Lab: Discrete Models: Fixed Points and Stability

1 A Few Basic Concepts

Through our examples of the Malthusian model and the discrete logistic model, we have witnessed population growth, decay and even stabilization, but how do you know if you can trust your simulations? There are a few mathematical techniques we will learn throughout the semester, which will tell us what to expect in the simulations. On a few occasions our techniques will even tell us that we should have no idea what to expect! One of the very basic techniques finds **fixed points**. These are values of our variable (in this case bacteria population) that will return to the exact same value in a finite number of time steps, but usually we will only consider the next time step.

2 Fixed Points

Let us consider the following experiment. We will start the petri dish (or chemostat) off with no bacteria. Assuming we have no outside contamination, what will the population size be in an hour? a day? a week? Zero! This is an example of a fixed point. If we start without any bacteria we will not have any bacteria at the next time step and so forth. This is usually called the "trivial fixed point" for obvious reasons (not to mention this is not a very exciting biological experiment either). You should also note, that this fixed point should be in both of our models, the Malthusian and Discrete Logistic.

How about a non-trivial fixed point? Well, let us think back to the Practice Problems from the previous lab. With the Malthusian model the population either exploded or decayed unless the growth rate was $\lambda = 1$. On the other hand, the Discrete Logistic model with either $r = 1$ or $r = 0.5$ resulted in a steady population size of 100 (or K). This was even true when we started with an initial population larger than the carrying capacity. What would expect will happen to the simulation if you started with no bacteria? What about the Malthusian model with an initial bacterial population of zero?

2.1 Malthusian Model Fixed Point

Mathematically, it appears the Malthusian model has only one fixed point (unless λ exactly equals one) and the Discrete Logistic model has two fixed points. We should learn how to find these mathematically. If we want a variable value to return to itself in one time step then we really want:

$$B_t = B_{t+1} = B_{t+2} = B_{t+3} = ...$$

What we really want to do is solve the Malthusian model:

$$B_{t+1} = \lambda B_t$$

when we have $B_t = B_{t+1}$, or

$$B_t = \lambda B_t$$

Which we can simplify by dropping the subscripts:

$$B = \lambda B$$

Now to use the MATLAB solvers, we must set one side of this equation to zero or:

$$B - \lambda B = 0$$

You may be familiar with solving these this type of equation or you may have noticed that the only solution to this equation occurs when $B = 0$. We should check this using MATLAB:

```
>> clear B lambda
>> syms B lambda
>> solve('B-lambda*B', 'B')
```

Again we see the only fixed point is when $B = 0$. Biologically this means that the only fixed point (assuming λ does NOT equal zero) for the Malthusian model is when there are no bacteria. When we have bacteria the population will not attain a fixed point, which implies the population will either grow infinitely or decline to zero (the fixed point) and remain there. Since this is not an interesting experiment, though it would be fairly straight forward to count no bacteria, let us move onto the logistic model.

2.2 Discrete Logistic Model Fixed Points

Using the same method as in the previous section, we find the fixed point for the discrete logistic model by dropping the subscripts on the variable, B and solving for it.

$$B = r(1 - \frac{B}{K})B + B.$$

When we set this equation equal to zero on one side we have:

$$0 = r(1 - \frac{B}{K})B + B - B,$$

and combining terns gives

$$0 = r(1 - \frac{B}{K})B.$$

If you have seen this type of equation before you know that either $B = 0$ or

$$r(1 - \frac{B}{K}) = 0,$$

which has the solution $B = K$. Using MATLAB to check this answer we would need to type

```
>> clear B r K
>> syms B r K
>> solve('r*(1-B/K)*B', 'B')
```

As you have noticed this gives us two different fixed points for this model. This confirms what we discovered through our simulations, though this is a more mathematically rigorous method for finding fixed points.

