

Complete Derivation of $(T_{T_m^2})^2$ Using Recursive Triangular Identity

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1 Introduction

This document derives mathematical expressions for $(T_{T_m^2})^2$ using the fundamental triangular number identity recursively:

$$i^2 = T_{i-1} + T_i \quad (1)$$

where $T_n = \frac{n(n+1)}{2}$ is the n -th triangular number.

The key insight is that we must apply this identity recursively until no squares remain on the right side of the equation.

2 The Triangular Number Identity

The triangular number identity states that any perfect square can be expressed as the sum of two consecutive triangular numbers:

$$i^2 = T_{i-1} + T_i \quad (2)$$

$$= \frac{(i-1)i}{2} + \frac{i(i+1)}{2} \quad (3)$$

$$= \frac{i}{2}[(i-1) + (i+1)] \quad (4)$$

$$= \frac{i}{2}[2i] \quad (5)$$

$$= i^2 \quad (6)$$

3 Complete Derivation

We want to find an expression for $(T_{T_m^2})^2$ using only terms of the form T_l , applying the identity $i^2 = T_{i-1} + T_i$ to every squaring operation.

3.1 Step 1: Apply Identity to T_m^2

For any natural number m , we have $T_m = \frac{m(m+1)}{2}$.

The expression $(T_{T_m^2})^2$ contains the inner square T_m^2 . Applying the identity where $i = T_m$:

$$T_m^2 = T_{T_m-1} + T_{T_m} \quad (7)$$

3.2 Step 2: Substitute into the Original Expression

Substituting this into $(T_{T_m^2})^2$:

$$(T_{T_m^2})^2 = (T_{T_{T_m-1}+T_{T_m}})^2 \quad (8)$$

3.3 Step 3: Apply Identity to the Outer Square

Now we apply the identity to the outer squaring operation. Let $j = T_{T_{T_m-1}+T_{T_m}}$, then:

$$(T_{T_{T_m-1}+T_{T_m}})^2 = j^2 \quad (9)$$

$$= T_{j-1} + T_j \quad (10)$$

$$= T_{T_{T_m-1}+T_{T_m}-1} + T_{T_{T_m-1}+T_{T_m}} \quad (11)$$

3.4 Final Expression

Therefore, the complete expression for $(T_{T_m^2})^2$ with no squares remaining is:

$$\boxed{(T_{T_m^2})^2 = T_{T_{T_m-1}+T_{T_m}-1} + T_{T_{T_m-1}+T_{T_m}}} \quad (12)$$

4 Specific Cases

4.1 Case 1: $m = 1$

$$T_1 = 1 \quad (13)$$

$$T_1^2 = 1^2 = T_0 + T_1 = 0 + 1 = 1 \quad (14)$$

$$T_{T_1^2} = T_1 = 1 \quad (15)$$

$$(T_{T_1^2})^2 = 1^2 = T_0 + T_1 = 0 + 1 = 1 \quad (16)$$

Since neither 0 nor 1 are perfect squares (except $1 = 1^2$, but $T_0 = 0$), we have:

$$(T_{T_1^2})^2 = T_0 + T_1 \quad (17)$$

4.2 Case 2: $m = 2$

$$T_2 = 3 \quad (18)$$

$$T_2^2 = 3^2 = T_2 + T_3 = 3 + 6 = 9 \quad (19)$$

$$T_{T_2^2} = T_9 = 45 \quad (20)$$

$$(T_{T_2^2})^2 = 45^2 = T_{44} + T_{45} \quad (21)$$

Since neither 44 nor 45 are perfect squares, we have:

$$(T_{T_2^2})^2 = T_{44} + T_{45} \quad (22)$$

4.3 Case 3: $m = 3$

$$T_3 = 6 \quad (23)$$

$$T_3^2 = 6^2 = T_5 + T_6 = 15 + 21 = 36 \quad (24)$$

$$T_{T_3^2} = T_{36} = 666 \quad (25)$$

$$(T_{T_3^2})^2 = 666^2 = T_{665} + T_{666} \quad (26)$$

Since neither 665 nor 666 are perfect squares, we have:

$$(T_{T_3^2})^2 = T_{665} + T_{666} \quad (27)$$

5 General Pattern

For any natural number m , the complete recursive application of the triangular number identity gives:

1. $T_m^2 = T_{T_m-1} + T_{T_m}$
2. $T_{T_m^2} = T_{T_{T_m-1}+T_{T_m}}$
3. $(T_{T_m^2})^2 = T_{T_{T_m^2}-1} + T_{T_{T_m^2}}$
4. If $T_{T_m^2} - 1$ or $T_{T_m^2}$ are perfect squares, apply the identity recursively

6 Conclusion

Using the triangular number identity $i^2 = T_{i-1} + T_i$ recursively, we can express $(T_{T_m^2})^2$ entirely in terms of triangular numbers with no squares remaining:

$$(T_{T_m^2})^2 = T_{T_{T_m^2}-1} + T_{T_{T_m^2}} \quad (28)$$

This demonstrates the power of recursive application of the triangular number identity in reducing complex expressions involving multiple squares to sums of triangular numbers.