Complainer's Dilemma

By: Greg Leo*

Technological innovations have made complaining easier than ever, but often where it is easy to complain, only problems meeting some large threshold of complaints are addressed. Why are these low-effort / high-threshold policies becoming more common? In this paper, I model the strategic environment facing complainers. I demonstrate that properties of the resulting games' equilibria justify the existence of high complaint thresholds and the trend away from high-effort / low-threshold policies. By setting costs and thresholds appropriately, the administrator can prevent complaints that are not worth addressing. Furthermore, leveraging novel results that rely on the behavior of a transcendental function known as Lambert-W, I demonstrate that policies that minimize the cost of complaining while requiring a suitably large threshold are universally more efficient for large constituencies. *(JEL* C72, C73, D82*)*

1 Introduction

"...but when complaints are freely heard, deeply consider'd and speedily reform'd, then is the utmost bound of civil liberty attain'd..."

-John Milton (Areopagitica, 1886).

Technology advancements make it easier to complain. National governments including those of the United States, Australia, South Korea, and several European countries have adopted internet petition platforms, known as e-petitions, to increase public engagement (Hagen et al., 2018). At the civic level, systems for reporting issues have evolved from call-in hotlines to smartphone applications (Gordon and Baldwin-Philippi, 2013). These "democratic crowd-sourcing" systems (Aitamurto, 2012) lower the cost of complaining. On the surface, this is beneficial to the democratic process as it attracts more complaints about a wider variety of issues.

However, reducing the cost of complaining leads to additional noise in the form of more complaints about issues that are not worth addressing because the cost to address the complaint is higher than the social benefit. When resources are limited, administrators must also consider the opportunity cost of not attending to other problems. In illustration of this, an early discussion of e-petitions in British Parliament raises concerns that online petition campaigns might result in "Parliamentary graffiti" interfering with more formal business (Miller, 2009).

To cope with the additional noise, many democratic crowdsourcing systems have adopted explicit policies of addressing only the problems that reach a certain threshold of complaints. For instance, the threshold for providing an official response in the United States e-petition platform "We the People" has increased from 5,000 to 25,000 to 100,000 signatures over the course of its implementation (Phillips, 2013). While the White House has been explicit about this

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threshold, other democratic crowdsourcing systems acknowledge a threshold but are not so explicit. For instance, the website for “Street Bump”, an app for reporting potholes in Boston, says: “We won’t inspect a roadway problem unless we get multiple reports” ("About Street Bump", 2020). Even in the absence of such acknowledgment, it is possible that complainers understand that such policies are in place.

This raises two key questions about the effectiveness of democratic crowd-sourcing. Do threshold policies effectively filter complaints to important matters? And, if they do, are the thresholds met frequently enough that the result is better than a system with a high cost of complaining but lower threshold?

This paper addresses these questions. First, I demonstrate that threshold policies can effectively filter out trivial complaints. I then show that, even with appropriate thresholds in place, policies with a lower cost of complaining are more effective in raising important issues.

These results arise from formal analysis of the strategic environment facing complainers. In this setting, there is an incentive to free-ride. To demonstrate this, suppose there are several who would benefit from having a particular problem addressed, with this benefit being greater than the cost of complaining. If the administrator’s policy is to address any problem that receives at least one complaint, then anyone would be willing to complain if they knew for sure no one else would. However, since only one complaint is needed, everyone prefers someone else do it.

Such scenarios are elegantly modeled by the Volunteer’s Dilemma game (Diekmann, 1985). In the symmetric equilibrium of the Volunteer’s Dilemma, all players volunteer (or complain in this case) with some probability that is decreasing in the volunteering cost. This suggests why a reduction of complaining costs is so appealing to the complainers. A reduction in cost eases the free-riding problem so there is a better chance *someone* will volunteer/complain.

However, the cost reduction also creates incentives to complain about new and potentially trivial problems, (as long as the personal benefit exceeds the, now smaller, cost). The reduction in free-riding, clearly beneficial for complainers, creates additional noise for the administrator, leading to higher threshold requirements. However, with a higher threshold, the traditional Volunteer’s Dilemma no longer captures the strategic incentives of the complainers. In section two of this paper, I present a generalization, the m-volunteers dilemma, that captures this strategic scenario. This game has an interesting property that implies that the threshold solution can be particularly effective.

When only one complaint is required, complaints can be received (in equilibrium) about *any* problem for which the individual benefit exceeds the cost of complaining. In contrast, when more than one complaint is required, there is a benefit level*, strictly above the cost of complaining*, below which complaints are *never* received in a symmetric equilibrium. That is, there are some problems that the complainers might like to have addressed (and in fact would complain about if only one complaint were required) which they will never complain about in equilibrium when *more than one* complaint is required.

This property of the strategic environment can be leveraged by the administrator. By carefully selecting the cost of complaining and the number of complaints required, the administrator can indirectly force the complainers to filter the problems they complain about. In fact, complaints about trivial problems (no matter how they are defined by the administrator) can be completely eliminated. However, it is possible to do this in many ways- using higher complaint costs with lower thresholds or lower complaint cost with higher thresholds.

In the key result of this paper, I demonstrate that among policies that rule out the same set of trivial complaints, problems deemed non-trivial by the administrator are more likely to be addressed under low cost and high threshold policies. Thus, from the standpoint of fixing non-trivial problems, lowering complaint costs and raising the complaint threshold is more efficient.[1]

[1]In large groups, the cost of complaints can largely be ignored in determining efficiency since the complaint costs, endured by only a small fraction of the group, are swamped by efficiency changes in addressing problems which affect the entire group.

The paper is structured as follows. Section 2 presents the m-volunteer's dilemma game, and presents it's relevant equilibrium properties. Section 3 focuses on the administrator's problem of choosing a threshold policy for a finite constituency. Section 4 extends the results in sections 2 and 3 to large constituencies. The novel results in this section rely on a useful transcendental function known as Lambert-W and demonstrate it's application in game theory. The paper concludes in section 5.

2 The Dilemma

2.1 Administrator's Environment and Complaint Policy

An administrator is responsible for n constituents. Problems arise that are only observed by the constituents but must be addressed by the administrator. Each problem is characterized by the value $b > 0$, which the constituents receive if the problem is addressed. The administrator deems problems with $b < \gamma$ too trivial to address. For instance, γ might be the cost of addressing problems in per-capita terms such that it is not socially efficient to address problems with $b < \gamma$. However, γ can be chosen arbitrarily, and the paper is otherwise agnostic to this value.

The administrator cannot directly observe problems but instead learns about them through complaints. Furthermore, even upon receiving complaints, it is assumed the administrator cannot directly assess the value b for the problem the complaints pertain to. This may be unrealistic in some environments. For instance, serious campaigns are easily picked out from humorous ones on e-petition platforms. However, this assumption maximizes the asymmetry of information between the complainers and the administrator and focuses the intuition we gain from this model on how changes in the strategic environment induced by policy choices can be leveraged by the policy-maker.

A complaint policy consists of two elements chosen by the administrator: the cost of complaining $c > 0$ and integer threshold m- the number of complaints the administrator requires before addressing a problem. We can assume that m is chosen freely and that the administrator has access to an ample catalog of paper forms, call-in hotline menus, and apps involving various levels of annoyance to choose from to manipulate c.

2.2 Strategic Environment

Complaint policies of the kind discussed above induce a strategic game on constituents. All who complain pay a cost of c. If there are at least m complaints, the administrator addresses the problem, and each player receives a benefit b.

Games of this kind are marked by free-riding incentives. When $m = 1$, the game is the classic Volunteer's Dilemma (Diekmann, 1985). When $b > c$, any player would be willing to complain if they knew no one else would, but if they believe someone else will complain, they would prefer not to. When $m > 1$, the constituents play a related game that I refer to as the m-volunteer's dilemma.[2]

2.3 Equilibria

A non-degenerate symmetric equilibrium[3] of the game has a simple form. By complaining, a player bears cost c, but complaining only changes the outcome when a player is pivotal, and a player is pivotal when $m-1$ out of the $n-1$ others have complained. Thus, the expected value of complaining is equal to the benefit b weighted by the pivotal

[2] This game appears to have first been considered by Archetti and Scheuring (2011) who demonstrate cooperators and defectors can coexist in a stable equilibrium among a population playing a generalized dilemma which includes the m-volunteer's dilemma.

