Problem Solutions

i) Find, with proof, all pairs of natural numbers (m, n) so that $m < n$ and $m^n < n^m$.

Consider the function $f(x) = \frac{\ln x}{x}$. The derivative $f'(x) = \frac{1-\ln x}{x^2}$ shows that f increases on $(1, e)$ and decreases on (e, ∞).

For $m < n$: - If $1 < m < n < e$, then $f(m) > f(n)$, so $m^n > n^m$. - If $m < e < n$, then $f(m) < f(n)$, so $m^n < n^m$. - If $e < m < n$, then $f(m) > f(n)$, so $m^n > n^m$.

Natural numbers with $m \geq 2$, $n \geq 3$: only $m = 2$, $n \geq 3$.

Check: $2^3 = 8 < 9 = 3^2$; $2^4 = 16 \not< 16$; $2^5 = 32 > 25$.

Thus, only $(2, 3)$.

ii) Use part (i) to find, with proof, the smallest odd number of the form $n = c^d f^g$ where c, d, f, g are distinct natural numbers greater than one, and find all such expressions for that particular n.

From part (i), $m^n < n^m$ only for $(m, n) = (2, 3)$.

To find the smallest odd $n = c^d f^g$ with c, d, f, g distinct integers > 1.

n must be odd, so c, f odd.

The smallest such n is $2025 = 5^2 \times 3^4$, with $c = 5, d = 2, f = 3, g = 4$.

All values $5, 2, 3, 4$ are distinct and > 1.

Is there a smaller odd n? Checking smaller values shows no others satisfy the conditions.

For 2025, the only factorization as two powers with distinct exponents and bases is $5^2 \times 3^4$.