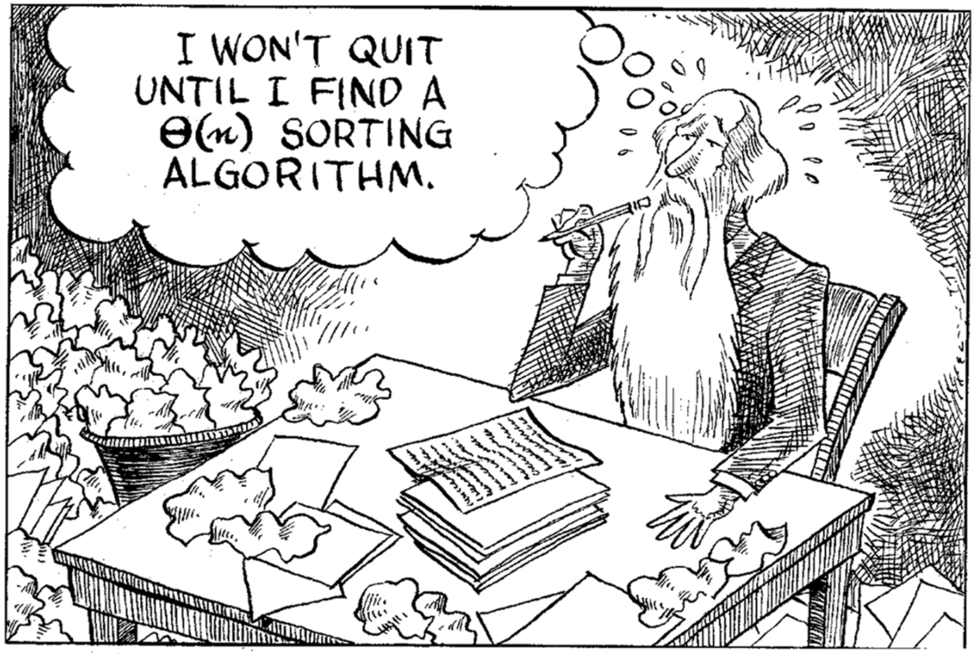
Introduction to Computational Complexity: The Sorting Problem

Computational Complexity



- Can we find a new sorting algorithm better than $O(n\log n)$?

Computational Complexity

- Computational complexity
  - A field of computer science studying a problem itself, not developing efficient algorithms solving the problem.
    - Prove that some problems cannot be solved by computers : halting problem
    - Prove a lower bound of problems
  - A computational complexity analysis tries to determine a ***lower bound*** on the efficiency of all algorithms for a given problem.

Computational Complexity

- For example, the lower bound of sorting problem is $\Omega(n\log n)$.
- This implies that it is ***impossible*** to develop an algorithm better than $O(n\log n)$.
- Therefore, merge-sort, quick-sort algorithms are the best algorithms solving the sort problem.

Computational Complexity

- Example: Matrix Multiplication Problem
  - How fast can we multiply two matrices of size $n \times n$?

- Design an efficient algorithm: basic operation is multiplication of two numbers
  - $O(n^3)$: simple
  - $O(n^{2.81})$: Strassen [1969]
  - $O(n^{2.38})$: Coppersmith, Winograd [1969]

- Develop a lower bound of this problem
  - $\Omega(n^2)$: easy

Computational Complexity

- What do we do next?
  - Fill the gap between the lower bound $\Omega(n^2)$ and the best upper bound $O(n^{2.38})$.
    - Develop a new algorithm better than $O(n^{2.38})$.
    - Prove a new lower bound of the problem better than $\Omega(n^2)$, for example $\Omega(n^{2.3456789})$.

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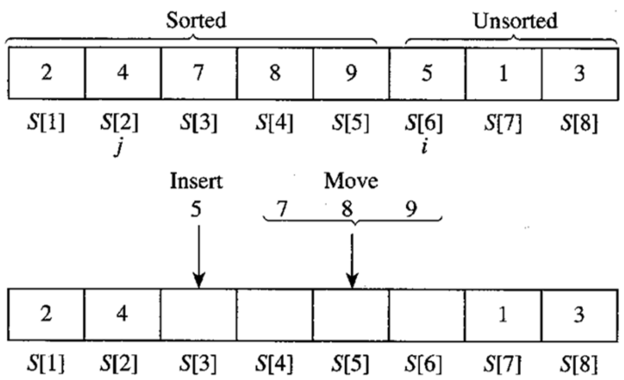
Insertion Sort

- Insertion sorting




```
void insertionsort (int n, keytype S[ ])
{
   index i, j;
   keytype x;

   for (i = 2; i <= n; i++) {
      x = S[i];
      j = i − 1;
      while (j > 0 && S[j] > x) {
         S[j + 1] = S[j];
         j− −;
      }
      S[j + 1] = x;
   }
}
```

Insertion Sort

- Analysis
  - Worst-case Time complexity Analysis of ***Number of Comparisons***
    - Basic operation : the comparison of *S[j]* with *x*.
    - Input size : *n*, the number of keys to be sorted

$$T(n) = \sum_{i=2}^{n} (i-1) = \frac{n(n-1)}{2}$$

Selection Sort

❑ Selection sorting

```
void selectionsort (int n, keytype S[ ])
{
   index i, j, smallest;

   for (i = 1; i <= n - 1; i++) {
      smallest = i;
      for (j = i + 1; j <= n; j++)
         if (S[j] < S[smallest])
            smallest = j;
      exchange S[i] and S[smallest];
   }
}
```

- Selection sorting is an $O(n\log n)$ algorithm.

Heapsort (Binary Heap)

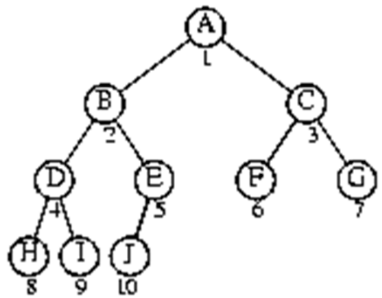
- Categories
  - A Dictionary:
    - Basic Operations
      - Insert
      - Delete
      - Search
  - Data Structures for Dictionary
    - Binary Search Tree,
    - Red-Black Tree,
    - Splay Tree, etc

Heapsort (Binary Heap)

- A Priority Queue
  - Basic Operations
    - Insert
    - Delete Min (or Delete Max)
- Data Structures for Priority Queue
  - The Binary Heap