[3] There is always a symmetric equilibrium for $m \geq 2$ where everyone volunteers with zero probability.

probability. When the cost of complaining c is equal to the expected benefit, the player is indifferent and willing to randomize between their pure strategies. This occurs when the pivotal probability is equal to $\frac{c}{b}$. Let X represent the number of players out of any $n-1$ who complain, the equilibrium condition is[4]:

$$Pr(X = m-1) = \frac{c}{b} \tag{1}$$

In a symmetric equilibrium, X is a binomial random variable. Letting p be the probability that an *individual* complains, the equilibrium condition can be written:

$$\binom{n-1}{m-1} p^{m-1} (1-p)^{n-m} = \frac{c}{b} \tag{2}$$

2.3.1 Geometry of Equilibrium

For $m \geq 2$, the left side of equation 2 is a polynomial equal to zero at both $p = 1$ and $p = 0$ and reaching a maximum at $p = \frac{m-1}{n-1}$. Since it has a single stationary point (a maximum) in the domain $p \in (0,1)$, it is quasiconcave.

The maximum of the left side of equation 2 can be interpreted as the largest probability that a player could possibly believe they are pivotal under symmetric strategies. This value proves useful for determining when mixed-strategy equilibria exist and plays a crucial role in choosing a threshold policy. This value is given by $z_{m,n} = \binom{n-1}{m-1}\left(\frac{m-1}{n-1}\right)^{m-1}\left(\left(\frac{n-m}{n-1}\right)\right)^{n-m}$.

2.3.2 Properties of Equilibrium

The next three results characterize the symmetric equilibria for this game. For fixed constituency size n and required complaint threshold m, the number and location of equilibria depend on the cost/benefit ratio $\frac{c}{b}$. If this ratio is large enough, (benefit is low relative to the cost of complaining) there are no symmetric equilibria involving non-zero probability of complaining (*proposition 1*).[5]

Importantly, non-existence of equilibria with complaints can occur *even if the benefit is higher than the cost of complaining*. In equilibrium the pivotal probability must equal the cost benefit ratio. However, the pivotal probability is capped at $z_{m,n}$ and so cannot be made arbitrarily close to 1 when $m \geq 2$.

On the other hand, if the benefit of having a problem fixed is high enough relative to the cost of complaining, so that it is possible to be pivotal with probability $\frac{c}{b}$, then there exist *two* symmetric equilibria involving non-zero probability of complaints (*proposition 2*). In this case, only one of the equilibria is stable, and it is the equilibrium that involves a higher probability of complaining (*proposition 3*).

These results are stated more formally below with additional discussion:

2.3.3 Existence of Equilibrium

Proposition 1. *There does not exist a symmetric equilibrium with non-zero probability of volunteering when $\frac{c}{b} > z_{m,n}$.*

Proof. An equilibrium occurs where $\binom{n-1}{m-1}p^{m-1}q^{n-m} = \frac{c}{b}$. The function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ has a maximum of $z_{m,n}$. If $\frac{c}{b} > z_{m,n}$ then there is no $p \in [0,1]$ which makes this true. This is demonstrated graphically in figure 1. ☐

[4] A player is indifferent when: $b[P(X \geq m-1)] - c = b[P(X \geq m)]$. This is equivalent to $b[P(X \geq m-1) - P(X \geq m)] = c$ which simplifies to $b[P(X = m-1)] = c$.

[5] When $m \geq 2$ there is always a pure-strategy symmetric Nash equilibrium in which no one complains.

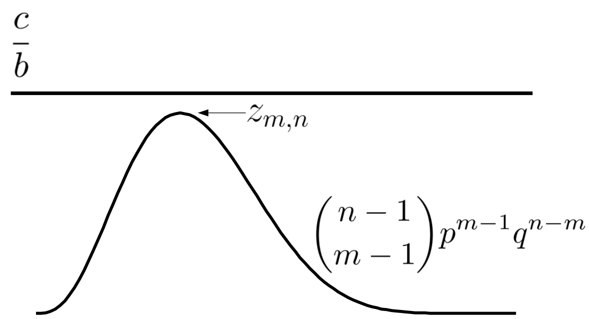

Figure 1: No Mixed Equilibrium Exist $\frac{c}{b} > z_{m,n}$

This puts a bound on the types of problems that constituents will complain about.[6] For example, when $m = 2$ and $c = 1$, constituents complain (at least probabilistically) about any problems for which $b \geq \left(\frac{n-1}{n-2}\right)^{n-2}$. When $n = 3$ this is $b \geq 2$. As $n \to \infty$, this converges to $b \geq e$. This limit is notable since this condition will arise again in the analysis of large constituencies in section 4.

The next result characterizes the equilibria that exist when the benefit b is high enough relative to the cost of complaining.

2.3.4 Multiple Equilibria

Proposition 2. *For $m \geq 2$ and any $\frac{c}{b} < z_{m,n}$, there are two symmetric non-degenerate mixed-strategy equilibria p_1^* and p_2^* with $p_1^* < \frac{m-1}{n-1}$ and $p_2^* > \frac{m-1}{n-1}$*

Proof. Since $\binom{n-1}{m-1} p^{m-1} (1-p)^{n-m}$ is continuous, strictly quasiconcave in p, equal to zero at $p = 0$ and $p = 1$, and reaches a maximum of $z_{m,n}$ on this domain, when $0 < \frac{c}{b} < z_{m,n}$, it must attain the value $\frac{c}{b}$ exactly twice on either side of the maximizer $p = \frac{m-1}{n-1}$. This is demonstrated graphically in figure 2. $\square$

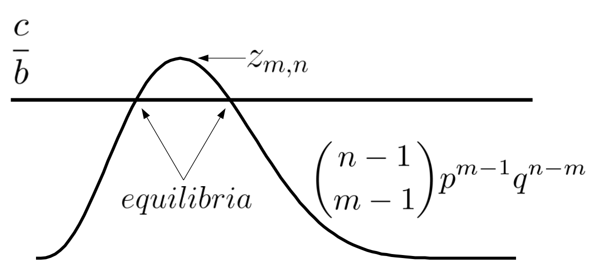

Figure 2: Two Mixed Equilibrium Exist $\frac{c}{b} > z_{m,n}$

For example, suppose a constituency consists of three people. The administrator sets the threshold to two, and the cost of complaining $c = 1$. For a problem with $b = 4$, in order to be indifferent between complaining and not, the

[6]Interestingly, since $z_{m,n}$ is decreasing in n (proof in appendix), larger constituencies complain about a smaller set of problems (under a fixed policy). This result is related in nature to the group size results of Bergstrom and Leo (2015); Diekmann (1985). Larger groups are more susceptible to the free-riding inherent in this environment.

probability of being pivotal must be $\frac{1}{4}$. That is, $2p(1-p) = \frac{1}{4}$. This has solutions approximately 14.6% and 85.4%. From the perspective of any player, there is a 25% chance of being pivotal. However, in the low probability case, this is because most of the time (about 73%) there will be too few complainers no matter what they choose to do. In the high probability case, there is about 73% chance her complaint would be redundant. So while there is a symmetry between these equilibria in terms of pivotal probability, there can be large asymmetries in terms of the number of complaints and the probability of reaching the threshold. Fortunately, the next result solves this multiple equilibrium problem in favor of the high-probability equilibrium by leveraging a stability argument.

2.3.5 Equilibrium Stability

Since groups encounter many issues to complain about, one way of thinking of the equilibrium conditions appearing in this paper is as the result of some adjustment process rather than as the active calculation and mutually-correct conjecturing involved in Nash equilibrium. However, of the two symmetric equilibria involving non-zero probability of complaining, only the equilibrium with a higher probability of complaining could be the result of such an adjustment process. The high-probability equilibrium is an *evolutionary stable strategy [ESS].*

This result is based on the extension of the ESS to many-player symmetric games by Palm (1984). Recent evidence for the predictive power of ESS has been found by Kuzmics and Rodenburger (2019) in reanalysis of a lab experiment on a fourteen-player voting game originally run by Forsythe et al. (1993) that has some features similar to the game analyzed here.

Proposition 3. *When there are two equilibria with $p > 0$, only the one with higher p is evolutionarily stable in the sense of Palm (1984). For this equilibrium, there is a neighborhood $N(p)$ such that the utility from complaining with probability p when others complain with probability $p' \in N(p)$ is larger than complaining with probability p'. (Proof in Appendix)*

For some intuition about why the low and high-probability equilibria are respectively unstable and stable, note that, based on the results in subsection 2.3.1, the pivotal probability is increasing in the probability of complaining when $p < \frac{m-1}{n-1}$. In this region, a small increase in the probability that others complain increases the pivotal probability. This increases the incentive for complaining pushing the complaint probability further from equilibrium. This positive feedback is responsible for the instability of the low-probability equilibrium.

On the other hand, since the pivotal probability is decreasing in p when $p > \frac{m-1}{n-1}$, the opposite occurs, and there is a negative, stabilizing feedback.

