

## Explicit Description of the Cotorsion Hull

Let  $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$  (the direct sum of cyclic 2-groups) and  $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$  (the full product). Define  $K = \text{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$ , the cotorsion hull of  $G$ .

Our goal is to characterize exactly which elements of  $\Pi$  belong to  $K$ . We do this by building up three conditions, each refining the previous one.

**Definition 1.** For  $x \in \Pi$  and a positive integer  $n$ , we say  $x$  is  $n$ -divisible modulo  $G$  if we can “almost divide  $x$  by  $n$ ” in the following sense: there exists some element  $g \in G$  such that when we compute  $n \cdot x - g$ , the result lies entirely in  $G$  (has only finitely many nonzero coordinates).

We call  $g$  an  $n$ -remainder of  $x$ . Think of it as the “leftover” when trying to divide  $x$  by  $n$ , but we’re allowed to adjust by an element of  $G$  to make it work.

**Example 1.** If  $x \in G$  itself, then for any  $n$ , we can take  $g = 0$  and have  $n \cdot x - 0 = n \cdot x \in G$ , so elements of  $G$  are  $n$ -divisible modulo  $G$  for all  $n$ .

The first condition requires that  $x$  can be divided by *every* positive integer (modulo  $G$ ):

**Proposition 1** (Condition 1: Universal Divisibility). An element  $x \in \Pi$  belongs to  $K$  only if  $x$  is  $n$ -divisible modulo  $G$  for every  $n \geq 1$ . That is, for each  $n$ , there exists  $g_n \in G$  with  $n \cdot x - g_n \in G$ .

However, this alone is not sufficient. We need the remainders to be *compatible*:

**Proposition 2** (Condition 2: Compatibility of Remainders). The remainders we choose for different divisors must be compatible with each other. Specifically, if we have an  $n$ -remainder  $g_n$  and an  $m$ -remainder  $g_m$ , they must satisfy:

$$n \cdot g_m - m \cdot g_n \in G$$

for all positive integers  $n$  and  $m$ .

**Why this matters:** Think of dividing  $x$  by  $nm$  in two different ways:

- First divide by  $n$  (getting remainder  $g_n$ ), then divide that result by  $m$
- First divide by  $m$  (getting remainder  $g_m$ ), then divide that result by  $n$

These two procedures should give the same final answer (modulo  $G$ ). The compatibility condition ensures this consistency.

The final condition ensures these compatible remainders don’t just exist arbitrarily, but actually come from the mathematical structure of  $K/G \cong \varprojlim^1 G$ :

**Proposition 3** (Condition 3:  $\varprojlim^1$  Compatibility). Even if remainders are compatible (satisfying Condition 2), they might not correspond to a real element of  $K$ . We need one more check to ensure they “fit together” in the way that elements of  $K/G$  must.

To verify this, we look at the factorial subsequence of remainders:  $(g_{1!}, g_{2!}, g_{3!}, g_{4!}, \dots)$ .

The requirement is that the “gaps” or “jumps” between consecutive terms follow a specific pattern. More precisely, for each  $n \geq 1$ , there must exist some  $h_n \in G$  such that:

$$g_{n!} - (n+1) \cdot g_{(n+1)!} = h_n - (n+1) \cdot h_{n+1}$$

**What this means:** The quotient  $K/G$  has a very specific structure (called  $\varprojlim^1 G$ ). This condition ensures that our sequence of remainders actually represents a valid element in that structure, not just any arbitrary compatible sequence. It’s like checking that a puzzle piece not only fits with its neighbors, but also belongs to the correct puzzle.

**Theorem 1.** An element  $x = (x_i)_{i \geq 1} \in \Pi$  belongs to  $K$  if and only if:

1.  $x$  is  $n$ -divisible modulo  $G$  for every  $n \geq 1$  (with some choice of  $n$ -remainders  $g_n \in G$ ).
2. The remainders  $(g_n)$  are compatible:  $n \cdot g_m - m \cdot g_n \in G$  for all  $n, m \geq 1$ .
3. For each  $n \geq 1$ , there exists a sequence  $(h_k)_{k \geq n}$  with  $h_k \in G$  such that:

$$g_{n!} - (n+1) \cdot g_{(n+1)!} = h_n - (n+1) \cdot h_{n+1}$$

This ensures the factorial subsequence  $(g_{n!})$  represents a well-defined element in  $\varprojlim^1 G = (\prod_{n=0}^{\infty} G)/\text{Im}(\partial)$ , where  $\partial((a_n)) = (a_n - (n+1) \cdot a_{n+1})$ .

*Proof.* **Why these conditions are necessary:** If  $x \in K$ , then  $x \bmod G$  is an element of  $K/G$ . Since  $K/G \cong \varprojlim^1 G$  is divisible (every element can be divided by any positive integer), for each  $n \geq 1$ , there exists some  $y \in K$  such that  $n \cdot y \equiv x \pmod{G}$ . Setting  $g_n = n \cdot y - x \in G$  gives us condition (1).

The compatibility condition (2) must hold because if we divide  $x$  by  $nm$  in two different orders, we must get the same result.

Condition (3) ensures that the remainders  $(g_{n!})$  actually represent the correct element of  $\varprojlim^1 G$  that corresponds to  $x \bmod G$ .

**Why these conditions are sufficient:** If  $x \in \Pi$  satisfies all three conditions, then the remainders  $(g_n)$  form a compatible system that represents a well-defined element of  $\varprojlim^1 G$ . Since  $K/G \cong \varprojlim^1 G$ , this means  $x \bmod G$  corresponds to an element of  $K/G$ , which implies  $x \in K$ .  $\square$