Complete Proof of the Sine Product Identity

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$$

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Abstract

We present a complete and rigorous proof of the sine product identity using roots of unity and the product formula for sine. The proof proceeds in two stages: first establishing a preliminary result for the product at $k\pi/n$, then using a clever doubling argument with complementary angles to obtain the desired result at $k\pi/(2n)$.

1 Main Result

Theorem 1. *For any positive integer $n \geq 2$:*

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right) = \frac{\sqrt{n}}{2^{n-1}} \tag{1}$$

2 Preliminary Lemma

We first establish a related identity that will be crucial for our proof.

Lemma 1. *For any positive integer $n \geq 2$:*

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \tag{2}$$

Proof. Consider the polynomial equation $x^n = 1$. Its solutions are the n-th roots of unity:

$$\omega_k = e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1 \tag{3}$$

We can factor $x^n - 1$ as:

$$x^n - 1 = (x-1) \prod_{k=1}^{n-1} (x - \omega_k) \tag{4}$$

Dividing both sides by $x - 1$:

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1} = \prod_{k=1}^{n-1} (x - \omega_k) \tag{5}$$

Taking the limit as $x \to 1$ (using L'Hôpital's rule on the left side):

$$\lim_{x \to 1} \frac{x^n - 1}{x - 1} = \lim_{x \to 1} nx^{n-1} = n \tag{6}$$

By continuity of the product:

$$\prod_{k=1}^{n-1}(1-\omega_k) = n \tag{7}$$

Now, we compute the modulus of each factor. For $\omega_k = e^{2\pi i k/n}$:

$$\begin{aligned}
|1-\omega_k|^2 &= |1-\cos(2\pi k/n) - i\sin(2\pi k/n)|^2 &(8)\\
&= (1-\cos(2\pi k/n))^2 + \sin^2(2\pi k/n) &(9)\\
&= 1 - 2\cos(2\pi k/n) + \cos^2(2\pi k/n) + \sin^2(2\pi k/n) &(10)\\
&= 2 - 2\cos(2\pi k/n) &(11)\\
&= 2(1-\cos(2\pi k/n)) &(12)
\end{aligned}$$

Using the half-angle formula $1-\cos\theta = 2\sin^2(\theta/2)$:

$$|1-\omega_k|^2 = 2\cdot 2\sin^2(\pi k/n) = 4\sin^2(\pi k/n) \tag{13}$$

Therefore:

$$|1-\omega_k| = 2\sin(\pi k/n) \tag{14}$$

where we use that $\sin(\pi k/n) > 0$ for $k = 1, 2, \ldots, n-1$.

Now, observe that $1-\omega_k$ has argument $\pi - \pi k/n$ for $k < n/2$ and the product $\prod_{k=1}^{n-1}(1-\omega_k)$ is a real positive number (equal to n). This is because the complex arguments cancel in pairs due to symmetry: ω_k and ω_{n-k} contribute conjugate factors.

Therefore:

$$\prod_{k=1}^{n-1}|1-\omega_k| = n \tag{15}$$

Substituting our expression for $|1-\omega_k|$:

$$\prod_{k=1}^{n-1} 2\sin(\pi k/n) = n \tag{16}$$

$$2^{n-1}\prod_{k=1}^{n-1}\sin(\pi k/n) = n \tag{17}$$

$$\prod_{k=1}^{n-1}\sin(\pi k/n) = \frac{n}{2^{n-1}} \tag{18}$$

□

3 Key Observation

The crucial insight is to relate the products at $k\pi/(2n)$ to those at $k\pi/n$ and $(2n-k)\pi/(2n)$.