Complete Binary Tree

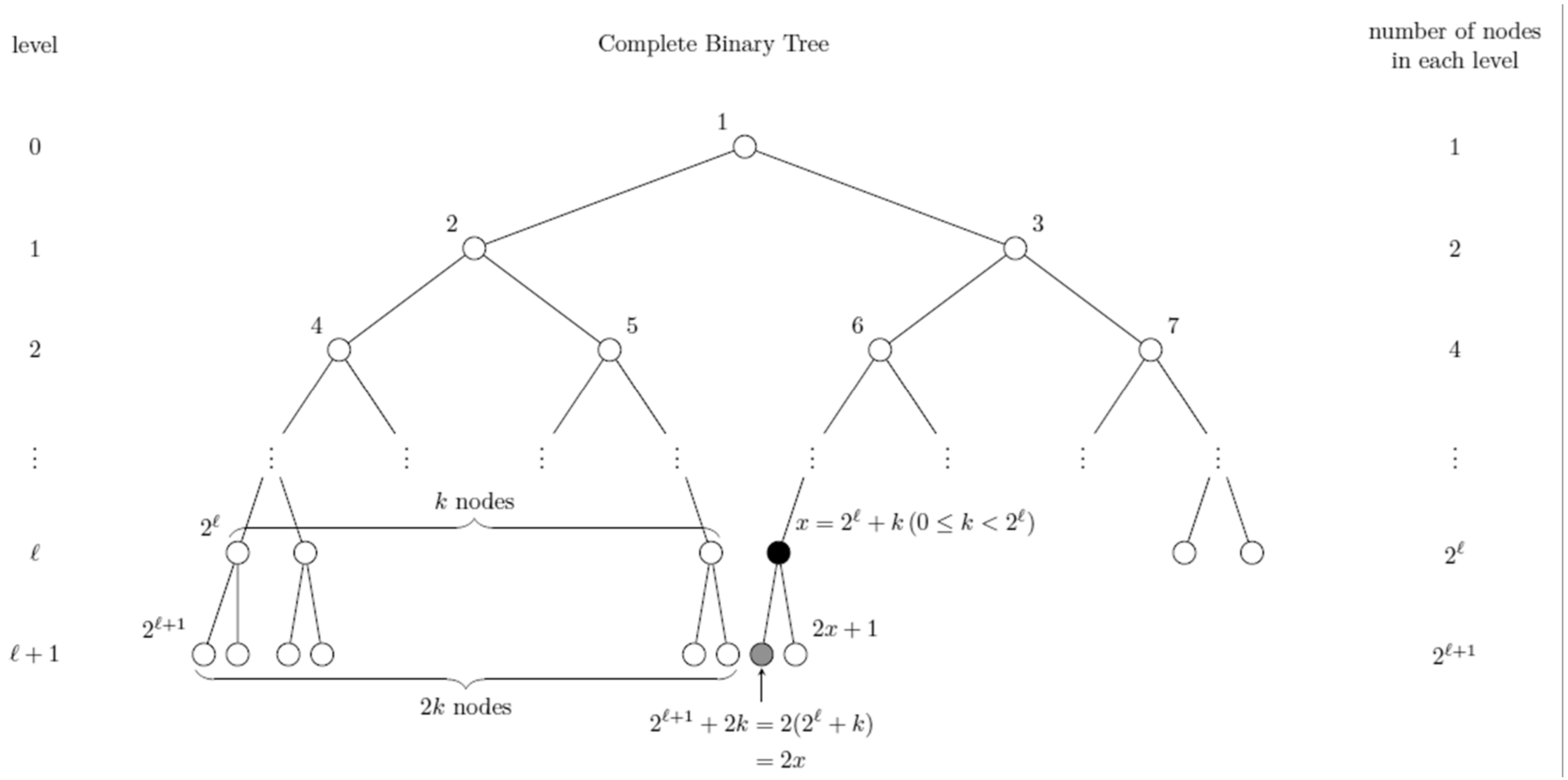
- The complete binary tree
  - A tree that is completely filled (called full binary tree), with the possible exception of the bottom level, which is filled from left to right.
  - An array of size *N* can represent a complete binary tree with *N* elements.



Complete Binary Tree

- Lemma:
  - The height (longest path length from the root) of a complete binary tree is $\lfloor \log N \rfloor$.
  - A complete binary tree of height H has between 2^H and 2^{H+1}-1 nodes.
  - In an array representation of a complete binary tree, for a node of position *k*,
    - the parent is in position $\lfloor k/2 \rfloor$.
    - the left child is in *2k*
    - the right child is in *2k+1*

Complete Binary Tree



Complete Binary Tree

The number of nodes at level k : 2^k

The height of a complete binary tree : max level of the tree

The total number N of nodes of *full binary tree* of height H

$$N = 2^0 + 2^1 + \cdots + 2^H$$
$$= 2^H - 1$$

Then total number N of nodes of *complete binary tree* of height H

$$2^H \leq N < 2^{H+1}$$

The height of a *complete binary tree* of N nodes

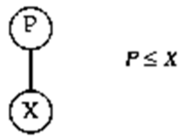
$$2^H \leq N < 2^{H+1}$$

$\Rightarrow H \leq \lg N < H + 1$

Therefore, $H = \lfloor \lg N \rfloor$

Binary Heap

- The binary heap
  - The binary heap has the following properties:
    - It is a complete binary tree
    - (**heap order property**) In a heap, for every node X with parent P, the key in P is smaller than or equal to the key in X.

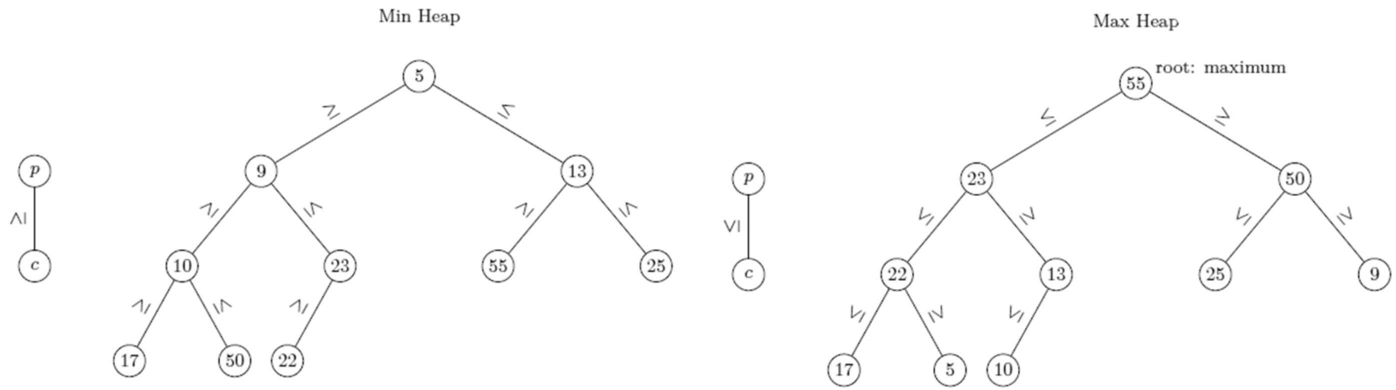


Heap order property

  - In this case, the heap is called a *min heap*.
  - *Max heaps* have the heap order property in the other way.

