Econ 3012

Greg Leo

November 13, 2021

Part I

Budget (2.1-2.7)

1 Bundles

Bundle: $x = (x_1, x_2)$

Example. Ice Cream Bowls. x_1 is the amount of vanilla. x_2 is the amount of chocolate.

$(1,1)$ one scoop of each flavor.

$(2,2)$ two scoops of each flavor.

$(0.28, 100)$ a lot of chocolate and a little vanilla.

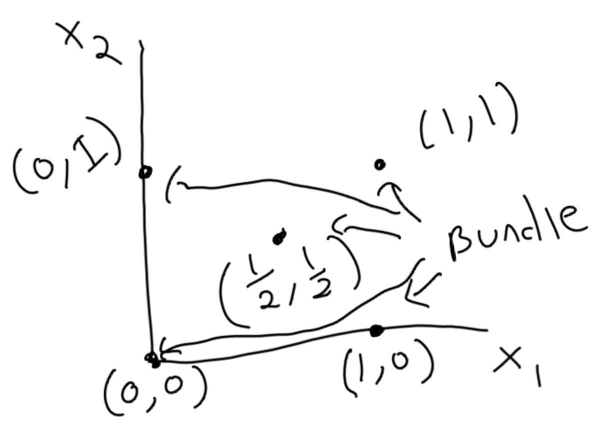

Figure 1.1: Bundles on Cartesian Plane.

2 Feasible Set

The Feasible Set: X is the "feasible" set of bundles.

The feasible set is the universe of bundles that might be relevant in a model. The feasible set defines the scope of a model.

3 Budget Set

Budget Set: B

The budget set is the set of bundles *available* to a particular consumer.

The budget set must be a subset of the feasible set.

In set notation: $B \subseteq X$

3.1 Budget Sets from Prices and Income

Prices: p_1, p_2: Price of good 1 and price of good 2.

Cost of a bundle: $p_1 x_1 + p_2 x_2$.

Income: m.

Budget set: $B = \{x | x \in X \,\&\, x_1 p_1 + x_2 p_2 \leq m\}$ **.**

In non-math language, this says the budget set is the set of bundles such that the price of the bundle is less than income.

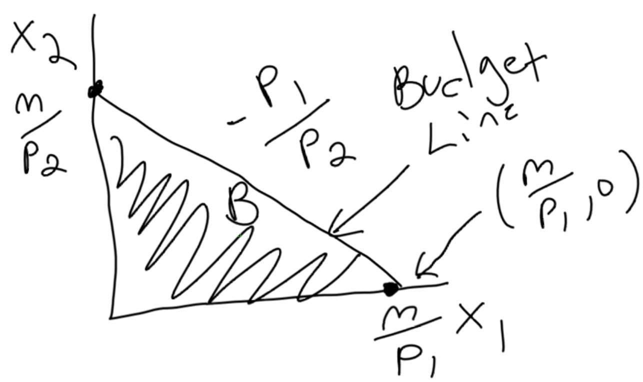

Figure 3.1: Graphical Representation of the Budget Set

3.2 Changing Prices and Income

Suppose income increases. m **changes.**

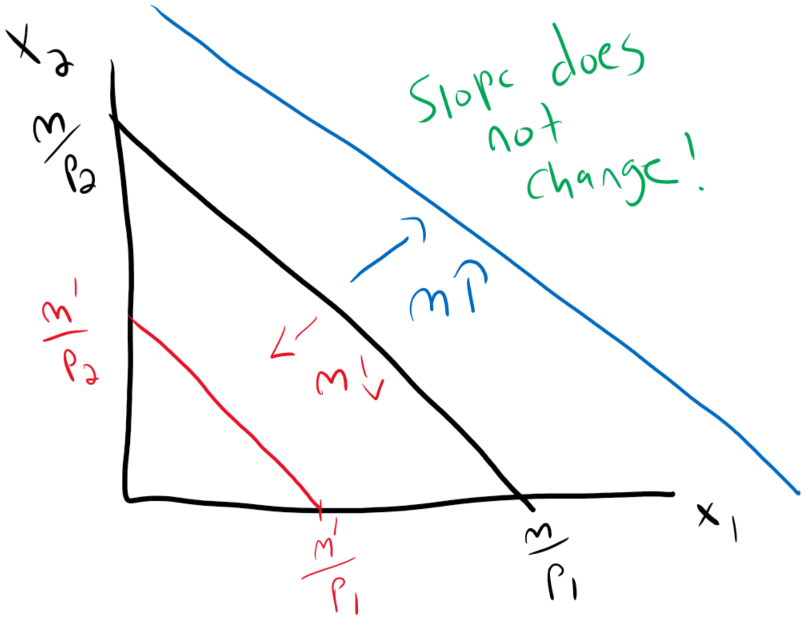

Figure 3.2: How Budget Changes with Income

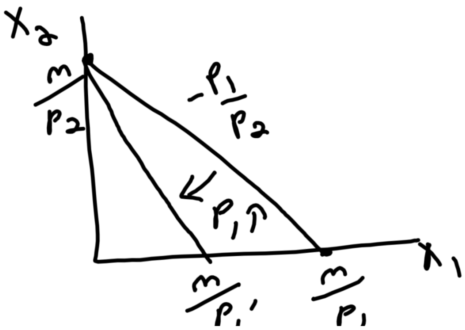

Figure 3.3: How Budget Changes with and increase in p_1.

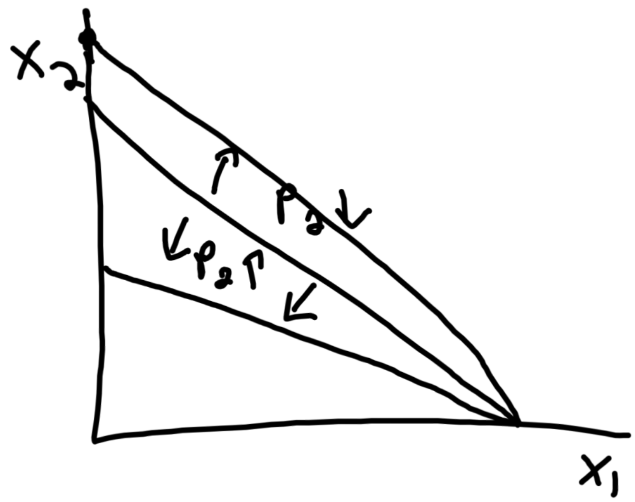

Figure 3.4: How Budget Changes with Changes to p_2

Both endpoints change. If m increaes, $\frac{m}{p_1}$ (the amount I can buy of good 1 changes) increases and $\frac{m}{p_2}$ (maximum affordable x_2) increases. The slope does not change. If m decreases, the opposite happens.

Suppose one of the prices changes.

p_1. If p_1 goes up, the slope decreases (more negative). If p_1 goes down, the slope increases. The x_2 intercept stays the same.

p_2. If p_2 goes up, the slope increases. In p_2 goes down the slope decreases (more negative). The x_1 intercept stays the same.

3.3 Taxes

Quantity tax on good 1:

$$p_1x_1 + tx_1 + p_2x_2 = m$$

$$(p_1 + t)\,x_1 + p_2x_2 = m$$

Ad Valorem Tax on good 1:

$$(p_1x_1) + \tau\,(p_1x_1) + p_2x_2 = m$$

$$(1 + \tau)\,(p_1x_1) + p_2x_2 = m$$

We will focus on quantity taxes.

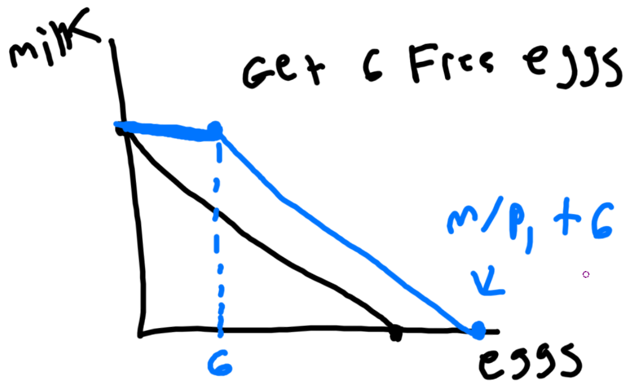

Figure 3.5: Six Free Eggs

3.4 More Complex Scenarios

3.5 Price Depends on Quantity

Part II
Preferences (3.1-3.8)

4 The Preference Relation

4.1 Definitions

The preference relation is a set of statements about **pairs** of bundles. The statement x is preferred to bundle x' is shorted to:

$$x \succsim x'$$

Ice Cream Example:

Suppose a consumer eats bowls of ice cream. The bundles (bowls) are written with the vanilla scoops first and chocolate second. For example: $(2, 0)$ is two scoops of vanilla and zero of chocolate.

A consumer who likes vanilla ice cream might have these preferences:

$$(2, 0) \succsim (0, 2)$$

$$(1, 0) \succsim (0, 1)$$

A consumer who like more ice cream to less might have these preferences:

$$(2,0) \succsim (1,0)$$

$$(2,2) \succsim (1,1)$$

For someone who gets sick of ice cream: (who wants to eat 100 scoops of ice cream?)

$$(1,0) \succsim (100,0)$$

For someone who does not care about flavor:

$$(1,0) \succsim (0,1) \,\&\, (0,1) \succsim (1,0)$$

Indifference Relation: $\sim$

When the following is true: $x \succsim y$ and $y \succsim x$ we say "x is indifferent to y" and write $x \sim y$.

Strict Preference Relation: $\succ$

When the following is true: $x \succsim y$ and **not** $y \backslash succsimx$ we say "x is strictly preferred to y" and write $x \succ y$.

4.2 Assumptions on $\succsim$

Axiom 1. Reflexive. *For all bundles. The bundle is at least as good as itself.*

In set notation:

$$\forall x \in X : x \succsim x$$

Axiom 2. Complete. *For every pair of distinct bundles. Either one is at least as good as the other or the consumer is indifferent.*

In set notation:

$$\forall x, y \in X \& x \neq y : x \succsim y \, or \, y \succsim x \, or \, both$$

A consumer can say "I'm indifferent." but not "I don't know".

Axiom 3. Transitivity.

$$x \succsim y, y \succsim z \,\textbf{implies}\, x \succsim z$$

Transitivity (along with the other assumptions) implies we can always put a set of objects into a ranking.

4.3 Example of Violating Transitivity

Example: Suppose there are three people on a dating app.

Person 1. Rich, Very Intelligent, Average Looking

Person 2. Financially Constrained, Genius, Good Looking

Person 3. Moderately Well Off, Average Intelligence, Best Looking

Suppose you prefer a person who is better in two aspects than another. We have a cycle:

$$1 \succ 3 \succ 2 \succ 1$$

This kind of multi-dimensional preference can easily cause intransitivity.

4.4 From Preference to Choice

Choice Function:

$$C : B \to B$$

$$C\left(B\right) \subseteq B$$

The choice function takes a budget set as input and returns the things the consumer would like to have from that set.

$C\left(B\right)$ is all the objects in B such that those objects are at least as good as everything else in the set.

$$C\left(B\right) = \{x | x \in B : \forall x' \in B, x \succsim x'\}$$

4.5 Indifference Curves and the Weakly Preferred Set

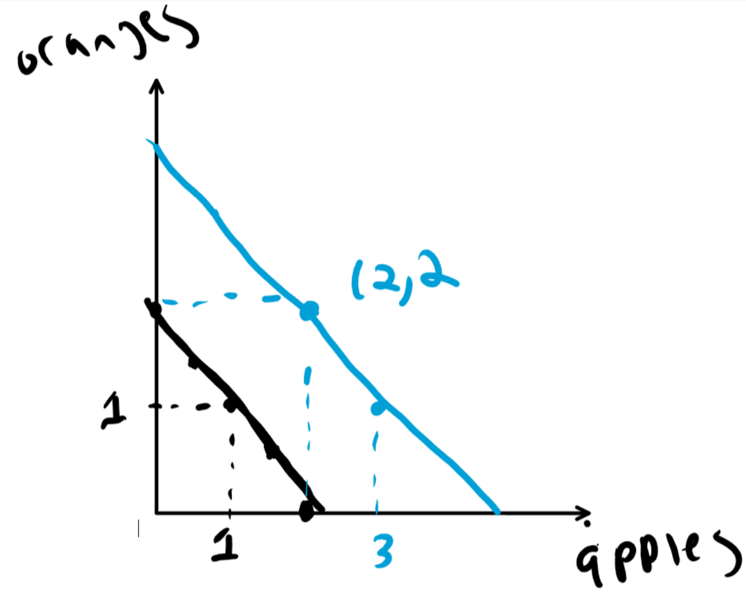

Figure 4.1: Indifference curves through the points $(1, 1)$ and $(2, 2)$ for a consumer who will always give up one orange to get one apple.

An indifference curve is a set bundles such that the consumer is indifferent between all of the bundles on the curve.

Note: There are many indifference curves. We only sketch a few to get an idea of the "shape" of preferences.

4.6 Indifference Curves Cannot Cross

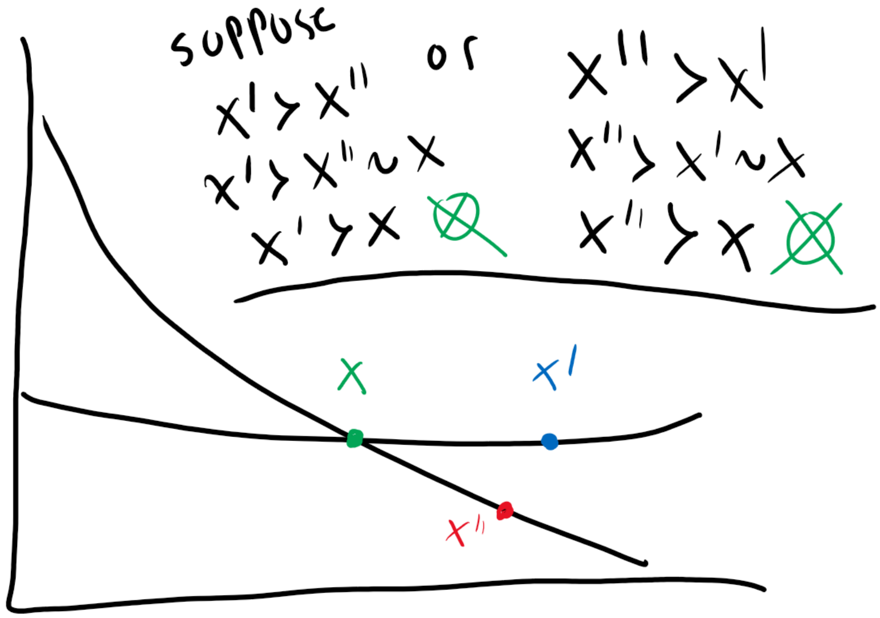

Figure 4.2: Indifference curves cannot cross if preferences are transitive.

4.7 Examples of Preferences

4.7.1 Perfect Substitutes

These preferences are such that my willingness to trade-off between the goods is the same everywhere.

The indifference curves are always downward sloping lines with the same slope. **The slope measures the amount of x_2 you are willing to give up to get 1 more unit of x_1.**

Steep slope: stronger preference for x_1.

Shallow slope: stronger preference for x_2.

Figure 4.3: Indifference curves for perfect substitutes preferences. This consumer would be willing to give up 2 units of x_2 in exchange for 1 unit of x_1.

4.7.2 Perfect Complements

Preferences where one good cannot substitute for the other.

Example: Left and right shoes.

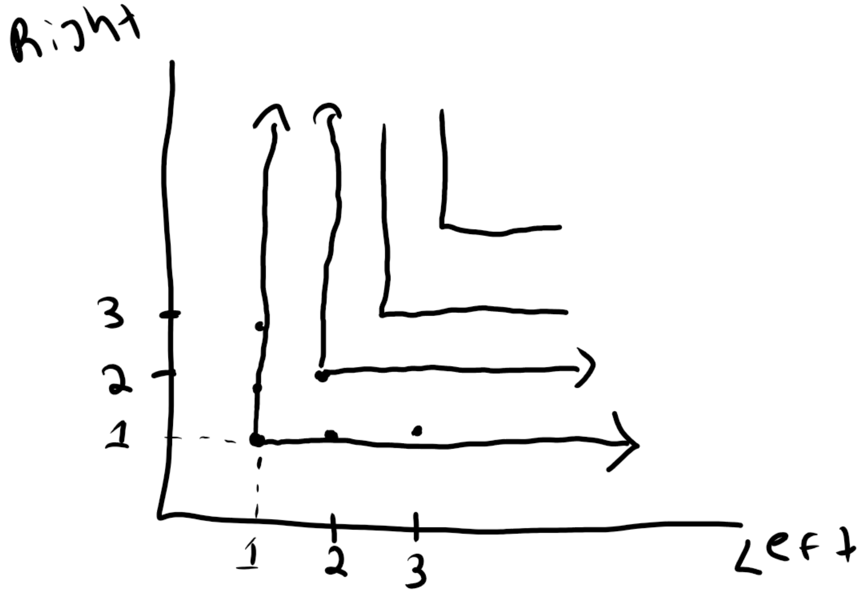

Figure 4.4: Indifference curves for perfect complements preferences where Left/Right shoes must be consumed in a 1-to-1 one ratio.

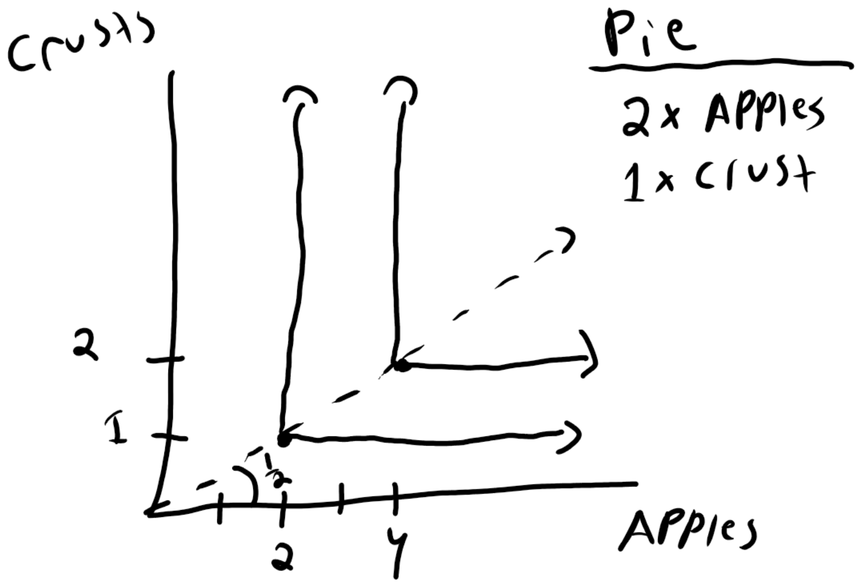

Figure 4.5: Indifference curves for perfect complements preferences where the goods are consumed in a 2-to-1 ratio. In this case, 2 apple and 1 crust make a pie.

