Lab: Introduction to Continuous Models

1 Introduction to Continuous Models

You have been working with discrete models in the previous labs. For these models time marches on in specific increments and the species involved reproduce and mature on a time schedule. This type of modeling works well for plants, insects and even some mammals such as bears that hibernate in the winter and have a specific breeding period. What about other species? For instance do humans have a definite breeding period and specific times of the year that everyone moves from a child to an adult? Of course not. In this situation discrete models would not be appropriate. We need to use a different type of model known as a continuous model. In continuous models, time is not broken up into intervals, but is instead continuous (hence the name)! Occasionally, we will revisit some of our earlier discrete models to see how they compare with the continuous ones.

1.1 Bacteria Growth

In our first lab about discrete models we looked at bacterial growth. In that case we had a bacteria population that doubled every hour, and we derived the following model before we even talked about discrete time steps:

$$B(t) = B_0 \times 2^t$$

where the variable t is the number of time intervals since the experiment began. As you may have seen in other biology or math courses a similar model in continuous time can be written as:

$$B(t) = B_0 e^{rt}$$

where the variable t is the time since the experiment began. This model is know as exponential growth or decay depending on whether the parameter r is positive or negative. When $r > 0$ this population will grow and when $r < 0$ this population will decline or decay. For simplicity of notation, the exponential model is often expressed as:

$$B = B_0 e^{rt}$$

where B is assumed to be a function of time. Now if our bacteria divides every two hours the growth rate will be $r = \frac{\ln 2}{2}$ or approximately $r = 0.3466$. Why is that? (You are asked to figure this out in the practice problems!)

1.1.1 Continuous Time

Biologist often set up experiments to see how a population (or protein expression, genetic expression, ...) is changing over time without having an equation to work with. Instead they are able to look at the way the species is changing and develop a differential equation. In fact we are presenting this concept to you backwards, by giving you the exponential model that measures the bacteria population as a function of time, not how it is actually changing over time. However, we can figure out the differential equation for this. In order to figure out how the population is changing in time we need to use some calculus. Let's take the derivative of both sides of the exponential model with respect to time:

$$\frac{d}{dt}[B] = \frac{d}{dt}[B_0 e^{rt}]$$

which can be reduced to

$$\begin{aligned} \frac{dB}{dt} &= B_0 r e^{rt} \\ &= r * (B_0 e^{rt}) \end{aligned}$$

but we already know that $B_0 e^{rt} = B$. This reduces the previous equation into:

$$\frac{dB}{dt} = rB$$

This is how continuous models are usually written and this is the differential form of the exponential equation which we will now be working with. Another name for this particular model is an "Ordinary Differential Equations " or ODE for short. The good news is that MATLAB has many built in functions to work with ODEs but the down side is that we have to be very careful with how we use these MATLAB functions. As before, we have some of the same biological concerns with this model as we did with the discrete model.

1.1.2 Carrying Capacity

Previously we talked about a carrying capacity, or stationary phase, of bacteria growing in a chemostat. We have the same concept with continuous models, though it may look a little bit different:

$$\frac{dB}{dt} = r(1 - \frac{B}{K})B$$

This model is known as the logistic equation where once again $r > 0$ is the growth rate uninhibited by the carrying capacity and K is the carrying capacity for the population. How does this model differ from the discrete model? Why do we not need a "$+B$" term at the end of the continuous model, like we needed with the discrete model? Let us look at this using MATLAB and see if we can better understand this model.

2 MATLAB and Discrete Population Models

Now that you have familiarized yourself with programing in MATLAB, we need to show you some new tricks for simulating continuous models. As you may have guessed, the code for simulating a discrete model is a bit more straight forward than the code for the simulation of a continuous model. For instance, with continuous models we will need to nest a program inside of another one. This creates a few problems we will need to work around. We can work around the first problem by using a command known as **global**. When you label a parameter as global, the computer will be able to move the parameter and more importantly its numerical value between programs. This will save us from having to change our parameters in a bunch of places later.

