# Derivation of $(T_{T_m^2})^2$ Using Triangular Number Identity

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## 1 Introduction

This document derives mathematical expressions for $(T_{T_m^2})^2$ using the fundamental triangular number identity:

$$i^2 = T_{i-1} + T_i \tag{1}$$

where $T_n = \frac{n(n+1)}{2}$ is the $n$-th triangular number.

## 2 The Triangular Number Identity

The key identity states that any perfect square can be expressed as the sum of two consecutive triangular numbers:

$$\begin{aligned} i^2 &= T_{i-1} + T_i && (2)\\ &= \frac{(i-1)i}{2} + \frac{i(i+1)}{2} && (3)\\ &= \frac{i}{2}[(i-1) + (i+1)] && (4)\\ &= \frac{i}{2}[2i] && (5)\\ &= i^2 && (6) \end{aligned}$$

## 3 Derivation of $(T_{T_m^2})^2$

We want to find an expression for $(T_{T_m^2})^2$ using only terms of the form $T_l$.

### 3.1 Step 1: Express $T_m$ and $T_m^2$

For any natural number $m$:

$$\begin{aligned} T_m &= \frac{m(m+1)}{2} && (7)\\ T_m^2 &= \left(\frac{m(m+1)}{2}\right)^2 && (8) \end{aligned}$$

### 3.2 Step 2: Apply the identity to $T_m^2$

Since $T_m^2$ is a perfect square, we can apply the identity $i^2 = T_{i-1} + T_i$:

Let $i = T_m$, then:

$$T_m^2 = T_{T_m-1} + T_{T_m} \tag{9}$$

### 3.3 Step 3: Calculate $T_{T_m^2}$

Now we need to find $T_{T_m^2} = T_{T_{m-1}+T_m}$:

$$\begin{aligned} T_{T_m^2} &= T_{T_{m-1}+T_{T_m}} && (10) \\ &= \frac{(T_{T_m-1} + T_{T_m})(T_{T_m-1} + T_{T_m} + 1)}{2} && (11) \end{aligned}$$

### 3.4 Step 4: Apply the identity to $(T_{T_m^2})^2$

To find $(T_{T_m^2})^2$, we apply the identity again. Let $j = T_{T_m^2}$, then:

$$(T_{T_m^2})^2 = j^2 = T_{j-1} + T_j = T_{T_{T_m^2}-1} + T_{T_{T_m^2}} \quad (12)$$

## 4 Specific Cases

### 4.1 Case 1: $m = 1$

$$\begin{aligned} T_1 &= \frac{1 \cdot 2}{2} = 1 && (13) \\ T_1^2 &= 1^2 = T_0 + T_1 = 0 + 1 = 1 && (14) \\ T_{T_1^2} &= T_1 = 1 && (15) \\ (T_{T_1^2})^2 &= 1^2 = T_0 + T_1 = 0 + 1 = 1 && (16) \end{aligned}$$

Therefore: $(T_{T_1^2})^2 = T_0 + T_1$

### 4.2 Case 2: $m = 2$

$$\begin{aligned} T_2 &= \frac{2 \cdot 3}{2} = 3 && (17) \\ T_2^2 &= 3^2 = T_2 + T_3 = 3 + 6 = 9 && (18) \\ T_{T_2^2} &= T_9 = \frac{9 \cdot 10}{2} = 45 && (19) \\ (T_{T_2^2})^2 &= 45^2 = T_{44} + T_{45} && (20) \end{aligned}$$

Therefore: $(T_{T_2^2})^2 = T_{44} + T_{45}$

### 4.3 Case 3: $m = 3$

$$\begin{aligned} T_3 &= \frac{3 \cdot 4}{2} = 6 && (21) \\ T_3^2 &= 6^2 = T_5 + T_6 = 15 + 21 = 36 && (22) \\ T_{T_3^2} &= T_{36} = \frac{36 \cdot 37}{2} = 666 && (23) \\ (T_{T_3^2})^2 &= 666^2 = T_{665} + T_{666} && (24) \end{aligned}$$

Therefore: $(T_{T_3^2})^2 = T_{665} + T_{666}$

## 5 General Pattern

For any natural number $m$:

1. $T_m^2 = T_{T_m - 1} + T_{T_m}$
2. $T_{T_m^2} = T_{T_{T_m-1}+T_{T_m}}$
3. $(T_{T_m^2})^2 = T_{T_{T_m^2}-1} + T_{T_{T_m^2}}$

## 6 Conclusion

Using the triangular number identity $i^2 = T_{i-1} + T_i$, we can express $(T_{T_m^2})^2$ entirely in terms of triangular numbers:

$$(T_{T_m^2})^2 = T_{T_{T_m^2}-1} + T_{T_{T_m^2}} \tag{25}$$

This demonstrates the power of the triangular number identity in reducing complex expressions involving squares to sums of triangular numbers.