Lab: Introduction to Discrete Models

1 Introduction to Biomathematics

Over the past weeks we have learned how to use MATLAB as a tool for exploring mathematics and even learned some basic programming and programming logic. Now it is time to use these tools to learn more about the use of mathematics in biology. The main goal of this course is to teach you how to use MATLAB and become comfortable using this program. Another goal in this course is for you to gain respect for how mathematical models can help scientists while learning about the limitations of these models and their results. In order to proceed with this we will introduce you to various types of models with different biological applications. Most of this lesson can be found in the first chapter of *Mathematical Models in Biology: An Introduction* by Elizabeth S. Allman and John A. Rhodes.

1.1 Bacteria Growth

Let us begin by assuming that the bacterium divides every hour and that we start with a single in a petri dish at 9 a.m. How many bacteria will there be at 10:00 a.m.? How about noon? You may have seen this before and know that you can find a mathematical model of this system by looking at a table.

Time	Time since experiment began	Number of bacteria	Model
9:00 a.m.	0 hours	1	1×2^0
10:00 a.m.	1 hours	2	1×2^1
11:00 a.m.	2 hours	4	1×2^2
12:00 p.m.	3 hours	8	1×2^3
1:00 p.m.	4 hours	16	1×2^4
2:00 p.m.	5 hours	32	1×2^5
...	$\mathbf{t}$ hours	...	$1 \times 2^{\mathbf{t}}$

The model can be written mathematically by allowing the number of bacteria at time $\mathbf{t}$ to be $B(\mathbf{t})$ and letting B_0 be the initial bacteria population at the beginning of the experiment (Note: In the table, $B_0 = 1$). Then this model becomes

$$B(t) = B_0 \times 2^t$$

where the variable t is the time since the experiment began.

1.1.1 Discrete Time

You may know about some of the limitations of this model, such as the bacteria will continue to grow and double forever, eventually taking over the world! We will address this issue in the next section. There are other limitations not necessarily of this model, but of the questions you can ask the model, that you may not have thought about before. For instance, how many bacteria are there at 11:30 a.m.? How about 11:51? Well, these questions do not make sense in this situation. How can you have decimal answers to whole organisms. There are many answers to this dilemma. One answer is to restrict time to only be full hours. This is known as discretizing time. Now our time will only have full (whole) hours and nothing in between. The notation for the model will change a bit with B_t being the bacteria population at time **t** and B_{t+1} will be the bacteria population an hour later. Here is the formula for the first bacterial division.

$$B_1 = 2 \times B_0$$

Notice, the bacteria population at the next time step is completely dependent on the bacteria population at the previous time. And we would have to step through this equation each time to figure out the population size (well, we could actualyy solve this one and write a general formula for B_{t+1} in terms of t but NOT in terms of B_t. Can you do this?).

Time	Time since experiment began	Number of bacteria	Formula
9:00 a.m.	0 hours	1	B_0
10:00 a.m.	1 hours	2	$B_1 = 2 \times B_0$
11:00 a.m.	2 hours	4	$B_2 = 2 \times B_1$
12:00 p.m.	3 hours	8	$B_3 = 2 \times B_2$
1:00 p.m.	4 hours	16	$B_4 = 2 \times B_3$
2:00 p.m.	5 hours	32	$B_5 = 2 \times B_4$
...	t+1 hours	...	$B_{t+1} = 2 \times B_t$

To look at the population size at the next step $(t+1)$ we need to know how many bacteria are at the time step (t). This will let us figure out the population by

$$\begin{aligned}
\text{Population}_{t+1} &= \text{Population}_t + \text{New Additions} \\
&= \text{Population}_t + (\text{Birth/Death rate}) \times \text{Population}_t \\
&= (1 + (\text{Birth/Death rate})) \times \text{Population}_t \\
&= \lambda \times \text{Population}_t
\end{aligned}$$

This model is know in its more general form as the Malthusian Model and is usually written as:

$$P_{t+1} = \lambda P_t$$

In the previous example our growth doubled at each time step implying $\lambda = 2$, but in general λ can be an number that relates to the growth rate. If the value for λ is greater than one the population will grow and if the value of λ is less than one the population will decline. What do you think should happen when $\lambda = 1$?