2.4 *Comparative Statics*

The comparative statics in this section form the basis for analyzing complaint policies. There are four relevant parameters for the game: the threshold m, the number of players n, the cost c, and the benefit b. Although comparative statics with regard to group size are commonly studied in the Volunteer's Dilemma and related games (see for instance Diekmann, 1985; Bergstrom and Leo, 2015), here I focus on the selection of policy for a fixed constituency. Furthermore, only the cost-benefit ratio is relevant in the equilibrium condition. Thus, I focus on how the equilibrium changes with the threshold m and the cost-benefit ratio $\frac{c}{b}$.

In the stable equilibrium (high probability equilibrium), the complaint probability for each player reacts in an intuitive way. Increasing the required number of complaints increases the probability of complaining. Increasing cost-benefit ratio $\frac{c}{b}$ by increasing the cost or decreasing the benefit decreases the probability of complaining.

While I use the stable equilibrium to derive the rest of the results in this paper, the unusual comparative statics of the unstable equilibrium provide some additional intuition for how the equilibria in these games behave. Similar to the stable equilibrium, players in the unstable equilibrium (low probability equilibrium) complain more often when the threshold is increased. Unlike in the stable equilibrium, they complain more often when the cost is increased or the benefit is decreased.

Ultimately, it is the fact that the pivotal probability is increasing in the individual probability of complaint that causes both the instability of this equilibrium and its unintuitive response to changes in costs and benefits. In equilibrium, if the cost is increased or the benefit is decreased, the pivotal probability must increase. To drive up the pivotal probability, each player has to complain more often.

A formal statement of each of the comparative statics, together with their proofs and figures providing some geometrical intuition for each result follows:

Proposition 4. *The probability of volunteering increases with m in the stable/unstable equilibrium. (Proof in Appendix)*

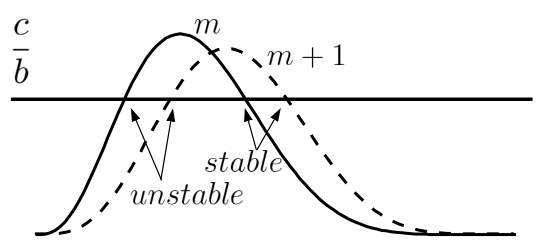

Figure 3: Comparative Statics in m

Proposition 4 is demonstrated graphically in the figure above. As m increases, the function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ shifts to the right. $\frac{c}{b}$ intersects the new function at higher p in both the unstable and stable equilibrium.

Proposition 5. *The probability of complaining decreases/increases with cost in the stable/unstable equilibrium. The probability of complaining decreases/increases with benefit in the stable/unstable equilibrium.*

Proof. The equilibrium is given by $\binom{n-1}{m-1}p^{m-1}q^{n-m} = \frac{c}{b}$. By the implicit function theorem, whether the equilibrium p^* is increasing/decreasing with c depends on whether $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ increasing/decreasing in p. $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ is increasing in p from 0 to $z_{m,n}$ and decreasing from $z_{m,n}$ to 1. The unstable equilibrium occurs below $z_{m,n}$ and the stable equilibrium occurs above $z_{m,n}$. Thus, the unstable equilibrium complaint probability is increasing with c and the stable equilibrium complaint policy is decreasing with c. By the same logic, the opposite is true for benefit b. $\square$

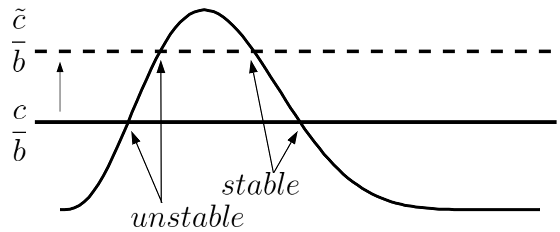

Figure 4: Comparative Statics in m

This proposition is demonstrated in the figure above. The quasi-concave shape of the function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ and the fact that the two equilibria occur on either side of the maximum imply that as $\frac{c}{b}$ increases, the function $\binom{n-1}{m-1}p^{m-1}q^{n-m}$ intersects it at a higher p below the maximizer $\frac{m-1}{n-1}$ (*unstable equilibrium*) and a lower p above the maximizer (*stable equilibrium*).

Of the four parameters, only the cost c and the threshold m are under control of the administrator. The next section demonstrates how the administrator can adjust these values to take advantage of the equilibrium structure to address only high-benefit complaints, despite not observing that benefit.

3 Complaint Policy

Any choice of c and m creates a particular m-volunteer's dilemma on potential complainers when a problem with value b arises. By manipulating c and m, the administrator can determine the set of benefit levels which would induce constituents to complain at all.

For instance, suppose as in the example used above, there are three constituents $n = 3$, the administrator requires two complaints $m = 2$ and the cost of complaining is $c = 1$. Any problems that receive complaints must have a value $b \geq 2$ (see section 2.3).[7] For lower benefit levels, there are no symmetric equilibria that involve complaining. The administrator can leverage this fact in choosing a policy.

Suppose the level of benefit above which the administrator deems it worthwhile to address a problem is $\gamma = 2$. The policy of requiring two complaints with a cost of complaining $c = 1$ guarantees that problems with a benefit below this level are never addressed.

This policy is not perfectly efficient since a problem worth fixing might not receive enough complaints. The actual efficiency depends on the distribution of benefits associated with addressing problems. However, as a simple example, suppose there are two types of problems. Some problems are trivial $b = 1$ and some are significant $b = 10$. Under this policy, trivial problems are never addressed while the probability an important problem is addressed is approximately 99%.

Since the administrator has two parameters to work with, there are many policies that will rule out the exact same complaints. In the next section, I derive this class of policies for any chosen benefit threshold γ.

[7] The administrator can also deduce the expected value of the benefit from the number of complaints. However, if the constituents know the administrator will use this information, it distorts their incentives, making the initial assessment unreliable.

3.1 Eliminating Trivial Complaints

The administrator can eliminate complaints about problems below a chosen benefit threshold $(b < \gamma)$ if c and m are chosen such that $\frac{c}{b} = z_{m,n}$ when $b = \gamma$. That is, the administrator chooses a cost and threshold so that problems with a benefit below their chosen threshold γ would require constituents to believe they are pivotal with a probability that is impossible under symmetric strategies. Doing so ensures there are no symmetric equilibria involving a non-zero complaint probability when $b < \gamma$. This is demonstrated in figure 5.

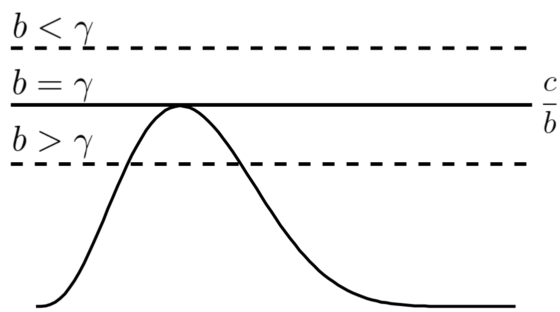

Figure 5: Eliminating Trivial Complaints.

For a given m, such a trivial-complaint-eliminating policy can be achieved by setting the following cost:

$$c_m = \gamma z_{m,n} = \gamma \binom{n-1}{m-1} \left(\frac{m-1}{n-1}\right)^{m-1} \left(\frac{n-m}{n-1}\right)^{n-m} \tag{3}$$

For instance, when $n = 10$, and the administrator wants to eliminate complaints with benefit below 1, they can choose a threshold of 2 with cost 0.390, a threshold of 3 with cost 0.306, a threshold of 4 with cost 0.273, and so forth. Notice that there is an inverse relationship between threshold and cost. This is true as long as $m < \frac{n+1}{2}$. In fact, there is a symmetry around $\frac{n+1}{2}$. This symmetry can be understood by recasting a game where many people must opt-in to complaining as a game where only a few may opt-out of complaining.

For the purposes of the results below, I will assume the threshold is small relative to the size of the constituency. In this case, the administrator will always be operating in the region where the cost of complaining and the complaint threshold are inversely related. This is demonstrated in figure 6. If the administrator decreases the cost of complaining, the threshold must be increased to eliminate the same trivial complaints. This demonstrates more formally the intuitive arguments discussed in the introduction.

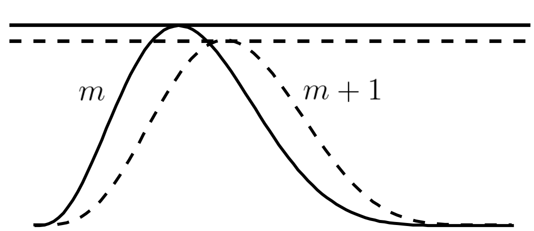

Figure 6: Higher m requires lower c when $m < \frac{n+1}{2}$.