Binary Heap

- The binary heap



Binary Heap

- Example:

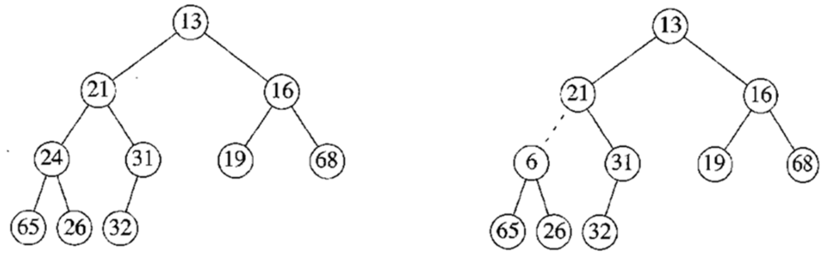


Figure 20.3 Two complete trees (only the left tree is a heap)

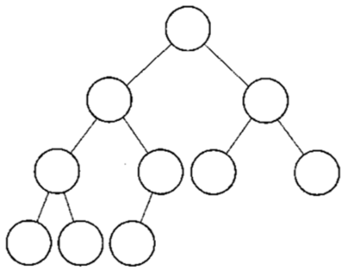


Figure 7.4 An essentially complete binary tree.

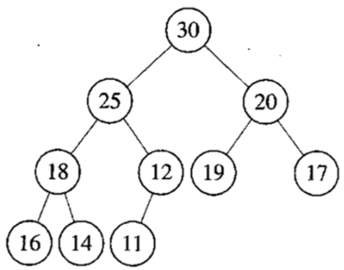
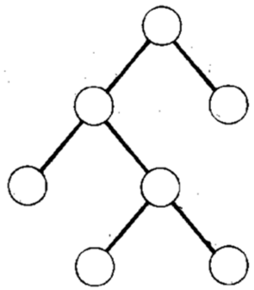


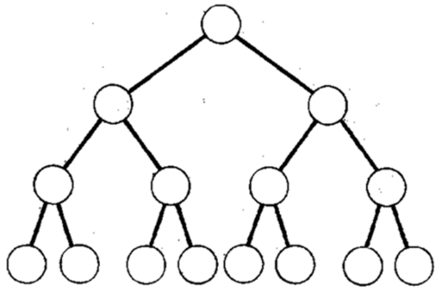
Figure 7.5 A heap.

Binary Heap

- Example:

