Short Proof of $\prod_{k=1}^{n-1} \sin \frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$

Proof. First, prove the preliminary identity using roots of unity:

$$x^n - 1 = (x-1)\prod_{k=1}^{n-1}(x - e^{2\pi i k/n})$$

At $x = 1$: $n = \prod_{k=1}^{n-1}(1 - e^{2\pi i k/n})$. Thus $\prod_{k=1}^{n-1} 2\sin(\pi k/n) = n$, so

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

Apply double-angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{2n}\cos\frac{k\pi}{2n} = \frac{n}{2^{n-1}}$$

$$\prod_{k=1}^{n-1} \sin\frac{k\pi}{2n} \cdot \prod_{k=1}^{n-1}\cos\frac{k\pi}{2n} = \frac{n}{2^{2n-2}}$$

By symmetry (complementary angles: $\cos\frac{k\pi}{2n} = \sin(\pi/2 - \frac{k\pi}{2n}) = \sin\frac{(n-k)\pi}{2n}$),

$$\prod_{k=1}^{n-1}\cos\frac{k\pi}{2n} = \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}.$$

Thus:

$$\left(\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\right)^2 = \frac{n}{2^{2n-2}}$$

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}.$$

$\square$