The second problem (and the last one we will address in this lab session) is best described as a change of variables. The built in MATLAB function we will be using is known as **ode45**. In order to use this command we have to have our initial conditions set up as a matrix and will have to use the variable name $y(1)$ for our variable when we code in the equations in the form of a matrix as well! This will change our function from:

$$\frac{dB}{dt} = r(1 - \frac{B}{K})B$$

to

$$\frac{dy(1)}{dt} = r(1 - \frac{y(1)}{K})y(1)$$

We will deal with turning these into matrices in the code itself.

Here is an example MATLAB code that will plot the bacteria population with logistic growth we have been working with. The parameters for this model assume the bacteria divides every two hours and that we start with a single bacterium. We have also set the carrying capacity to 100 for the logistic model. Be sure to save this file as "logistic1.m".

```
% Your Name
% Today's Day
% Course Number
% Logistic Model 1: From lecture notes

function [] = logistic1()

clc

% Number of time iterations (in hours)
n = 50;

% Parameter and Initial Conditions
global r ; % growth rate
```

```
global k; % carrying capacity
r = log(2)/2;
k = 100;
intBac = 1; % Initial bacteria population

% Solution Population
Bac = zeros(1,n+1);

% Set initial conditions and place in the proper form
intcond = [intBac];

% Simulation
% Note: this is the built in MATLAB function known as ode45
[time, Ans] = ode45(@systemlogistic, 0:n, intcond);
% Pulling out the solutions from the MATLAB function
for i=1:(n+1),
    Bac(i) = Ans(i);
end

% Plot
figure(1)
clf
hold on
set(gca,'fontsize',20)
plot(time, Bac,'g-','linewidth',3)
xlabel('Time (hours)')
ylabel('Population')
title(['Continuous Logistic Model with r = ', num2str(r)])
hold off

%%% Differential Equation (Model)

function dy = systemlogistic(t,y)
% Parameters
global r;
global k;

% Note: The model is entered as a matrix using y(1) in place of B.

dy = [r*(1-(y(1)/k))*y(1)];

% Model
% dB/dt = r(1-B/k)B
```

This is the basic code for simulating continuous models. Now let us look at a few practice problems.

3 Practice Problems

1. When we use the exponential model $B = B_0 e^{rt}$ why is our growth rate $r = \frac{\ln 2}{2}$ for our bacteria population that is doubling every two hours?

2. Use the Logistic model to plot the following scenarios for a time period of one day:

   (a) Initial bacteria population of 1, $K = 100$ and $r = 1$

   (b) Initial bacteria population of 1, $K = 100$ and $r = 0.5$

   (c) Initial bacteria population of 1, $K = 100$ and $r = 0.05$

   Did anything interesting happen? Is this what you expected?

3. What happens in the Logistic model with an initial bacteria population of 1 and $r = 1$ over the course of 10 days? What about $r = 0.5$? Or $r = 0.05$?

4. Write a code for the exponential differential equation $\frac{dB}{dt} = rB$, by altering the logistic1.m file. Label this file "exponentialdiff1.m".

5. Using your exponential differential equation code plot the following scenarios for the time period of one day:

   (a) Initial bacteria population 1 and $r = 1$

   (b) Initial bacteria population 1 and $r = 0.5$

   (c) Initial bacteria population 1 and $r = 0.05$

   Did anything interesting happen? Was this what you expected?

6. What happens in the exponential differential equation model with an initial bacteria population of 1 and $r = 1$ over the course of 10 days? What about $r = 0.5$? Or $r = 0.05$?

7. Compare your answers from problems 3 and 6. How is the end behavior different for these two models? What would you think would happen to each model if we ran the simulation for 100 days? 1,000 days?

8. Just for fun, run the Logistic model under the following conditions:

   (a) Initial bacteria population of 100, $K = 100$ and $r = 0.5$

   (b) Initial bacteria population of 150, $K = 100$ and $r = 0.5$

   Is this what you thought would happen? Explain your results.