4.7.3 Bads

Figure 4.6: When one good is a "bad", indifference curves slope upward!

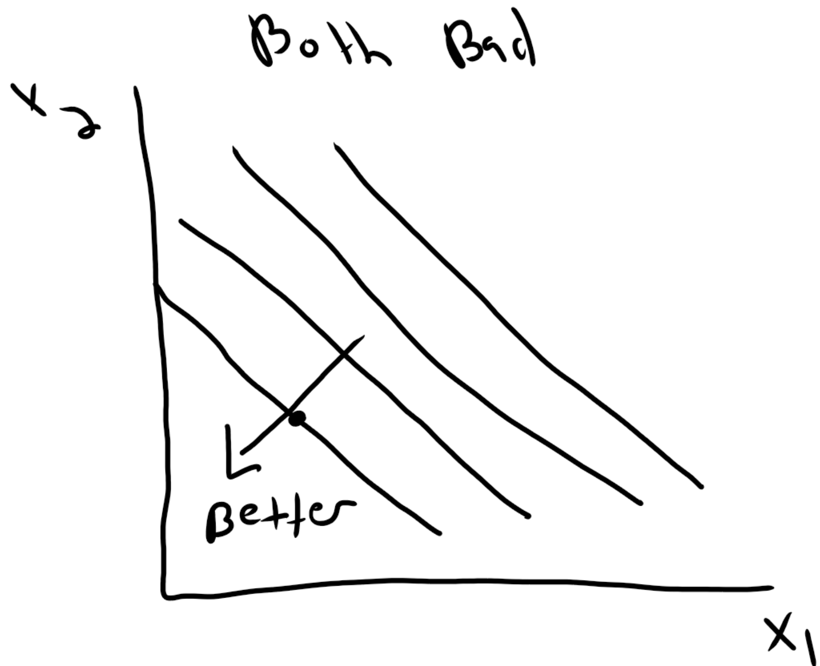

Figure 4.7: When both goods are bad, indifference curves slope down, but preference "increases" towards the origin (to the south west).

4.8 Further Assumptions: "Well Behaved Preferences"

4.8.1 Monotonicity

The assumption that everything is a "good".

There are two forms of this assumption. *Strict* and *Weak.*

Strict Monotonicity: For two bundles (x_1, x_2) and (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. $(x_1, x_2) \succ (y_1, y_2)$ if either $x_1 > y_1$ or $x_2 > y_2$

Perfect substitutes are strictly monotonic.

Perfect complements are ***not*** *strictly monotonic.*

Weak Monotonicity. (AKA "Monotonic"): For two bundles (x_1, x_2) and (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. $(x_1, x_2) \succ (y_1, y_2)$ if **both** $x_1 > y_1$ **and** $x_2 > y_2$

Perfect substitutes are weakly monotonic.

Perfect complements ***are*** *weakly monotonic.*

4.8.2 What does monotonicity imply about the indifference curves?

1. Weakly downward sloping. We cannot have upward sloping indifference curves.

2. Preference increases to the north east (away from the origin).

4.8.3 Convexity and Strict Convexity

The assumption that mixtures are better than extremes.

There are two forms of this assumption. *Strict* and *Weak.*

Strictly Convex: For two indifferent bundles $(x_1, x_2) \sim (y_1, y_2)$, for any $t \in (0, 1)$, the mixture given by $(tx_1 + (1-t)\,y_1, tx_2 + (1-t)\,y_2) \succ (x_1, x_2)$ and $(tx_1 + (1-t)\,y_1, tx_2 + (1-t)\,y_2) \succ (y_1, y_2)$.

Weakly Convex: For two indifferent bundles $(x_1, x_2) \sim (y_1, y_2)$, for any $t \in [0, 1]$, the mixture given by $(tx_1 + (1-t)\,y_1, tx_2 + (1-t)\,y_2) \succsim (x_1, x_2)$ and $(tx_1 + (1-t)\,y_1, tx_2 + (1-t)\,y_2) \succsim (y_1, y_2)$.

The geometry of convex indifference curves. Assume preferences are monotonic:

If preferences are strictly convex, then the indifference curve always lies strictly below a line between any two points on that indifference curve.

If preferences are weakly convex, then the indifference curve always lies weakly below a line between any two points on that indifference curve.

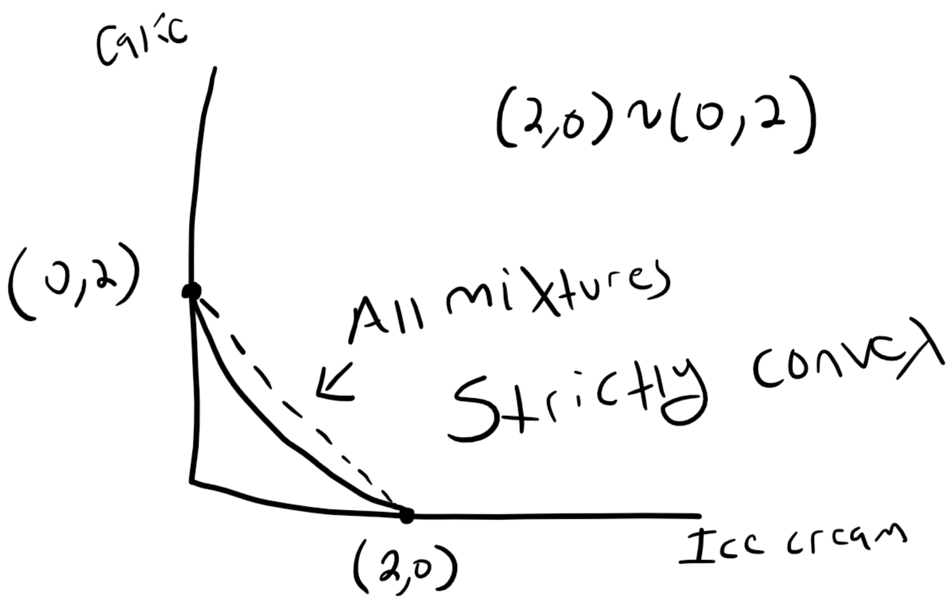

Figure 4.8: Example of Convex Indifference Curves

Perfect substitutes, and perfect complements are both **weakly** convex because their indifference curves include "flat" portions.

4.9 Marginal Rates of Substitution and Slope of the Indifference Curve

The MRS is defined as the slope of the indifference curve. We need this because optimal bundles will often occur where the slope of the indifference curve is the same as the slope of the budget line.

The MRS measures my willingness to trade off between good 1 and good 2. Specifically, it's how much x_2 I would give up to get one more unit of x_1.

5 Utility (4.1-4.5)

5.1 Definition

A utility function is a way of assigning "scores" to bundles, such that better bundles according to $\succsim$ get a higher score. For example, suppose a consumer's preferences are:

$$A \succ B \succ C \sim D$$

Some utility functions that represent these preferences:

$U(A) = 10$, $U(B) = 5$, $U(C) = U(D) = 0$

$U(A) = 12$, $U(B) = 1$, $U(C) = U(D) = -100$

Definition. A utility function $U(x)$ represents preferences $\succsim$ when for every pair of bundles x and y, $U(x) \geq U(y)$ if and only if $x \succsim y$.

That is, if x is better than y according to $\succsim$ it gets a higher utility according to $U()$.

Utility is Ordinal:

Because the magnitude of the numbers are meaningless, and only the relationships matter, we say that these utility functions are "**ordinal**" rather than "**cardinal**". There is no sense in which two times higher utility means that the preference is two times stronger. We can only infer the ranking of things, but not how strong the preferences are from $\succsim$ and a utility function that represents $\succsim$.

5.2 Monotonic Transformations

Because utility is ordinal, we are free transform one utility function into another, as long as it maintains the same preferences. Any **strictly increasing** function of a utility function represents the same preferences as the original utility function. For example, suppose:

$$U(x_1, x_2) = x_1 + x_2$$

This represents the preferences of someone who only cares about the total amount of stuff, but not the composition. In fact, this utility function represents *perfect substitutes preferences*. Here are some other utility functions that represent the same preferences:

$$U'(x_1, x_2) = x_1 + x_2 + 100 = U(x_1, x_2) + 100$$

$$U'(x_1, x_2) = (x_1 + x_2)^2 = (U(x_1, x_2))^2$$

Since the functions $f(u) = u + 100$ and $f(u) = u^2$ are strictly increasing for $u \geq 0$ (which is always true for the original utility function). These are monotonic transformations of the original utility function.

5.3 MRS from Utility Function

*The **Marginal Rate of Substitution** (MRS) is the slope of the indifference curve.*

The **Marginal Utility** of good i is $mu_i = \frac{\partial u(x_1, x_2)}{\partial x_i}$.

$$MRS = -\frac{mu_1}{mu_2} = -\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

Note that MRS is an **ordinal** property since it represents the slope of indifference curves. Because two preferences that are the same have the same indifference curves, they will also have the same MRS. **Same MRS, same preferences**.

5.4 Examples of Utility Functions

5.4.1 Perfect Substitutes

$$u(x_1, x_2) = ax_1 + bx_2$$

$$MRS = -\frac{a}{b}$$

Constant MRS implies a constant willingness to trade off between the two goods.

$$u(x_1, x_2) = (ax_1 + bx_2)^2$$

$$-\frac{a}{b}$$

5.4.2 Quasi-Linear

Consumer only gets tired of one of the two goods.

One common quasi-linear utility function:

$$u\left(x_1, x_2\right) = x_1 + ln\left(x_2\right)$$

$$MRS = -\frac{\frac{\partial\left(x_1 + ln\left(x_2\right)\right)}{\partial x_1}}{\frac{\partial\left(x_1 + ln\left(x_2\right)\right)}{\partial x_2}} = -x_2$$

Another example of a quasi-linear utility function:

$$u\left(x_1, x_2\right) = 10x_1 + \sqrt{x_2}$$

Practice taking the MRS of this function. Notice that it only depends on the amount of x_2!

5.4.3 Cobb-Douglass

Consumer Gets Tired of Both Goods.

$$u\left(x_1, x_2\right) = x_1^{\alpha} x_2^{\beta}$$

$$mu_1 = \frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta}$$

$$mu_2 = \frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1}$$

$$MRS = -\frac{MU_1}{MU_2} = -\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}}$$

$$= -\frac{\alpha x_1^{\alpha} x_1^{-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta} x_2^{-1}} = -\frac{\alpha x_1^{-1}}{\beta x_2^{-1}} = -\frac{\alpha}{\beta}\frac{x_2}{x_1}$$

Let's compare two CD Functions:

Increasing the exponent on either good will increase the consumers desire for that good. They will still get tired of it, but between two consumers, one with a larger exponent on a good, that consumer will have a stronger desire for the good at the same bundle.

$$u\left(x_1, x_2\right) = x_1 x_2$$

$$MRS = -\frac{x_2}{x_1}$$

At the point $(1,1)$: $MRS = -1$.

$$\tilde{u}(x_1, x_2) = x_1^{10} x_2$$

$$MRS = -10\frac{x_2}{x_1}$$

At the point $(1,1)$: $MRS = -10$

Notice that the consumer with $\tilde{u}(x_1, x_2) = x_1^{10} x_2$ would be willing to give up ten-times more x_2 to get the same amount of x_1 as the consumer with utility function $u(x_1, x_2) = x_1 x_2$.

5.4.4 Perfect Complements

$$u(x_1, x_2) = min\{ax_1, bx_2\}$$

Examples.

Left\Right Shoes

$$u(l, r) = min\{l, r\}$$

Notice that $min\{1,1\} = min\{2,1\} = min\{1,2\} = 1$. That is, the bundles $(1,1), (2,1), (1,2)$ are all indifferent. For this consumer, indifference curves are "L-shaped" and the kinks of these curves follow the path $l = r$.

Apple pies. 2 Apples, 1 Crust makes a pie.

$$u(a, c) = min\left\{\frac{1}{2}a, c\right\}$$

Notice that $min\left\{\frac{1}{2}2, 1\right\} = min\left\{\frac{1}{2}2, 2\right\} = min\left\{\frac{1}{2}3, 1\right\} = 1$. That is, the bundles $(2,1), (2,2), (3,1)$ are all indifferent. For this consumer, indifference curves are "L-shaped" but the "kinks" of these curves follow the line $\frac{1}{2}a = c$.

Notice $\frac{1}{2}a$ is the most pies we could possible make with a apples and c is the most possible pies we could make with c apples.

5.4.5 Max Preferences

$$u(x_1, x_2) = max\{x_1, x_2\}$$

Notice here, the consumer is indifferent between $(2,2), (1,2), (2,1)$. The indifference curves are *backwards L-shaped* curves. The bend the opposite direction to prefect complements indifference curves.

6 Choice (Chapter 5.1-5.3,5.5)

6.1 Three Possibilities

We are going to find the optimal bundle**s** in a budget set. We are going to look for bundles that are all at least as good as everything else in the budget set.

Assume $\succsim$ is reflexive, complete, transitive and $\succsim$ **monotonic**.

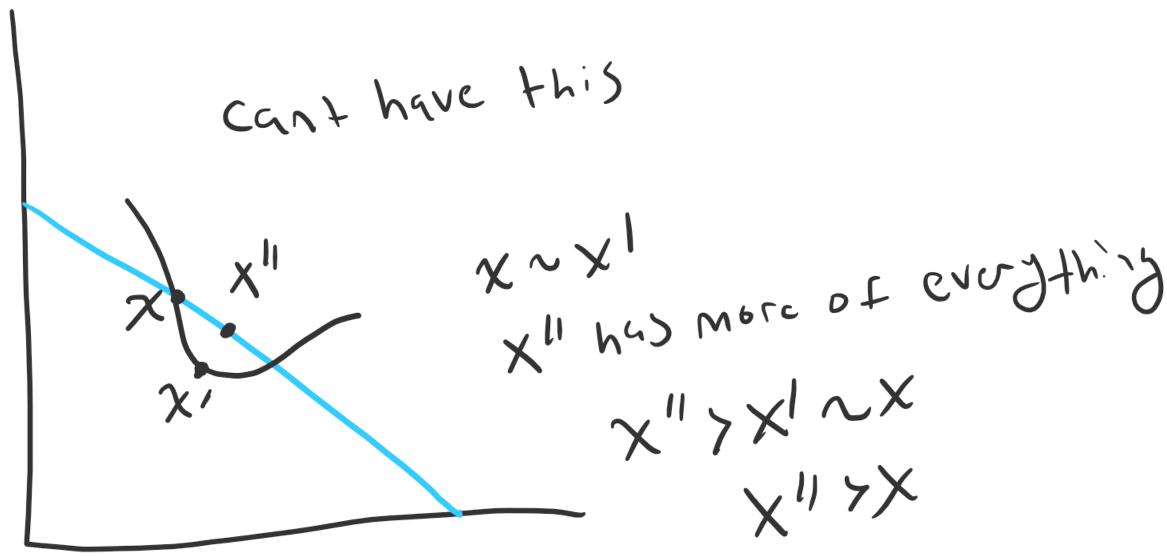

Figure 6.1: An optimal bundle cannot be on an indifference curve that passes "into" the budget set.

There are only three possibilities for an optimal bundle:

1. (Tangent) It is at bundle where the indifference curve at that bundle had the same slope as the budget line.

2. (Touching but not tangent) The bundle is a "non-smooth" point on the indifference curve, but the that point just touches the budget line.

3. (Boundary) We are at one of the boundaries ($x_1 = 0$ or $x_2 = 0$) in this case the slope of the indifference curve and the slope of the budget need not be equal.

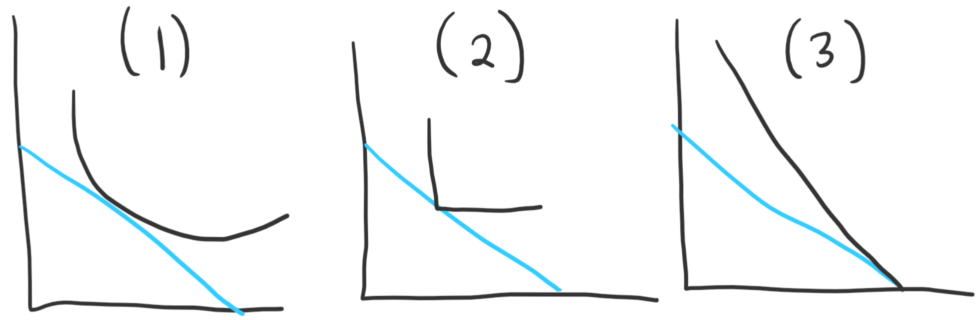

Figure 6.2: Graphical Examples of the Three Possibilities

Under some weak conditions (we can take derivatives of the utility function). The tangency condition is necessary for an *interior* optimum (involves consuming some of both things).