3.2 Large Constituencies

While small constituencies are relevant, for instance in modeling the administration of a condo building or a single academic department, administrators often oversee very large constituencies such as is the case for governments, large corporations, universities, etc. For the rest of the paper, I focus on large constituencies. Beyond the relevance for these important environments, there are several benefits to analyzing complaint policy for large constituencies.

In large constituencies, it is easier to assess the efficiency of a particular policy. In small constituencies the aggregate costs associated with complaining may be relatively meaningful in comparison to the expected net social benefits of problems addressed. In this case, optimizing over c and m to find the most efficient policy would require knowledge of the distribution of benefits among the possible problems about which constituents may complain. Even then, it would be primarily a numerical exercise given the polynomial form of the equilibrium condition in equation 2.

However, as n increases, the probability of complaining shrinks to zero (this is discussed further section 4) and the effect of complaint costs are dominated by efficiency changes through the probability of having problems addressed. For large constituencies, it is thus convenient (and approximately correct) to ignore the costs of complaining in assessing the efficiency of policies. This simplifies the administrator's objective.

Further, for large constituencies, the equilibria of the m-volunteer's dilemma game can be approximated by an equation involving the Poisson distribution and can be solved in closed form.[8] This permits analytical analysis that yields additional insight leading to the ultimate result of this paper that low cost / high threshold policies are most efficient.

4 Large Constituency Equilibrium and The Lambert-W Correspondence

In this section, I derive the properties of the limit of m-volunteer dilemma equilibria as the group size grows large. In the limit, the equilibrium expected number of volunteers can be solved in closed-form, where solving for these values in finite groups would generally involve numerical methods to find the roots of high-order polynomials. The practical motivation for working with the limit of these games is that this limit can serve as an approximation for the behavior of large finite groups, such as a large institution. Fortunately, much of the structure of the finite constituency analysis applies here as well.

4.1 Equilibria

In finite constituencies, each player assesses the likelihood that their complaint is pivotal, and the number of other people who complain is a binomial random variable. As n grows large, the probability any one individual complains shrinks to zero. By the *law of rare events,* the number of others that complain can be approximated by a Poisson random variable. The following condition thus provides an approximation of the equilibrium condition for large group sizes.

$$\frac{((n-1)\,p)^{m-1}\,e^{-(n-1)p}}{(m-1)!} = \frac{c}{b} \tag{4}$$

[8] The solution involves the Lambert-W correspondence, an exponential function closely related to log. Whether this function is establitheyd well enough to qualify the solution as "closed form" is probably subject to debate.

Proposition 6. *The solutions to equation (4) are given by the expression: (Proof in Appendix)*

$$p = -\frac{m-1}{n-1} W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{5}$$

The W in the expression above is worth remark. W, the Lambert-W correspondence, provides the inverse solution of $y = xe^x$. More discussion and some relevant properties of this function are given in the next subsection.

The limit of (5) is uninformative since the right side approaches zero as n increases. That is, the probability any one person complains approaches zero in large groups, otherwise receiving at least m complaints would approach certainty, eliminating the incentive to complain– a disequilibrium outcome. It is more informative to consider the limit of the approximation of the expected number of complainers:

$$p(n-1) = -(m-1)W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{6}$$

Here, the right side is invariant in n, and the left side approaches the expected number of complaints as n grows large. Denote this expectation by $\lambda_{m,b}$. In the limit:

$$\lambda_{m,b} = -(m-1)W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{7}$$

Recall that for small-groups, the game has two equilibria with volunteering (when any equilibria exist). This is true in the limit as well. The two equilibria correspond with the two branches of W as seen in the figure 7. Along the upper branch, the expected number of volunteers approaches zero as b grows. This corresponds to the limit of the low-probability / unstable family of equilibria discussed in the previous section. Along the lower branch, the expected number of volunteers approaches infinity as b grows. This corresponds to the stable family of equilibria discussed in the previous section.

To demonstrate some properties of the equilibrium, suppose $m = 2$ where the equilibrium takes the following simple form:

$$\lambda_{2,b} = -W\left(-\frac{c}{b}\right) \tag{8}$$

Note that W is not defined below $-\frac{1}{e}$. This corresponds to the fact that as $n \to \infty$ for $m = 2$ and $c = 1$, the benefit level below which no equilibrium with volunteering exists e, as previously noted in an example in section 2.3.2. It is interesting to compare this to the limit (in n) of the standard Volunteer's Dilemma, which has a the following expected number of volunteers/complaints:

$$\lambda_{1,b} = -log\left(\frac{c}{b}\right) \tag{9}$$

This provides some context to the relationship between *log* and *Lambert-W* in the context of this particular problem.

4.2 *The Lambert-W Function*

The remainder of this paper relies heavily on properties of the Lambert-W function. As mentioned above, this function is the inversion of $y = xe^x$. A graph of Lambert-W is provided in figure 7.

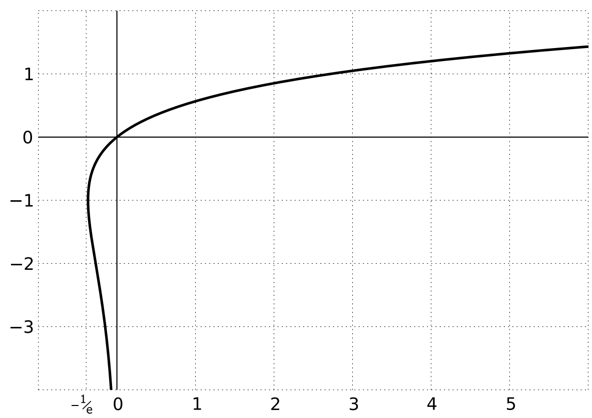

Figure 7: Lambert-W Function

Note that the function is double-valued below zero (the part of the domain relevant in this paper) and single-valued above. This function is also known as "omega" and "product log" (Corless et al., 1996; Weisstein, 2002).

Like its close cousin the *log* function, it is a convenient way to represent the inversion of a transcendental equation. In the case of *log,* this equation is of the form $y = e^x$, and for Lambert-W, it is of the form $y = xe^x$. Importantly, this inversion cannot be expressed in terms of elementary functions (Bronstein et al., 2008).

Like *log,* Lambert-W is useful to the extent that it is useful in understanding the properties of solutions to equations. While the form $xe^x = y$ may seem specialized, many transcendental equations can be transformed into this pattern and thus have solutions that can be expressed in terms of Lambert-W. This includes equations of the form $xa^x = y$, $x^{x^a} = y$, and $a^x - x = y$ among others (Corless et al., 1996). Interestingly, the value $x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}}}$ can also be expressed as $-\frac{W(-ln(x))}{ln(x)}$(Weisstein, 2002).

In this paper, Lambert-W is used to find the the parameter values where a Poisson probability is equal to some constant. In fact, Lambert-W can be to solve any equation of this form.

In economics, Lambert-W has been used in a wide range of applications. It arises in the a Condorcet Jury voting model of Morgan and Várdy (2012), a principal agent problem using tournaments to elicit effort (Imhof and Kräkel, 2016), macroeconomic models in a monopolistic competition framework (Rodriguez-Lopez, 2011; Behrens et al., 2014; Jensen, 2016), the solution of a continuous time problem for calculating the optimal refinancing policy of a fixed rate loan (Agarwal et al., 2013), and the optimal provision of medicine to risk-averse agents (Borges et al., 2011). Lambert-W is also used in solving for similar limits to those used here in a game related to the one studied here called the *Coordinated Volunteer's Dilemma* (Bergstrom and Leo, 2015). Disney and Warburton (2012) demonstrate the usefulness of the Lambert-W function through its applications in optimal inventory management. The function also many applications outside of economics from solving fuel-consumption problems, to modeling the spread of epidemics (Corless et al., 1996).

Further, the usefulness of this function is improved by its understood properties. In this paper, its derivative plays a key role. The derivative of Lambert-W is given by $W'(x) = \frac{1}{x}\frac{W(x)}{1+W(x)}$ for $x \neq \left\{0, -\frac{1}{e}\right\}$ and $W'(0) = 1$.

4.3 Eliminating Trivial Complaints

Eliminating trivial complaints for large constituencies relies on the same logic used in section 3.1 on eliminating trivial complaints for finite constituencies. The equilibrium condition (7) is only defined on the domain where the term inside the Lambert-W correspondence is larger than $-\frac{1}{e}$. When the term is smaller than $-\frac{1}{e}$, there are no equilibria with a non-zero expected number of complaints. Since this term is increasing in b, if c is set such that this term is exactly $-\frac{1}{e}$ for a problem just meeting the benefit threshold γ, then the administrator can eliminate complaints about all problems that do not meet this threshold ($b < \gamma$). This is the cost c that solves:

$$-\frac{1}{m-1}\left[(m-1)!\frac{c}{\gamma}\right]^{\frac{1}{m-1}} = -\frac{1}{e} \tag{10}$$

This solution can be given explicitly by:

$$c_m = \gamma\frac{\left((m-1)\frac{1}{e}\right)^{m-1}}{(m-1)!} \tag{11}$$

Equation (11) corresponds to the limit of the analogous expression given for finite constituencies in equation (3). It is noted that c_m is decreasing in m and approaches 0 as $m \to \infty$. Again, to maintain a policy where problems with benefit just less than γ are ruled out as c decreases, the threshold must be raised accordingly.