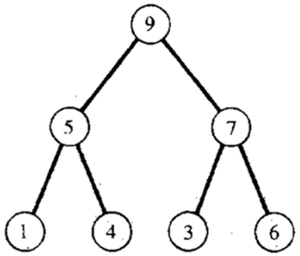


A 2-tree

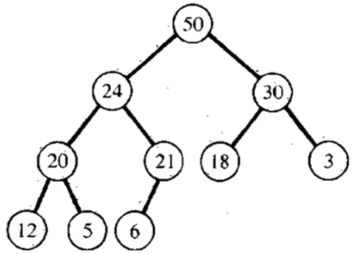


A complete binary tree

a full binary tree



Heap 1



Heap 2

Binary Heap

- Allowed operations in heap
  - Insert an element in a heap

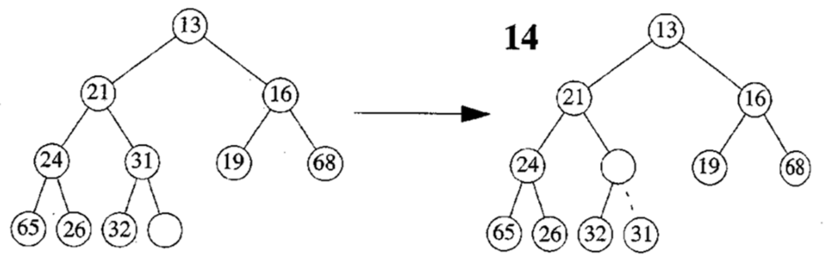


Figure 20.9 Attempt to insert 14, creating the hole and bubbling the hole up

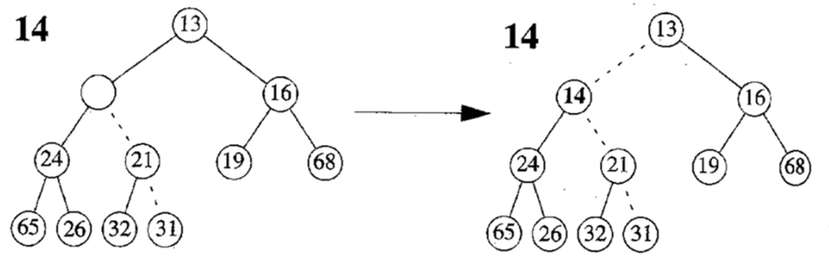


Figure 20.10 The remaining two steps to insert 14 in previous heap

Binary Heap

- Delete a minimum element from a heap

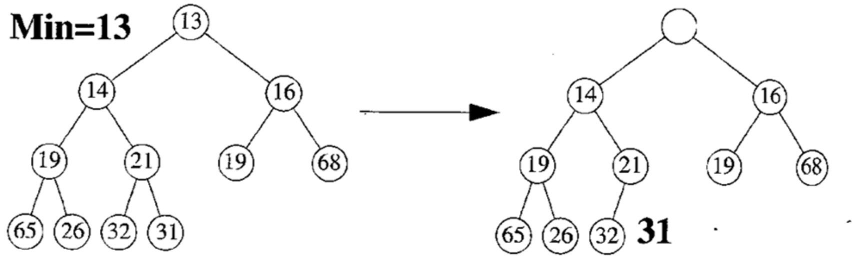


Figure 20.13 Creation of the hole at the root

Binary Heap

- Delete a minimum element from a heap

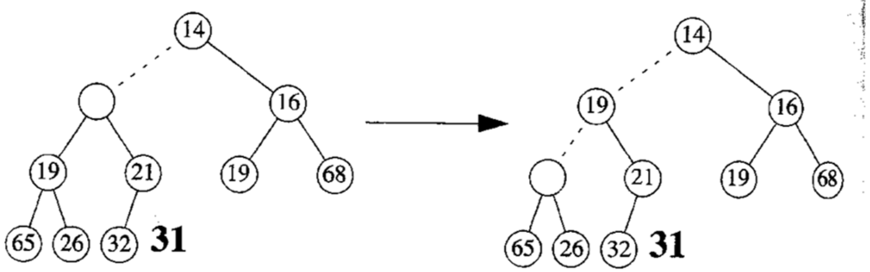


Figure 20.14 Next two steps in `DeleteMin`

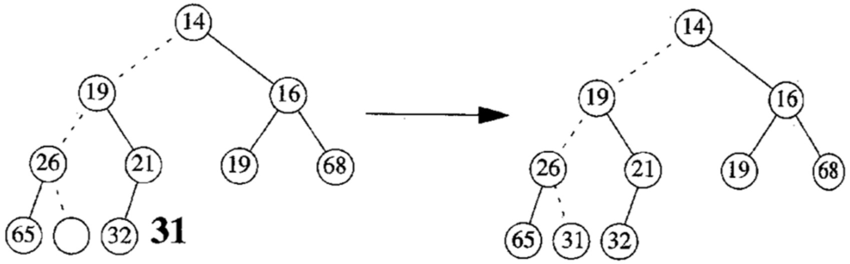


Figure 20.15 Last two steps in `DeleteMin`

Bina

- Delete a maximum element from a heap.

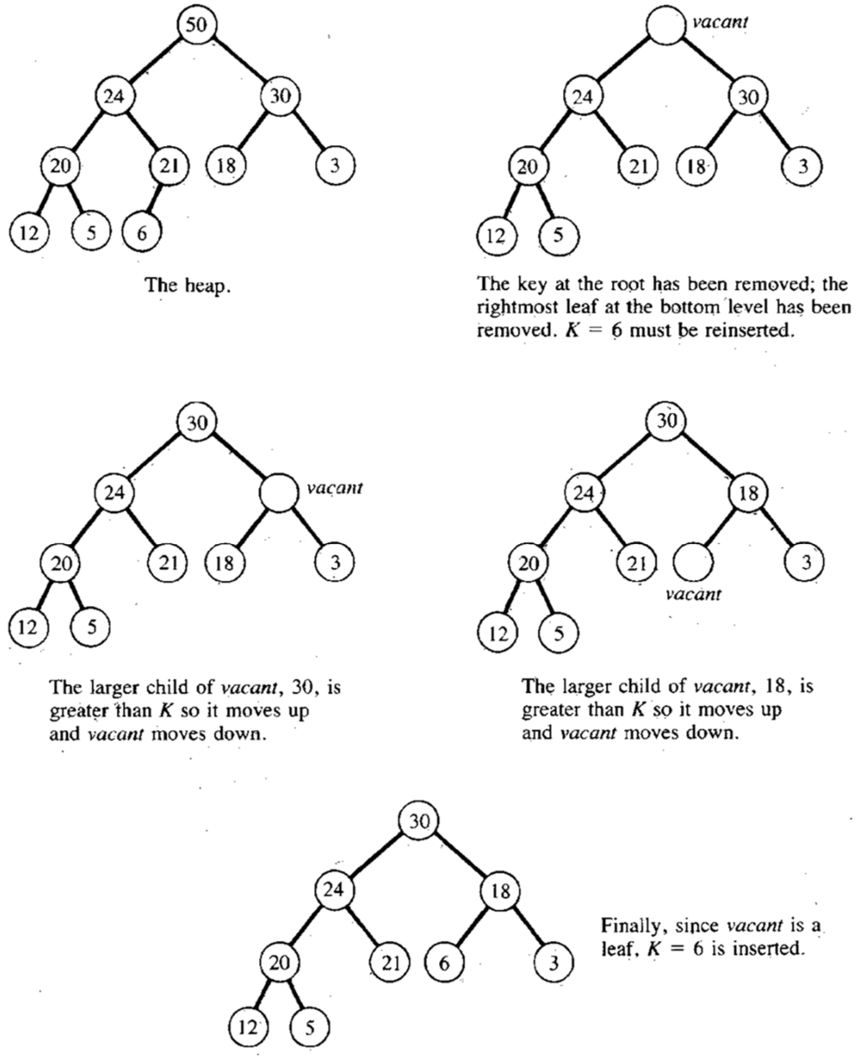


Figure 2.15 Deleting the key at the root and reestablishing the heap property.

Heap Construction

- Heap construction
  - If we are given a complete tree that does not have heap order, we are going to construct a heap.
  - FixHeap Operation:
    - We are given a complete tree that only the root violates the heap order property.

Heap Construction

- Fix heap operation
  - Example for a max heap

Algorithm 2.8 FixHeap

Input: The *root* of a heap and a key K to be inserted.

Output: The heap with keys properly rearranged.

```
procedure FixHeap (root: Node; K: Key);
var
    vacant, largerChild : Node;
begin
    vacant := root;
    while vacant is not a leaf do
        largerChild := the child of vacant with the larger key;
        if K < largerChild's key
            then
                copy largerChild's key to vacant;
                vacant := largerChild
            else exitloop
        end { if }
    end { while };
    put K in vacant
end { FixHeap }
```

Heap Construction

- Fix heap operation
  - The FixHeap operation takes $2\lfloor \log N \rfloor$ time, if there are N elements in the heap.

Heap Construction

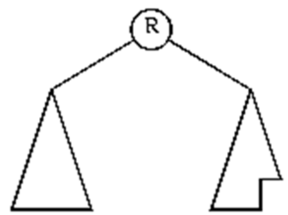
- Sift down operation
  - siftdown() in the text book is the same as FixHeap() .

```
void siftdown (heap& H)                       // H starts out having the
{                                             // heap property for all
   node parent, largerchild;                  // nodes except the root.
                                              // H ends up a heap.
   parent = root of H;
   largerchild = parent's child containing larger key;
   while (key at parent is smaller than key at largerchild) {
      exchange key at parent and key at largerchild;
      parent = largerchild;
      largerchild = parent's child containing larger key;
   }
}
```

Heap Construction

- Heap construction by divide and conquer

procedure *ConstructHeap* (*root*: *Node*);
begin
  if *root* is not a leaf **then**
    ConstructHeap (left child of *root*);
    ConstructHeap (right child of *root*);
    FixHeap (*root*, key in root)
  end { if }
end { ConstructHeap }



Recursive view of the heap

Heap Construction

- Heap construction by divide and conquer
  - Analysis:

$$T(N) = \begin{cases} 1 & \text{if } N = 1 \\ 2T(N/2) + \log N & \text{otherwise} \end{cases}$$

  - Can you represent the recurrence equation in closed form? (In this case, we cannot apply the master's theorem. Why?)
  - Next time we will show that *T(N)=O(N)*.

