Complete Derivation of $(T_{T_m^2})^2$ Using Recursive Triangular Identity

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1 Introduction

This document derives mathematical expressions for $(T_{T_m^2})^2$ using the fundamental triangular number identity recursively:

$$i^2 = T_{i-1} + T_i \tag{1}$$

where $T_n = \frac{n(n+1)}{2}$ is the n-th triangular number.

The key insight is that we must apply this identity recursively until no squares remain on the right side of the equation.

2 The Triangular Number Identity

The triangular number identity states that any perfect square can be expressed as the sum of two consecutive triangular numbers:

$$\begin{aligned}
i^2 &= T_{i-1} + T_i && (2)\\
&= \frac{(i-1)i}{2} + \frac{i(i+1)}{2} && (3)\\
&= \frac{i}{2}[(i-1) + (i+1)] && (4)\\
&= \frac{i}{2}[2i] && (5)\\
&= i^2 && (6)
\end{aligned}$$

3 Complete Derivation

We want to find an expression for $(T_{T_m^2})^2$ using only terms of the form T_l, applying the identity $i^2 = T_{i-1} + T_i$ to every squaring operation.

3.1 Step 1: Apply Identity to T_m^2

For any natural number m, we have $T_m = \frac{m(m+1)}{2}$.

The expression $(T_{T_m^2})^2$ contains the inner square $T_m^2 = m^2$. Applying the identity where $i = m$:

$$T_m^2 = m^2 = T_{m-1} + T_m \tag{7}$$

3.2 Step 2: Substitute into the Original Expression

Substituting this into $(T_{T_m^2})^2$:

$$(T_{T_m^2})^2 = \left(T_{T_{m-1}+T_m}\right)^2 \tag{8}$$

3.3 Step 3: Apply Identity to the Outer Square

Now we apply the identity to the outer squaring operation. Let $j = T_{T_{m-1}+T_m}$, then:

$$\begin{align} \left(T_{T_{m-1}+T_m}\right)^2 &= j^2 \tag{9} \\ &= T_{j-1} + T_j \tag{10} \\ &= T_{T_{T_{m-1}+(T_m)}-1} + T_{T_{T_{m-1}+T_m}} \tag{11} \end{align}$$

3.4 Final Expression

Therefore, the complete expression for $(T_{T_m^2})^2$ with no squares remaining is:

$$\boxed{(T_{T_m^2})^2 = T_{T_{m-1}+(T_m)-1} + T_{T_{m-1}+T_m}} \tag{12}$$

4 Specific Cases

4.1 Case 1: $m = 1$

$$\begin{align} T_1 &= 1 \tag{13} \\ T_1^2 &= 1^2 = T_0 + T_1 = 0 + 1 = 1 \tag{14} \\ T_{T_1^2} &= T_1 = 1 \tag{15} \\ (T_{T_1^2})^2 &= 1^2 = T_0 + T_1 = 0 + 1 = 1 \tag{16} \end{align}$$

Since neither 0 nor 1 are perfect squares (except $1 = 1^2$, but $T_0 = 0$), we have:

$$(T_{T_1^2})^2 = T_0 + T_1 \tag{17}$$

4.2 Case 2: $m = 2$

$$\begin{align} T_2 &= 3 \tag{18} \\ T_2^2 &= 3^2 = T_2 + T_3 = 3 + 6 = 9 \tag{19} \\ T_{T_2^2} &= T_9 = 45 \tag{20} \\ (T_{T_2^2})^2 &= 45^2 = T_{44} + T_{45} \tag{21} \end{align}$$

Since neither 44 nor 45 are perfect squares, we have:

$$(T_{T_2^2})^2 = T_{44} + T_{45} \tag{22}$$

4.3 Case 3: $m = 3$

$$\begin{align} T_3 &= 6 \tag{23} \\ T_3^2 &= 6^2 = T_5 + T_6 = 15 + 21 = 36 \tag{24} \\ T_{T_3^2} &= T_{36} = 666 \tag{25} \\ (T_{T_3^2})^2 &= 666^2 = T_{665} + T_{666} \tag{26} \end{align}$$

Since neither 665 nor 666 are perfect squares, we have:

$$(T_{T_3^2})^2 = T_{665} + T_{666} \tag{27}$$

5 General Pattern

For any natural number m, the complete recursive application of the triangular number identity gives:

1. $T_m^2 = T_{T_m - 1} + T_{T_m}$
2. $T_{T_m^2} = T_{T_{T_m-1} + T_{T_m}}$
3. $(T_{T_m^2})^2 = T_{T_{T_m^2} - 1} + T_{T_{T_m^2}}$
4. If $T_{T_m^2} - 1$ or $T_{T_m^2}$ are perfect squares, apply the identity recursively

6 Conclusion

Using the triangular number identity $i^2 = T_{i-1} + T_i$ recursively, we can express $(T_{T_m^2})^2$ entirely in terms of triangular numbers with no squares remaining:

$$(T_{T_m^2})^2 = T_{T_{T_m^2} - 1} + T_{T_{T_m^2}} \tag{28}$$

This demonstrates the power of recursive application of the triangular number identity in reducing complex expressions involving multiple squares to sums of triangular numbers.