Problem Solutions

i) Find, with proof, all pairs of natural numbers (m, n) so that $m < n$ and $m^n < n^m$.

Consider the function $f(x) = \frac{\ln x}{x}$. The derivative $f'(x) = \frac{1-\ln x}{x^2}$ shows that f increases on $(1, e)$ and decreases on (e, ∞).

For $m < n$: - If $1 < m < n < e$, then $f(m) > f(n)$, so $m^n > n^m$. - If $m < e < n$, then $f(m) < f(n)$, so $m^n < n^m$. - If $e < m < n$, then $f(m) > f(n)$, so $m^n > n^m$.

Natural numbers with $m \geq 2$, $n \geq 3$: only $m = 2$, $n \geq 3$.

Check: $2^3 = 8 < 9 = 3^2$; $2^4 = 16 \not< 16$; $2^5 = 32 > 25$.

Thus, only $(2, 3)$.

ii) Use part (i) to find, with proof, the smallest odd number of the form $n = c^d f^g$ where c, d, f, g are distinct natural numbers greater than one, and find all such expressions for that particular n.

From part (i), $m^n < n^m$ only for $(m, n) = (2, 3)$.

For odd $n = c^d f^g$ with c, d, f, g distinct integers > 1, c, f must be odd.

The exponents d, g must be distinct from each other and from c, f.

Since $(2, 3)$ is the only pair with $m^n < n^m$, the minimal odd analog uses the next available exponent after the conflicting ones.

The smallest bases are $c = 3, f = 5$, and to avoid $d = 2$ conflicting with bases, use $d = 4$ (next after 3), $g = 2$.

Thus $n = 3^4 \times 5^2 = 81 \times 25 = 2025$.

All values $3, 4, 5, 2$ are distinct and > 1.

Any smaller n would require smaller bases or exponents, but $3, 5$ are the smallest odd primes, and exponents $2, 3$ conflict, so $2, 4$ is minimal.

For 2025, the only such expression is $3^4 \times 5^2$.