That is, if there is an optimal bundle that involves consuming some of both goods, it must have the property that the slope of the indifference curve at that optimal bundle is the same as the slope of the budget line. T

This condition is formalized by the familiar equation:

$$MRS = -\frac{p_1}{p_2}$$

6.2 Examples

6.2.1 Cobb Douglass:

$$u(x_1, x_2) = x_1 x_2$$

$$p_1 x_1 + p_2 x_2 = m$$

Tangency condition:

$$MRS = -\frac{\frac{\partial(x_1 x_2)}{\partial x_1}}{\frac{\partial(x_1 x_2)}{\partial x_2}} = -\frac{p_1}{p_2}$$

$$-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$$

Tangency Condition:

$$* x_1 p_1 = x_2 p_2$$

Budget Condition:

$$** \; x_1p_1 + x_2p_2 = m$$

Plug Tangency Condition into Budget Condition:

$$x_1p_1 + x_1p_1 = m$$

$$2x_1p_1 = m$$

$$x_1^* = \frac{1}{2}\frac{m}{p_1}$$

Plug this back into one of the two equations:

$$x_1p_1 = x_2p_2$$

$$\frac{1}{2}\frac{m}{p_1}p_1 = x_2p_2$$

$$\frac{1}{2}\frac{m}{p_1}\frac{p_1}{p_2} = x_2$$

$$x_2^* = \frac{1}{2}\frac{m}{p_2}$$

The optimal bundle:

$$\left(\frac{\frac{1}{2}m}{p_1}, \frac{\frac{1}{2}m}{p_2}\right)$$

6.2.2 Perfect Substitutes

$$u\left(x_1, x_2\right) = 2x_1 + x_2$$

$p_1 = 1$, $p_2 = 1$, $m = 10$

$$1x_1 + 1x_2 = 10$$

Tangency Condition:

$$-\frac{2}{1} = -\frac{1}{1}$$

$$-2 = -1$$

This is never true. There can't be an interior solution. There has to be a boundary solution. Let's check the utility of both.

$$u(x_1, x_2) = 2x_1 + x_2$$

Only consume x_1:

$$\left(\frac{m}{p_1}, 0\right) = (m, 0)$$

$$u(m, 0) = 2m = 20$$

Only consume x_2:

$$\left(0, \frac{m}{p_2}\right) = (0, m)$$

$$u(0, m) = m = 10$$

Since consuming only x_1 gives me more utility, that is the optimal bundle:

$$(m, 0)$$

6.2.3 Anything is Optimal

$$u(x_1, x_2) = 2x_1 + x_2$$

$p_1 = 2$, $p_2 = 1$, $m = 10$

$$2x_1 + 1x_2 = 10$$

$$-\frac{2}{1} = -\frac{2}{1}$$

$$-2 = -2$$

As long as I spend all of my money, any bundle is optimal.
All of the bundles such that:

$$p_1x_1 + p_2x_2 = m$$

6.2.4 Perfect Complements

$$u\left(x_1, x_2\right) = min\left\{x_1, x_2\right\}$$

$$2x_1 + x_2 = 15$$

We still know the budget condition must be true:

$$** \; 2x_1 + x_2 = 15$$

What is the other condition?
"No Waste Condition". (Equation for the "kink" points).

$$* \, x_1 = x_2$$

Plug one condition into the other:

$$2x_1 + x_2 = 15$$

$$2x_1 + x_1 = 15$$

$$3x_1 = 15$$

$$x_1 = 5$$

Plug back into one of the equations:

$$x_2 = 5$$

6.2.5 Perfect Complements (2 Apples, 1 Crust)

$$u\left(x_1, x_2\right) = min\left\{\frac{1}{2}x_1, x_2\right\}$$

$$2x_1 + x_2 = 15$$

We still know the budget condition must be true:

$$** \; 2x_1 + x_2 = 15$$

What is the other condition?
"No Waste Condition". (Equation for the "kink" points).

$$\frac{1}{2}x_1 = x_2$$

Combine the conditions:

$$2x_1 + x_2 = 15$$

$$2x_1 + \frac{1}{2}x_1 = 15$$

$$\frac{5}{2}x_1 = 15$$

$$x_1 = 6$$

$$x_2 = 3$$

6.2.6 Max Preferences

$$u\left(x_1, x_2\right) = max\left\{x_1, x_2\right\}$$

$$2x_1 + x_2 = 15$$

Try this one at home: what is the optimal bundle?
Solution:

$$x_1 = 0, x_2 = 15$$

7 Demand (6.1-6.8)

$$x_1 x_2$$

$$p_1, p_2, m$$

$$-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$$

$$\frac{x_2}{p_1} = \frac{x_1}{p_2}$$

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

In this form, we have "the marginal utility of a dollar spent on both goods is the same"

$$x_1 p_1 = x_2 p_2$$

$$x_1^* = \frac{\frac{1}{2}m}{p_1}$$

$$x_2^* = \frac{\frac{1}{2}m}{p_2}$$

7.1 Marshallian Demand

The Mashallian demand is the optimal amount of a good, given prices and income.

$$x_1^* \left(p_1, p_2, m\right)$$

$$x_2^* \left(p_1, p_2, m\right)$$

7.2 Changes in Income

$$\frac{\partial x_1^* \left(p_1, p_2, m\right)}{\partial \left(m\right)}?$$

How does demand change with income?

7.2.1 Normal/Inferior

If demand *increases* when income increases, we say the good is **"Normal".**

If demand *decreases* when income increases, we say the good is **"Inferior".**

Example: Cobb-Douglass $x_1 x_2$

$$x_1 = \frac{\frac{1}{2}m}{p_1}$$

Example: Inferior.

$$x_1 = \frac{10}{m p_1}$$

7.2.2 Income Offer Curve

Income offer curve is a plot of optimal bundles (x_1^*, x_2^*) as income changes but prices are fixed.

Example:$u(x_1, x_2) = x_1 x_2$. $p_1 = 2, p_2 = 1$:

$x_1 = \frac{\frac{1}{2}m}{2} = \frac{1}{4}m$, $x_1 = \frac{\frac{1}{2}m}{1} = \frac{1}{2}m$

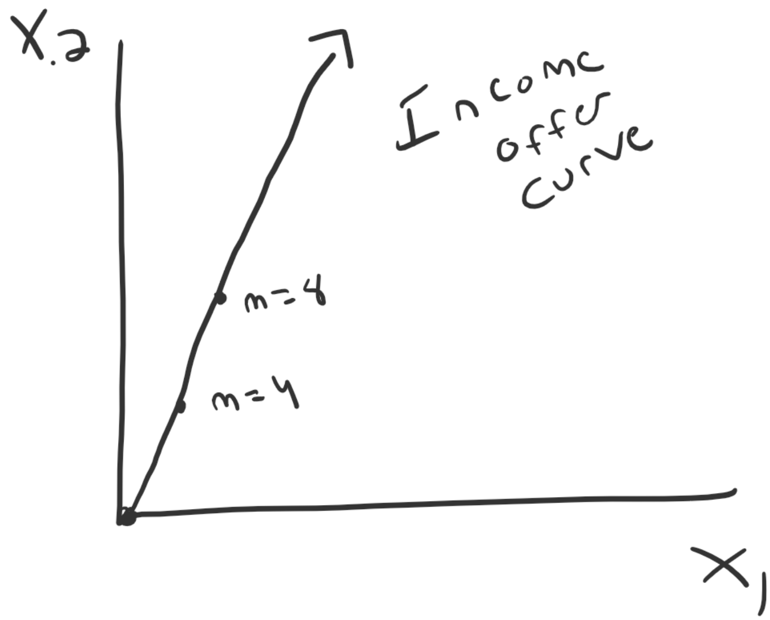

Figure 7.1: Income offer curve for Cobb Douglass preference example.

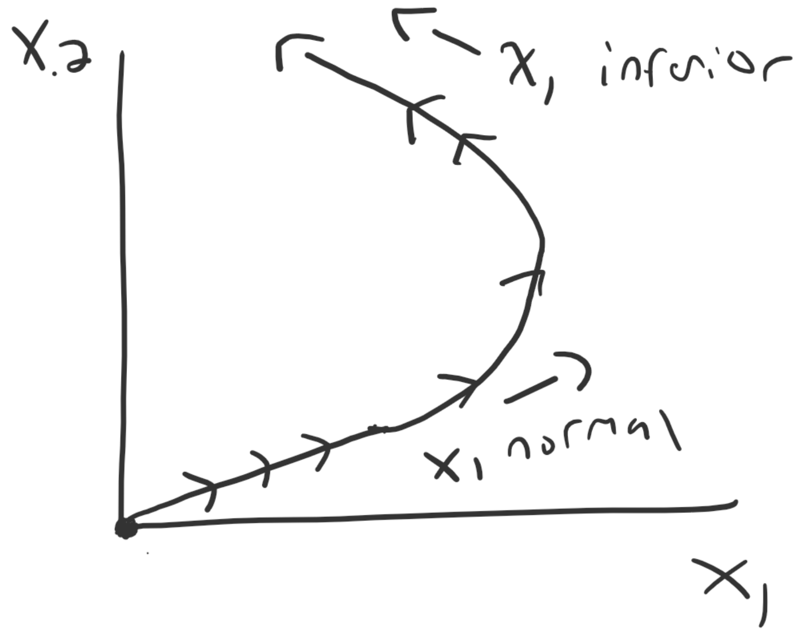

Figure 7.2: Example of an income offer curve where x_1 is initially normal but becomes inferior as m grows.

7.2.3 Engle Curve

The relationship between income and a **single** good. Plotting m on the y-axis against x_i on the x-axis.

$$x_1 = \frac{1}{4}m$$

$$m = 4x_1$$

Really what we are plotting is the amount of income a consumer would need to have to demand some amount x_1 of good 1.

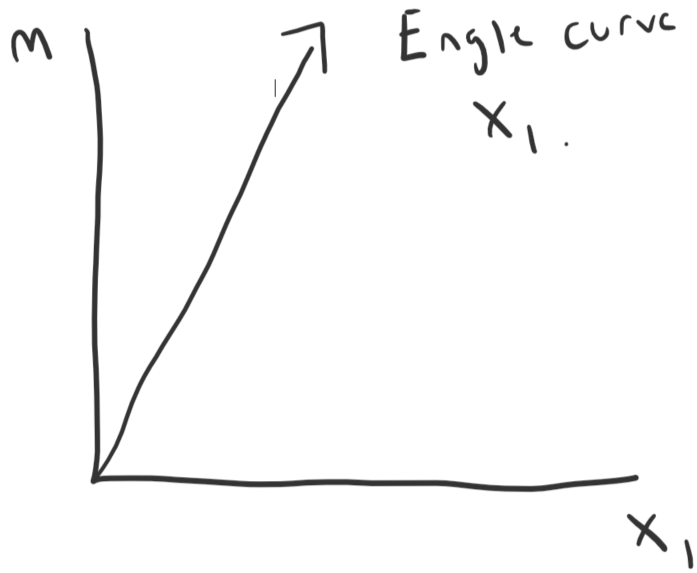

Figure 7.3: Engle curve for $x_1 = \frac{1}{4}m$.

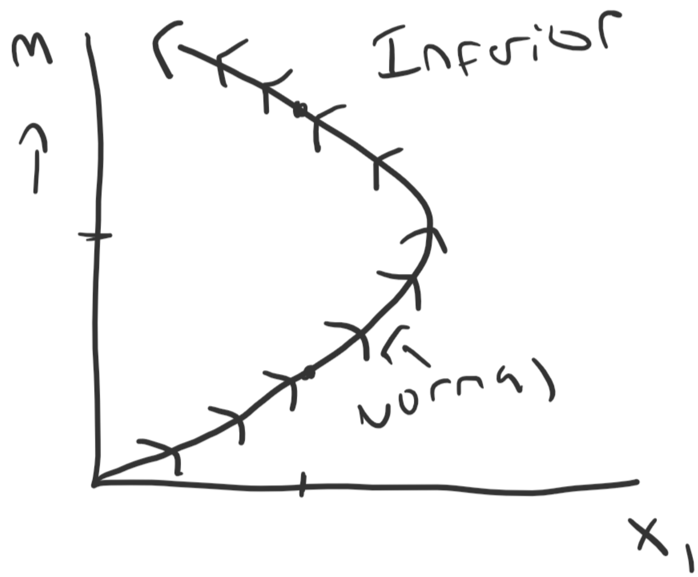

Figure 7.4: Engle curve for a good that is normal for low income and inferior for high income.

7.2.4 Example: Perfect Complements

$U(x_1, x_2) = min\{x_1, x_2\}$. $p_1 = 2$, $p_2 = 1$.
At the optimum, $x_1 = x_2$ (no waste condition).
$2x_1 + 1x_2 = m$ (budget condition)

$$2x_1 + x_1 = m$$

$$x_1 = \frac{m}{3}$$

$$x_2 = \frac{m}{3}$$

Income Offer Curve:

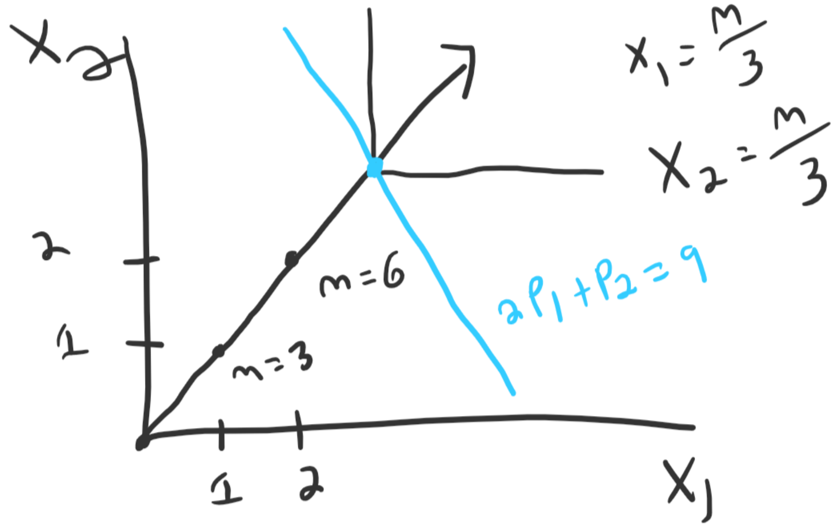

Figure 7.5: Income offer curve for $min\{x_1, x_2\}$ with $p_1 = 2$ and $p_2 = 1$

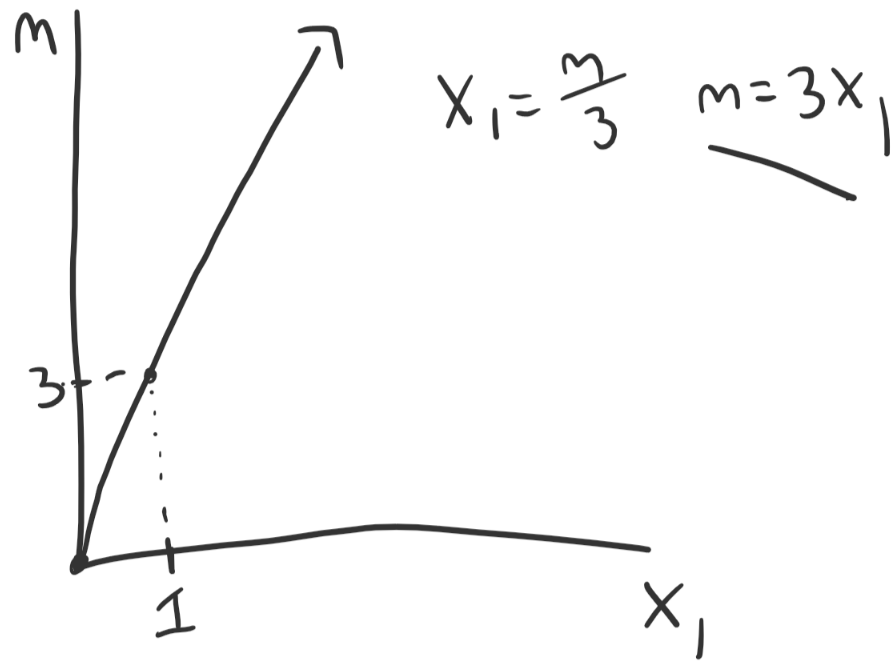

Figure 7.6: Engle Curve of x_1 for $min\{x_1, x_2\}$ with $p_1 = 2$ and $p_2 = 1$

7.3 Changes in "Own" Price

What happens to demand for a good when the price for that good changes.

$$\frac{\partial x_1 (p_1, p_2, m)}{\partial p_1}?$$

7.3.1 Ordinary/Giffen

When the price of a good goes up, and demand goes *down*, we say the good is **ordinary.**

When the price of a good goes up, and demand goes *up*, we say the good is **giffen.**

(We will see this more later). A giffen good has to be inferior.

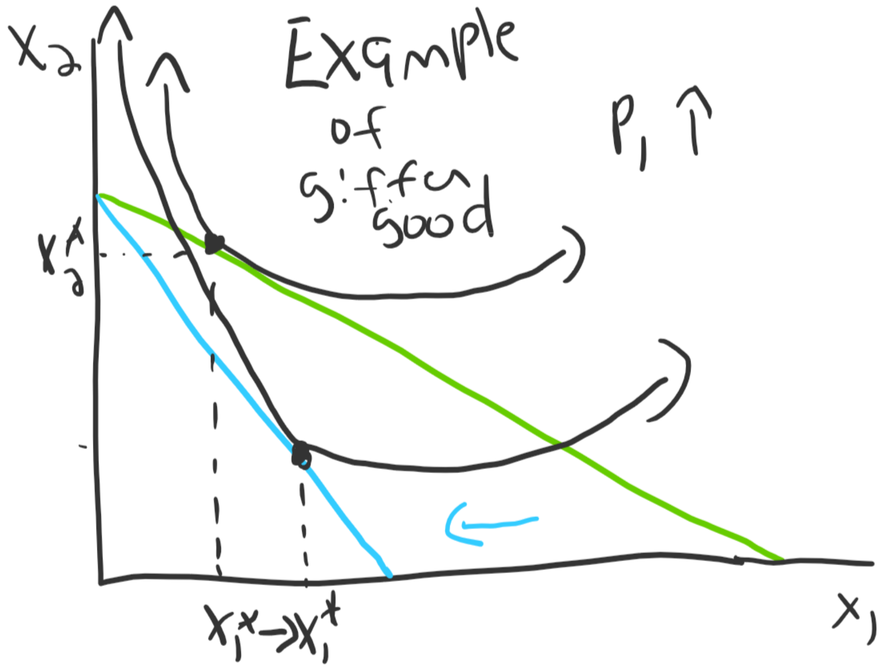

Figure 7.7: Indifference curves and budget for a "giffen" good. Note that as the price p_1 increases from the green to blue budget, the optimal amount of x_1 increases.

7.3.2 Price Offer Curve

Hold income and one of the prices fixed, the price offer curve is the set of bundles x_1^*, x_2^* that are optimal at each level of the other price.

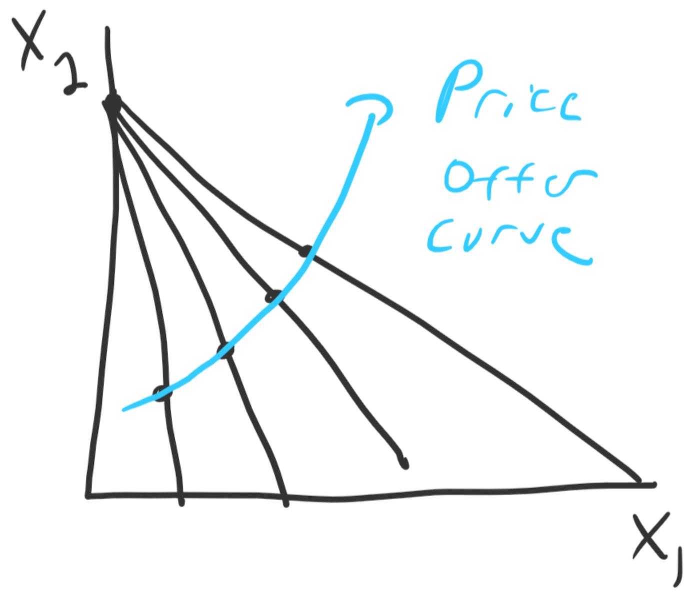

Figure 7.8: Example of a price offer curve (blue). The price offer curve plots the bundles that are optimal along each of the budget lines (black) as p_1 changes.