Summation

$$\text{Let } S = \sum_{k=1}^{d-1} k2^k$$
$$= (d-1)2^{d-1} + (d-2)2^{d-2} + \cdots + 2 \cdot 2^2 + 1 \cdot 2^1$$

$$\text{Then } 2S = (d-1)2^d + (d-2)2^{d-1} + \cdots + 2 \cdot 2^3 + 1 \cdot 2^2$$

$$2S - S = (d-1)2^d - 2^{d-1} - \cdots - 2^3 - 2^2 - 2^1 - 2^0 + 1$$
$$= (d-1)2^d - (2^d - 1) + 1$$
$$= (d-1)2^d - 2^d + 2$$
$$S = d2^d - 2^{d+1} + 2$$

$$\text{Let } R = \sum_{k=0}^{d-1} (d-k)2^k$$
$$= d\sum_{k=0}^{d-1} 2^k - \sum_{k=0}^{d-1} k2^k$$
$$= d(2^d - 1) - (d2^d - 2^{d+1} + 2)$$
$$= 2^{d+1} - d - 2$$
$$= 2N - \log N - 2$$

Heap Construction

$$T(N) = \begin{cases} 1 & \text{if } N = 1 \\ 2T(N/2) + \log N & \text{otherwise} \end{cases}$$

Let $N = 2^d$

Then $T(2^d) = 2T(2^{d-1}) + d$

$= 2\{2T(2^{d-2}) + (d-1)\} + d = 2^2T(2^{d-2}) + 2(d-1) + d$

$= 2^2\{2T(2^{d-3}) + (d-2)\} + 2(d-1) + d = 2^3T(2^{d-3}) + 2^2(d-2) + 2(d-1) + d$

$\dots$

$= 2^dT(2^{d-d}) + 2^{d-1}(d-(d-1)) + \cdots + 2^2(d-2) + 2(d-1) + d$

$= 2^dT(1) + \sum_{k=0}^{d-1} 2^k(d-k)$

$= 2^d + 2^{d+1} - d - 2$

$= 3 \cdot 2^d - d - 2$

$= 3N - \log N - 2$

$= O(N)$

Heap Construction

- Iterative version of Heap construction:

Algorithm 2.9 Heap Construction

Input: A heap structure (Property (1)) with keys in arbitrary nodes.

Output: The same structure satisfying the heap-ordering property (Property (2)).

for *level* := *depth*−1 **to** 0 **by** −1 **do**
    for each non-leaf *node* at level *level* **do**
        K := the key at *node*;
        FixHeap(*node*, *K*)
    end { for }
end { for }

Heap Construction

- Iterative version of Heap construction:

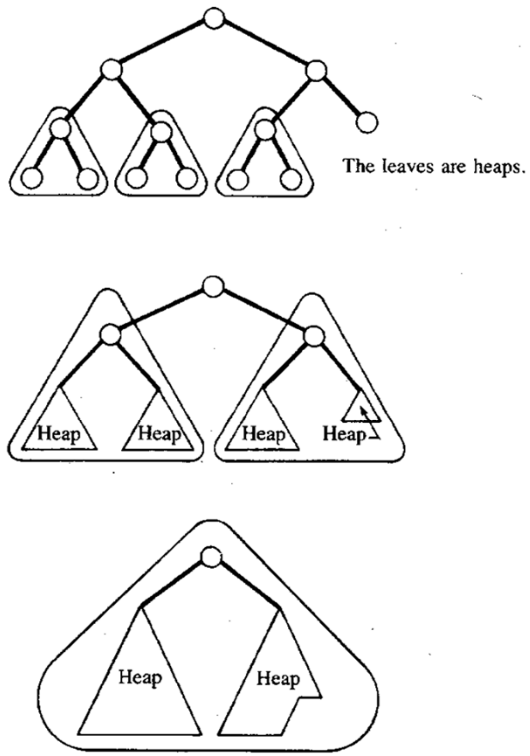


Figure 2.16 Constructing the heap. (*FixHeap* is called for each circled subtree.)

Heap Construction

❑ Example

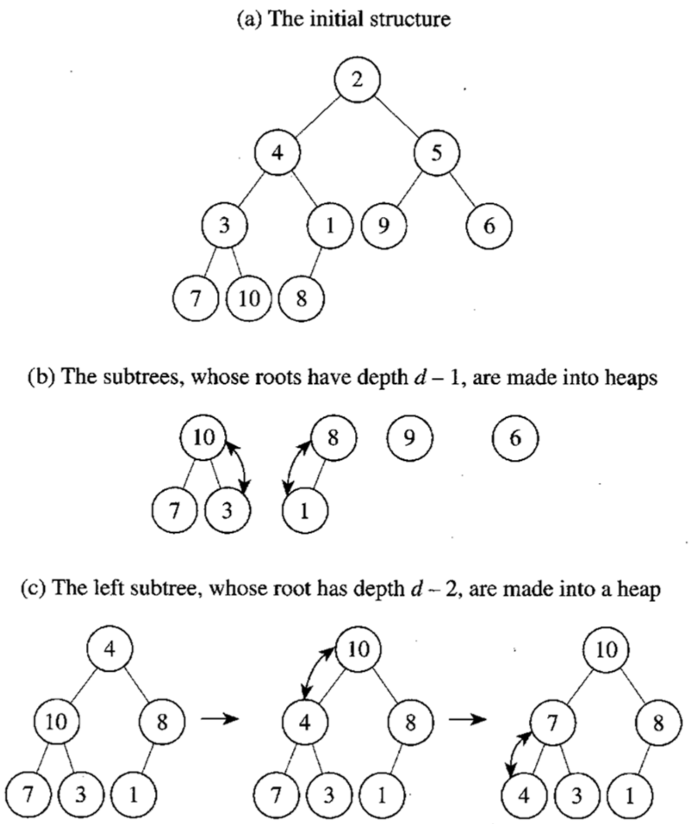


Figure 7.7 Using *siftdown* to make a heap from an essentially complete binary tree. After the steps shown, the right subtree, whose root has depth $d - 2$, must be made into a heap, and finally the entire tree must be made into a heap.

