Short Proof of $\prod_{k=1}^{n-1} \sin \frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$

Proof. Consider the identity from roots of unity:

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$$

Apply double-angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\prod_{k=1}^{n-1} 2\sin\frac{k\pi}{2n}\cos\frac{k\pi}{2n} = \frac{n}{2^{n-1}}$$

$$\prod_{k=1}^{n-1} \sin\frac{k\pi}{2n} \cdot \prod_{k=1}^{n-1} \cos\frac{k\pi}{2n} = \frac{n}{2^{2n-2}}$$

By symmetry, $\prod_{k=1}^{n-1} \cos \frac{k\pi}{2n} = \prod_{k=1}^{n-1} \sin \frac{k\pi}{2n}$ (complementary angles).
Thus:

$$\left(\prod_{k=1}^{n-1} \sin\frac{k\pi}{2n}\right)^2 = \frac{n}{2^{2n-2}}$$

$$\prod_{k=1}^{n-1} \sin\frac{k\pi}{2n} = \sqrt{\frac{n}{2^{2n-2}}} = \frac{\sqrt{n}}{2^{n-1}}$$

$\square$