7.3.3 Plotting the Demand Curve

The **demand** for a good is $x_i(p_1, p_2, m)$ that is, the optimal amount that a consumer chooses given the prices and income. When we talk about "plotting" the demand curve of x_1 we usually mean holding p_2 and m fixed and plotting how the demand for x_1 changes as p_1 changes. For this, we put p_1 on the vertical axis and x_1 on the horizontal axis.

For example, suppose demand for x_1 is:

$$x_1 = \frac{\frac{1}{2}m}{p_1}$$

Let's plug in an income $m = 10$ and hold that fixed. We get $x_1 = \frac{5}{p_1}$. To plot this with p_1 on the vertcal axis, it is useful to solve for p_1. When we do this we get $p_1 = \frac{5}{x_1}$ this is what we call the **inverse demand.** It is the price that would be responsible for the consumer buying some amount x_1 of the good. This is actually what we plot when we are asked to plot the "demand".

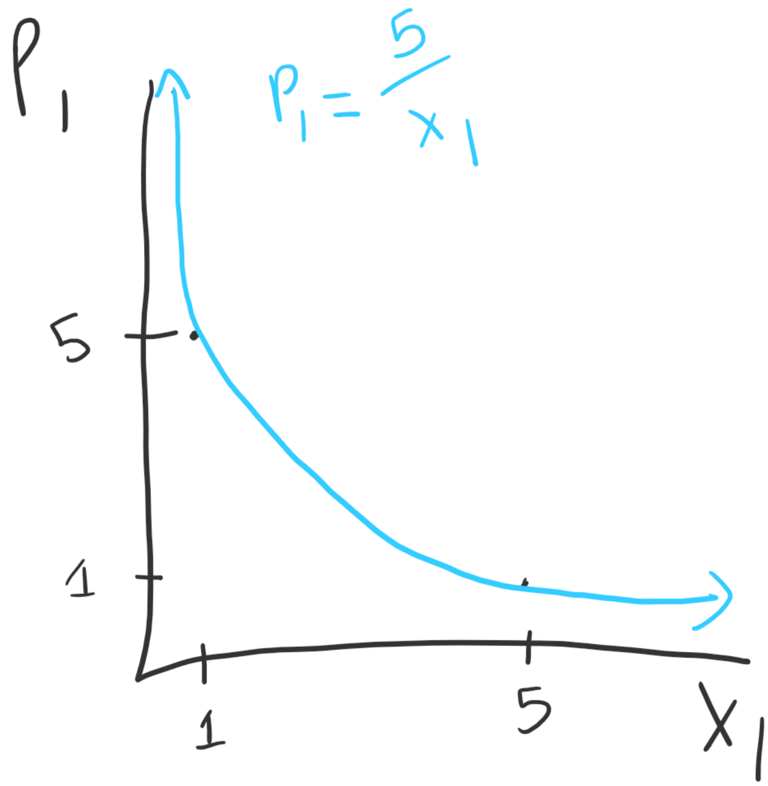

Figure 7.9: Plotting demand for $x_1 = \frac{5}{p_1}$.

7.4 Changes in "Other" Price

So far we have looked at what happens to a good when we change income and it's own price. We might also be interested in how demand changes for a good when the change the price of another good.

7.4.1 Complements/Substitutes

If the demand for a good goes **down** when the price of the other good goes up, we say the goods are **complements**.

If the demand for a good goes **up** when the price of the other good goes up, we say the goods are **substitutes**.

If demand for a good does not change when the price of the other good goes up, we say the goods are neither complements nor substitutes.

7.4.2 Examples Perfect Complements

$u = min\{x_1, x_2\}$ *has* demand $x_1 = \frac{m}{p_1+p_2}$ and $x_2 = \frac{m}{p_1+p_2}$. For both goods, as you increase the price of the other good, the demand goes down. They are **complements** (*hopefully this is not a surprise*).

7.4.3 Examples Perfect Substitutes

$u = x_1 + x_2$ has demand $x_1 = \frac{m}{p_1}$ $x_2 = 0$ if $p_1 < p_2$ and $x_1 = 0$ $x_2 = \frac{m}{p_2}$ if $p_1 > p_2$. Let's look at the change in p_1. If $p_1 < p_2$ and p_1 increases, then if it increases enough to that $p_1 > p_2$ the demand for x_2 increases from 0 to $x_2 = \frac{m}{p_2}$. So, as long as the change in price p_1 has any effect on the demand for p_2 (it might not if it does not change which price is higher in this example) then the goods are **substitutes.**

7.4.4 Examples Cobb Douglass

Suppose $u = x_1 x_2$. Demand is $x_1 = \frac{\frac{1}{2}m}{p_1}$ and $x_2 = \frac{\frac{1}{2}m}{p_2}$. Neither good's demand depends on the price of the other good. **They are neither complements nor substitutes.**

8 Slutsky Decomposition

Decomposes the change in demand for a good into two parts:

Substitution Effect: *Price went up, so I will demand less because I buy other things instead. This will always lead to a decrease in demand.*

Income Effect: *Price went up, so what I continue to buy is now more expensive. My effective income is now lower and my demand will change. May be positive or negative.*

Law of Demand:
For a change in price of good i the substitution effect (on good i) will always lead to a decrease or no change in demand x_i.
Thus, if price of a **normal** good increases, demand will decrease.

Three things that can happen.

Ordinary/Normal- *Both effects decrease demand.*

Ordinary/Inferior- *Substitution decreases demand (it always does) and income effect increases demand, but not enough to overcome the decrease due to substitution.*

Giffen/Inferior- *Substitution decreases demand (it always does) and income effect increases demand so much that it overcomes the decrease due to substitution and increases demand overall.*

8.1 The Slutsky Decomposition.

This decomposition is a thought experiment. Suppose price of a good increases, we go from the budget:

$$p_1 x_1 + p_2 x_2 = m$$

To a new budget:

$$p_1^{'} x_1 + p_2 x_2 = m$$

The **total effect** is:

$$x_1^* \left(p_1, p_2, m\right) - x_1^* \left(p_1^{'}, p_2, m\right)$$

How can we decompose demand? To study substitution effect only, we need to know what the consumer would choose if the price had changed, but their demand could not change due to income. Thus, we think about how much income would they need at the new prices to afford the old bundle? If we were to give the consumer this extra income and ask what they buy at the new prices, however their demand is different than the original bundle could not be due to income effect! It is due only to substitution.

To find this, we calculate the:

compensating income: cost of the original bundle under the new prices.

If we are analyzing a change in p_1 this would be:

$$\tilde{m} = p_1^{'} x_1^* \left(p_1, p_2, m\right) + p_2 x_2^* \left(p_1, p_2, m\right)$$

Now we construct a new budget:

$$p_1^{'} x_1 + p_2 x_2 = \tilde{m}$$

We ask: what does the consumer choose on this budget?

$$x_1^* \left(p_1^{'}, p_2, \tilde{m}\right)$$

The **substitution effect** is:

$$x_1^* \left(p_1, p_2, m\right) - x_1^* \left(p_1^{'}, p_2, \tilde{m}\right)$$

That is the difference between the original demand and the demand on this thought experiment budget (new prices, extra income).

The **income effect** is the remainder:

$$x_1^* \left(p_1^{'}, p_2, \tilde{m}\right) - x_1^* \left(p_1^{'}, p_2, m\right)$$

That is, the difference between what they choose on the thought experiment budget (new prices, extra income) and what they choose under the new prices with their actual income.

8.2 Graphically Decomposing Demand

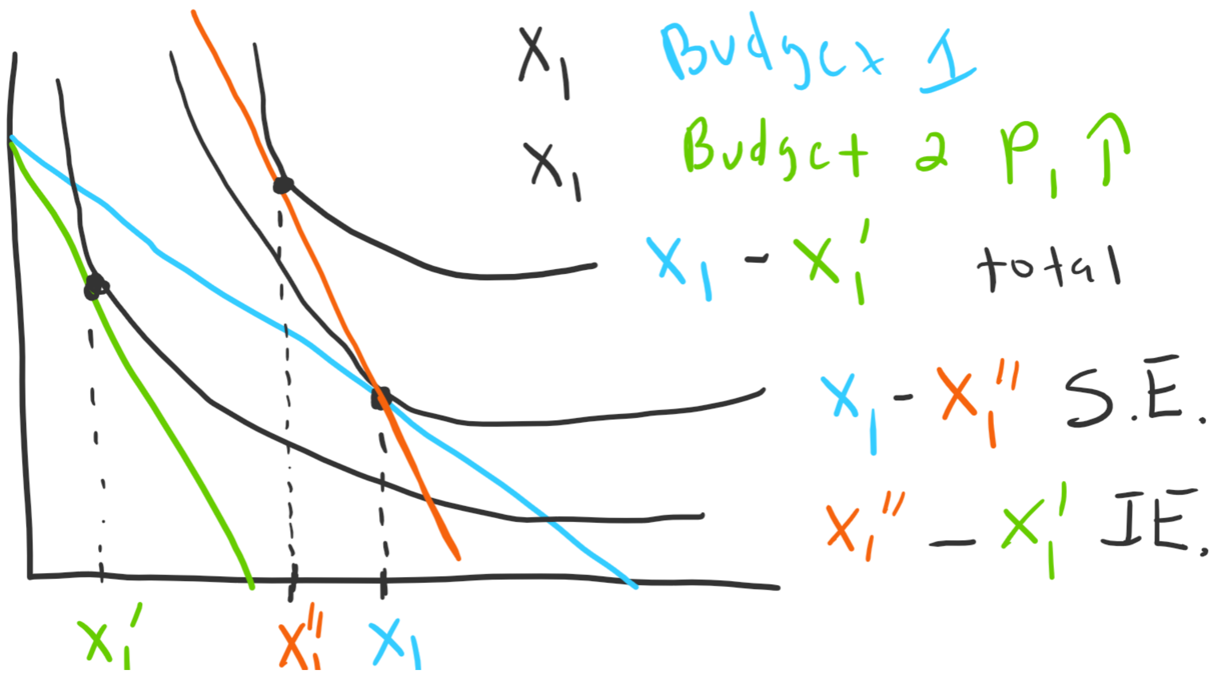

Figure 8.1: The Slutsky Decomposition for an Ordinary/Normal Good.

Figure 8.2: Slutsky Decomposition for an inferior good that is not Giffen. Notice that, while the income effect increases demand (orange to green) after the price increases, it does not increase enough to overcome the decrease in demand due to substitution (blue to orange).

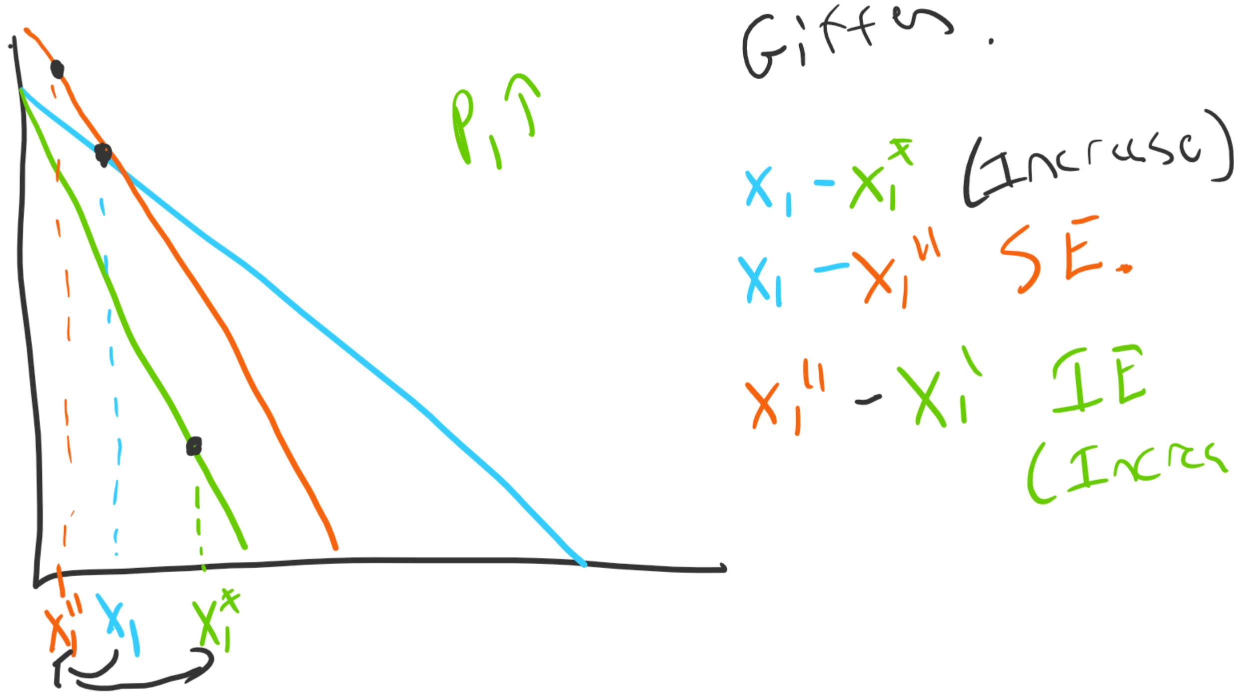

Figure 8.3: The Slutsky decomposition for a Giffen good. Notice the total effect is positive (demand increases when price increases). Demand decreases (blue to orange demand) due to substitution, but increases enough due to income (orange to green) to overcome this and lead to an overall increase in demand.

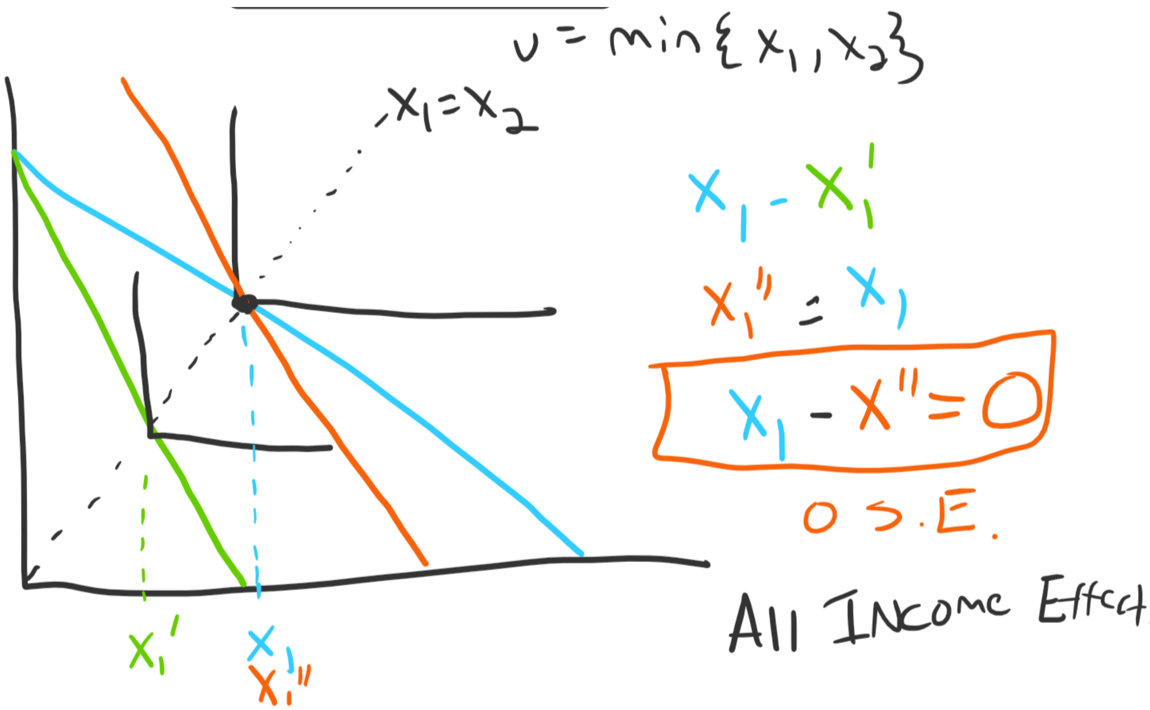

Figure 8.4: The Slutsky decomposition for perfect complements. There is **only** substitution effect.

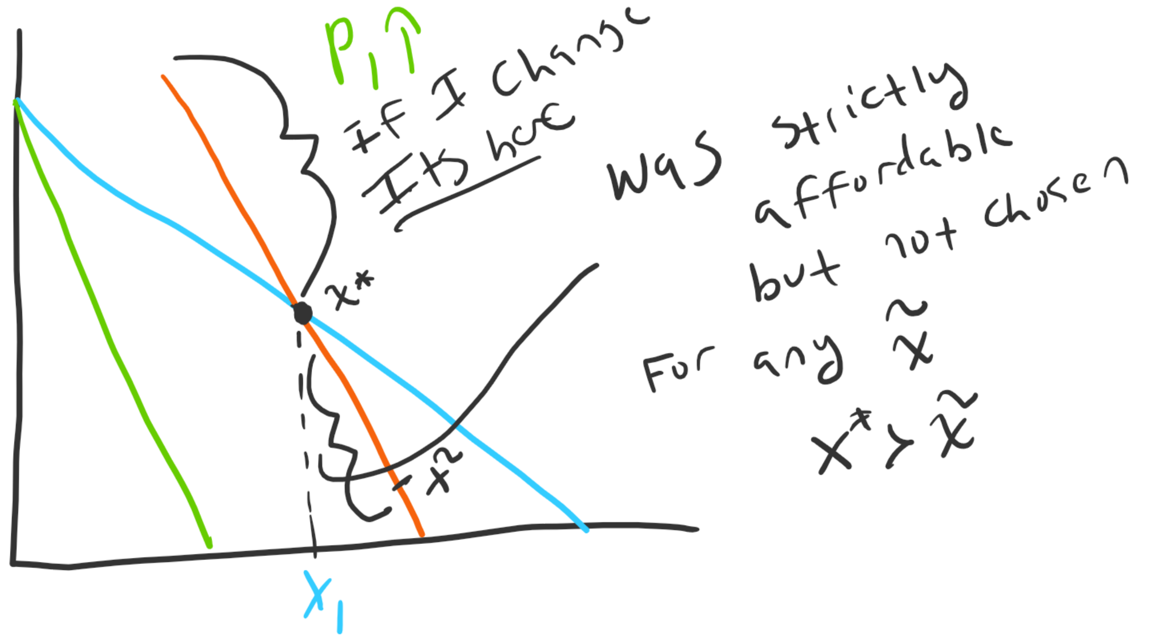

Figure 8.5: **The substitution effect must be negative.** Everything on the lower portion of the orange budget (which determines the substituion effect) was available under the original (blue) budget and was **strictly affordable**. Thus, the point chosen on the blue budget x^* must be strictly better than any of those points. Thus, no point on the lower point of the orange budget can be chosen. This implies demand must decrease due to substitution.