Heap Construction

❑ Example

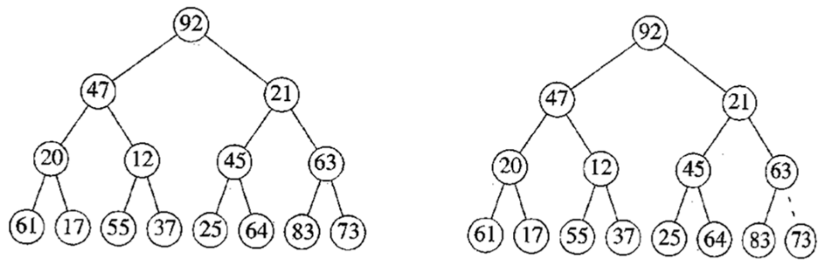


Figure 20.20 Initial heap (left); after `PercolateDown(7)` (right)

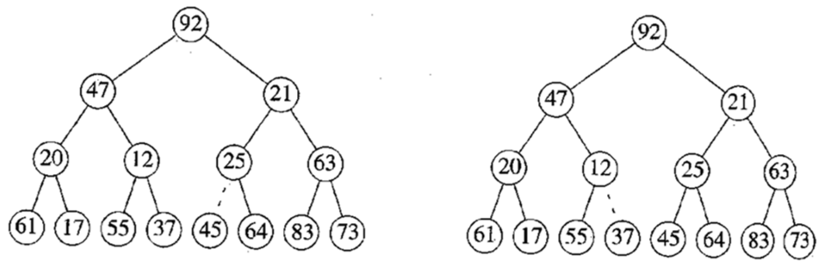


Figure 20.21 After `PercolateDown(6)` (left); after `PercolateDown(5)` (right)

Heap Construction

- Example

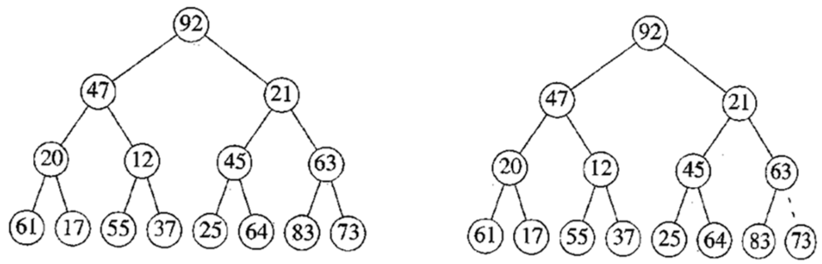


Figure 20.20 Initial heap (left); after `PercolateDown(7)` (right)

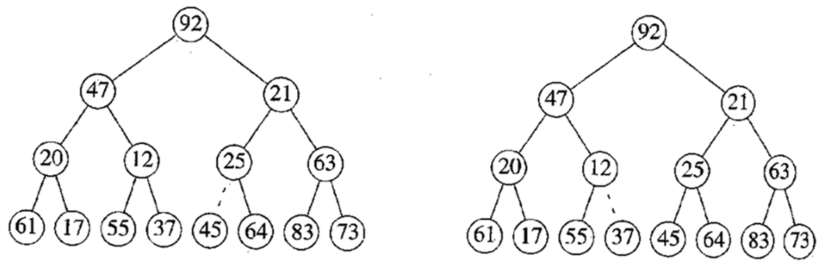


Figure 20.21 After `PercolateDown(6)` (left); after `PercolateDown(5)` (right)

Heap Construction

- Analysis of the heap construction:
  - Let $d = \lfloor \log N \rfloor$
  - Then,

$$T(N) = \sum_{k=0}^{d-1} 2(d-k)(\text{the number of nodes at level } k)$$

$$= \sum_{k=0}^{d-1} 2(d-k)2^k$$

$$= 2d\sum_{k=0}^{d-1} 2^k - 2\sum_{k=0}^{d-1} k2^k$$
$$= 2d(2^d - 1) - 2(d2^d - 2^{d+1} + 2)$$
$$= 2^{d+2} - 2d - 4$$
$$= 4N - 2\log N - 4$$

  - Thus the heap is constructed in *T(N) = O(N)*, linear time!

Heap Sorting

- Heapsort:
  - The priority queue can be used to sort *N* items as follows:
    - Put all the elements in an array of size *N*.
    - Construct a heap
    - Extract every item by calling DeleteMin *N* times. The result is sorted.

- Heapsort:
  - The priority queue can be used to sort *N* items as follows:
    - Initially, make an empty priority queue
    - insert all the elements in an array of size N (call insert() N times)
    - delete min for N times from priority queue (call deletemin() N times)

Heap Sorting (simple, not correct)

- **Heapsort:**
  - The priority queue can be used to sort *N* items as follows:
    - Initially, make an empty priority queue
    - insert all the elements in an array of size N (call insert() N times)
    - delete min for N times from priority queue (call deleteMin() N times)

```
void heapSort(int data[], int size)
{
    Heap minHeap;   // min heap
    for(int i=0; i<size; i++)
        minHeap.insert(data[i]);

    for(int i=0; i<size; i++
        data[i] = minHeap.deleteMin();
}
```

Heap Sorting

- Heapsort:

Algorithm 2.10 Heapsort

Input: L, an unsorted array, and $n \geq 1$, the number of keys.

Output: L, with keys in nondecreasing order.

```
procedure Heapsort (var L: Array; n: integer);
var
    i, heapsize : Index;
    max : Key;
begin
    { Heap Construction }
    for i := ⌊n/2⌋ to 1 by −1 do
        FixHeap (i, L[i], n)
    end { for };

    { Repeatedly remove the key at the root and rearrange the heap. }
    for heapsize := n to 2 by −1 do
        max := L[1];
        FixHeap (1, L[heapsize], heapsize−1);
        L[heapsize] := max
    end { for }
end { Heapsort }
```

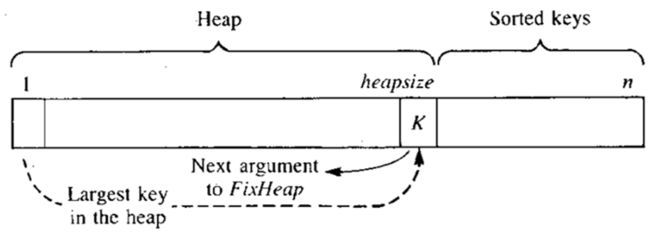


Figure 2.18 The heap and sorted keys in the array.

Heap Sorting

- Save the array space:
  - In heapsort, we construct a max heap, and retract a max value from the heap and put it in the end of the heap. Then we sort elements in increasing order.