4.4 Expected Number of Complaints in Equilibrium Under Policies That Eliminate Trivial Complaints

Plugging (11) into equation (7) provides the expected number of complainers under the policies of this class, which can be simplified into the following expression:

$$\lambda_{m,b} = -(m-1)\,W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right) \tag{12}$$

One interesting property of this game is that complainers overcompensate in the stable equilibrium. Increasing the complaint threshold by one leads the expected number of complaints to increase by at least one.

Proposition 7. *(Overcompensation)* In the stable equilibrium, $\lambda_{m,b} - \lambda_{m-1,b} \geq 1$.

Proof. From equation 4 the ratio of pivotal probabilities when the threshold is $m+1$ and m respectively must equal the ratio of costs that are given by equation 11. The equilibrium rates $\lambda_{m-1,b}$ and $\lambda_{m,b}$ must obey:

$$\frac{\frac{e^{-\left(\lambda_{m+1,b}\right)}\left(\lambda_{m+1,b}\right)^m}{(m!)}}{\frac{e^{-\lambda_{m,b}}\lambda_{m,b}{}^{m-1}}{(m-1)!}} = \frac{1}{e}\left(\frac{m}{m-1}\right)^{m-1} \tag{13}$$

As discussed in section 2.4, in the stable equilibrium, the pivotal probability is decreasing in the volunteering rate. This is true here as well and $\frac{e^{-\left(\lambda_{m+1,b}\right)}\left(\lambda_{m+1,b}\right)^m}{(m!)}$ is decreasing in $\lambda_{m+1,b}$.

Suppose the assertion is not correct and $\lambda_{m+1,b} < \lambda_{m,b}+1$. If this is true then replacing each instance of $\lambda_{m+1,b}$ with $\lambda_{m,b}+1$ in the ratio above must result in a smaller ratio. That is:

$$(14) \qquad \frac{\frac{e^{-\left(\lambda_{m+1,b}\right)}\left(\lambda_{m+1,b}\right)^{m}}{(m!)}}{\frac{e^{-\lambda_{m,b}}\lambda_{m,b}{}^{m-1}}{(m-1)!}} > \frac{\frac{e^{-\left(\lambda_{m,b}+1\right)}\left(\lambda_{m,b}+1\right)^{m}}{(m!)}}{\frac{e^{-\lambda_{m,b}}\lambda_{m,b}{}^{m-1}}{(m-1)!}}$$

To derive a contradiction, I demonstrate that the right side of this inequality must be weakly larger than the ratio of costs $\frac{1}{e}\left(\frac{m}{m-1}\right)^{m-1}$, which is sufficient to demonstrate that the original ratio of pivotal probabilities strictly exceeds the cost ratio– a contradiction. Thus, $\lambda_{m+1,b} \geq \lambda_{m,b}+1$. To derive this contradiction, simplify the following inequality as follows:

$$(15) \qquad \frac{\frac{e^{-\left(\lambda_{m,b}+1\right)}\left(\lambda_{m,b}+1\right)^{m}}{(m!)}}{\frac{e^{-\lambda_{m,b}}\lambda_{m,b}{}^{m-1}}{(m-1)!}} \geq \frac{1}{e}\left(\frac{m}{m-1}\right)^{m-1}$$

Simplifying (15):

$$(16) \qquad \frac{\left(\lambda_{m,b}+1\right)^{m}}{\lambda_{m,b}^{m-1}} \geq \frac{m^{m}}{(m-1)^{m-1}}$$

This inequality is true since the left expression is increasing in $\lambda_{m,b}$ when $\lambda_{m,b} \geq m-1$. Taking $\lambda_{m,b} = m-1$ as a sufficient condition, it is met with equality. $\square$

The overcompensation result puts a lower bound on how the expected number of complaints increases when the threshold is increased. The next result puts an upper bound on the proportional increase in expected complaints. Note that at $b=\gamma$ the ratio $\frac{\lambda_{m+1,b}}{\lambda_{m,b}}$ is $\frac{m}{m-1}$. The following result demonstrates that this is the largest this ratio can be.

Proposition 8. (Ratio Bound) In the stable equilibrium, $\frac{m}{m-1} \geq \frac{\lambda_{m+1,b}}{\lambda_{m,b}}$.

Proof. Substituting in 12, this is equivalent to $W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right) \geq W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)$. Since W is decreasing along its lower branch, this inequality is true when $-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}} \geq -\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}$. Since $\gamma \leq b$, this is true since $\frac{1}{m-1} \geq \frac{1}{m}$. $\square$

Putting these results together, as the threshold is increased from m to $m+1$, the expected number of complaints increases by at least one but not by so much as to make the new expected number of complaints more than $\frac{m}{m-1}$ times larger.

4.5 Policy Choice: Minimizing The Failure Rate

Since the expected number of complaints remains finite as group size grows, the aggregate welfare effects of complaint costs are swamped by changes in the probabilities of having problems addressed. Thus, the administrator's objective is simply a function of the probability that enough complaints are received for a particular problem. In a perfectly efficient outcome, relative to the chosen benefit threshold, the complaint threshold is always met for problems meeting or exceeding the benefit threshold and never otherwise.

By construction, in the class of policies defined by (11), problems that do not meet the benefit threshold receive no complaints. Thus, optimizing within this class of policies requires the administrator to maximize the probability of reaching the complaint threshold for problems where $b \geq \gamma$.[9]

On the surface, the over-compensation property proved in the previous section hints at the efficiency of low-cost high threshold policies. Increasing the threshold by one and lowering the cost appropriately leads to an increase in expected complaints that is larger than one. However, what really matters is not the number of complaints but rather the probability that there are not enough complaints to reach the threshold m. Let $Q(x, \lambda)$ denote the probability a Poisson random variable with rate λ takes a value x or lower.

A higher threshold policy is universally better if it has a lower failure probability for all problems meeting the benefit threshold. That is when $Q\left(m-1, \lambda_{m,b}\right) > Q\left(m, \lambda_{m+1,b}\right)$ for all $b \geq \gamma$.

These failure probabilities can be expressed as follows:

$$Q\left(m-1, \lambda_{m,b}\right) = \sum_{k=0}^{m-1} \frac{e^{-\lambda_{m,b}} \left(\lambda_{m,b}\right)^k}{k!} \tag{17}$$

4.5.1 The Case of $b = \gamma$

Take as an example a problem $b = \gamma$: the most trivial of problems that meet the benefit threshold and receive complaints in equilibrium. We can compare policies that require m and $m+1$ complaints. By equation (12), $\lambda_{m,\gamma} = m-1$, and $\lambda_{m,\gamma} = m$. In equilibrium, the *expected* number of complaints is exactly equal to one less than the threshold. Note that while this does not violate the overcompensation proposition, the complainers *just* compensate for the additional required complaint.

Is this increase in complaining enough to reduce the failure probability? This would be true if $Q(m-1, m-1) > Q(m, m)$ for the Poisson distribution, which has been shown to be true by Teicher (1955); Adell and Jodra (2005).

In this case, the constituents' exact compensation for the increased threshold reduces the probability of failure. At least for $b = \gamma$, higher thresholds and lower costs are better.

However, the addition of *one* expected complaint is not always enough to reduce the probability of failure. For example suppose $m = 4$ and b is such that 6 complaints are produced on average in equilibrium. The probability of failure is $Q(3, 6) \approx 15.1\%$. Now suppose an additional complaint is required, and the constituents exactly compensate by increasing the expected complaints to 7. The probability of failure is $Q(4, 7) \approx 17.3\%$. In this case, an increase in the number of expected complaints by one *is not enough* to reduce the probability of failure. The complainers need to *over-compensate*, increasing the expected complaints by 1.25 to reduce the failure rate.

In fact, in equilibrium they produce around 1.36 extra complaints, reducing the failure probability to 14.3%. Does this pattern hold up? Yes, players always overcompensate by enough to reduce the failure probability in the stable equilibrium.