8.3 Example Problem

$$u = x_1 x_2$$

$$x_1^* = \frac{\frac{1}{2}m}{p_1}$$

$$x_2^* = \frac{\frac{1}{2}m}{p_2}$$

$p_1 = 1$, $p_2 = 2$, $m = 10$

Optimal Bundle (original prices):

$$x_1^* = \frac{\frac{1}{2}10}{1} = 5$$

$$x_2^* = \frac{\frac{1}{2}10}{2} = 2.5$$

Price of good 1 changes to $p_1^{'} = 2$

$$x_1^* = \frac{\frac{1}{2}10}{2} = 2.5$$

$$x_2^* = \frac{\frac{1}{2}10}{2} = 2.5$$

Total effect:

$$x_1\left(1, 2, 10\right) = 5$$

$$x_1\left(2, 2, 10\right) = 2.5$$

$$Total\,Effect = (5 - 2.5) = 2.5$$

Let's calculate the income needed to afford old bundle at the new prices.
Old Bundle: $(5, 2.5)$
Cost of this under the new prices: $p_1 = 2$, $p_2 = 2$

$$5\left(2\right) + 2.5\left(2\right) = 15$$

Compensating Income.
We need to construct a budget that has the **new prices** but enough income to afford the old bundle.

$$p_1 = 2, p_2 = 2, m = 15$$

What does the consumer actually demand here?

$$x_1\left(2, 2, 15\right) = \frac{\frac{1}{2}15}{2} = 3.75$$

The demand for good 1 under the "thought experiment" is 3.75.
Total Effect: $5 - 2.5 = 2.5$
Substitution Effect: $5 - 3.75 = 1.25$
Income Effect:
(Total Effect-Substitution): 2.5 - 1.25 = 1.25
Thought Experiment Demand - Demand After Change:

$$3.75 - 2.5 = 1.25$$

The price change decreases demand by 2.5. Demand is decreased by 1.25 due to substitution and decreased by 1.25 due to the income effect.

9 Buying and Selling (9.1-9.4)

9.1 Income to Endowments

Until this point our consumers had income in terms of money. $m = \$10$ for instance.
Now we will think of the consumers as having an **endowment** of goods to start with.

$$y \rightarrow w_1, w_2$$

w_1 is the endowment of good 1. It is the amount the start with.
Apple Farmer grows apples. x_1 apples. x_2 crusts.

$$w_1 = 10$$

$$w_2 = 0$$

New Budget Equation:
Cost of chosen bundles equals the value of the endowment:

$$p_1 x_1 + p_2 x_2 = p_1 w_1 + p_2 w_2$$

Income now reacts to changes in prices.

9.2 Gross Demand vs. Net Demand

Gross Demand: x_i
Net Demand: $x_i - w_i$
We can also write the budget equation in terms of net demand:
Rearrange the budget equation:

$$p_1 x_1 + p_2 x_2 = p_1 w_1 + p_2 w_2$$

$$p_1 x_1 - p_1 w_1 = p_2 w_2 - p_2 x_2$$

$$p_1 (x_1 - w_1) = p_2 (w_2 - x_2)$$

$$p_1 (x_1 - w_1) - p_2 (w_2 - x_2) = 0$$

Budget Balance in terms of Net Demand:

$$p_1 (x_1 - w_1) + p_2 (x_2 - w_2) = 0$$

If I am a net demander of one good, I am a net supplier of the other.

9.3 Net Buyer/Net Seller

Net Buyer or Net Demand of good 1: $x_1 - w_1 > 0, x_1 > w_1$
Net Seller or Net Supplier of good 1: $x_1 - w_1 < 0, x_1 < w_1$

9.4 Drawing the Budget Line and Changes to Price

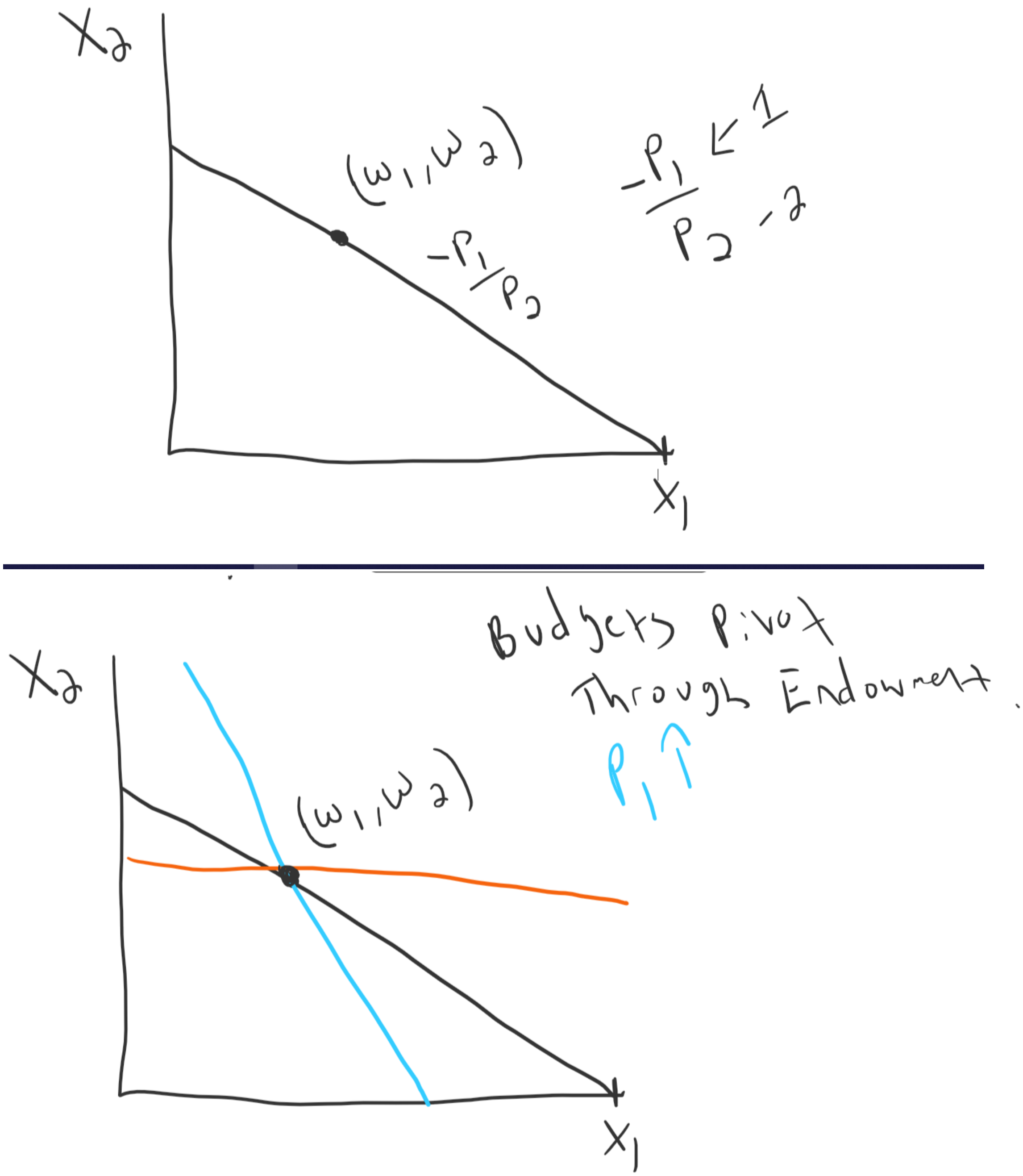

Figure 9.1: The budget line always passes through the endowment (w_1, w_2). If prices change, the slope changes, and the budget pivots through this point. p_1 increase (or p_2 decrease) is shown in blue. p_1 decrease (or p_2 increase) is shown in orange.

x_1 intercept: "the amount of x_1 afford if I only buy x_1"

$$\frac{p_1 w_1 + p_2 w_2}{p_1} = w_1 + \frac{p_2 w_2}{p_1}$$

x_2 intercept:

$$\frac{p_1 w_1 + p_2 w_2}{p_2} = \frac{p_1 w_1}{p_2} + w_2$$

9.5 Price Changes and Net Buyers/Sellers

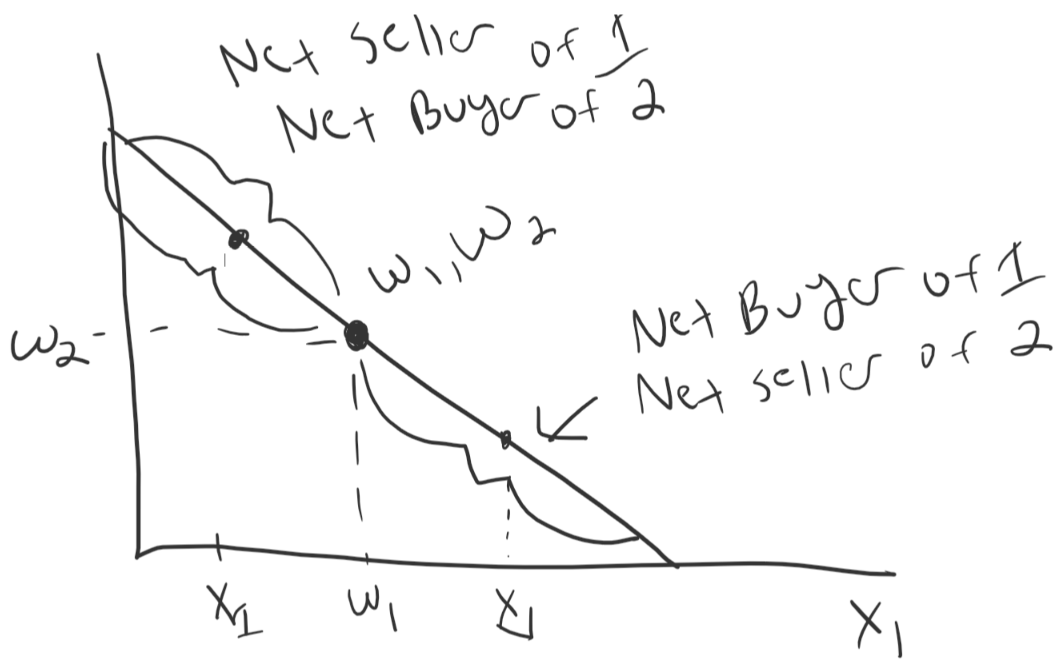

Figure 9.2: Regions where a consumer is a net buyer/seller.

A consumer who is a **net buyer of a good**, if the price of that good **decreases**, they will remain a **net buyer** and be **strictly better off**.

A consumer who is a **net seller of a good**, if the price of that good **increases**, they will remain a **net seller** and be **strictly better off**.

9.6 Example Problem

Apple farmer. $w_1 = 10$, $w_2 = 0$

$p_1 = 1$, $p_2 = 1$. $u = min\left\{\frac{1}{2}x_1, x_2\right\}$

$$p_1x_1 + p_2x_2 = p_1w_1 + p_2w_2$$

$$1x_1 + 1x_2 = 1(10) + 1(0)$$

$$x_1 + x_2 = 10$$

No Waste Condition:

$$\frac{1}{2}x_1 = x_2$$

Budget Condition:

$$x_1 + x_2 = 10$$

$$x_1 + \frac{1}{2}x_1 = 10$$

$$\frac{3}{2}x_1 = 10$$

$$x_1 = \frac{20}{3}$$

$$x_2 = \frac{1}{2}(x_1) = \frac{10}{3}$$

10 Intertemporal Choice (Chapter)

10.1 Bundles (Consumption Today, Consumption Tomorrow)

Borrowing and Saving Behavior.

Two period model.

c_1, c_2- consumption on in period 1 and consumption in period 2. A bundle is (c_1, c_2). How much money to use for consumption in period 1 and 2.

m_1, m_2- Income in period 1 and 2. This is the "endowment" of money in both periods.

Because this is the endowment, it will always be "affordable"

10.2 Prices (Interest Rate)

"Price" become the interest rate.

r is the interest rate.

If want to borrow \$1000 in period 1, you pay back $1000\,(1+r)$ in period 2.

$$1000 + 1000\,(r)$$

If you save \$1000 in period 1, you get back:

$$1000 + 1000\,(r)$$

10.3 Budget Constraint (Future Value Version)

If I saved money: my consumption in period 2 is my income in period 2 (m_2) plus how much I saved in period 1 ($m_1 - c_1$) multiplied my $1 + r$

$$c_2 = m_2 + (1+r)\,(m_1 - c_1)$$

Suppose borrowed money in period 1. My consumption in period 2 is my income in period 2 minus the amount I have to pay back to cover my loan from period 1.

$$c_2 = m_2 - (1+r)\,(c_1 - m_1)$$

$$c_2 = m_2 + (1+r)\,(m_1 - c_1)$$

The **future value of income**. How much c_2 can I consume if I only consume c_2 :

$$c_2 = m_2 + (1+r)\,m_1$$

10.4 Budget Constraint (Present Value Version)

$$c_2 = m_2 + (1+r)\,(m_1 - c_1)$$

$$c_2 = m_2 + (1+r)\,m_1 - (1+r)\,c_1$$

$$(1+r)\,c_1 + c_2 = m_2 + (1+r)\,m_1$$

$$(1+r)\,c_1 = m_2 + (1+r)\,m_1 - c_2$$

$$c_1 = \frac{m_2}{(1+r)} + m_1 - \frac{c_2}{(1+r)}$$

$$c_1 = m_1 + \frac{m_2 - c_2}{(1+r)}$$

Suppose consumption in period 2 is zero. I am borrowing as much as I possibly can:

$$c_1 = m_1 + \frac{m_2}{(1+r)}$$

Notice if I take out a loan of $\frac{m_2}{(1+r)}$ I will owe the bank m_2 which is exactly my income. This is the biggest loan I can take out.
Present value of income:

$$m_1 + \frac{m_2}{(1+r)}$$

10.5 Plotting the Budget Equation.

We are used to seeing:

$$p_1 x_1 + p_2 x_2 = p_1 w_1 + p_2 w_2$$

Out budget equation is:

$$c_1 = m_1 + \frac{m_2 - c_2}{(1+r)}$$

$$(1)\, c_1 + \left(\frac{1}{1+r}\right) c_2 = (1)\, m_1 + \left(\frac{1}{1+r}\right) m_2$$

Multiply both sides by $(1+r)$

$$(1+r)\, c_1 + c_2 = (1+r)\, m_1 + m_2$$

For both of these, the ratio of prices:

$$(1+r)$$

This is the slope of the budget equation.

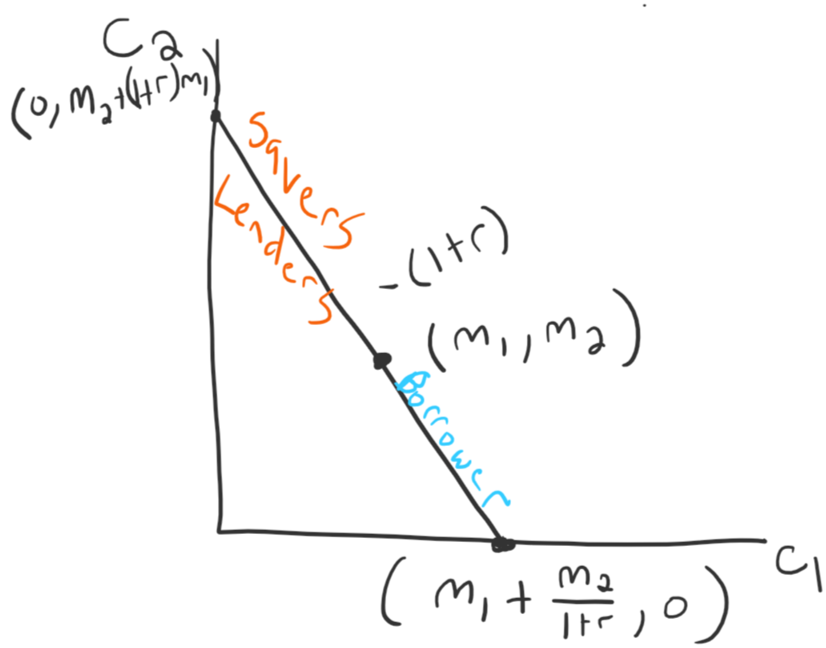

Figure 10.1: The budget equation for an intertemporal choice problem.

10.6 Comparative Statics

A borrow, when the interest rate goes down.

Because you are a "net buyer" of "good 1" and an interest rate decrease is really a decrease in the price of consumption in period 1.

Remains a borrower. And must be strictly better off.

A lender (saver), interest rate goes up.

A lender is really a net seller of c_1. If the price of c_1 goes up because the interest rate increased,

Remain a lender (saver). And must be strictly better off.

10.7 Example Problem

$m_1 = 200$. $m_2 = 600$. $r = \frac{1}{2}$. Utility: $u(c_1, c_2) = c_1 c_2$.

Write down the budget equation:

$$(1 + r) c_1 + c_2 = (1 + r) m_1 + m_2$$

Only consume this month (set $c_2 = 0$):

$$c_1 = m_1 + \frac{m_2}{(1+r)} = 600.$$

Only consume next month (set $c_1 = 0$):

$$c_2 = (1+r)\, m_1 + m_2 = 900.$$

Finding the optimal consumption:

$$MRS = -\,(1+r)$$

$$-\frac{c_2}{c_1} = -1.5$$

The optimality condition is:

$$c_2 = (1.5)\, c_1$$

Plug this back into the budget equation and solve:

$$1.5c_1 + c_2 = (1.5)\, 200 + 600$$

$$1.5c_1 + 1.5c_1 = (1.5)\, 200 + 600$$

$$3c_1 = 900$$

$$c_1 = \frac{900}{3} = 300$$

$$c_2 = (1.5)\, 300 = 450$$

At this interest rate the consumer is a borrower since $c_1 = 300 > 200 = m_1$.
If the interest rate were to decrease to $\frac{1}{4}$ we know that he will remain a borrower.
Solve this at home and check that this is the case.

