Derivation of $(T_{T_m^2})^2$ Using Consecutive Triangular Sum Property

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1 Introduction

This document derives mathematical expressions for $(T_{T_m^2})^2$ using the fundamental property of triangular numbers:

$$T_n + T_{n+1} = (n+1)^2 \tag{1}$$

where $T_n = \frac{n(n+1)}{2}$ is the n-th triangular number.

2 The Consecutive Triangular Sum Property

The consecutive triangular sum property states that the sum of two consecutive triangular numbers equals the square of the larger subscript:

$$\begin{aligned}
T_n + T_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} && (2)\\
&= \frac{(n+1)}{2}[n + (n+2)] && (3)\\
&= \frac{(n+1)}{2}[2n+2] && (4)\\
&= \frac{(n+1)}{2} \cdot 2(n+1) && (5)\\
&= (n+1)^2 && (6)
\end{aligned}$$

3 Derivation of $(T_{T_m^2})^2$

We want to find an expression for $(T_{T_m^2})^2$ using the consecutive sum property.

3.1 Step 1: Express T_m and T_m^2

For any natural number m:

$$T_m = \frac{m(m+1)}{2} \tag{7}$$

$$T_m^2 = \left(\frac{m(m+1)}{2}\right)^2 = \frac{m^2(m+1)^2}{4} \tag{8}$$

3.2 Step 2: Calculate $T_{T_m^2}$

$$T_{T_m^2} = \frac{T_m^2(T_m^2+1)}{2} \tag{9}$$
$$= \frac{\frac{m^2(m+1)^2}{4}\left(\frac{m^2(m+1)^2}{4}+1\right)}{2} \tag{10}$$
$$= \frac{m^2(m+1)^2}{8}\left(\frac{m^2(m+1)^2}{4}+1\right) \tag{11}$$
$$= \frac{m^2(m+1)^2}{8}\cdot\frac{m^2(m+1)^2+4}{4} \tag{12}$$
$$= \frac{m^2(m+1)^2[m^2(m+1)^2+4]}{32} \tag{13}$$

3.3 Step 3: Apply Consecutive Sum Property

If $T_{T_m^2}$ is a perfect square, say $T_{T_m^2} = k^2$ for some k, then we can apply the consecutive sum property:

$$T_{T_m^2} = T_{k-1} + T_k = k^2 \tag{14}$$

Therefore:

$$(T_{T_m^2})^2 = (k^2)^2 = k^4 \tag{15}$$

3.4 Step 4: General Expression

For the general case, we have:

$$(T_{T_m^2})^2 = \left(\frac{m^2(m+1)^2[m^2(m+1)^2+4]}{32}\right)^2 \tag{16}$$

4 Specific Cases

4.1 Case 1: $m = 1$

$$T_1 = \frac{1\cdot 2}{2} = 1 \tag{17}$$
$$T_1^2 = 1 \tag{18}$$
$$T_{T_1^2} = T_1 = 1 = 1^2 \tag{19}$$

Since $T_1 = 1^2$, we can apply the consecutive sum property:

$$T_1 = T_0 + T_1 = 0 + 1 = 1 \tag{20}$$
$$(T_{T_1^2})^2 = (T_1)^2 = (1^2)^2 = 1^4 = 1 \tag{21}$$

4.2 Case 2: $m = 2$

$$T_2 = \frac{2\cdot 3}{2} = 3 \tag{22}$$
$$T_2^2 = 9 \tag{23}$$
$$T_{T_2^2} = T_9 = \frac{9\cdot 10}{2} = 45 \tag{24}$$

Since $T_9 = 45$ is not a perfect square, we cannot directly apply the consecutive sum property. The general formula gives:

$$(T_{T_2^2})^2 = (T_9)^2 = 45^2 = 2025 \tag{25}$$

5 Conclusion

The consecutive triangular sum property $T_n + T_{n+1} = (n+1)^2$ provides a powerful tool for deriving expressions involving triangular numbers. For $(T_{T_m^2})^2$:

- When $T_{T_m^2}$ is a perfect square, we can express $(T_{T_m^2})^2$ as k^4 where $k^2 = T_{T_m^2}$
- In the general case, we have the explicit formula involving $m^2(m+1)^2$
- The case $m = 1$ is special, allowing direct application of the consecutive sum property

This derivation demonstrates the deep connections between triangular numbers and perfect squares, revealing the elegant mathematical structure underlying these sequences.