Lemma 2 (Complementary Angles)**.** *For $1 \le k \le n-1$:*

$$\sin\left(\frac{k\pi}{2n}\right)\cdot\sin\left(\frac{(2n-k)\pi}{2n}\right) = \sin\left(\frac{k\pi}{2n}\right)\cdot\sin\left(\pi - \frac{k\pi}{2n}\right) = \sin^2\left(\frac{k\pi}{2n}\right) \tag{19}$$

Also:

$$\sin\left(\frac{(n+k)\pi}{2n}\right) = \sin\left(\frac{\pi}{2} + \frac{k\pi}{2n}\right) = \cos\left(\frac{k\pi}{2n}\right) \tag{20}$$

4 Proof of Main Theorem

Proof. Consider the product over all k from 1 to $2n-1$:

$$\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) \tag{21}$$

We can split this product into two parts:

$$\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) = \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right) \cdot \sin\left(\frac{n\pi}{2n}\right) \cdot \prod_{k=n+1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) \tag{22}$$

Note that $\sin(n\pi/(2n)) = \sin(\pi/2) = 1$.
For the second product, substitute $k = n + j$ where $j = 1, 2, \ldots, n-1$:

$$\prod_{k=n+1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) = \prod_{j=1}^{n-1} \sin\left(\frac{(n+j)\pi}{2n}\right) = \prod_{j=1}^{n-1} \sin\left(\frac{\pi}{2} + \frac{j\pi}{2n}\right) \tag{23}$$

Using $\sin(\pi/2 + \alpha) = \cos\alpha$:

$$\prod_{j=1}^{n-1} \sin\left(\frac{\pi}{2} + \frac{j\pi}{2n}\right) = \prod_{j=1}^{n-1} \cos\left(\frac{j\pi}{2n}\right) \tag{24}$$

Therefore:

$$\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) = \prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right) \cdot \prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{2n}\right) \tag{25}$$

Now, we apply Lemma 1 with n replaced by $2n$. Note that:

$$\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) = \frac{2n}{2^{2n-1}} \tag{26}$$

Using the double-angle formula $\sin(2\alpha) = 2\sin\alpha\cos\alpha$:

$$\sin\left(\frac{k\pi}{n}\right) = 2\sin\left(\frac{k\pi}{2n}\right)\cos\left(\frac{k\pi}{2n}\right) \tag{27}$$

From Lemma 1:

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \prod_{k=1}^{n-1} 2\sin\left(\frac{k\pi}{2n}\right)\cos\left(\frac{k\pi}{2n}\right) = 2^{n-1}\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right)\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{2n}\right) \tag{28}$$

Substituting from Lemma 1:

$$\frac{n}{2^{n-1}} = 2^{n-1}\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right)\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{2n}\right) \tag{29}$$

Let $P = \prod_{k=1}^{n-1} \sin(k\pi/(2n))$ and $Q = \prod_{k=1}^{n-1} \cos(k\pi/(2n))$. Then:

$$PQ = \frac{n}{2^{2(n-1)}} \tag{30}$$

From the earlier observation with $n \to 2n$ in Lemma 1:

$$\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) = PQ = \frac{2n}{2^{2n-1}} = \frac{n}{2^{2n-2}} \tag{31}$$

Therefore:

$$P \cdot Q = \frac{n}{2^{2n-2}} \tag{32}$$

But from Lemma 1 applied directly:

$$2^{n-1} P \cdot Q = \frac{n}{2^{n-1}} \tag{33}$$

which gives:

$$P \cdot Q = \frac{n}{2^{2n-2}} \tag{34}$$

Now we use the symmetry property. Note that:

$$\cos\left(\frac{k\pi}{2n}\right) = \sin\left(\frac{\pi}{2} - \frac{k\pi}{2n}\right) = \sin\left(\frac{(n-k)\pi}{2n}\right) \tag{35}$$

Actually, let's use a different approach. We know:

$$P^2 = \prod_{k=1}^{n-1} \sin^2\left(\frac{k\pi}{2n}\right) \tag{36}$$

Using $\sin^2 \alpha = \frac{1-\cos(2\alpha)}{2}$:

$$P^2 = \prod_{k=1}^{n-1} \frac{1-\cos(k\pi/n)}{2} = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \left(1 - \cos\frac{k\pi}{n}\right) \tag{37}$$

Using $1 - \cos\theta = 2\sin^2(\theta/2)$:

$$\prod_{k=1}^{n-1} \left(1 - \cos\frac{k\pi}{n}\right) = \prod_{k=1}^{n-1} 2\sin^2\left(\frac{k\pi}{2n}\right) = 2^{n-1} P^2 \tag{38}$$

Therefore:

$$P^2 = \frac{2^{n-1} P^2}{2^{n-1}} = P^2 \tag{39}$$

This is circular. Let me use the correct approach with products over different ranges.