11 Market Demand (15.1-15.2,15.5-15.6,15.8,15.11)

11.1 Adding Demand Curves

n consumers, each with a demand for good 1 and a demand for good 2.
Demand of consumer i for good 1: $x_i^1(p_1, p_2, m_i)$

$$X^1(p_1, p_2, m_1, ..., m_n) = \sum_{i=1}^{n} x_i^1(p_1, p_2, m_i)$$

Market demand for good 2:

$$X^2(p_1, p_2, m_1, ..., m_n) = \sum_{i=1}^{n} x_i^2(p_1, p_2, m_i)$$

11.2 Example Cobb Douglass Demand

Suppose we have Cobb Douglass consumers.
All have the utility function:

$$u_i\left(x_i^1, x_i^2\right) = \left(x_i^1\right)^1 \left(x_i^2\right)^1$$

(Note: the 1 and 2 superscripts are no exponents, but rather the label for the good.)

$$x_i^1 = \frac{\frac{1}{2}m_i}{p_1}$$

$$x_i^2 = \frac{\frac{1}{2}m_i}{p_2}$$

Market demand for good 1 is the sum of the individual demands:

$$\sum_{i=1}^{n}\left(x_i^1\right) = \sum_{i=1}^{n}\left(\frac{\frac{1}{2}m_i}{p_1}\right)$$

Suppose $p_1 = 1$ and $m_1 = 10$, $m_2 = 20$, $m_3 = 30$.

$$\left(\frac{\frac{1}{2}10}{1}\right) + \left(\frac{\frac{1}{2}20}{1}\right) + \left(\frac{\frac{1}{2}30}{1}\right) \quad = \quad 30$$

Notice: let $M = \sum_{i=1}^{n} m_i$. aggregate income.

$$\sum_{i=1}^{n}\left(\frac{\frac{1}{2}m_i}{p_1}\right) = \frac{1}{2}\frac{1}{p_1}\sum_{i=1}^{n} m_i = \frac{\frac{1}{2}M}{p_1}$$

11.3 Homothetic Preferences.

If

$$x \succsim y$$

Then

$$tx \succsim ty$$

Suppose $(1,2) \succsim (2,1)$. Then, $(2,4) \succsim (4,2)$.
Cobb Douglass Preferences are homothetic:

$$u = x_1^{\alpha} x_2^{\beta}$$

$$x_1^{\alpha} x_2^{\beta} > \tilde{x}_1^{\alpha} \tilde{x}_2^{\beta}$$

So $(x_1, x_2) \succ (\tilde{x}_1, \tilde{x}_2)$
We know:

$$(tx_1)^{\alpha} (tx_2)^{\beta} > (t\tilde{x}_1)^{\alpha} (t\tilde{x}_2)^{\beta}$$

Because:

$$t^{\alpha} t^{\beta} \left(x_1^{\alpha} x_2^{\beta} \right) > t^{\alpha} t^{\beta} \left(\tilde{x}_1^{\alpha} \tilde{x}_2^{\beta} \right)$$

$$\left(x_1^{\alpha} x_2^{\beta} \right) > \left(\tilde{x}_1^{\alpha} \tilde{x}_2^{\beta} \right)$$

There is an easier way to test if preferences are homothetic. If preferences are homothetic, then MRS depends on the ratio of goods but not the amount.

$$-\frac{\frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_1}}{\frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_2}} \quad = \quad -\frac{\alpha x_2}{\beta x_1}$$

MRS for $(1,1)$ is $-\frac{\alpha}{\beta}$ and the MRS for $(2,2)$ is $-\frac{\alpha}{\beta}$. This only depends on the ratio of goods.

Here are some non-homothetic preferences.

$$u = x_1 + \sqrt{x_2}$$

$$-\frac{\frac{\partial(x_1+\sqrt{x_2})}{\partial x_1}}{\frac{\partial(x_1+\sqrt{x_2})}{\partial x_2}} = - \quad 2\sqrt{x_2}$$

The indifference curves are parallel along a ray through the origin.
Homothetic preferences will always have linear Engle curves.

11.4 Elasticity

Suppose the price of a good changes from 1 to 2. Consumer 1's demand changes from 100 to 50 and consumer 2's changes from 10 to 5. Their behavior in terms of absolute changes in demand $\frac{\Delta x_i}{\Delta p_i}$ is wildly different, but their behavior in terms of percentage terms $\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}}$ is identical. Elasticity is simply a way of quantifying comparative statics in **unit-free percentage terms.**

$$\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}} = \frac{\frac{100-50}{100}}{\frac{1-2}{1}} = -\frac{1}{2}$$

Interpret this as: for every 1% increase in price, demand goes down by $\frac{1}{2}$%.

11.5 Price Elasticity

For very small changes in price, we measure this through derivatives:
"Price elasticity"

$$\epsilon_{i,i} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_i}{p_i}} = \frac{\partial x_i}{\partial p_i}\frac{p_i}{x_i}$$

"Cross-price elasticity"

$$\epsilon_{i,j} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$$

Cobb-Douglass Example: Suppose utility is $u = x_1 x_2$
Demand:

$$x_1 = \frac{\frac{1}{2}m}{p_1}$$

Price Elasticity:

$$\epsilon_{1,1} = \frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial p_1}\frac{p_1}{\frac{\frac{1}{2}m}{p_1}} = -1$$

$$= -\left(\frac{\frac{1}{2}m}{p_1^2}\right)\frac{p_1}{\frac{\frac{1}{2}m}{p_1}}$$

$$= -\left(\frac{\frac{1}{2}m}{p_1^2}\right)\frac{p_1^2}{\frac{1}{2}m}$$

$$= -1$$

Constant unit-elastic demand. elasticity is -1. A 1% increase in price leads to a 1% decrease in demand.

If elasticity is less than -1. For instance -2. Demand is "**Elastic**".

If elasticity is more than -1. For instance $-\frac{1}{2}$. Demand is "**Inelastic**".

11.6 Cross-Price Elasticity:

$$\epsilon_{1,2} = \frac{\partial\left(x_1\right)}{\partial p_2}\frac{p_2}{x_1}$$

Cross price-elasticity for good 1 with respect to price 2.

Example: Cobb-Douglass demand.

If $u = x_1 x_2$ then demand for good 1 is: $\frac{\frac{1}{2}m}{p_1}$.

$$\epsilon_{1,2} = \frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial p_2}\frac{p_2}{x_1} = 0$$

Because demand for one good does not depend on the price of the others, the cross-price elasticities are zero for **Cobb Douglass.**

Example: suppose utility is: $u = min\left\{x_1, x_2\right\}$.

Optimal demand for good 1:

$$x_1 = \frac{m}{p_1 + p_2}$$

$$\epsilon_{1,2} = \frac{\partial\left(\frac{m}{p_1+p_2}\right)}{\partial p_2}\frac{p_2}{\frac{m}{p_1+p_2}}$$

$$\epsilon_{1,2} \quad = \quad -\frac{p_2}{p_1 + p_2}$$

When price p_2 increases by 1%, demand for x_1 goes down by $\frac{p_2}{p_1+p_2}$%.

11.7 Income Elasticity

$$\eta_i = \frac{\partial x_i}{\partial m} \frac{m}{x_i}$$

This is the percent that demand changes when we increase income by 1%.
Example. Cobb Douglass Demand

$$x_i = \frac{\frac{1}{2}m}{p_i}$$

$$\eta_i = \frac{\partial x_i}{\partial m} \frac{m}{x_i}$$

$$\eta_i = \frac{\partial \left(\frac{\frac{1}{2}m}{p_i} \right)}{\partial m} \frac{m}{\frac{\frac{1}{2}m}{p_i}}$$

$$\eta_i = 1$$

Example: $u = x_1^{\alpha} x_2^{\beta}$

$$x_1 = \frac{\frac{\alpha}{\alpha+\beta} * m}{p_i}$$

$$\eta_1 = \frac{\partial \left(\frac{\frac{\alpha}{\alpha+\beta} * m}{p_1} \right)}{\partial m} \frac{m}{\frac{\frac{\alpha}{\alpha+\beta} * m}{p_1}} = 1$$

$$\eta_2 = \frac{\partial \left(\frac{\frac{\beta}{\alpha+\beta} * m}{p_2} \right)}{\partial m} \frac{m}{\frac{\frac{\beta}{\alpha+\beta} * m}{p_2}} = 1$$

Example: $u = 2x_1 + x_2$. $p_1 = 1, p_2 = 1$
Spend all money on good 1.

$$x_1 = \frac{m}{p_1}$$

$$\frac{\partial \left(\frac{m}{p_1} \right)}{\partial m} \frac{m}{\frac{m}{p_1}} \quad = \quad 1$$

For the other good we either say the income elasticity is zero because the demand doesn't change or we can say it is undefined.

Example:

Suppose demand is $\frac{\frac{1}{2}m^2}{p_1}$:

$$\frac{\partial\left(\frac{\frac{1}{2}m^2}{p_1}\right)}{\partial m}\frac{m}{\left(\frac{\frac{1}{2}m^2}{p_1}\right)} = 2$$

Constant income elasticity.

Another *Example:*

Suppose demand is $\frac{\log(m)}{p_1}$:

$$\frac{\partial\left(\frac{\log(m)}{p_1}\right)}{\partial m}\frac{m}{\left(\frac{\log(m)}{p_1}\right)} = \frac{1}{\log(m)}$$

Non-constant income elasticity of demand.

12 Equilibrium (Chapter 16.1-16.9)

12.1 Market Demand/Supply

We focus on one good at a time. This is called: *Partial Equilibrium.*

Market demand $Q_d(p)$ (what is the total amount demanded at price p).

Market supply $Q_s(p)$ (what is the total amount supplied at price p).

Inverse market demand: $p_d(Q)$ (at what price are Q units demanded?)

Inverse market supply: $p_s(Q)$ (at what price are Q units supplied?)

Example.

If Market Demand is:

$$Q_d = 1000 - p$$

Inverse market demand is:

$$p = 1000 - Q_d$$

Example. Cobb Douglass:

Suppose everyone had utility $x_1 x_2$. They all demand $\frac{\frac{1}{2}m_i}{p_1}$ units of x_1. In total..

Market Demand for Cobb Douglass Consumers:

$$Q_d = \frac{\frac{1}{2}M}{p}$$

Inverse Market Demand for Cobb Douglass Consumers:

$$p = \frac{\frac{1}{2}M}{Q_d}$$

12.2 What is an equilibrium?

An equilibrium is defined as a price p^* such that:

$$Q_d(p^*) = Q_s(p^*)$$

Suppose at some price p, supply exceeds demand:

$$Q_s(p) > Q_d(p)$$

In this case price is too high. There are **surplus** units of the good, and any firm with a surplus unit would be willing to sell at a lower price. There is **downward pressure** on prices.

Suppose demand exceeds supply:

$$Q_d(p) > Q_s(p)$$

In this case price is too low. There is a **shortage** and consumers willing to buy at a higher price. There is **upward pressure** on prices.

12.3 Example

$$Q_d = \frac{500}{p}$$

$$Q_s = 100p$$

We look for a price p^* such that $Q_d = Q_s$.

$$\frac{500}{p} = 100p$$

$$500 = 100p^2$$

$$\frac{1}{200}(1000) = p^2$$

$$p^* = \sqrt{5}$$

This is the equilibrium price. To get equilibrium quantity, plug into either supply or demand. We should get the same thing:

$$Q_s = 100\left(\sqrt{5}\right)$$

$$Q_d = \frac{\frac{1}{2}1000}{\sqrt{5}} = \frac{500}{\sqrt{5}} = 100\frac{5}{\sqrt{5}} = 100\sqrt{5}$$

$$q^* = 100\sqrt{5}$$

12.4 Example. Fixed Supply

With fixed supply, the Q_s is constant for any price. The inverse supply curve (the thing we plot on the "equilibrium graph" is a vertical line). This would be the case, for instance, with concert tickets. The size of the venue is fixed regardless of the price of tickets.

$$Q_s = 1000$$

$$Q_d = \frac{500}{p}$$

Find the equilibrium price:

$$1000 = \frac{500}{p}$$

$$p^* = \frac{1}{2}$$

12.5 Effect of a Tax

The government imposes a tax of t per unit of good. If we think of p as being the "sticker price" then we have:

$$Q_s(p) = Q_d(p+t)$$

Consumers pay the "sticker price" plus the tax and firms get the sticker price. We could alternatively think of p as the price consumers actually pay (this would be the case when tax is included in the posted price). Then the firm gets $p - t$. It turns out, these will be exactly the same. I recommend sticking with one. Since the "sticker price" is what we are used to (we pay the stricker price plus tax), I will use that for the examples.

Example.

$$Q_s = 100p$$

$$Q_d = 300 - 50p$$

The inverse supply and demand:

$$p_s = \frac{Q}{100}$$

$$p_d = \frac{300 - Q}{50} = 6 - \frac{Q}{50}$$

The equilibrium without a tax. Set $Q_s = Q_d$:

$$100p = 300 - 50p$$

$$150p = 300$$

$$p^* = 2$$

$$q^* = 200$$

Suppose the government adds a tax of $t = 1$. Consumers will pay $p + 1$ since they pay the "sticker price" p plus the tax. We solve for the "sticker price" p the equates supply and demand:

$$300 - 50\,(p + 1) = 100p$$

$$300 - 50p - 50 = 100p$$

$$250 = 150p$$

$$\frac{5}{3} = p$$

Suppliers get $p = \frac{5}{3}$ per unit and consumers pay $p + t = \frac{5}{3} + 1 = \frac{8}{3}$.

To the market quantity, plug this into the supply function (we will also check that the same quantity is demanded):

$$Q_s = 100\left(\frac{5}{3}\right) = \frac{500}{3}$$

$$Q_d = 300 - 50\left(\frac{8}{3}\right) = 300 - \frac{400}{3} = \frac{900 - 400}{3} = \frac{500}{3}$$

The effect of the tax is that the new equilibrium has a lower quantity. Consumers pay more then they used to and suppliers receive less than they used to. Both are worse off. How should we quantify "worse off"? We use surplus.

12.6 Surplus

Consumer surplus is a measure of welfare that tells us how much "better-off" the consumers are because the market sells them quantity q at price p. This is measured by the area under the inverse demand curve but above price. The producer surplus is the area above inverse supply but below price. These are motivated by thinking of the height of the inverse demand function at a point being the price some consumer is willing to pay for a unit of that good. The difference between that height and the price is the difference between what they would pay and what they have to pay. That difference is a measure of "consumer surplus". "Summing" over all the consumers gives that area below the inverse demand curve and above price. The same argument motivates the area above the inverse demand and below price as being the producer surplus.

In the case above, the consumer surplus is:

$$\frac{1}{2}(4 * 200) = 400$$

The producer surplus:

$$\frac{1}{2}(2 * 200) = 200$$

Total welfare is the sum of consumer and producer surplus.

$$600$$

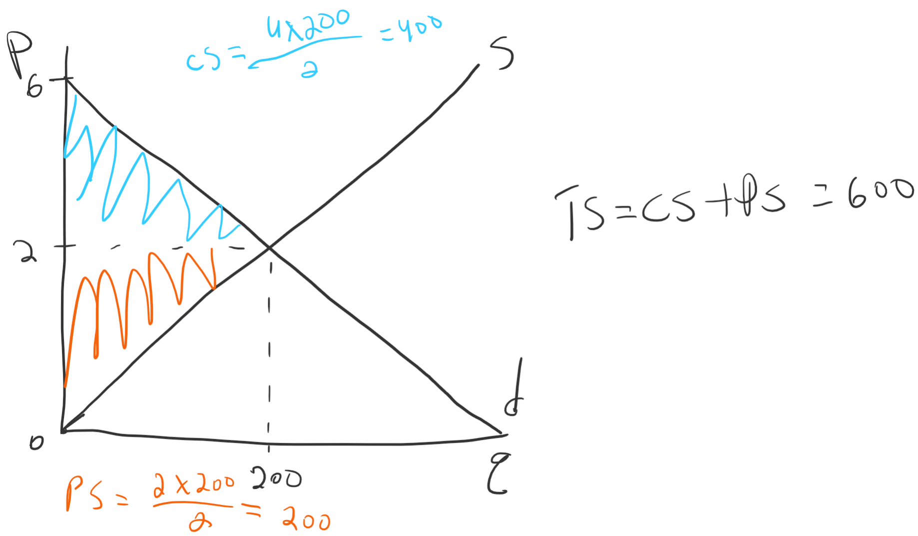

Figure 12.1: Calculating Consumer and Producer Surplus

12.7 Pareto Efficiency

The concept that you can't make anyone better off without making someone worse off. A market in equilibrium is **Pareto efficient.** Total surplus is maximized.

12.8 Deadweight Loss

Deadweight loss is the measurement of how much total surplus is lost due to a tax.

For instance, from the previous problem, we calculated the equilibrium under a tax of size 1.

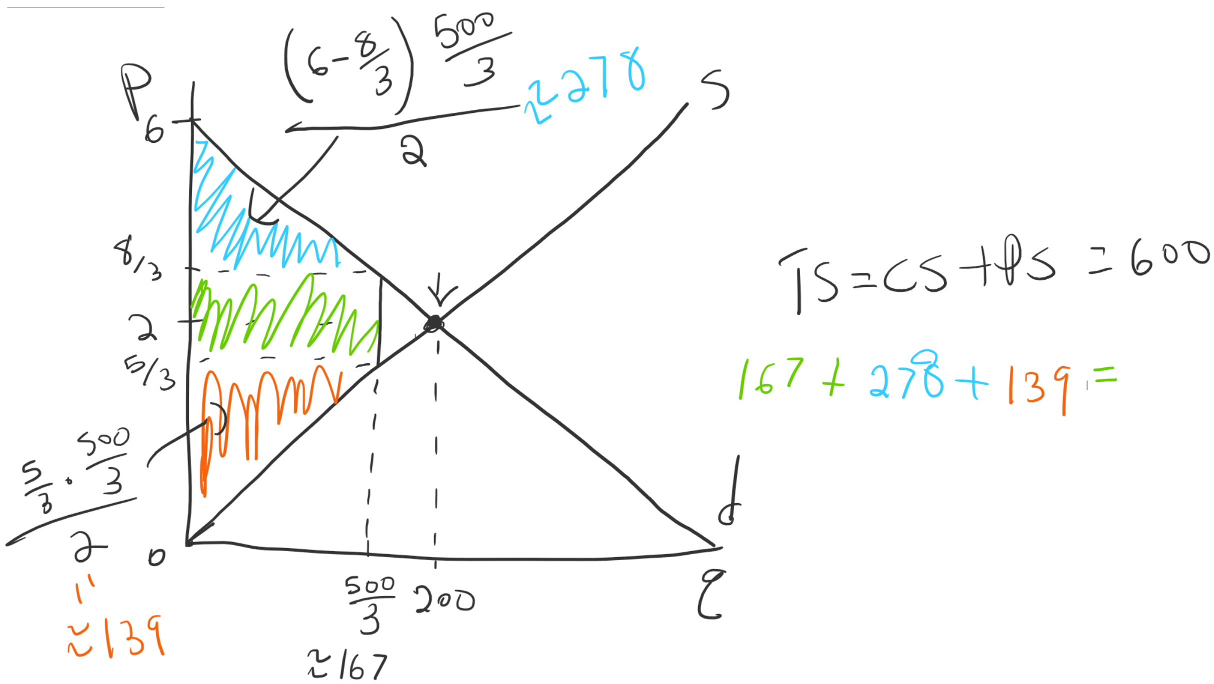

Figure 12.2: Calculating Surplus For Previous Example

The consumer surplus under the tax:

$$\frac{\left(6-\frac{8}{3}\right)\frac{500}{3}}{2} \quad = \quad \frac{2500}{9} = 277.778$$

The producer surplus under the tax:

$$\frac{\left(\frac{5}{3}\right)\frac{500}{3}}{2} \quad = \quad \frac{1250}{9} = 138.889$$

The government revenue under the tax:

$$\frac{500}{3} = 166.667$$

The total surplus under the tax:

$$\frac{2500}{9} + \frac{1250}{9} + \frac{500}{3.0} \quad = \quad 583.333$$

Compare this to the original surplus which was 600. The dead-weight loss is the difference.

$$600 - 583.333 \quad = \quad 16.667$$

We can also find this by the area of the missing triangle in the chart above:

$$\frac{\left(\frac{8}{3} - \frac{5}{3}\right)\left(200 - \frac{500}{3}\right)}{2} \quad = \quad 16.6667$$

12.9 Tax Burden

Calculating **tax burden** or **tax incidence** is calculating who ends up "paying" for the tax. After a tax is imposed, consumers will pay more and producers will receive less than they did without the tax.

