Explicit Description of the Cotorsion Hull

Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$ and $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$. Define $K = \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, G)$, the cotorsion hull of G.

Our goal is to characterize exactly which elements of Π belong to K. We do this by building up three conditions, each refining the previous one.

Definition 1. *For $x \in \Pi$ and a positive integer n, we say x is* n-divisible modulo G *if we can "almost divide x by n" in the following sense: there exists some element $g \in G$ such that when we compute $n \cdot x - g$, the result lies entirely in G (has only finitely many nonzero coordinates).*

We call g an n-remainder *of x.*

The first condition requires that x can be divided by *every* positive integer (modulo G):

Proposition 1 (Condition 1: Universal Divisibility)**.** *An element $x \in \Pi$ belongs to K only if x is n-divisible modulo G for every $n \geq 1$. That is, for each n, there exists $g_n \in G$ with $n \cdot x - g_n \in G$.*

However, this alone is not sufficient. We need the remainders to be *compatible*:

Proposition 2 (Condition 2: Compatibility of Remainders)**.** *The remainders we choose for different divisors must be* compatible *with each other. Specifically, if we have an n-remainder g_n and an m-remainder g_m, they must satisfy:*

$$n \cdot g_m - m \cdot g_n \in G$$

for all positive integers n and m.

Why this matters: *Think of dividing x by nm in two different ways:*

- *First divide by n (getting remainder g_n), then divide that result by m*
- *First divide by m (getting remainder g_m), then divide that result by n*

These two procedures should give the same final answer (modulo G). The compatibility condition ensures this consistency.

The final condition ensures these compatible remainders don't just exist arbitrarily, but actually come from the mathematical structure of $K/G \cong \varprojlim^1 G$:

Proposition 3 (Condition 3: $\lim^1$ Compatibility)**.** *Even if remainders are compatible (satisfying Condition 2), they might not correspond to a real element of K. We need one more check to ensure they "fit together" in the way that elements of K/G must.*

To verify this, define the sequence $\hat{g}_n = g_{n!}$ for $n \geq 1$, which extracts the factorial subsequence of remainders. The requirement is that the "gaps" or "jumps" between consecutive terms follow a specific pattern. More precisely, there exists a sequence $(h_n)_{n\geq 1}$ with $h_n \in G$ and a sequence $(d_n)_{n\geq 1}$ with $d_n \in \Pi$ such that:

1. *$d_n = \hat{g}_n - h_n$ for all $n \geq 1$*

2. $d_n = (n+1) \cdot d_{n+1}$ *for all* $n \geq 1$ *(equivalently,* $d_1 = n! \cdot d_n$ *for all* n*)*

Why $d_n \in \Pi$***:*** *Since* $d_1 = n! \cdot d_n$ *for all* n*, the element* d_1 *must be divisible by* $n!$ *for all* n*. This is not possible in* G *(a direct sum of cyclic 2-groups), so* d_n *must lie in the larger group* $\Pi = \prod_{i=1}^{\infty} \mathbb{Z}/2^i\mathbb{Z}$*. The condition* $d_n = \hat{g}_n - h_n$ *means that* d_n *is congruent to an element of* G *(modulo the natural embedding), but* d_n *itself must be in* Π *to satisfy the divisibility condition* $d_n = (n+1) \cdot d_{n+1}$*.*

What this means: *The quotient* K/G *has a very specific structure (called* $\varprojlim^1 G$*). This condition ensures that our sequence of remainders actually represents a valid element in that structure, not just any arbitrary compatible sequence.*

Theorem 1. *An element* $x = (x_i)_{i \geq 1} \in \Pi$ *belongs to* K *if and only if:*

1. x *is* n*-divisible modulo* G *for every* $n \geq 1$ *(with some choice of* n*-remainders* $g_n \in G$*).*

2. *The remainders* (g_n) *are compatible:* $n \cdot g_m - m \cdot g_n \in G$ *for all* $n, m \geq 1$*.*

3. *Define* $\hat{g}_n = g_{n!}$ *for* $n \geq 1$*. There exists a sequence* $(h_n)_{n \geq 1}$ *with* $h_n \in G$ *and a sequence* $(d_n)_{n \geq 1}$ *with* $d_n \in \Pi$ *such that:*

   (a) $d_n = \hat{g}_n - h_n$ *for all* $n \geq 1$

   (b) $d_n = (n+1) \cdot d_{n+1}$ *for all* $n \geq 1$ *(equivalently,* $d_1 = n! \cdot d_n$ *for all* n*)*

   This ensures the sequence $(\hat{g}_n)$ *represents a well-defined element in* $\varprojlim^1 G = (\prod_{n=0}^{\infty} G)/\mathrm{Im}(\partial)$*, where* $\partial((a_n)) = (a_n - (n+1) \cdot a_{n+1})$*.*

Proof. **Why these conditions are necessary:** If $x \in K$, then $x \bmod G$ is an element of K/G. Since $K/G \cong \varprojlim^1 G$ is divisible (every element can be divided by any positive integer), for each $n \geq 1$, there exists some $y \in K$ such that $n \cdot y \equiv x \pmod{G}$. Setting $g_n = n \cdot y - x \in G$ gives us condition (1).

The compatibility condition (2) must hold because if we divide x by nm in two different orders, we must get the same result.

Condition (3) ensures that the sequence $(\hat{g}_n) = (g_{n!})$ actually represents the correct element of $\varprojlim^1 G$ that corresponds to $x \bmod G$.

Why these conditions are sufficient: If $x \in \Pi$ satisfies all three conditions, then the remainders (g_n) form a compatible system that represents a well-defined element of $\varprojlim^1 G$. Since $K/G \cong \varprojlim^1 G$, this means $x \bmod G$ corresponds to an element of K/G, which implies $x \in K$. $\square$