Complete Derivation of $(T_{T_m^2})^2$ Using Recursive Triangular Identity

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1 Introduction

This document derives mathematical expressions for $(T_{T_m^2})^2$ using the fundamental triangular number identity recursively:

$$i^2 = T_{i-1} + T_i \tag{1}$$

where $T_n = \frac{n(n+1)}{2}$ is the n-th triangular number.

The key insight is that we must apply this identity recursively until no squares remain on the right side of the equation.

2 The Triangular Number Identity

The triangular number identity states that any perfect square can be expressed as the sum of two consecutive triangular numbers:

$$\begin{align}
i^2 &= T_{i-1} + T_i \tag{2}\\
&= \frac{(i-1)i}{2} + \frac{i(i+1)}{2} \tag{3}\\
&= \frac{i}{2}[(i-1) + (i+1)] \tag{4}\\
&= \frac{i}{2}[2i] \tag{5}\\
&= i^2 \tag{6}
\end{align}$$

3 Complete Recursive Derivation

We want to find an expression for $(T_{T_m^2})^2$ using only terms of the form T_l, with no squares remaining.

3.1 Step 1: Apply Identity to T_m^2

For any natural number m:

$$\begin{align}
T_m &= \frac{m(m+1)}{2} \tag{7}\\
T_m^2 &= \left(\frac{m(m+1)}{2}\right)^2 \tag{8}
\end{align}$$

Since T_m^2 is a perfect square, we apply the identity $i^2 = T_{i-1} + T_i$ where $i = T_m$:

$$T_m^2 = T_{T_m-1} + T_{T_m} \tag{9}$$

3.2 Step 2: Calculate $T_{T_m^2}$

Now we calculate:

$$T_{T_m^2} = T_{T_{T_m-1}+T_{T_m}} \tag{10}$$

This gives us the triangular number whose subscript is the sum of two consecutive triangular numbers.

3.3 Step 3: Apply Identity to $(T_{T_m^2})^2$

To find $(T_{T_m^2})^2$, we apply the identity again. Let $j = T_{T_m^2}$, then:

$$(T_{T_m^2})^2 = j^2 = T_{j-1} + T_j \tag{11}$$

3.4 Step 4: Eliminate Remaining Squares

Now we must check if $j-1$ or j are perfect squares and apply the identity recursively:

- If $j-1 = k^2$ for some k, then $T_{j-1} = T_{k-1} + T_k$
- If $j = l^2$ for some l, then $T_j = T_{l-1} + T_l$

We continue this process until no squares remain in any subscript.

4 Specific Cases

4.1 Case 1: $m = 1$

$$T_1 = 1 \tag{12}$$
$$T_1^2 = 1^2 = T_0 + T_1 = 0 + 1 = 1 \tag{13}$$
$$T_{T_1^2} = T_1 = 1 \tag{14}$$
$$(T_{T_1^2})^2 = 1^2 = T_0 + T_1 = 0 + 1 = 1 \tag{15}$$

Since neither 0 nor 1 are perfect squares (except $1 = 1^2$, but $T_0 = 0$), we have:

$$(T_{T_1^2})^2 = T_0 + T_1 \tag{16}$$

4.2 Case 2: $m = 2$

$$T_2 = 3 \tag{17}$$
$$T_2^2 = 3^2 = T_2 + T_3 = 3 + 6 = 9 \tag{18}$$
$$T_{T_2^2} = T_9 = 45 \tag{19}$$
$$(T_{T_2^2})^2 = 45^2 = T_{44} + T_{45} \tag{20}$$

Since neither 44 nor 45 are perfect squares, we have:

$$(T_{T_2^2})^2 = T_{44} + T_{45} \tag{21}$$

4.3 Case 3: $m = 3$

$$T_3 = 6 \tag{22}$$
$$T_3^2 = 6^2 = T_5 + T_6 = 15 + 21 = 36 \tag{23}$$
$$T_{T_3^2} = T_{36} = 666 \tag{24}$$
$$(T_{T_3^2})^2 = 666^2 = T_{665} + T_{666} \tag{25}$$

Since neither 665 nor 666 are perfect squares, we have:

$$(T_{T_3^2})^2 = T_{665} + T_{666} \tag{26}$$

5 General Pattern

For any natural number m, the complete recursive application of the triangular number identity gives:

1. $T_m^2 = T_{T_m - 1} + T_{T_m}$
2. $T_{T_m^2} = T_{T_{T_m-1} + T_{T_m}}$
3. $(T_{T_m^2})^2 = T_{T_{T_m^2} - 1} + T_{T_{T_m^2}}$
4. If $T_{T_m^2} - 1$ or $T_{T_m^2}$ are perfect squares, apply the identity recursively

6 Conclusion

Using the triangular number identity $i^2 = T_{i-1} + T_i$ recursively, we can express $(T_{T_m^2})^2$ entirely in terms of triangular numbers with no squares remaining:

$$(T_{T_m^2})^2 = T_{T_{T_m^2} - 1} + T_{T_{T_m^2}} \tag{27}$$

This demonstrates the power of recursive application of the triangular number identity in reducing complex expressions involving multiple squares to sums of triangular numbers.