2.3 Fixed Points In General

We found out how to find the fixed point or fixed points for the Malthusian and Discrete Logistic Models. We can generalize this for discrete equations. To find the fixed point(s) for a discrete model you

1. Drop the time subscripts off of the variables.

2. Solve for the variables.

When we use MATLAB to help, be sure and set one side of the equation to zero! Please remember there may be more than one fixed point and in some situations there will not be any fixed point, though usually the trivial fixed point is present in biological models. There is also another situation you should be aware of with biological models. Mathematically the models are valid for positive and negative populations, but biologically negative populations do not make sense. If you should come across a model that had a negative fixed point, it is not considered to be a biological fixed point. For example, let us assume we mathematically (through MATLAB or by hand) found three fixed points: the trivial point, a positive point and a negative point. The only biologically relevant points would be the trivial point and the positive fixed point. We would exclude the negative point from consideration, unless we needed to look at the model mathematically instead of biologically.

3 Stability of Fixed Points

Now that we have a basic understanding of a fixed point, we can begin to ask some important questions. For instance, let us assume that the population is at a fixed point what happens if I perturb the population slightly? Add one more bacterium? Kill off one bacterium? What will happen to the population then? These question can be answered by understanding how small perturbations effect the fixed points. We say small perturbations, because large perturbations can not easily be dealt with mathematically.

If a small perturbation does nothing and the system returns to the same fixed point it started at, we say that fixed point is **stable**. This implies that small perturbations will not effect the long term behavior. However, if a small perturbation cause the system NOT to return to the fixed point it started at, the fixed point is classified as **unstable**. This implies that small perturbations will change the dynamics (long-term behavior) of the system. A fixed point may be classified as **semistable**. This is where a small perturbation which increases the population is stable and a small perturbation which decreases the population is unstable or vice versa.

When we consider small perturbations we use the derivatives you learned in calculus to understand how the system will respond. You may find that you are familiar enough with taking derivative to work through the next section by hand, however MATLAB has the ability to find derivatives for us! The command to take derivative is **diff**. Let us practice a straight forward derivative and evaluation of these derivatives.
The derivative of $f(x) = x^2$ with respect to x can be denoted as $f'(x) = 2x$. If we wanted to evaluate the derivative at $x = 3$ we would find $f'(3) = 2 * 3$ or $f'(3) = 6$. Also if we wanted to then evaluate the derivative at $x = 4$, we would have $f'(4) = 2 * 4$ which simplifies to $f'(x) = 8$. Here is the same problem and solution using MATLAB with the derivative of function f denoted as df:

```
>> clear f x df
>> syms f x df
>> f = x∧2
>> df = diff('x∧2', 'x')
>> x = 3;
>> eval(df)
>> x = 4;
>> eval(df)
```

3.1 Stability for General Discrete Models of One Variable

If we have a discrete model:

$$P_{t+1} = f(P_t)$$

and we know that there is a fixed point $P = a$ then the stability of that fixed point can be reduced to the following three cases:

- If the absolute value of the derivative with respect to P_t and evaluated at $P_t = a$ is LESS THAN ONE, then the fixed point is **stable**.
- If the absolute value of the derivative with respect to P_t and evaluated at $P_t = a$ is GREATER THAN ONE, then the fixed point is **unstable**
- If the absolute value of the derivative with respect to P_t and evaluated at $P_t = a$ is EXACTLY EQUAL TO ONE, then we can not tell if the fixed point is stable or not with this particular method.

One quick note, when finding the derivative of the right hand side with respect to P_t, it is usually more clear to drop the subscripts and just find the derivative of $f(P)$ with respect to P and evaluating the derivative at $P = a$.
In the next two sections, we will look at the stability of the Malthusian and Discrete Logistic Models.

3.2 Stability of the Malthusian Model

At the beginning of this lab, we found the only fixed point of the Malthusian Model:

$$B_{t+1} = \lambda B_t,$$

is the trivial state, $B_t = 0$. We now know how to find the stability of this point. Before we do this, what do you think will happen if we perturb the fixed point by adding a few bacteria? Notice, in this case we can not perturb the fixed point by killing a few bacteria due to the fact that we can not have negative bacteria.

