Homework 06

Math 4101

Due October 19, 2025(**Sunday**) by 10pm

Directions

Complete the following problems—neatly and nicely written with lots of margin space. Points may be deducted for illegible or poorly presented solutions. Turn them in via gradescope.com by 10pm on October 19, 2025. The homework is worth a total of 60 points. Note we use $\subset$ and $\subseteq$ interchangeably. If we want to denote a strict subset we will use the notation $\subsetneq$.

Solutions

Questions

1. This exercise works through proving that every open set in $\mathbb{R}$ is a countable disjoint union of open intervals. Let $A \subset \mathbb{R}$ be open and nonempty.

   (a) (10 pts) Define a relation on elements of A by writing $a \sim b$ if there is an open interval (c, d) such that $a, b \in (c, d) \subset A$. **Prove** that this is an equivalence relation.

   Proof. (Reflexivity) Since A is open, for any $a \in A$ there is an interval (c, d) such that $a \in (c, d) \subset A$. This means $a \sim a$.

   (Symmetry) This is clear because if $a, b \in (c, d)$ then $b, a \in (c, d)$. So, $a \sim b$ implies $b \sim a$.

   (Transitivity) Suppose $a_1 \sim a_2$ and $a_2 \sim a_3$. Then, there exist $(b_1, b_2), (b_3, b_4)$ inside A such that $a_1, a_2 \in (b_1, b_2)$ and $a_2, a_3 \in (b_3, b_4)$. We claim that $(b_1, b_2) \cup (b_3, b_4) = (\min\{b_1, b_2\}, \max\{b_2, b_4\})$. First note that if $x \in (b_1, b_2) \cup (b_3, b_4)$ then $b_1 < x$ or $b_3 < x$ implies $\min\{b_1, b_3\} < x$ and similarly $x < b_2$ or $x < b_4$ implies $x < \max\{b_2, b_4\}$. This proves $(b_1, b_2) \cup (b_3, b_4) \subset (\min\{b_1, b_3\}, \max\{b_2, b_4\})$ In the other direction, if $x \in (\min\{b_1, b_3\}, \max\{b_2, b_4\})$, then supposing without loss of generality that $b_1 \leq b_3$ then $b_1 < x$. If $x < b_2$ we are done (namely $x \in (b_1, b_2)$), so suppose $b_2 \leq x$. Then, $b_3 < a_2 < b_2 \leq x$. Since $x < \max\{b_2, b_4\}$ we must have $b_4 = \max\{b_2, b_4\}$ so that $x \in (b_3, b_4)$. This proves
   $$(\min\{b_1, b_3\}, \max\{b_2, b_4\}) \subset (b_1, b_2) \cup (b_3, b_4).$$
   This implies that $a_1, a_3 \in (\min\{b_1, b_3\}, \max\{b_2, b_4\}) \subset A$ so that $a_1 \sim a_3$. $\square$

   (b) (10 pts) **Prove** that equivalence classes under this equivalence relation are open intervals. (We allow open intervals with $\pm\infty$ as endpoints.)

   Proof. Suppose B is an equivalence class of A under the above equivalence relation. We claim that $B = (\inf B, \sup B)$.

   First we claim $\inf B \notin B$. If $\inf B = -\infty$ this is automatic, so suppose $\inf B > -\infty$. If $\inf B \in B$, then since A is open there exists an interval (a, b) with $\inf B \in (a, b) \subset A$. In particular, there exists $c \in (a,b) \subset A$ satisfying $a < c < \inf B$ contrary to the fact that $\inf B$ is a lower bound. Similarly, $\sup B \notin B$.

This proves that $B \subset (\inf B, \sup B)$ since for all $b \in B$ we have $\inf B \le b \le \sup B$ and we just proved that equality cannot hold.

Conversely, suppose $x \in (\inf B, \sup B)$. Since $x > \inf B$, x is not a lower bound for B meaning there exists $c \in B$ such that $x > c$. Similarly, there exists $d \in B$ such that $x < d$. Since $c, d \in B$, we have $c \sim d$ and there must exist an interval (a,b) such that $c, d \in (a,b) \subset B$. Because of this, $a < c < x < d < b$ implies that $x \in B$ as desired. Therefore, $B = (\inf B, \sup B)$. □

(c) (Recommended; do not hand in) Deduce that A is a disjoint union of open intervals. Also, deduce that the number of open intervals is countable by showing that each contains a rational number.

2. (10 pts) Given $A \subset \mathbb{R}$, the *closure* of A is the set A unioned with its set of limit points. We write closure(A) for this set. **Prove** that the closure of any set is closed.

Proof. Recall that being closed means a set contains its limit points. Contrapositively, this is equivalent to saying that if a point is not in the set then the point is not a limit point.