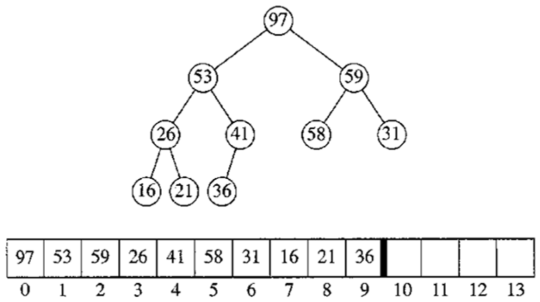


Figure 20.28 (Max) Heap after `FixHeap` phase

Heap Sorting

- ❑ Save the array space:

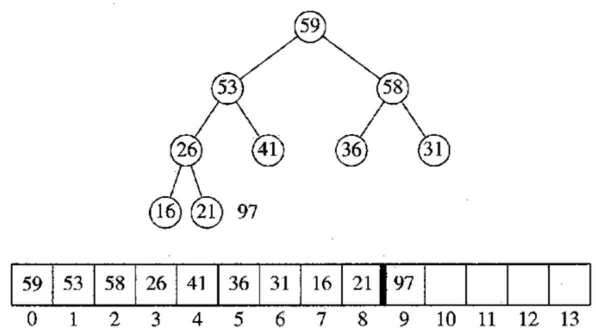
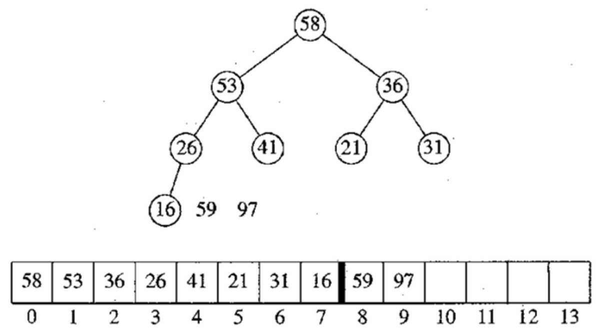
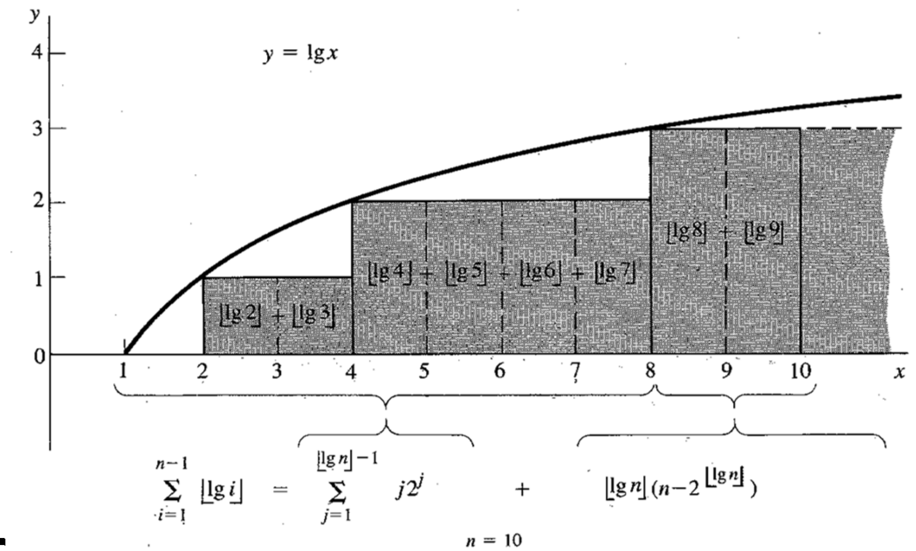


Figure 20.29 Heap after first `DeleteMax`



Heap Sorting

- Analysis of heapsort:
  - Since the number of comparison done by FixHeap on a heap with k elements is at most $2\lfloor \log k \rfloor$, so the total for all deletions is at most $2\sum_{k=1}^{n-1}\lfloor \log k \rfloor$



Heap Sorting

- Analysis of heapsort:
  - Let $d = \lfloor \log N \rfloor$
  - The sum is

$$\sum_{k=1}^{d-1} k2^k + d(N - 2^d)$$

$$= (d2^d - 2^{d+1} + 2) + d(N - 2^d)$$
$$= Nd - 2^{d+1} + 2$$
$$= Nd - 2N + 2$$
$$= O(N\log N)$$

  - Therefore heapsort takes *O(N log N)* time to sort *N* elements!

Question

- 1부터 n까지의 모든 bit 수의 합 T(n) 은?
  - 정수 n의 bit수: $\lfloor \lg n \rfloor + 1$, let $d = \lfloor \lg n \rfloor$

1	1
2	1 0
3	1 1
4	1 0 0

n	1 0 1 1 0 ··· 0

Question

- 1부터 n까지의 모든 bit 수의 합 T(n) 은?
  - 정수 n의 bit수: $\lfloor \lg n \rfloor + 1$, let $d = \lfloor \lg n \rfloor$

1	1
2	10
3	11
4	100
n	1 0 1 1 0 ⋯ 0

bit수가 k 개인 정수의 수 :2^{k-1}

$$T(n) = \sum_{k=1}^{\lfloor \lg n \rfloor} k\, 2^{k-1} + (\lfloor \lg n \rfloor + 1)\left(n - 2^{\lfloor \lg n \rfloor} + 1\right)$$

$$= \frac{1}{2}\sum_{k=1}^{d} k2^k + (d+1)\left(n - 2^d + 1\right)$$

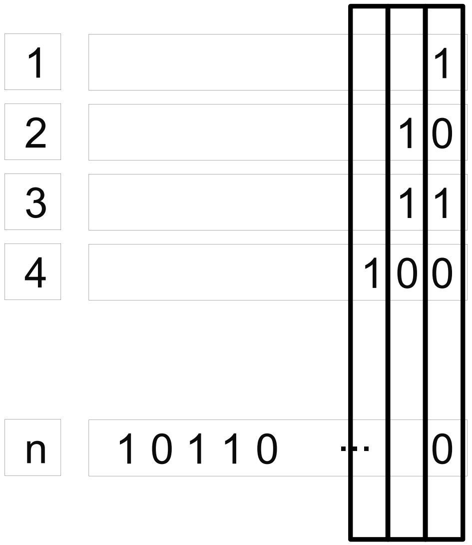
$$= \frac{1}{2}(d2^d - 2^{d+1} + 2) + \frac{1}{2}d2^d + (d+1)(n - 2^d + 1)$$

$$= n(d+1) - 2^{d+1} + d + 2$$

$$T(n) = n(\lfloor \lg n \rfloor + 1) - 2^{\lfloor \lg n \rfloor + 1} + \lfloor \lg n \rfloor + 2$$

Question

- 1부터 n까지의 모든 bit 수의 합 T(n) 은?
  - 정수 n의 bit수: $\lfloor \lg n \rfloor + 1$, let $d = \lfloor \lg n \rfloor$