But how much more and how much less? This is called the **burden** of the tax.

From the previous problem, after the tax is imposed, consumers pay $\frac{8}{3}$. They paid only 2 before the tax. The burden on consumers is $\frac{8}{3} - 2 = \frac{2}{3}$.

Producers used to get 2 and now they get $\frac{5}{3}$. The burden of the tax on producers is $\frac{1}{3}$.

Notice that the burden on consumers plus the burden on producers adds to 1, the size of the tax. This will always be the case. This also allows us to calculate the proportion of burden. Just divide the burden on each size by the size of the tax.

In this case the consumers bear 66.7% of the tax. Producers bear 33.3% of the tax.

12.10 Tax Burden and Elasticicies

The burden of the tax is determined by the relative elasticities of supply and demand. If demand is relative elastic and supply is relatively inelastic, then most of burden will be on producers. Suppliers do not "pass on" much of the tax. When demand is relatively inelastic compared to supply, most of the burden of the tax will be on the consumers. The supplier's "pass on" most of the tax to consumers.

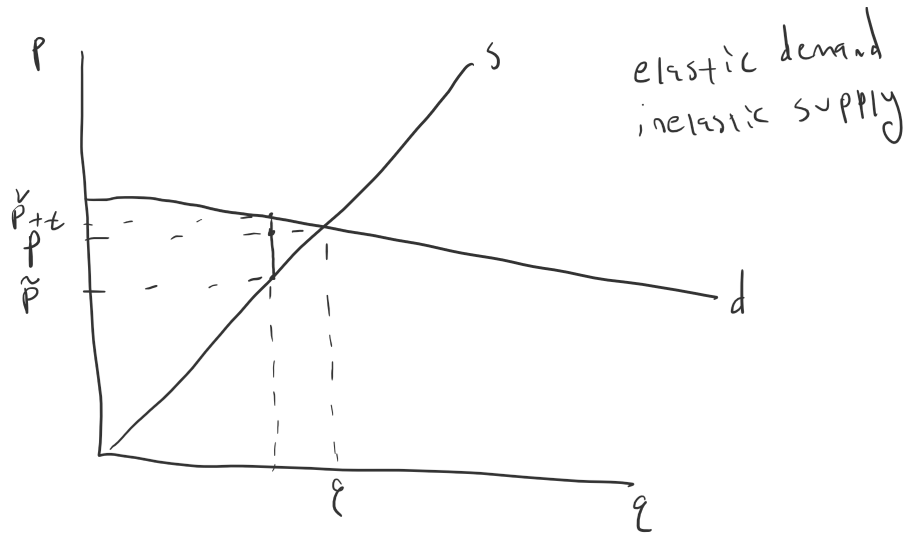

Figure 12.3: Elastic Demand/Inelastic Supply. Most of the burden is on producers.

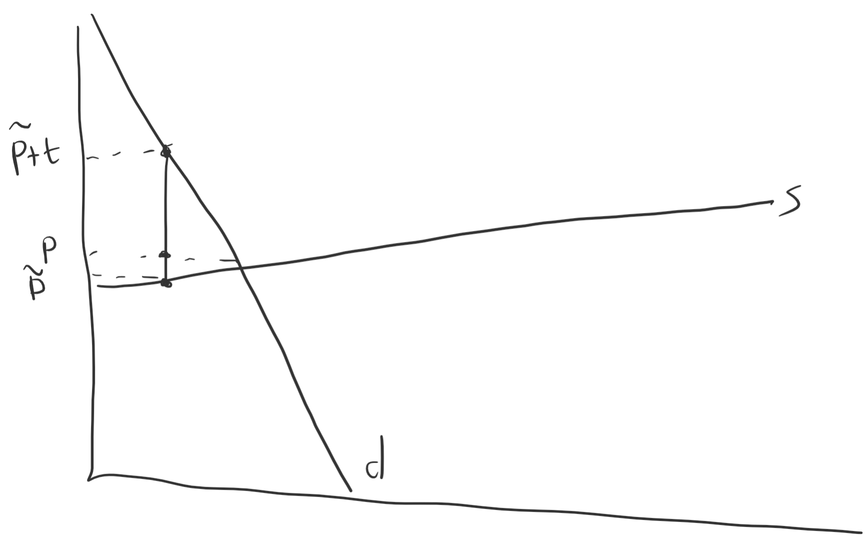

Figure 12.4: Inelastic Demand/Elastic Supply. Most of the burden is on consumers.

13 Technology (19.1-19.10)

Technology is made up of inputs x_1, x_2 and an output y. For instance, x_1 and x_2 might be apples and crusts and y would be pies. Technology is about describing how the inputs turn into output.

We do this throug production functions:

13.1 Production Functions

The pie producer takes 1 crust and 2 apples to make each pie.

The production function looks lie:

$$f(1,2) = 1$$

$$f(2,4) = 2$$

$$f(3,6) = 3$$

$$f(x_1, x_2) = min\left\{x_1, \frac{1}{2}x_2\right\}$$

This is the production function for pie. We can't take transformations of this function. For instance,

$$2min\left\{x_1, \frac{1}{2}x_2\right\}$$

This is a more productive technology and produces two times more pies for every combination of inputs.

$$2 * f(1,2) = 2$$

$$2 * f(2,4) = 4$$

This is a different technology- a more productive one. When we are given a production function, that's the one we are stuck with. No transformations allowed.

13.2 Isoquants

Isoquants are combinations of input that give you the same amount of output.

These are recipes for the same output.

For instance, $(1,2),(1,3),(1,4),(2,2),(3,2)$... all give 2 pie. They are on the same isoquant.

13.3 Example - Fixed Proportions/Perfect Complements

$f(x_1,x_2) = min\left\{x_1, \frac{1}{2}x_2\right\}$.

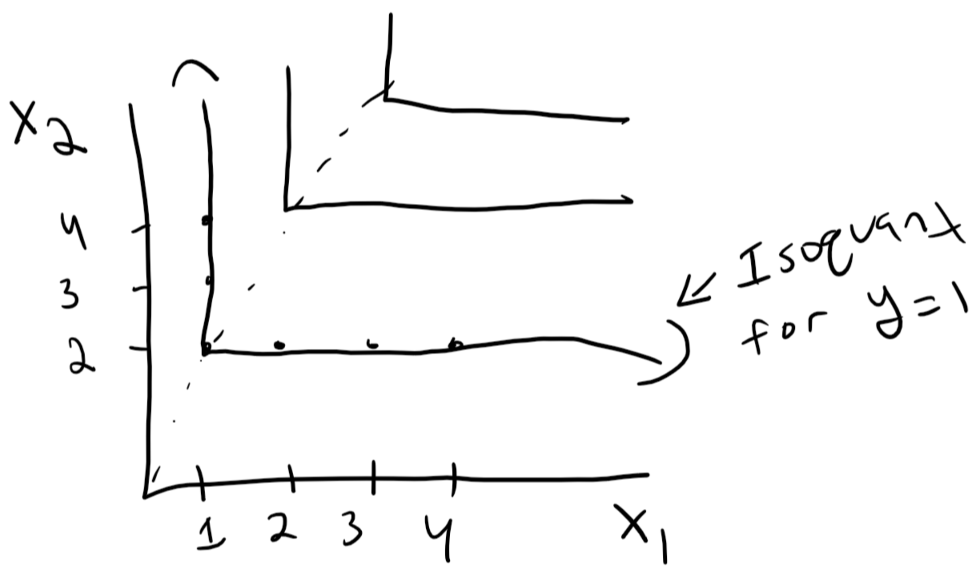

Figure 13.1: Isoquants for fixed-proportion technology.

13.4 Example - Perfect Substitutes

$f(x_1,x_2) = 2x_1 + x_2$

$$f(1,0) = 2$$

Other bundles that produce 2 units: $(0,2),\left(\frac{1}{2},1\right)$

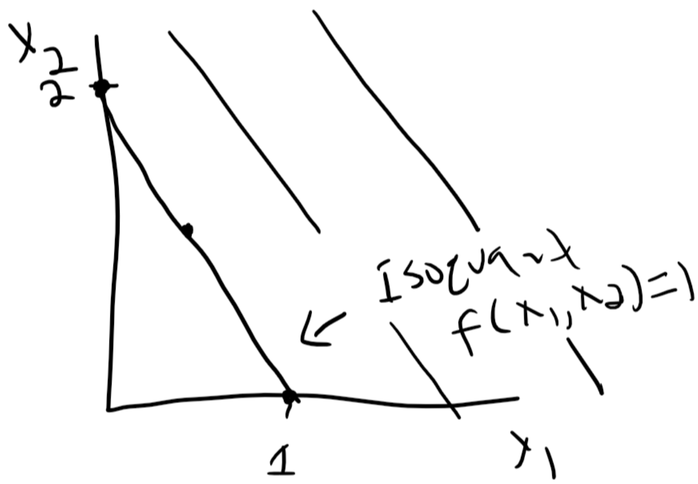

Figure 13.2: Isoquants for perfect substitutes technology.

13.5 Marginal Products

The marginal products themselves are meaningful, unlike for the consumer where the marginal utility is not meaningful by itself.

$$MP_i = \frac{\partial f(x_1, x_2)}{\partial x_i}$$

This is the partial derivative of the production function with respect to input i. It is analogous to the marginal utility.

Example $f(x_1, x_2) = 2x_1 + x_2$

$$MP_1 = 2, MP_2 = 1$$

Example $f(x_1, x_2) = (x_1 + x_2)^{\frac{1}{2}}$ *CES (Constant Elasticity of Substitution)*

$$MP_1 = \frac{\partial\left((x_1 + x_2)^{\frac{1}{2}}\right)}{\partial x_1} = \frac{1}{2}(x_1 + x_2)^{-\frac{1}{2}}$$

$$MP_2 = \frac{1}{2}(x_1 + x_2)^{-\frac{1}{2}}$$

Example $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ *(Cobb Douglass Production)*

$$MP_1 = \frac{\partial\left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}\right)}{\partial x_1} = \frac{1}{2} x_1^{\frac{1}{2}-1} x_2^{\frac{1}{2}} = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} = \frac{1}{2} \frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} = \frac{\sqrt{x_2}}{2\sqrt{x_1}}$$

$$MP_2 = \frac{\sqrt{x_1}}{2\sqrt{x_2}}$$

13.6 Diminishing Marginal Product

Diminishing marginal product is the idea that if you increase one of the inputs while holding the other fixed, the extra output you get will decrease. That is, each input becomes less productive as you increase only that input.

The definition is that,

$$\frac{\partial\left(MP_i\right)}{\partial x_i} = \frac{\partial f\left(x_1, x_2\right)}{\partial x_i \partial x_i} < 0$$

This is the second derivative of f with respect to x_i. For marginal product to be diminishing in both inputs, the second partial derivative of f has to be negative for both inputs.

From the cobb douglass example:

$$MP_1 = \frac{\sqrt{x_2}}{2\sqrt{x_1}}$$

$$\frac{\partial\left(\frac{\sqrt{x_2}}{2\sqrt{x_1}}\right)}{\partial x_1} = -\frac{\sqrt{x_2}}{4x_1^{3/2}} < 0$$

Example $f\left(x_1, x_2\right) = \left(x_1 + x_2\right)^{\frac{1}{2}}$ *CES (Constant Elasticity of Substitution)*

$$MP_1 = \frac{\partial\left(\left(x_1 + x_2\right)^{\frac{1}{2}}\right)}{\partial x_1} = \frac{1}{2}\left(x_1 + x_2\right)^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{\sqrt{x_1 + x_2}}$$

By inspection, this decreases as x_1 increses since it only appears in the denominator. Thus we have diminishing marginal product.

$$\frac{\partial\left(\frac{1}{2} \frac{1}{\sqrt{x_1 + x_2}}\right)}{\partial x_1} = -\frac{1}{4\left(x_1 + x_2\right)^{3/2}}$$

Example $x_1^2 x_2^2$

$$\frac{\partial\left(x_1^2 x_2^2\right)}{\partial x_1} \quad = \quad 2x_1x_2^2$$

This has increasing marginal product since the marginal product of x_1 is increasing with x_1.

13.7 Technical Rate of Substitution

Along a particular isoquant, the slope of the isoquant measures how much x_2 you can give up if you add 1 unit of x_1 so that you continue producing the same amount of output.

This slope is given by the **Technical Rate of Substitution**

$$TRS = -\frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}} = -\frac{MP_1}{MP_2}$$

This is measuring tradeoffs a firm is *willing* to make to produce the same amount. Eventually it will play a key role in finding optimal input bundles.

13.8 Returns to Scale

The idea of returns to sale is about asking what happens to output when we "double" (or change by some other proportion) input.

Pie Example: 1 crust, 2 apples make 1 pie.

$$f(1,2) = 1$$

Double input: 2 crusts, 4 apples makes 2 pies.

$$f(2,4) = 2$$

Doubling the inputs, doubled the output.

$$f(x_1,x_2) = min\left\{x_1, \frac{1}{2}x_2\right\}$$

$$f(2x_1,2x_2) = min\left\{(2)\,x_1, (2)\,\frac{1}{2}x_2\right\} = 2min\left\{x_1, \frac{1}{2}x_2\right\}$$

In general doubling the inputs doubles the output. **Linear (constant) returns to scale**.

Liner (constant) returns to scale:

$$f(tx_1, tx_2) = tf(x_1, x_1)$$

Decreasing returns to scale:

$$f(tx_1, tx_2) < tf(x_1, x_1)$$

Example: $f(x_1, x_2) = (x_1 + x_2)^{\frac{1}{2}}$.

$$f(2, 2) = (2 + 2)^{\frac{1}{2}} = 2$$

$$f(4, 4) = (4 + 4)^{\frac{1}{2}} = 2.82843 < 4$$

This has decreasing returns to scale.
Increasing returns to scale:

$$f(tx_1, tx_2) > tf(x_1, x_1)$$

14 Profit Maximization / Cost Minimization

14.1 Profit Maximization

p the price of output.
Revenue is given by $pf(x_1, x_2)$
Cost is given by $(w_1x_1 + w_2x_2)$

$$\pi(x_1, x_2) = pf(x_1, x_2) - (w_1x_1 + w_2x_2)$$

Caveat: The assumption that p is fixed no matter how much output the firm produces is not realistic in most markets. This is called the *price taking* assumption. We will relax this later.

14.2 Short-Run Profit Maximization

If x_2 is fixed, then profit is only a function of x_1. x_2 is fixed at some value.

$$\pi(x_1, \bar{x_2}) = pf(x_1, x_2) - w_1x_1 - w_2x_2$$

In this case profit maximization only involves choosing the optimal level of the variable input x_1.

14.3 Example

Suppose $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ but x_2 is fixed at $x_2 = 10$. Price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$.

The short-run production function is given by:

$$f(x_1, 10) = x_1^{\frac{1}{2}} (10)^{\frac{1}{2}} = \sqrt{10} x_1^{\frac{1}{2}}$$

$$MP_1 = \frac{\partial \sqrt{10} x_1^{\frac{1}{2}}}{\partial x_1} \quad = \quad \frac{\sqrt{\frac{5}{2}}}{\sqrt{x_1}}$$

The short-run profit function is:

$$\pi(x_1, 10) = 100\sqrt{10} x_1^{\frac{1}{2}} - (x_1 + 10)$$

We need to look for a point where the slope is zero. This will be where this function is maximized.

$$\frac{\partial \left(100\sqrt{10} x_1^{\frac{1}{2}} - (x_1 + 10)\right)}{\partial x_1} \quad = \quad \frac{50\sqrt{10}}{\sqrt{x_1}} - 1$$

At the optimum this has to be true (first-order condition):

$$\frac{50\sqrt{10}}{\sqrt{x_1}} - 1 = 0$$

$$\left(50\sqrt{10}\right)^2 = x_1$$

$$25000 = x_1$$

What is the maximum profit the firm can earn in the short run? Plug the optimal x_1 into the short run profit function:

$$\pi(x_1, 10) = 100\sqrt{10.0}\sqrt{25000} - (25000 + 10)$$

$$\pi = 24990.$$

14.4 Long-Run Profit Maximization

Suppose $f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$. Price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$.
Whenever the production has increasing returns to scale, there is no profit max.
Fortunately, this one is decreasing returns to scale.

$$\left((tx_1)^{\frac{1}{3}} (tx_2)^{\frac{1}{3}}\right) = \left(t^{\frac{1}{3}} x_1^{\frac{1}{3}} t^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) = t^{\frac{2}{3}} \left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right)$$

Long-run profit function:

$$\pi(x_1, x_2) = p x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (w_1 x_1 + w_2 x_2)$$

At the profit maximizing point, both of these will have to be true:

$$\frac{\partial \pi(x_1, x_2)}{\partial x_1} = 0$$

$$\frac{\partial \pi(x_1, x_2)}{\partial x_2} = 0$$

Suppose either of these is non-zero. For instance, suppose:

$$\frac{\partial \pi(x_1, x_2)}{\partial x_i} > 0$$

If I increase x_i profit will go up.

$$\frac{\partial \pi(x_1, x_2)}{\partial x_i} < 0$$

If I decrease x_i profit will go up.
Both of these will have to be zero at the maximum.

14.5 Example

Suppose $f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$. Price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$.

$$Max_{x_1 x_2} x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)$$

$$\frac{\partial \left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)\right)}{\partial x_1} = \frac{\sqrt[3]{x_2}}{3x_1^{2/3}} - 1$$

$$\frac{\sqrt[3]{x_2}}{3x_1^{2/3}} - 1 = 0$$

$$\frac{\partial\left(x_1^{\frac{1}{3}}x_2^{\frac{1}{3}}-(x_1+x_2)\right)}{\partial x_2} = \frac{\sqrt[3]{x_1}}{3x_2^{2/3}}-1$$

$$\frac{\sqrt[3]{x_1}}{3x_2^{2/3}}-1=0$$

$$\frac{1}{27}=x_1=x_2$$

To solve for the profit max, we need to solve these two equations for the two unknowns x_1, x_2.