4.5.2 A Numerical Example

Table 1 reports computed values of $Q\left(m-1, \lambda_{m,b}\right)$ for various benefit levels with $\gamma = 2$. Note that the failure rate is decreasing monotonically in each row as the required number of complaints is increased. At moderate levels of b,

[9] This is just one class of policies available. With more detailed knowledge of the distribution of problem values b, the administrator could choose to lower costs below those in condition 3. This would decreases Q_m for all non-trivial problems but require the administrator to occasionally address trivial problems. It is an interesting question whether it might sometimes be efficient to make such a trade-off. On the other hand, setting complaint cost according to condition 3 and eliminating complaints about all trivial problems is likely to be most efficient for large groups when the distribution of problems is such that trivial problems are much more frequent than non-trivial problems.

failure to reach the complaint threshold is rare for all policies. Still, moving from $m = 2$ to $m = 10000$ offers substantial relative improvement. The failure rate is cut in half for most of the reported benefit levels.

For a rough interpretation of these numbers, note that, referencing formula (11), moving from threshold of 2 to 10000 is associated with reducing the complaint cost approximately one–hundred–fold. In terms of time, this is equivalent to the difference between a task that takes an hour and a task that takes 30 seconds. This might be, roughly, the difference in effort between writing a letter and signing an online petition.

	$m = 2$	3	5	10	20	50	100	1000	10000
$b = 2$	0.736	0.677	0.629	0.587	0.561	0.538	0.527	0.508	0.503
3	0.357	0.310	0.274	0.244	0.225	0.210	0.202	0.190	0.186
5	0.196	0.166	0.143	0.125	0.113	0.103	0.099	0.091	0.089
10	0.092	0.076	0.065	0.055	0.049	0.044	0.042	0.038	0.037
100	0.008	0.007	0.006	0.005	0.004	0.003	0.003	0.003	0.003
1000	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000

Table 1: Computed Failure Probabilities for $\gamma = 2$.

While table 1 provides further evidence of the monotonic nature of the failure probability in m, it is just one set of parameters. Below, I prove this is a general result. Reducing cost and raising the threshold is *always* more efficient within this class of policies.

4.6 Higher Thresholds are Universally Better

Below, I state and prove the main theorem of this paper. Higher threshold / lower cost policies are universally more effective. Every increase in threshold together with an appropriate reduction in cost reduces the rate of failure for all $b \geq \gamma$.

Theorem 1. *Among the class of policies that rule out complaints about problems with $b < \gamma$, the probability not enough complaints are received in the stable equilibrium is decreasing in m for any fixed b.*

This proof is detailed and relies on several lemmas. In the following subsections, I outline the proof and present each lemma. The proof relies heavily on properties of the tails of Poisson random variables and the Lambert-W function.

4.6.1 Proof Outline

This proof is equivalent to $Q\left(m-1, \lambda_{m,b}\right) - Q\left(m, \lambda_{m+1,b}\right) > 0$ for all m and $b \in [\gamma, \infty)$.

For convenience, define this difference as function g:

$$g\left(m,b\right) \equiv Q\left(m-1, \lambda_{m,b}\right) - Q\left(m, \lambda_{m+1,b}\right) \tag{18}$$

The proof proceeds by showing that g is strictly positive for any fixed m as b changes over the relevant domain $[\gamma, \infty)$. The complex nature of this function makes it difficult to work with directly, however, several more indirect facts about the geometry of g imply that it is strictly positive on this domain.

First, at $b = \gamma, \lambda_{m,b} = m - 1$. As previously discussed, for the most trivial problem meeting the benefit threshold, the expected number of complaints is one less than the amount needed for the complaint to be addressed. For this most trivial type of complaint, $g\left(m, \gamma\right) = Q\left(m-1, m-1\right) - Q\left(m, m\right) > 0$ by a result about the Poisson distribution due to

Teicher (1955); Adell and Jodra (2005), as mentioned in section 4.5.1. The rest of the proof demonstrates this is also true for other values of b.

We now look to the other side of the domain. As $b \to \infty$, any policy results in a near certainty that the complaint will be addressed. This is because as $b \to \infty$,the expected number of complaints increases to infinity. Thus, the probability of not receiving enough complaints decreases to zero for any policy. Because of this, $\lim_{b\to\infty} g(m,b) = 0$.

These two facts about g show little about the behavior of the function away from these extreme types of complaints. What happens on the interior of the domain? The first step in demonstrating that g never dips below zero is to show that at $b = \gamma$, g is increasing. That is, moving away from the most trivial problems, the improvement in the failure rate that results from increasing the required number of complaints, at least initially, becomes even larger. However, since g eventually approaches zero, g must shrink at some point. However, since the function is initially positive, initially increasing, and approaching zero at infinity, for it to drop below zero at some point, it would have to change directions twice. The final step of the proof is to demonstrate that this cannot happen because g has only a single stationary point.

Proof. This result is equivalent to $Q(m-1,\lambda_{m,b}) - Q(m,\lambda_{m+1,b}) > 0$ for all m and $b \in [\gamma,\infty)$. Let $g(m,b) \equiv Q(m-1,\lambda_{m,b}) - Q(m,\lambda_{m+1,b})$. By lemma [1], the function g is positive at $b = \gamma$ and by lemma [2] approaches 0 as $b \to \infty$. By lemma [3], g is increasing at $b = \gamma$, and by lemma [4] it has only one stationary point on $b \in [\gamma,\infty)$. Combining these four lemmas, the function g must be strictly positive and increasing to some stationary point, then decreasing and approaching zero from above, remaining strictly positive on the domain.

□

The four parts of this proof are demonstrated graphically below:

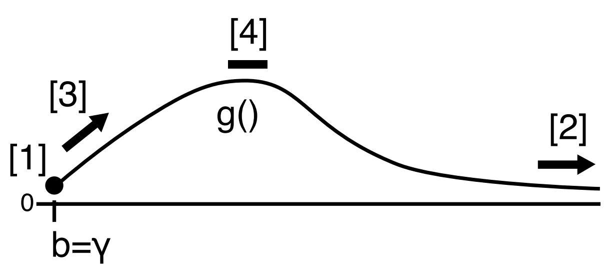

Figure 8: Graphical representation of the four lemmas regarding the geometry of the function g.

What remains is to prove the four lemmas used in this proof:

[1] $g(m,\gamma) > 0$.

Proof. See Teicher (1955); Adell and Jodra (2005).

[2] $lim_{b\to\infty} g(m,b) = 0$.

Proof. From equation (12) $\lambda_{m,b}$ and $\lambda_{m+1,b}$ both approach ∞ as $b \to \infty$ since W approaches $-\infty$ along its lower branch (in the stable equilibrium). Since $lim_{\lambda\to\infty} Q(m-1,\lambda) = 0$ and $lim_{\lambda\to\infty} Q(m,\lambda) = 0$, g also approaches 0. □

[3] $g(m,b)$ is increasing at $b = \gamma$

Proof. See subsection 4.6.2

[4] g has a single stationary point for $b > \gamma$.

Proof. See subsection 4.6.3

4.6.2 Lemma 3. g is increasing at γ.

In this subsection, I prove $g(m,b) = Q(m-1,\lambda_{m,b}) - Q(m,\lambda_{m+1,b})$ is increasing in b at $b=\gamma$ for all m.

Proof. The derivative of g with respect to b is given by the difference of two similar expressions.

$$\frac{\partial g(m,b)}{\partial b} = \frac{e^{mW_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}}{e}\right)} W_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}}{e}\right)\left(-mW_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}}{e}\right)\right)^{m}}{bm!\left(W_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}}{e}\right)+1\right)} \tag{19}$$

$$-\frac{e^{(m-1)W_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}}{e}\right)} W_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}}{e}\right)\left(-(m-1)W_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}}{e}\right)\right)^{m-1}}{b(m-1)!\left(W_{-1}\left(-\frac{\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}}{e}\right)+1\right)} \tag{20}$$

g is increasing in b when $\frac{\partial g(m,b)}{\partial b} > 0$. Simplifying the resulting inequality:

$$\frac{e^{-\lambda_{m+1,b}}\left(\lambda_{m+1,b}\right)^{m}}{m!}\frac{\frac{\lambda_{m+1,b}}{m}}{\left(\frac{\lambda_{m+1,b}}{m}-1\right)} > \frac{e^{-\lambda_{m,b}}\left(\lambda_{m,b}\right)^{m-1}}{(m-1)!}\frac{\frac{\lambda_{m,b}}{m-1}}{\left(\frac{\lambda_{m,b}}{m-1}-1\right)} \tag{21}$$

Taking the limit of equilibrium condition 4 as $n \to \infty$:

$$\frac{\left(\lambda_{m,b}\right)^{m-1}e^{-\lambda_{m,b}}}{(m-1)!} = \frac{c}{b} \tag{22}$$