오른쪽에서 k $(0 \le k \le \lfloor \lg n \rfloor - 1)$번째 bit를 가진 정수의 수 :

$$n - (2^k - 1)$$

$$T(n) = \sum_{k=0}^{\lfloor \lg n \rfloor - 1} (n - (2^k - 1)) \quad + \left(n - (2^{\lfloor \lg n \rfloor} - 1)\right)$$

$$= n(\lfloor \lg n \rfloor + 1) - \sum_{k=0}^{\lfloor \lg n \rfloor} 2^k + \lfloor \lg n \rfloor + 1$$

$$= n(\lfloor \lg n \rfloor + 1) - 2^{\lfloor \lg n \rfloor + 1} + \lfloor \lg n \rfloor + 2$$

Lower Bounds for Sorting

- Lower bounds for sorting only by comparisons of keys
- Decision trees for sorting algorithms
  - An algorithm for sorting three distinct numbers:

```
void sortthree (keytype S[ ])    // S is indexed from 1 to 3.
{
   keytype a, b, c;

   a = S[1]; b = S[2]; c = S[3];
   if (a < b)
      if (b < c)
         S = a, b, c;            // This means S[1] = a; S[2] = b; S[3] = c;
      else if (a < c)
         S = a, c, b;
      else
         S = c, a, b;

   else if (b < c)
      if (a < c)
         S = b, a, c;
      else
         S = b, c, a;
   else
      S = c, b, a;
}
```

Lower Bounds for Sorting

- Lemma 7.1
  - To every algorithm for sorting distinct numbers, there corresponds a decision tree containing exactly *n*! keys.
- Example:
  - The decision tree corresponding to exchange sort when sorting three numbers.

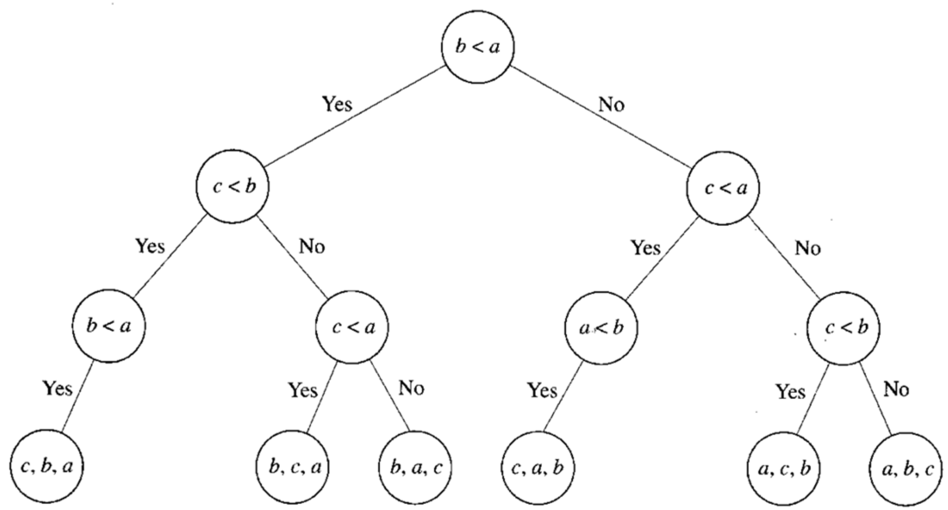


Figure 7.12 The decision tree corresponding to Exchange Sort when sorting three keys.

Lower Bounds for Sorting

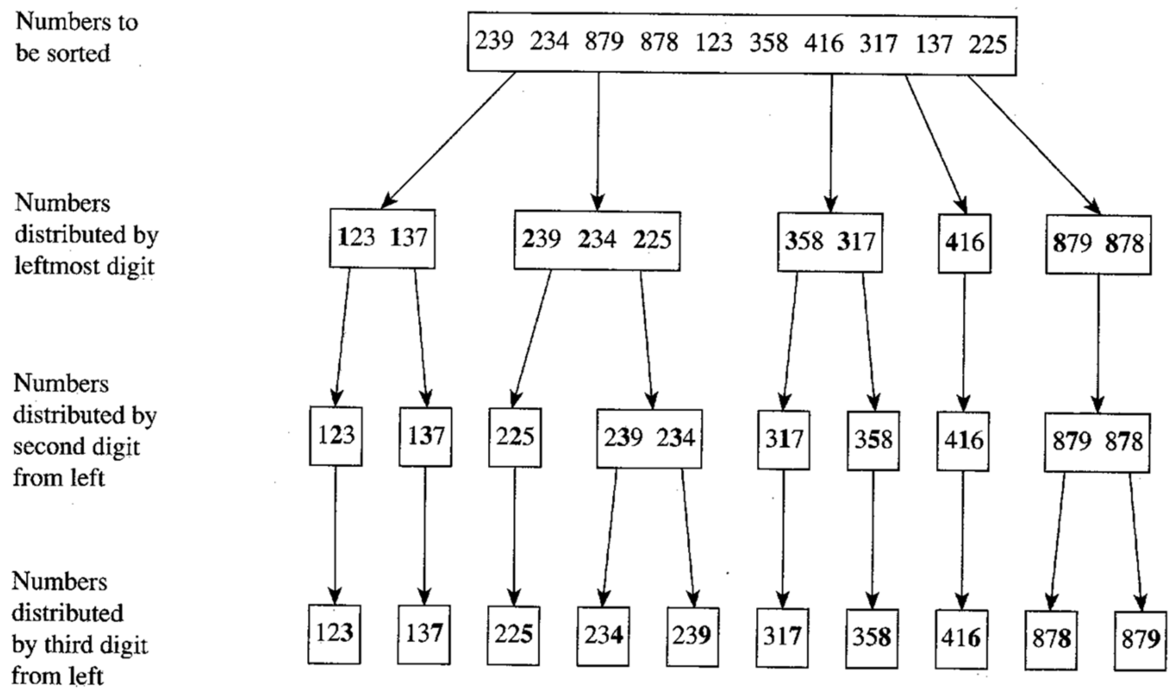
- Theorem 7.2
  - Any algorithm that sorts distinct numbers only by comparison of numbers must in the worst case do at least $\lceil \lg(n!) \rceil = O(n\log n)$ comparison of numbers.
- Proof:
  - By lemma 7.1 the decision tree has $n!$ leaf nodes
  - Then the depth of the tree is greater than or equal to $\lceil \lg(n!) \rceil$.
  - Note that

$$\begin{aligned}\lg(n!) &= \lg[n(n-1)(n-1)\cdots(2)1] \\ &= \sum_{i=1}^{n} \lg i \\ &\geq \int_1^n \lg x dx = \frac{1}{\ln 2}(n \ln n - n + 1) \\ &\geq n \lg n - 1.45n\end{aligned}$$

Sorting by Distribution (Radix Sort)

- Method 1

Figure 7.14 Sorting by distribution while inspecting the digits from left to right.



Sorting by Distribution (Radix Sort)

❑ Method 2

Figure 7.15 Sorting by distribution while inspecting the digits from right to left.