14.6 Example

Suppose $f(x_1, x_2) = x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}$. Price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$.

$$\frac{\partial\left(x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}-(x_1+x_2)\right)}{\partial x_1} = \frac{\sqrt{x_2}}{2\sqrt{x_1}}-1$$

$$\frac{\partial\left(x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}-(x_1+x_2)\right)}{\partial x_2} = \frac{\sqrt{x_1}}{2\sqrt{x_2}}-1$$

At the optimum both of those have to be equal to zero. Setting them to zero and rearranging we get:

$$x_2 = 4x_1$$

$$x_1 = 4x_2$$

The only solution to this is $x_1 = x_2 = 0$

Something weird is going on here, but it is hard to see what went wrong since we did not calculate the optimal level of production y.

14.7 Why We Rarely Maximize Profits in One Step

14.8 Cost Minimization

Profit maximization implies cost minimization. If a firm was not minimizing cost of producing what they thought was the profit maximizing level of output, there is a cheaper way to earn the same revenue, and thus get more profit.

This let's us break down the profit maximization problem into two steps:

1. Calculate the cheapest way to produce any level of output y.
2. Calculate the most profitable y.

Step 1 looks like this:

$Min_{x_1,x_2} w_1x_1 + w_2x_2$ subject to $f(x_1, x_2) = y$.

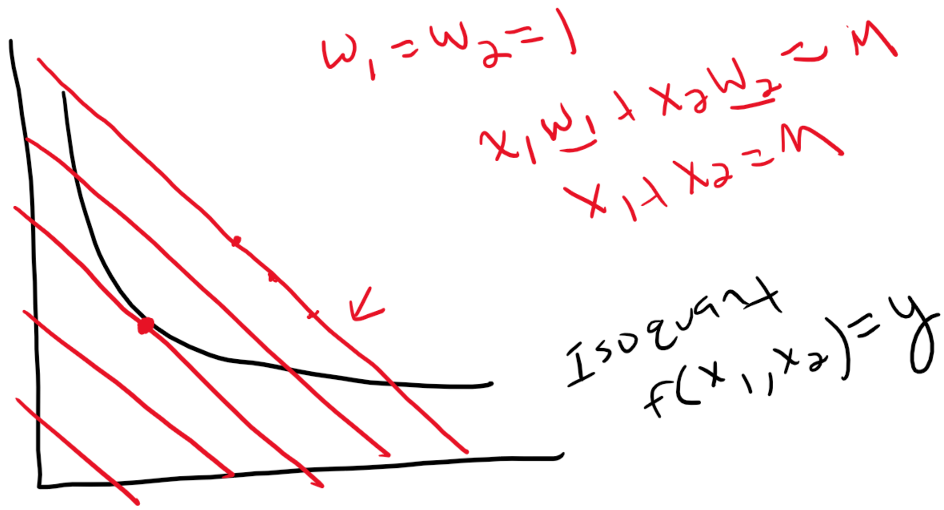

Figure 14.1: Cost minimization requires finding the lowest isocost curve on the fixed isoquant.

We have already seen problems like this. We know we need to look for points where the curves are tangent. The slope of the isoquant has to be equal to the slope of the isocost.

$$TRS = -\frac{w_1}{w_2}$$

$$-\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = -\frac{w_1}{w_2}$$

14.9 Example- Minimizing Cost for a Cobb Douglass Production Function

Minimize the cost of producing y units of output with production function f(x_1, x_2) $=x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}$.

The TRS is:

$$TRS = -\frac{\frac{\partial\left(x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}\right)}{\partial x_1}}{\frac{\partial\left(x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}\right)}{\partial x_2}} = -\frac{x_2}{x_1}$$

Equal-slope condition:

$$-\frac{x_2}{x_1} = -\frac{w_1}{w_2}$$

Solving this condition for x_1:

$$x_1 w_1 = x_2 w_2$$

$$x_1 = \frac{x_2 w_2}{w_1}$$

Instead of plugging this into a budget equation like we would for the consumer utility maximization, we need to plug it into the producer's constraint: the production constraint: $x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} = y$:

$$x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} = y$$

Plug in the condition above for x_1 :

$$\left(\frac{x_2 w_2}{w_1}\right)^{\frac{1}{4}} x_2^{\frac{1}{4}} = y$$

Solve for x_2 to get **conditional factor demand for x_2:**

$$x_2 = y^2 \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}$$

Plug this back into the equal slope condition above to get x_1:

$$x_1 = y^2 \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}$$

To calculate the cost function (what is the cheapest way to produce y): plug these conditional factor demands into the cost equation:

$$w_1 x_1 + w_2 x_2$$

Plug in the conditional factor demands:

$$w_1\left(y^2\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}\right)+w_2\left(y^2\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}\right)=2w_1^{\frac{1}{2}}w_2^{\frac{1}{2}}y^2$$

$$c\,(y)=2w_1^{\frac{1}{2}}w_2^{\frac{1}{2}}y^2$$

14.10 Profit Maximization Through Cost Minimization

Replacing the cost function in terms of x_1, x_2 with $c\,(y)$, we can write the highest profit a firm can possibly earn by producing y units of output.

$$\pi\,(y)=py-c\,(y)$$

This is very easy to maximize since it is just one-dimensional. It only depends on y.

14.11 Example

Maximize profit of f(x_1,x_2) $=x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}$ using cost minimization and then profit maximization. We have already found $c\,(y)$ for this firm:

$$c\,(y)=2w_1^{\frac{1}{2}}w_2^{\frac{1}{2}}y^2$$

The profit function is now in terms of only y is (this is often referred to as the *conditional profit* function).

$$\pi\,(y)=py-2w_1^{\frac{1}{2}}w_2^{\frac{1}{2}}y^2$$

For an interior maximum (y is some number other than 0), the slope of this will have to be zero. Otherwise, the firm could increase or decrease output and increase profit. The first order condition is:

$$\frac{\partial\left(\pi\,(y)\right)}{\partial y}=0$$

$$\frac{\partial\left(py-2w_1^{\frac{1}{2}}w_2^{\frac{1}{2}}y^2\right)}{\partial y}\quad=\quad p-4\sqrt{w_1}\sqrt{w_2}y$$

$$p=4\sqrt{w_1}\sqrt{w_2}y$$

In general, it is always true that this condition is:

$$MR=MC$$

Under the price taking assumption (that price p does not depend on y) the marginal revenue is just p and we have:

$$p = MC$$

Returning to the example, we can solve y to get the optimal y for any set of prices:

$$y^* (p, w_1, w_2) = \frac{p}{4\sqrt{w_1}\sqrt{w_2}}$$

This the optimal (profit maximizing) level of output for any price. We can also write the "profit function" take this optimal level of output and plugs it back into the "conditional profit function". We found previously that this conditional profit function is:

$$\pi (y) = py - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$$

Plugging in the optimal level of production yields the profit function:

$$\pi (y^*) = p \left(\frac{p}{4\sqrt{w_1}\sqrt{w_2}} \right) - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} \left(\frac{p}{4\sqrt{w_1}\sqrt{w_2}} \right)^2 =$$

$$\frac{p^2}{4\sqrt{w_1}\sqrt{w_2}} - \frac{p^2}{8\sqrt{w_1}\sqrt{w_2}} = \frac{p^2}{8\sqrt{w_1}\sqrt{w_2}}$$

Suppose $p = 10$ and $w_1 = w_2 = 1$ the maximum profit the firm can earn is (plug prices into the profit function above):

$$\pi^* = \frac{100}{8} \quad = \quad \frac{25}{2}$$

Find the optimal level of output by plugging prices into the optimal output function $y^* (p, w_1, w_2) = \frac{p}{4\sqrt{w_1}\sqrt{w_2}}$.

$$y^* = \frac{10}{4} = \frac{5}{2}$$

14.12 What can go wrong?

If returns to scale are linear or increasing then if we can find any output level y where the firm earns positive profit **then there is no profit maximizing level of y.** The firm wants to produce as much as possible.

This is because with linear or increasing returns to scale, doubling inputs will double cost and **at least** double output- so profit will at least double. Thus, if we can find a point were profit is positive, we can always use of all inputs and increase profit.

14.13 Linear Returns to Scale Example

Maximize profit of f(x_1, x_2) =$min\left\{\frac{1}{2}x_1, x_2\right\}$ using cost minimization and then profit maximization.

To minimize costs, the firm should use:

$$\frac{1}{2}x_1 = x_2$$

Plug this back into the production function to get the conditional factor demands:

$$x_2 = y$$

$$x_1 = 2y$$

The cost function:

$$c\left(y\right) = w_1 2y + w_2 y = \left(2w_1 + w_2\right) y$$

The conditional profit function:

$$\pi\left(y\right) = py - \left(2w_1 + w_2\right) y = \left(p - 2w_1 - w_2\right) y$$

If $p > 2w_1 + w_2$ there is no profit maximizing level. I want to produce as much as possible.

If $p < 2w_1 + w_2$ optimal level is $y = 0$ and profit is 0.

If they are equal the profit is always zero and the firm can choose whatever they want.

15 Cost Curves (22.1-22.3,22.5)

$$\pi\left(y\right) = Rev\left(y\right) - Cost\left(y\right)$$

Under price taking:

$$py - c\left(y\right)$$

$$p = \frac{\partial c\left(y\right)}{\partial y} = mc\left(y\right)$$

15.1 The Cost Function / Variable and Fixed Costs

For example, one cost function we have seen so far is:

$$c(y) = 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$$

This function only has one "part" a part that is variable in y. However, suppose we had the same production, but the firm also has to pay 100 in rent every month.

$$c(y) = 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2 + 100$$

Geneally, we decompose cost function into two parts:
Variable Cost: Changing in y
Fixed Cost: Not Changing in y

$$vc(y) = 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$$

$$fc = 100$$

Total costs are the sum of variable and fixed costs.

$$tc(y) = vc(y) + fc$$

A firm operating in the "long run" is a firm that has no fixed costs. All costs are variable costs.
A firm operating in the "short run" may have fixed costs.

15.2 Average Costs.

Average costs are costs divided by the output.

$$AC(y) = \frac{TC(y)}{y}$$

If the cost of producing 1000 units if 10000, the average cost of producing the units is 10.
From our example above:

$$tc(y) = 2y^2 + 100$$

$$ac(y) = \frac{2y^2 + 100}{y} = \frac{2y^2}{y} + \frac{100}{y} = 2y + \frac{100}{y}$$

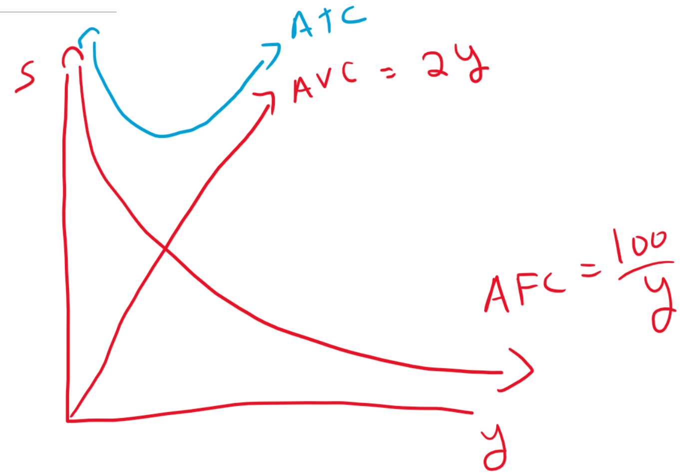

Figure 15.1: ATC,AVC, and AFC for the cost function $c(y) = 2y^2 + 100$

The average variable cost:

$$avc\,(y) = \frac{vc\,(y)}{y}$$

In our example: $avc\,(y) = 2y$
The average fixed cost:

$$afc\,(y) = \frac{fc}{y}$$

In our example: $afc\,(y) = \frac{100}{y}$

$$Table[\{2y + \frac{100}{y}, 2y, \frac{100}{y}\}, \{y, \{1, 5, 10, 100\}\}$$

$$ATC\,(1) = 2 + \frac{100}{1} = 102$$

$$AVC\,(1) = 2$$

$$AFC\,(1) = 100$$

$$ATC\,(10) = 2\,(10) + \frac{100}{10} = 30$$

$$AVC(1) = 20$$

$$AFC(1) = 10$$

15.3 Marginal Costs.

The marginal cost is the derivative of the cost function with respect to y. "What is the additional cost of producing 1 more unit".

$$mc(y) = \frac{\partial(tc(y))}{\partial y} = \frac{\partial(vc(y))}{\partial y}$$

For example: for $tc(y) = 2y^2 + 100$

$$mc(y) = 4y$$

For instance at $y = 1$ and $y = 10$:

$$mc(1) = 4, mc(10) = 40$$

15.4 Visualizing The Relationships

When the marginal cost is below average total/variable cost, the average total/variable cost is decreasing.

When the marginal cost is above average total/variable cost, the average total/variable cost is increasing.

When mc=avc/atc, the are neither increasing or decreasing. This is the point where they reach their minimum.

Thus, marginal cost passes through the minimum of average total and average variable cost.

$$tc(y) = 2y^2 + 100$$

$$avc = 2y$$

$$afc = \frac{100}{y}$$

$$mc = 4y$$

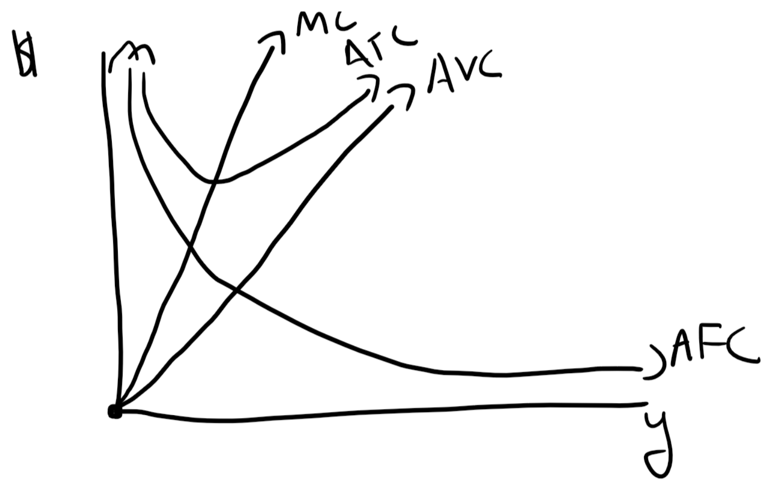

Figure 15.2: Adding Marginal Cost (MC) to the graph for $c(y) = 2y^2 + 100$

15.5 Long-run vs. Short-run Average Costs.

If we multiply average total costs by y, we get total cost. If average total costs, and thus total cost for producing the same y is lower for one firm than another, then it is more efficient.

In the short run, a firm might be operating with inefficient levels of some fixed input. For instance, a factory that is too small or too large for it's chosen level of output.

But in the long run, the firm can always choose whatever factor is most efficient for it's chosen level of output. Thus, in the long run, the average total cost/total cost of a firm for producing some level of output will be the whatever the lowest possible short-run level would be.

If a firm happened to, by chance, have the factory size optimal for it's chosen level of output, then in the long run, it would not change the factory size. The long run and short run average total cost coincide for that single optimal factory size. But the average total cost is higher for every other factory size. And thus the short run average total cost functions associated with those factory sizes are **above** the long-run average total cost.

This creates a situation where the long run ATC curve is "lower envelope" of the short run ATC cost curves. For every point on the long run ATC, there is some short run ATC curve that touches the long run ATC curve at that point.

Here is an example with three possible factory sizes and thus three short-run ATC curves. Think about which factory the firm would choose in the long run and trace out the costs of using those optimal factory sizes and we get the red long run average total cost curve below. Again, it is the **lower envelope** of all of the short run average total cost function.

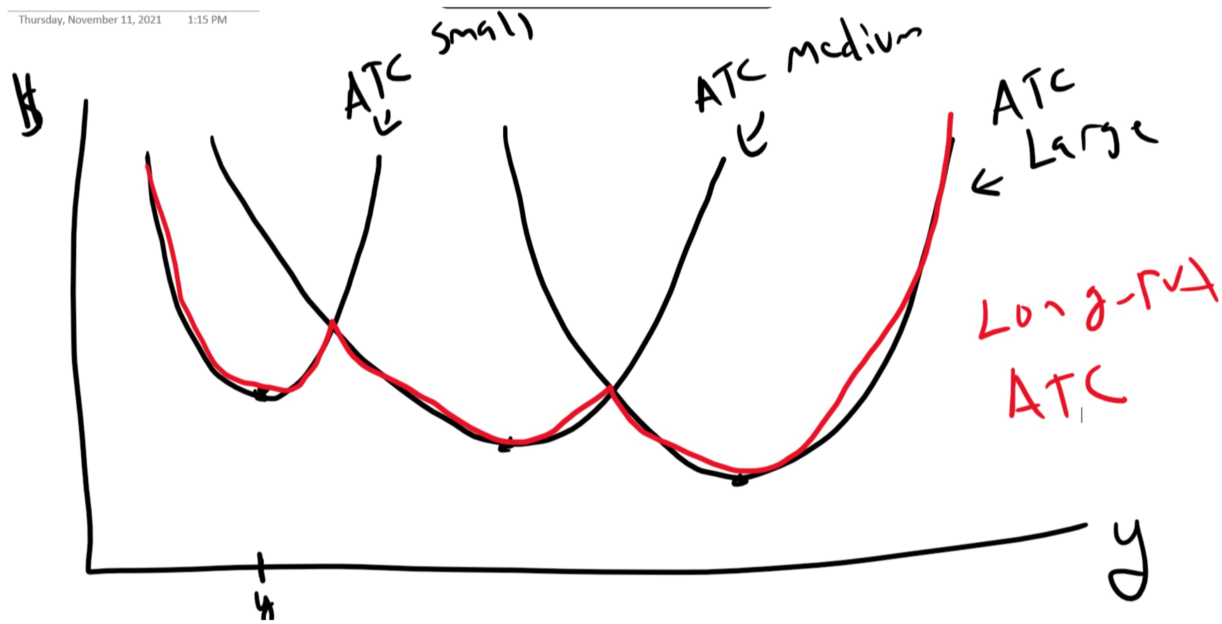

Figure 15.3: An example of the short function average cost functions and long run average cost function for a scenario where there are three possible sizes of factories: small, medium, and large. The long run average cost (red) is the lower envelope of the short run cost functions for each factory size.