So, suppose x is not in the closure of A. Since x is not a limit point and not in A, there must be a neighborhood U of x that does not contain any points of A. Namely, $x \in U$ and $U \cap A = \varnothing$.

The points of U cannot be limit points of A because they have the property that they have a neighborhood (U itself) that does not intersect A. Therefore, U does not intersect the closure of A. Therefore, x cannot be a limit point of the closure of A because x has a neighborhood (again U) that does not intersect the closure.

This proves the closure is indeed closed. □

3. (10 pts) Given $A \subset \mathbb{R}$, we say $x \in A$ is in the *interior* of A if there exists a neighborhood U of x such that $U \subset A$. We write int(A) for the interior. **Prove** that for any $A \subset \mathbb{R}$

$$\mathbb{R} \setminus \operatorname{int}(A) = \operatorname{closure}(\mathbb{R} \setminus A).$$

Proof. Suppose $x \in \mathbb{R} \setminus \operatorname{int}(A)$; namely x is not in the interior of A. This means that every neighborhood of x must contain points not in A. This implies x is a limit point of $\mathbb{R} \setminus A$ if x is not in A. Namely, either x is a limit point of $\mathbb{R} \setminus A$ or $x \in \mathbb{R} \setminus A$ which exactly means x is in the closure of A.

Conversely, suppose x is in the closure of $\mathbb{R} \setminus A$. If $x \in \mathbb{R} \setminus A$, then $x \notin A$ implies $x \notin \operatorname{int}(A)$ since $\operatorname{int}(A) \subset A$. Otherwise, $x \in A$ and x is a limit point of $\mathbb{R} \setminus A$. So, every neighborhood of x contains a point of $\mathbb{R} \setminus A$ and in particular x has no neighborhood contained in A. This implies $x \notin \operatorname{int}(A)$. Namely, $x \in \mathbb{R} \setminus \operatorname{int}(A)$. □

4. (10 pts) Given $B \subset A \subset \mathbb{R}$, we say that B is *dense* in A if $A \subset \operatorname{closure}(B)$.

Prove that every subset of $\mathbb{R}$ has either a countable or finite dense subset. *

Proof. Let $A \subset \mathbb{R}$. Let $(I_n)_{n=1}^{\infty}$ be an enumeration of the open intervals with rational endpoints. For each n such that $I_n \cap A \neq \varnothing$ choose $x_n \in I_n \cap A$. Let S be the set of n such that $I_n \cap A \neq \varnothing$.

Define

$$B = \{x_n : n \in S\}.$$

This set is at most countably infinite (although it could be finite). Let us prove that $A \subset \operatorname{closure}(B)$.

*Hint: the set of open intervals with rational endpoints is a countable set since its cardinality is at most that of $\mathbb{Q} \times \mathbb{Q}$ which is countable.

Let $x \in A$. We must show every neighborhood of A intersects B. Let U be a neighborhood of x. In particular, there exists an open interval (a, b) such that $x \in (a, b) \subset U$. By density of the rationals we know there exist rationals c, d such that $a < c < x < d < b$. Then, (c, d) is an open interval with rational endpoints that intersects A. So, $(c, d) = I_n$ for some n and $x_n \in B \cap U$. Thus, the neighborhood U intersects B as desired.

$\square$

5. (10 pts) A set $A \subset \mathbb{R}$ is *nowhere dense* if

$$\text{int}(\text{closure}(A)) = \varnothing.$$

Prove that if A is nowhere dense, then every open set $U \subset \mathbb{R}$ contains a (nonempty) open set V that is disjoint from A.

Proof. Since int(closure(A)) is empty this means that closure(A) contains no open sets. Therefore, every open U contains a point not in closure(A). Let $x \in U \setminus \text{closure}(A)$. Since closure($A$) is closed, its complement is open. Therefore, there exists a neighborhood V' of x such that $V' \subset \mathbb{R} \setminus \text{closure}(A)$. Then, $V := U \cap V'$ is open, contains x, and is disjoint from A which is what we want. $\square$

6. (Challenge problem; do not hand in) Prove that a countable union of nowhere dense sets in $\mathbb{R}$ has empty interior.

7. (Recommended problem; do not hand in) Let $(x_j)_{j=1}^{\infty} \in SEQ([0, \infty))$ and suppose the sequence has the property that for every $\epsilon > 0$ there exists a sequence $(y_j)_{j=1}^{\infty} \in SEQ([0, \infty))$ converging to zero such that

$$x_j \leq y_j + \epsilon$$

holds for all $j \in \mathbb{N}$. Prove that $(x_j)_{j=1}^{\infty}$ converges to 0. (There is a quick proof using $\limsup$.)