Perron's Theorem

If $\mathbf{A}_{n\times n} > \mathbf{0}$ with $r = \rho(\mathbf{A})$, then the following statements are true.

- $r > 0.$ (8.2.14)
- $r \in \sigma(\mathbf{A})$ (r is called the ***Perron root***). (8.2.15)
- $alg\ mult_{\mathbf{A}}(r) = 1.$ (8.2.16)
- There exists an eigenvector $\mathbf{x} > \mathbf{0}$ such that $\mathbf{Ax} = r\mathbf{x}$. (8.2.17)
- The ***Perron vector*** is the unique vector defined by

$$\mathbf{Ap} = r\mathbf{p}, \qquad \mathbf{p} > \mathbf{0}, \quad \text{and} \quad \|\mathbf{p}\|_1 = 1,$$

  and, except for positive multiples of $\mathbf{p}$, there are no other nonnegative eigenvectors for $\mathbf{A}$, regardless of the eigenvalue.

- r is the only eigenvalue on the spectral circle of $\mathbf{A}$. (8.2.18)
- ***The Collatz–Wielandt formula*** says $r = \max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x})$, where

$$f(\mathbf{x}) = \min_{\substack{1\le i\le n \\ x_i \neq 0}} \frac{[\mathbf{Ax}]_i}{x_i} \quad \text{and} \quad \mathcal{N} = \{\mathbf{x} \,|\, \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}.$$