Substituting this into the inequality above:

$$\frac{c_{m+1}}{b}\frac{\frac{\lambda_{m+1,b}}{m}}{\left(\frac{\lambda_{m+1,b}}{m}-1\right)} > \frac{c_m}{b}\frac{\frac{\lambda_{m,b}}{m-1}}{\left(\frac{\lambda_{m,b}}{m-1}-1\right)} \tag{23}$$

Substituting from equation 11 $c_m = \gamma\frac{\left((m-1)\frac{1}{e}\right)^{m-1}}{(m-1)!}$:

$$\gamma\frac{\left(m\frac{1}{e}\right)^{m}}{(m)!}\frac{\frac{\lambda_{m+1,b}}{m}}{\left(\frac{\lambda_{m+1,b}}{m}-1\right)} > \gamma\frac{\left((m-1)\frac{1}{e}\right)^{m-1}}{(m-1)!}\frac{\frac{\lambda_{m,b}}{m-1}}{\left(\frac{\lambda_{m,b}}{m-1}-1\right)} \tag{24}$$

Simplifying this expression:

$$\left(\frac{m}{m-1}\right)^{m-1}\frac{1}{e} > \frac{1-\frac{m}{\lambda_{m+1,b}}}{1-\frac{m-1}{\lambda_{m,b}}} \tag{25}$$

By (12), $\lambda_{m,b} = -(m-1)W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)$, substituting this into the inequality above:

$$\left(\frac{m}{m-1}\right)^{m-1}\frac{1}{e} > \frac{1+\frac{1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}}{1+\frac{1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}} = \frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1}\frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)} \tag{26}$$

To demonstrate that $\frac{\partial g(m,b)}{\partial b} > 0$ for $b=\gamma$ take the limit of the right side:

$$Lim_{b\to\gamma}\frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1}\frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)} = Lim_{b\to\gamma}\frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1}1$$

This limit can be found using the first term of the Taylor expansion:

$$Lim_{b\to\gamma}\frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1} = Lim_{b\to\gamma}\frac{1-1-\sqrt{2\left(1-\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}+O\left(\left(\frac{\gamma}{b}-1\right)^2\right)}{1-1-\sqrt{2\left(1-\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}+O\left(\left(\frac{\gamma}{b}-1\right)^2\right)} \tag{27}$$

$$= \sqrt{Lim_{b\to\gamma}\frac{\left(1-\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}{\left(1-\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}} \tag{28}$$

Using l'Hopital's rule:

$$\sqrt{Lim_{b\to\gamma}\frac{\left(1-\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}{\left(1-\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}} = \sqrt{\frac{m-1}{m}} \tag{29}$$

Plugging this limit into inequality (26) and rearranging provides the following inequality:

$$\frac{1}{e} > \left(\frac{m-1}{m}\right)^{m-\frac{1}{2}}$$

The right side is strictly increasing in m above 1 and $lim_{m\to\infty}\left(\frac{m-1}{m}\right)^{m-\frac{1}{2}} = \frac{1}{e}$. Thus, the condition is true for all finite m. □

Equation (25) is worth comment. This inequality provides a condition on when g is increasing/decreasing. This is determined by the value $\frac{1-\frac{m}{\lambda_{m+1,b}}}{1-\frac{m-1}{\lambda_{m,b}}}$, which may be rewritten $\frac{\frac{\lambda_{m+1,b}-m}{\lambda_{m+1,b}}}{\frac{\lambda_{m,b}-m-1}{\lambda_{m,b}}}$. We may think of $\frac{\lambda_{m+1,b}-m}{\lambda_{m+1,b}}$ as the proportion of excess complaining. The ratio $\frac{\frac{\lambda_{m+1,b}-m}{\lambda_{m+1,b}}}{\frac{\lambda_{m,b}-m-1}{\lambda_{m,b}}}$ is then the *ratio* of excess complaining. Denote this ratio by $h(b,m)$. Note that as b approaches γ, the ratio of excess complaining approaches $\sqrt{\frac{m-1}{m}}$, and as b approaches ∞,this ratio approaches 1, since excess complaining under any policy approaches 1. In the next lemma, I demonstrate that this ratio is monotone in b, implying that g has only one stationary point.

4.6.3 Lemma 4. g has a single stationary point.

In this subsection, I prove $\frac{\partial g(m,b)}{\partial b}$ has a single stationary point in b for all m.

Let $h(b,m) = \frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1} \frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}$. From inequality 26, g is stationary when $\left(\frac{m}{m-1}\right)^{m-1}\frac{1}{e} = h(b,m)$. From the previous lemma, $lim_{b\to\gamma} h(b,m) < \left(\frac{m}{m-1}\right)^{m-1}\frac{1}{e}$. Furthermore, $lim_{b\to\infty} h(b,m) = 1 > \left(\frac{m}{m-1}\right)^{m-1}\frac{1}{e}$ for $m>1$.
In the sub-lemma below, I demonstrate that h is increasing on the domain $[\gamma,\infty)$, and thus $h(b,m) = \left(\frac{m}{m-1}\right)^{m-1}\frac{1}{e}$ can only occur once, which implies there is a unique stationary point for g.

Sublemma. *$h(b,m)$ is increasing on $[\gamma,\infty)$.*

Proof. $h(b,m) = \frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1} \frac{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}{W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}$ is of the form $\frac{b(x)+1}{a(x)+1}\frac{a(x)}{b(x)}$ which has derivative $\frac{b(x)a'(x)-a(x)b'(x)+b(x)^2a'(x)-a(x)^2b'(x)}{(a(x)+1)^2b(x)^2}$ and is positive when $b(x)a'(x)-a(x)b'(x) \geq (-b(x))[b(x)a'(x)] - (-a(x))[a(x)b'(x)]$. This is true when [1] $-b(x) \leq -a(x)$ and [2] $b(x)a'(x)-a(x)b'(x) \geq 0$.
[1] $-b(x) \leq -a(x)$ is equivalent to $W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right) \geq W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)$. Substituting from (12) and rearranging the resulting inequality, this is equivalent to $\frac{m}{m-1} \geq \frac{\lambda_{m+1,b}}{\lambda_{m,b}}$ and true by proposition 8.
[2] substituting in the definition of $a()$ and $b()$,

$$b(x)a'(x)-a(x)b'(x) = W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)\frac{\partial W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}{\partial b} - W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)\frac{\partial W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}{\partial b}.$$

Taking these derivatives:

$$= W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)\left(-\frac{W\left(--\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}{b(m-1)\left(W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1\right)}\right) - W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)\left(-\frac{W\left(--\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}{b(m)\left(W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1\right)}\right)$$

This is positive when:

$$W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)\left(-\frac{W\left(--\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)}{b(m-1)\left(W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)+1\right)}\right) \geq W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)\left(-\frac{W\left(--\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)}{b(m)\left(W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1\right)}\right)$$

This is equivalent to:

$$mW\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m}}\right)+1 \leq (m-1)W\left(-\frac{1}{e}\left(\frac{\gamma}{b}\right)^{\frac{1}{m-1}}\right)$$

Substituting from (12):

$$1 \leq \lambda_{m+1,b} - \lambda_{m,b}$$

This is true by the over-compensation result in proposition 7. □

5 Discussion

The results of this paper suggest that the trend toward democratic crowdsourcing is a trend in the right direction. Among policies that eliminate trivial complaints, those that lower the cost of complaining are universally better at attracting complaints about non-trivial issues.

There are several interesting extensions possible. With assumptions about the distribution of benefit levels, it would be possible to compare policies that eliminate trivial complaints to policies attract some trivial complaints in order to increase the probability of meeting the complaint threshold for non-trivial problems. In what situations would administrators want to accept the prospect of occasionally addressing trivial issues?

Second, most of the analysis related to policy efficiency in this paper pertains to large groups. In large groups, the costs of complaining can be safely ignored because increases in the probability of having issues addressed benefit everyone, while the costs of complaining are borne by a vanishingly small proportion of the constituency. Again, with assumptions about the distribution of benefit values among potential issues, it would be possible to account for these costs and to analyze the efficiency of these policies in small groups. While the equilibrium condition does not generally have closed-form solutions (it is a polynomial equation), numerical and potentially some analytical results may be possible.

In this paper, I assume that complainers make decisions simultaneously. This is natural when the complainers do not know the number of complaints that have already been filed. However, this number is often visible as is the case with many e-petition platforms. Dynamic games related to the volunteer's dilemma have been studied in Bergstrom (2017); Shapira and Eshel (2000); Bilodeau and Slivinski (1996); Weesie (1994); Bliss and Nalebuff (1984). Extending the m-volunteer's dilemma game in this way may provide additional insights. Another interesting area to explore is asymmetric players, in terms of costs and benefits. This has been studied in the standard volunteer's dilemma by Diekmann (1993); Bergstrom and Leo (2015), and in an experimental setting by Pate and Healy (2016).