Correct Final Approach:

From Lemma 1 with $2n$ instead of n:

$$\prod_{k=1}^{2n-1} \sin\left(\frac{k\pi}{2n}\right) = \frac{2n}{2^{2n-1}} = \frac{n}{2^{2n-2}} \tag{40}$$

We separate this into three parts: $k \in \{1, \ldots, n-1\}$, $k = n$, and $k \in \{n+1, \ldots, 2n-1\}$:

$$\frac{n}{2^{2n-2}} = \left[\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right)\right] \cdot 1 \cdot \left[\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{2n}\right)\right] \tag{41}$$

Also, from Lemma 1 with n:

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \tag{42}$$

Using $\sin(2\alpha) = 2\sin\alpha\cos\alpha$:

$$\frac{n}{2^{n-1}} = 2^{n-1} \left[\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right)\right] \left[\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{2n}\right)\right] \tag{43}$$

Dividing these two equations:

$$\frac{n/2^{2n-2}}{n/2^{n-1}} = \frac{P \cdot Q}{2^{n-1} P \cdot Q} \tag{44}$$

$$\frac{1}{2^{n-1}} = \frac{1}{2^{n-1}} \tag{45}$$

This confirms consistency. From the second equation:

$$P \cdot Q = \frac{n}{2^{2n-2}} \tag{46}$$

We also have $P = Q$ by symmetry (the cosine product equals the sine product due to complementary angles). Therefore:

$$P^2 = \frac{n}{2^{2n-2}} \tag{47}$$

$$P = \sqrt{\frac{n}{2^{2n-2}}} = \frac{\sqrt{n}}{2^{n-1}} \tag{48}$$

□

5 Verification and Examples

5.1 Small Values

- **$n = 2$:**

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2^1} \quad \checkmark \tag{49}$$

- **$n = 3$:**

$$\sin\left(\frac{\pi}{6}\right) \cdot \sin\left(\frac{2\pi}{6}\right) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2^2} \quad \checkmark \tag{50}$$

- **$n = 4$:**

$$\sin\left(\frac{\pi}{8}\right) \cdot \sin\left(\frac{2\pi}{8}\right) \cdot \sin\left(\frac{3\pi}{8}\right) \tag{51}$$

$$= \sin\left(\frac{\pi}{8}\right) \cdot \frac{\sqrt{2}}{2} \cdot \cos\left(\frac{\pi}{8}\right) \tag{52}$$

$$= \frac{\sqrt{2}}{2} \sin\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{8}\right) \tag{53}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \sin\left(\frac{\pi}{4}\right) \tag{54}$$

$$= \frac{\sqrt{2}}{4} \cdot \frac{\sqrt{2}}{2} = \frac{2}{8} = \frac{1}{4} = \frac{\sqrt{4}}{2^3} \quad \checkmark \tag{55}$$

5.2 Numerical Verification for $n = 5$

For $n = 5$, we compute:

$$\prod_{k=1}^{4} \sin\left(\frac{k\pi}{10}\right) = \sin(18) \cdot \sin(36) \cdot \sin(54) \cdot \sin(72) \tag{56}$$

$$\approx 0.309 \times 0.588 \times 0.809 \times 0.951 \tag{57}$$

$$\approx 0.1399 \tag{58}$$

The formula gives:

$$\frac{\sqrt{5}}{2^4} = \frac{2.236}{16} \approx 0.1398 \quad \checkmark \tag{59}$$

6 Conclusion

We have rigorously proven that:

$$\boxed{\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{2n}\right) = \frac{\sqrt{n}}{2^{n-1}}} \tag{60}$$

This beautiful identity connects the product of sine values at equally spaced angles to a simple closed form involving a square root and a power of 2. The proof relies on the theory of roots of unity and clever manipulation of trigonometric identities.