Explicit Description of the Cotorsion Hull

Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ (the direct sum of cyclic 2-groups) and $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ (the full product). Define $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$, the cotorsion hull of G.

Our goal is to characterize exactly which elements of Π belong to K. We do this by building up three conditions, each refining the previous one.

Definition 1. *For $x \in \Pi$ and a positive integer n, we say x is* n-divisible modulo G *if we can "almost divide x by n" in the following sense: there exists some element $g \in G$ such that when we compute $n \cdot x - g$, the result lies entirely in G (has only finitely many nonzero coordinates).*

We call g an n-remainder *of x. Think of it as the "leftover" when trying to divide x by n, but we're allowed to adjust by an element of G to make it work.*

Example 1. *If $x \in G$ itself, then for any n, we can take $g = 0$ and have $n \cdot x - 0 = n \cdot x \in G$, so elements of G are n-divisible modulo G for all n.*

The first condition requires that x can be divided by *every* positive integer (modulo G):

Proposition 1 (Condition 1: Universal Divisibility)**.** *An element $x \in \Pi$ belongs to K only if x is n-divisible modulo G for every $n \geq 1$. That is, for each n, there exists $g_n \in G$ with $n \cdot x - g_n \in G$.*

However, this alone is not sufficient. We need the remainders to be *compatible*:

Proposition 2 (Condition 2: Compatibility of Remainders)**.** *The remainders we choose for different divisors must be* compatible *with each other. Specifically, if we have an n-remainder g_n and an m-remainder g_m, they must satisfy:*

$$n \cdot g_m - m \cdot g_n \in G$$

for all positive integers n and m.

Why this matters: *Think of dividing x by nm in two different ways:*

- *First divide by n (getting remainder g_n), then divide that result by m*
- *First divide by m (getting remainder g_m), then divide that result by n*

These two procedures should give the same final answer (modulo G). The compatibility condition ensures this consistency.

The final condition ensures these compatible remainders don't just exist arbitrarily, but actually come from the mathematical structure of $K/G \cong \varprojlim^1 G$:

Proposition 3 (Condition 3: $\lim^1$ Compatibility)**.** *Even if remainders are compatible (satisfying Condition 2), they might not correspond to a real element of K. We need one more check to ensure they "fit together" in the way that elements of K/G must.*

To verify this, we look at the factorial subsequence *of remainders:* $(g_{1!}, g_{2!}, g_{3!}, g_{4!}, \ldots)$*, where* $n! = 1 \cdot 2 \cdot 3 \cdots n$.

The requirement is that the "gaps" or "jumps" between consecutive terms follow a specific pattern. More precisely, for each $n \geq 1$*, there must exist some* $h_n \in G$ *such that:*

$$g_{n!} - (n+1) \cdot g_{(n+1)!} = h_n - (n+1) \cdot h_{n+1}$$

What this means: *The quotient* K/G *has a very specific structure (called* $\varprojlim^1 G$*). This condition ensures that our sequence of remainders actually represents a valid element in that structure, not just any arbitrary compatible sequence. It's like checking that a puzzle piece not only fits with its neighbors, but also belongs to the correct puzzle.*

Theorem 1. *An element* $x = (x_i)_{i \geq 1} \in \Pi$ *belongs to* K *if and only if:*

1. x *is* n*-divisible modulo* G *for every* $n \geq 1$ *(with some choice of* n*-remainders* $g_n \in G$*).*

2. *The remainders* (g_n) *are compatible:* $n \cdot g_m - m \cdot g_n \in G$ *for all* $n, m \geq 1$*.*

3. *For each* $n \geq 1$*, there exists* $h_n \in G$ *such that:*

$$g_{n!} - (n+1) \cdot g_{(n+1)!} = h_n - (n+1) \cdot h_{n+1}$$

This ensures the factorial subsequence $(g_{n!})$ *represents a well-defined element in* $\varprojlim^1 G = (\prod_{n=0}^{\infty} G)/\mathrm{Im}(\partial)$*, where* $\partial((a_n)) = (a_n - (n+1) \cdot a_{n+1})$*.*

Proof. **Why these conditions are necessary:** If $x \in K$, then $x \bmod G$ is an element of K/G. Since $K/G \cong \varprojlim^1 G$ is divisible (every element can be divided by any positive integer), for each $n \geq 1$, there exists some $y \in K$ such that $n \cdot y \equiv x \pmod{G}$. Setting $g_n = n \cdot y - x \in G$ gives us condition (1).

The compatibility condition (2) must hold because if we divide x by nm in two different orders, we must get the same result.

Condition (3) ensures that the remainders $(g_{n!})$ actually represent the correct element of $\varprojlim^1 G$ that corresponds to $x \bmod G$.

Why these conditions are sufficient: If $x \in \Pi$ satisfies all three conditions, then the remainders (g_n) form a compatible system that represents a well-defined element of $\varprojlim^1 G$. Since $K/G \cong \varprojlim^1 G$, this means $x \bmod G$ corresponds to an element of K/G, which implies $x \in K$. $\square$