The complaint scenario studied here provides a rich and interesting environment in which to study the m-volunteer's dilemma game, and the results can also inform analysis in other environments where the effort of several participants is needed to provide a public good. For instance, in fund-raising, matching funds are sometimes contingent on reaching a certain number of donations. Gee and Schreck (2016) call this matching procedure a "threshold match" and study behavior of contributors under this procedure in field and laboratory experiments. They find that subjects' behavior is consistent with the kind of pivotal calculations that are key to the equilibrium analysis of the m-volunteer's dilemma. The results here suggest that when using a threshold match to raise funds, a higher threshold with a lower minimum donation may be most effective.

In fact, the success of crowd-funding businesses like Kickstarter and Indigogo is built on the ask-less-from-more model of collective action (Agrawal et al., 2014). For an example of a successful campaign using minimal cost and large thresholds, *Radiotopia*, an online public radio collective, set up a fundraising drive where \$20,000 in matching funds would be provided if at least 10,000 fans donated. However, the minimum donation was only one dollar. They met this goal and a similar one the following year requiring 20,000 backers. Most interestingly, the total donations amounted to more than \$375,000 from 11,693 backers in the first campaign and \$620,000 from 21,808 backers in the second campaign (PRX, Inc., 2020). Without a counterfactual, it is impossible to determine the exact impact of the threshold match in these campaigns. However, the fact that the number of donations was close to the threshold in both cases suggests that the matching funds were motivating.

Beyond the results for complaint policies and related institutions, this paper provides an example of how the properties of the Lambert-W function can be used in economics to help pry results out of equations that, otherwise, cannot be solved in closed-form. Lambert-W arises in this paper in an equation where a Poisson probability is equal to a constant. This does not appear to be a very unusual structure. Given the function's known properties and flexibility in describing the solution to a large family of transcendental equations, familiarity with Lambert-W may be beneficial for economists working on a wide range of topics.

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7 Appendix

7.1 Proof that $z_{m,n}$ is decreasing in n.

This is true if $z_{m,n} < z_{m,n-1}$, expanding these expressions:

$$\binom{n}{m-1}\left(\frac{m-1}{n}\right)^{m-1}\left(\frac{n-m}{n}\right)^{n-m+1} < \binom{n-1}{m-1}\left(\frac{m-1}{n-1}\right)^{m-1}\left(\left(\frac{n-m}{n-1}\right)\right)^{n-m} \tag{30}$$

This simplifies to:

$$\frac{n-m}{n-m+1} < \left(\frac{n}{n-1}\right)^{n-1} \tag{31}$$

This inequality is true since $\frac{n}{n-1} > 1$ and $\frac{n-m}{n-m+1} < 1$.

7.2 Proof of Proposition 3

Let $\pi_i(p_i, p_{-i})$ be the expected payoff to a player as a function of the probability each complains with p_{-i} being the vector of probabilities that players other than i complain. By Palm (1984) (prop 4), in a symmetric $n-$player game stability of symmetric mixed equilibrium p is equivalent to $\pi(p,x,p,...,p) > \pi(x,x,p,...,p)$ for every $x \in [0,1] \setminus p$.

Letting $B(m,n,p)$ be the binomial probability of m successes in n trials with success probability p, the probability that $m-1$ complain out of $n-1$ when one player complains with probability x and the others with probability p is:

$$xB(m-2,n-2,p) + (1-x)B(m-1,n-2,p)$$

$B(m-1,n-2,p) > (<) B(m-2,n-2,p)$ when $p > (<) \frac{m-1}{n-1}$.

For the unstable equilibrium p_l, $p_l < \frac{m-1}{n-1}$. To show this equilibrium is not stable, suppose one other complains with probability $x < p_l$. In this case, $B(m-1,n-2,p_l) < B(m-2,n-2,p_l)$. Thus, the probability that $m-1$ complain out of $n-1$ so that the remaining player is pivotal is smaller than in equilibrium where all of the $n-1$ complain with probability p_l. Because the pivotal probability is smaller than what is required to make the remaining player indifferent, their best response is to not complain. Since $x < p_l$ puts more weight on not complaining: $\pi(x,x,p_l,...,p_l) > \pi(p_l,x,p_l,...,p_l)$.

For the stable equilibrium $p_h > \frac{m-1}{n-1}$, $B(m-1,n-2,p_h) > B(m-2,n-2,p_h)$ when one of the $n-1$ players complains with probability $x < p_h$ the probability that $m-1$ complain out of $n-1$ is higher than in equilibrium where all complain with probability p_h. Since they are pivotal with a probability higher than that which makers them indifferent between complaining and not, their best response is to complain. Since p_h puts more weight on complaining, $\pi(p_h,x,p_h,...,p_h) > \pi(x,x,p_h,...,p_h)$. Similarly if we take $x > p_h$, the best response is to not complain. Since p_h puts less weight on complaining, $\pi(p_h,x,p_h,...,p_h) > \pi(x,x,p_h,...,p_h)$ here as well. Thus, p_h is stable $\square$

7.3 *Proof of Proposition 4.*

Denote the stable equilibrium when m complaints are required p_m^*. The probability that a player is pivotal ($m-1$ of the others complain) must be $\frac{c}{b}$ in equilibrium. As shown below, when $m+1$ complaints are required rather than m, if the players continue to complain with probability p_m^*, the pivotal probability is *larger* than $\frac{c}{b}$.

Since the pivotal probability is *decreasing* in the probability an individual complains in the stable equilibrium, this implies that for the stable equilibrium, $p_{m+1}^* > p_m^*$.

Conversely, in the unstable equilibrium, p_m^* makes the pivotal probability too low when $m+1$ complaints are required. However, since in the unstable equilibrium the pivotal probability is *increasing* in the individual complaint probability, it is still true that $p_{m+1}^* > p_m^*$.

To prove this, note that for m and $m+1$ required complaints, holding the individual complaint probability fixed, the pivotal probability is strictly larger (respectively smaller) for m required complaints rather than $m+1$ required complaints when p is strictly less than (respectively strictly larger than) $\frac{m}{n}$.

$$\binom{n-1}{m-1} p^{m-1}(1-p)^{n-m} > (<) \binom{n-1}{m} p^{m}(1-p)^{n-m-1} \tag{32}$$

$$\frac{m}{n-m} > (<) \frac{p}{1-p} \tag{33}$$

$$p < (>) \frac{m}{n} \tag{34}$$

Consider the unstable equilibrium $p_m^* \leq \frac{m-1}{n-1}$. Since $\frac{m-1}{n-1} < \frac{m}{n}$, when the required complaints are increased to $m+1$, if players continue complain with the unstable equilibrium probability for when m complaints are required, the pivotal probability is too low: $\binom{n-1}{m}\left(p_{m-1}^*\right)^m\left(1-\left(p_{m-1}^*\right)\right)^{n-m-1} < \frac{c}{b}$. Since this function is increasing in p on $p < \frac{m}{n-1}$, then if an unstable equilibrium p_{m+1}^* exists, it must be below $\frac{m}{n-1}$. Thus $p_{m+1}^* > p_m^*$.

Now consider the stable equilibrium. First, suppose $\frac{m}{n} \geq p_m^* > \frac{m-1}{n-1}$. If a stable equilibrium p_{m+1}^* exists it must be that $p_{m+1}^* \geq \frac{m}{n-1} > \frac{m}{n}$. Thus, $p_{m+1}^* > p_m^*$.

Now suppose $p_m^* > \frac{m}{n}$. By a similar argument to the one used above, the pivotal probability when m complaints are required but players use the stable equilibrium from when $m-1$ are required is too high $\binom{n-1}{m}\left(p_{m-1}^*\right)^m\left(1-\left(p_{m-1}^*\right)\right)^{n-m-1} > \frac{c}{b}$. Since this function is decreasing in p, if a stable equilibrium p_{m+1}^* exists it must be that $p_{m+1}^* > p_m^*$.

7.4 *Proof of Proposition 6.*

The equilibrium condition is:

$$\frac{((n-1)p)^{m-1} e^{-(n-1)p}}{(m-1)!} = \frac{c}{b} \tag{35}$$

This is equivalent to:

$$-\frac{((n-1)\,p)}{m-1}e^{-\frac{(n-1)p}{m-1}} = -\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}} \tag{36}$$

The equation is of the form $xe^x = z$ and can be inverted using the Lambert-W function:

$$-\frac{(n-1)\,p}{m-1} = W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{37}$$

Isolating p provides the following solution:

$$p = -\frac{m-1}{n-1}W\left(-\frac{1}{m-1}\left[(m-1)!\frac{c}{b}\right]^{\frac{1}{m-1}}\right) \tag{38}$$