For this model, the function we need to take the derivative of is the right hand side of the equation. To avoid confusion, especially with MATLAB, we should drop the subscripts. Now we want to find the derivative of:

$$\lambda B,$$

with respect to B. Then we want to evaluate that result at $B = 0$ and find if the absolute value is greater than, less then or equal to one. In the code below, the function will be labeled f and the derivative will be labeled *df*

```
>> clear f B lambda df
>> syms f B lambda df
>> f = lambda*B
>> df = diff('lambda*B', 'B')
>> B = 0;
>> eval(df)
```

We have found the derivative is λ and the derivative evaluated at $B = 0$ is still λ. Now what is the absolute value of λ? Well that depends on λ. Think back to the simulations from the previous lab. What happened when $\lambda < 1$? Our stability analysis tells us the $B = 0$ fixed point should be **stable** and that our population should decline back to nothing. When we changed the simulation and ran the population with $\lambda > 1$ what happened? This time our stability analysis concludes that the $B = 0$ fixed point is **unstable** and any small perturbation will repel us from the $B = 0$ state. In the simulation we noticed unbounded bacteria growth. What happens if $\lambda = 1$, well we can not conclude anything using this method. We will use this same method to learn more about the stability of the Discrete Logistic Model.

3.3 Stability of the Discrete Logistic Model

With the Discrete Logistic Model,

$$B_{t+1} = r(1 - \frac{B_t}{K})B_t + B_t,$$

we found two fixed points, the trivial state at $B = 0$ and a nontrivial fixed point at $B = K$. Using the same method as before, we can find the stability of both these points, by just repeating the process twice.

For this model, the function we need to take the derivative of is the right hand side of the equation. To avoid confusion, especially with MATLAB, we should drop the subscripts. Now we want to find the derivative of:

$$r(1-\frac{B}{K})B+B,$$

with respect to B. Then we want to evaluate that result at both fixed points. We will start with the stability of $B = 0$ and then find the stability of $B = K$. We will use a similar notation to the previous example, with the function labeled f and the derivative labeled df

```
>> clear f B r K df
>> syms f B r K df
>> f = r*(1-B/K)*B+B
>> df = diff('r*(1-B/K)*B+B', 'B')
>> B = 0;
>> eval(df)
>> B = K;
>> eval(df)
```

When we consider the derivative evaluated at the trivial fixed point, $B = 0$ we get $r+1$. Now looking at the absolute value of this, we notice that since r is always greater than zero, our result will always be greater than one. This means the trivial fixed point is always **unstable**. (Remember, r can be thought of as a growth rate and it is defined to be a positive value between zero and one.)
When we work with the other fixed point, $B = K$ or biologically when the population is at carrying capacity, we find the derivative evaluated gives $-r+1$. Now some interesting things occur. Looking at the absolute value of this, we see that if $0 < r \leq 2$, this fixed point is **stable**. However, if $r > 2$ then this fixed point becomes **unstable**. Does this agree with the simulations from your previous lab? You now have enough mathematical information and MATLAB experience to look at and analyze one-dimensional discrete difference equations!

4 Practice Problems

1. Find the derivative of the following functions and then evaluate the function at the specified value
   (a) $f(x) = 3x^4 - 2x^2 + 10$; evaluate the derivative at $x = -1$ and $x = 2$
   (b) $g(y) = \sqrt{y+4}$; evaluate the derivative at $y = 0$ and $y = 1$
   (c) $f(x) = \frac{ax}{x+c}$; evaluate the derivative at $x = 0$ and $x = 1$
2. Find all of the fixed points for the following model and then tell under what conditions is each point biologically relevant. (Hint: biologically relevant fixed points must be either zero or positive. What are the restrictions on the parameters so that is true?)

$$X_{t+1} = \frac{aX_t}{1+X_t}$$

3. Find the stability of both the fixed points from the previous problem. For the trivial fixed point, what conditions make this point stable or unstable. For the nontrivial fixed point, remember that the parameter a has some restrictions in order to make this point biologically relevant.

4. Using a model from the book *Mathematical Models in Biology* by Leah Edelstein-Keshet:

$$X_{t+1} = -X_t^2(1 - X_t),$$

answer the following questions.

   (a) Find all three fixed points.

   (b) Which of these points (if any) are biologically relevant.

   (c) Find the stability of each of the biologically relevant fixed points.

   (d) Simulate this model starting with the following different initial conditions:

      i. initial X value $= 0$

      ii. initial X value $= 0.1$

      iii. initial X value $= 1$

      iv. initial X value $= 4$

      v. initial X value $= \frac{1}{2}\sqrt{5} + \frac{1}{2}$

      vi. initial X value $= 5$

   (e) Do your simulation results match your stability analysis? Why or Why not.