1.1.2 Carrying Capacity

In reality, if you placed bacteria in a petri dish and let it go, the population would eventually become stable, often labeled stationary state in biology, before the nutrients ran out and the population crashed. If the scientist ran this experiment in a chemostat, where nutrients could be regularly added the population size would become stable without declining. The chemostat experiment is much more straight forward to explore mathematically than the petri dish experiment. For now, let us assume our bacteria is growing and thriving in a chemostat. We know the bacteria population size will reach a stationary state and remain there as long as we remember to replenish the nutrients.

Mathematically, when the population is low and far below the carrying capacity, the overall growth is essentially r (or λ from before). As the population grows and becomes closer to the carrying capacity the overall growth slows until it reaches zero when the population reaches the carrying capacity. If the population exceeds the carrying capacity threshold, the overall growth rate will become negative until the population has decreased below or at the threshold.

This type of model is traditionally known as the discrete logistic equation and is usually written as:

$$B_{t+1} = r(1 - \frac{B_t}{K})B_t + B_t,$$

where $0 < r$ is the growth rate uninhibited by the carrying capacity and K is the carrying capacity for the population. We will use MATLAB in the Practice Problems to graph this and will gain some basic understanding of these population models in the next section.

2 MATLAB and Discrete Population Models

We will need to use our knowledge of programming logic and the for loop in order to proceed. Let us start by programming the Malthusian Model. Here is an example MATLAB code that will plot the bacteria population we have been working with that doubles every hour. Be sure to save this file as "malthusian1.m".

```
% Your Name
% Today's Day
% Course Number
% Malthusian Model: From lecture notes

function [] = malthusian1()
```

```
% Number of time steps (hours)
n = 24;
nn = n+1;
% This is to store our initial population plus the ones we find

% Parameter and Initial Conditions
lambda = 2; %growth rate
intBac = 1; % Initial bacteria population
inthours = 0; % Initial time (hours)

%Solution Population and Time
Bac = zeros(1,nn);
hours = zeros(1,nn);

%Set inital conditions
Bac(1) = intBac;
hours(1) = inthours;

% Equation
% Note: the "ii" index is for the previous time (t) and the "jj" is for the next time (t+1)
for ii=1:1:n,
    jj = ii+1;
    Bac(jj) = lambda*Bac(ii); % Malthusian Equation
    hours(jj) = ii; % Keeping track of the time (hours)
end

% Plot
figure(1)
clf
hold on
set(gca,'fontsize',20)
plot(hours, Bac,'g-','linewidth',3)
xlabel('Time (hours)')
ylabel('Population')
title('Malthusian Model with \lambda = 2')
hold off
```

Use this model to answer the practice problems.

3 Practice Problems

1. Use the Malthusian model to plot the following scenarios for a time period of one day:
   (a) Initial bacteria population of 1 and $\lambda = 1.5$
   (b) Initial bacteria population of 1 and $\lambda = 1$
   (c) Initial bacteria population of 1 and $\lambda = 0.5$

   Did anything interesting happen? Is this what you expected?

2. What happens in the Malthusian model with an initial bacteria population of 1 and $\lambda = 1.5$ over the course of 10 days? What about $\lambda = 1$? Or $\lambda = 0.5$?

3. Write a code for the discrete logistic equation, by altering the maltusain1.m file. Label this file "discretelogistic1.m".

4. Using your discrete logistic code plot the following scenarios for the time period of one day:
   (a) Initial bacteria population 1, $K = 100$, and $r = 2.5$
   (b) Initial bacteria population 1, $K = 100$, and $r = 2$
   (c) Initial bacteria population 1, $K = 100$, and $r = 1.5$

   Did anything interesting happen? Was this what you expected?

5. What happens in the discrete logistic model with an initial bacteria population of 1, $K = 100$ and $r = 2.5$ over the course of 10 days? What about $r = 2$? Or $r = 1.5$?

6. Compare your answers from problems 1 and 4. Are the growth rates λ and r the same, different, or similar under specific conditions?

7. Compare your answers from problems 2 and 5. How is the end behavior different for these two models? What do you think would happen to each model if we ran the simulation for 100 days? 1,000 days?

8. Just for fun, run the Discrete Logistic model under the following conditions:
   (a) Initial bacteria population of 100, $K = 100$ and $r = 1.5$
   (b) Initial bacteria population of 150, $K = 100$ and $r = 1.5$

   Is this what you thought would happen? Explain your results.