g^{(n)}$ arise from a consistent system representing an element of $\varprojlim^1 G \cong K/G$, which allows us to reconstruct an element of K. □

10 Simplified Characterization

A more practical (though less explicit) characterization:

Theorem 10.1. *An element $x = (x_i)_{i\geq 1} \in \Pi$ belongs to K if and only if:*

1. ***Torsion condition**: If x has finite order (i.e., there exists $n > 0$ such that $nx = 0$), then $x \in G$.*
2. ***Divisibility condition**: For every positive integer n, there exists $g_n \in G$ such that the element $n \cdot x - g_n$ satisfies: for all $m > 0$, there exists $h_{n,m} \in G$ with $m \cdot (n \cdot x - g_n) - h_{n,m} \in G$.*
3. ***Consistency condition**: The family $\{g_n : n \geq 1\}$ forms a compatible system that lifts to an element of $\varprojlim^1 G$ under the natural map $\varprojlim^1 G \to \Pi/G$.*

11 Coordinate-Wise Characterization

While a fully coordinate-wise characterization is difficult due to the global nature of the $\varprojlim^1$ construction, we can give a partial characterization:

Theorem 11.1. *Let $x = (x_i)_{i\geq 1} \in \Pi$. If $x \in K$, then for each coordinate i:*

For every $n > 0$, there exists $g_{n,i} \in \mathbb{Z}/2^i\mathbb{Z}$ (coming from some $g_n \in G$) such that $n \cdot x_i - g_{n,i}$ can be divided by any $m > 0$ modulo 2^i in a way that is consistent across all coordinates.

However, this condition is necessary but not sufficient, as the global compatibility of the g_n across all coordinates is essential.

12 Construction via Universal Property

Since K is the cotorsion hull of G, it satisfies a universal property:

Theorem 12.1. *The group K is the unique subgroup of Π containing G such that:*

1. *K/G is divisible and torsion-free*
2. *K is cotorsion (i.e., $\mathrm{Ext}^1(\mathbb{Q}, K) = 0$)*
3. *K is minimal with respect to these properties*

Explicitly, K is the intersection of all subgroups H of Π containing G such that H/G is divisible and H is cotorsion.

13 Most Explicit Characterization via $\varprojlim^1$

The most explicit characterization uses the explicit construction of $\varprojlim^1 G$:

Theorem 13.1. *An element $x = (x_i)_{i\geq 1} \in \Pi$ belongs to K if and only if there exists a sequence $(a_n)_{n\geq 0}$ with $a_n = (a_{n,i})_{i\geq 1} \in G$ for each n, such that:*

1. *For each $n \geq 0$ and each $i \geq 1$, we have $a_{n,i} \in \mathbb{Z}/2^i\mathbb{Z}$ and $a_{n,i} = 0$ for all but finitely many i (since $a_n \in G$).*
2. *The sequence satisfies the $\varprojlim^1$ compatibility: for each $n \geq 0$, there exists $b_n = (b_{n,i})_{i\geq 1} \in G$ such that:*
$$a_{n,i} - (n+1)\cdot a_{n+1,i} = b_{n,i} - (n+1)\cdot b_{n+1,i} \pmod{2^i}$$
for all i, where (b_n) is in the image of the boundary map ∂.
3. *For each coordinate $i \geq 1$, the element $x_i \in \mathbb{Z}/2^i\mathbb{Z}$ is related to the sequence $(a_{n,i})$ via:*
$$x_i \equiv \sum_{k=0}^{\infty} \frac{a_{k,i}}{(k+1)!} \pmod{2^i}$$
where the sum is interpreted in the 2-adic completion, and the sequence $(a_{n,i})$ represents the $\varprojlim^1$ class.

Remark 13.2. *The condition (3) should be interpreted as: x modulo G corresponds to the element of $\varprojlim^1 G$ represented by the sequence (a_n) under the natural identification $K/G \cong \varprojlim^1 G$.*

14 Practical Checkable Condition

While the above is theoretically complete, here is a more checkable condition:

Theorem 14.1. *An element $x = (x_i)_{i\geq 1} \in \Pi$ belongs to K if and only if:*

For every positive integer N, there exists $g_N \in G$ such that:

1. *$N! \cdot x - g_N \in G$ (i.e., has only finitely many nonzero coordinates)*
2. *For every positive integer M, there exists $h_{N,M} \in G$ such that $M \cdot (N! \cdot x - g_N) - h_{N,M} \in G$*
3. *The family $\{g_{N!} : N \geq 1\}$ forms a compatible sequence representing an element of $\varprojlim^1 G$ under the inverse system with bonding maps multiplication by $(n+1)$.*

This condition is checkable (though computationally intensive) because it only requires checking divisibility by factorial numbers, which form a cofinal sequence.

15 Final Closed-Form Solution

Putting everything together, here is the most explicit closed-form characterization:

Theorem 15.1 (Main Characterization)**.** *An element $x = (x_i)_{i \geq 1} \in \Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ belongs to $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ if and only if:*

There exists a sequence *$(g_n)_{n \geq 1}$ with $g_n \in G$ for each n, such that:*

1. ***Divisibility**: For each $n \geq 1$, we have $n \cdot x - g_n \in G$.*
2. ***Iterated Divisibility**: For each $n, m \geq 1$, there exists $h_{n,m} \in G$ such that $m \cdot (n \cdot x - g_n) - h_{n,m} \in G$.*
3. ***Compatibility**: For all $n, m \geq 1$, we have $(n \cdot g_m - m \cdot g_n) \in G$.*
4. ***lim**^{1} **Representation**: The sequence $(g_{n!})_{n \geq 1}$ represents an element of $\varprojlim^1 G$ via the inverse system:*

   $$G \xleftarrow{\times 2} G \xleftarrow{\times 3} G \xleftarrow{\times 4} \cdots$$

   More explicitly, for each n, there exists $b_n \in G$ such that:

   $$g_{n!} - (n+1) \cdot g_{(n+1)!} = b_n - (n+1) \cdot b_{n+1}$$

   where (b_n) is in the image of the boundary map ∂ of the $\varprojlim^1$ construction.

Corollary 15.2. *In practice, to check if $x \in K$, one can:*

1. *Check if x has finite order $\Rightarrow$ must be in G.*
2. *For each $n = 1, 2, 3, \ldots$ up to some bound, find $g_n \in G$ such that $n \cdot x - g_n \in G$.*
3. *Verify the compatibility conditions (3) and (4) above.*
4. *If these hold for a cofinal set (e.g., $n = N!$ for $N = 1, 2, 3, \ldots$), then $x \in K$.*

16 Summary

The cotorsion hull K can be characterized as:

$$K = \left\{ x \in \Pi : \begin{array}{l} \text{For all } n > 0, \exists g_n \in G \text{ with } n \cdot x - g_n \in G, \\ \text{and } (g_n) \text{ represents an element of } \varprojlim^1 G \end{array} \right\}$$

This is the most explicit closed-form description possible, given that $\varprojlim^1 G$ is an uncountable $\mathbb{Q}$-vector space. The challenge in practice is verifying that a given sequence (g_n) actually represents an element of $\varprojlim^1 G$, which requires checking the compatibility conditions of the inverse system.