Intermediate Microeconomics*

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These notes are based on my Vanderbilt Economics Course 3012. **They are preliminary.** If you find any typos or errors in this text, please e-mail me at g.leo@vanderbilt.edu.

Contents

Part I

Bundles & Budget

1 Bundles

Bundles are the fundamental object of study in microeconomics. In our models, when a consumer makes a choice, they choose a **bundle** from the set of bundles available to them (the **budget set**). Bundles can be anything or combination of things you can think of. In this course, however, bundles are *usually* going to be amounts of some things we call **goods** and very often we will just look at two goods.

Bundle: $x = (x_1, x_2)$

Example. Ice Cream Bowls (the bundles) are made of up two goods: scoops of vanilla ice cream and scoops of chocolate ice cream. x_1 is the amount of vanilla. x_2 is the amount of chocolate. $(1, 1)$ represents one scoop of each flavor, $(2, 2)$ two scoops of each flavor, and $(0.28, 100)$ a lot of chocolate (100 scoops) and a little vanilla (0.28 scoops).

Since bundles with two goods are represented by ordered pairs, we can plot bundles on and x_1, x_2 axis. An example of this is shown below.

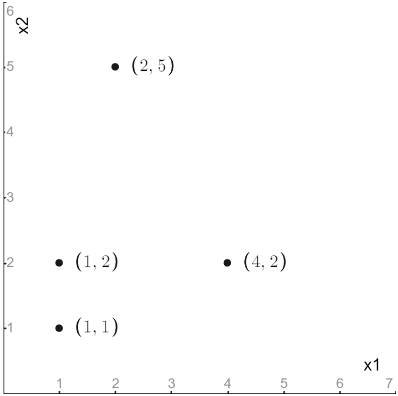


Figure 1.1: A Few Bundles on The Cartesian Plane.

2 Feasible Set

The set of all bundles relevant to a model is called the **Feasible Set**. The feasible set defines the scope of a model.

The Feasible Set: X is the "feasible" set of bundles.

Example. The feasible set for a model about choosing ice cream bowls is the set of all ordered pairs possible ice cream bowls: (x_1, x_2). Of course, it does not make sense to have a negative amount of ice cream, so in this case we might say $X = \mathbb{R}^2_+$. (This notation says that the feasible set is made up of 2 real numbers that are non-negative.)

3 Budget Set

Budget Set: B

The budget set is the set of bundles *available* to a particular consumer. The budget set must be a subset of the feasible set. In set notation we write: $B \subseteq X$

3.1 Budget Sets from Prices and Income

Not everything in the feasible set is going to be achievable for every consumer. Some bundles are affordable and others are not. The set of bundles that a consumer can *actually choose from* is called the **budget set**. Our budget sets will be constructed by assuming consumers have some income and that each good has a price.

Prices: p_1, p_2: Price units of good 1 and good 2.
Income: m.

With these, we can define the cost of a bundle:

Cost of a bundle: $p_1x_1 + p_2x_2$

The set of all bundles that a consumer can afford is called the **Budget Set.** We can define if formally this way:

Budget set: $B = \{x | x \in X \,\&\, x_1p_1 + x_2p_2 \leq m\}$ **.**[a]

[a] In "normal" language, this says the budget set is the set of bundles such that the price of the bundle is less than income.

Since we are able to plot bundles, we can also plot the budget set. To do this, it is easiest to first, we draw the **Budget Line**. This is the set of bundles that are "just affordable".

Budget Line: $x_1p_1 + x_2p_2 = m$

Now we can plot this on an x_1, x_2 plane. Let's put x_2 of the vertical axis. In this case, it is useful to rewrite the budget line into a form we are more familiar with:

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$$

This is now clearly an equation for a line with intercept $\frac{m}{p_2}$ and slope $-\frac{p_1}{p_2}$. Before we plot it, let's interpret it a little. Notice that if $x_1 = 0$ we get $x_2 = \frac{m}{p_2}$. This says "If I were only to buy x_2, I could afford $\frac{m}{p_2}$ units of x_2. Furthermore, for every unit that we increase x_1 by, x_2 goes down by $-\frac{p_1}{p_2}$. This says "If I am spending all my money, if I want to buy one more unit of x_1, I have to give up $-\frac{p_1}{p_2}$ units of x_2. This is a very important thing to know about the slope of the budget line. **The slope of the budget line represents the trade-off between x_1 and x_2 at the market prices.** We are now ready to plot the budget set. It is the budget line and all of the bundles "below" the budget line.

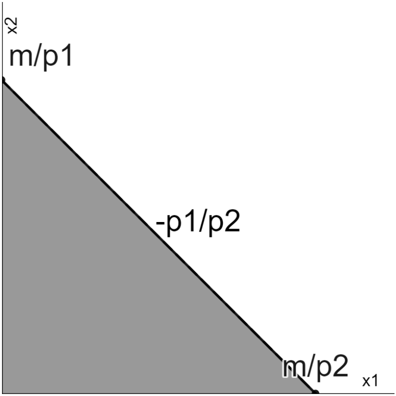


Figure 3.1: Graphical Representation of the Budget Set with slope $-\frac{p_1}{p_2}$ and intercepts $\frac{m}{p_1}$ and $\frac{m}{p_2}$.

3.2 Changing Prices and Income

We are often interested in looking at how the budget set changes when we change on of the parameters of the model: m, p_1, or p_2.

We can work out how the budget set changes by looking at changes in the budget line. There are three key elements to the budget line: the slope $-\frac{p_1}{p_2}$ and the intercepts $\frac{m}{p_1}$ and $\frac{m}{p_2}$.

When income changes, notice that only the intercepts change. If m increases, both intercepts increase. This should be intuitive. Since the intercepts represent how much of a good we can buy if we only buy that good, then if income increases, we can afford more. When income decreases, the opposite happens.

Importantly, when income changes, the slope of the budget line does not change. This is because the trade-off between the goods stays the same regardless of income (as long as the price remain the same).

When a price changes on the other hand, the slop of the budget line changes and **one** of the intercepts changes. For instance, if p_1 goes up, the slope of the budget line becomes steeper (because more x_2 has to be given up to get an extra unit of x_1). Furthermore, the x_1 intercept decreases because less x_1 can be afforded if we only buy x_1.

Some of the possible changes are demonstrated in the graphs below.

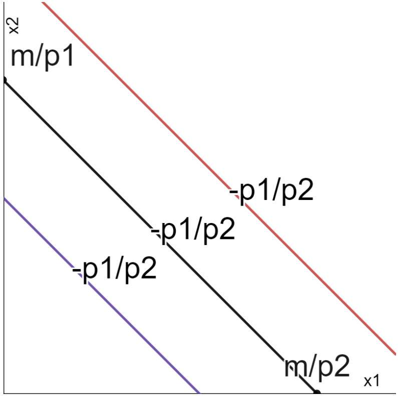


Figure 3.2: How Budget Changes with Income.

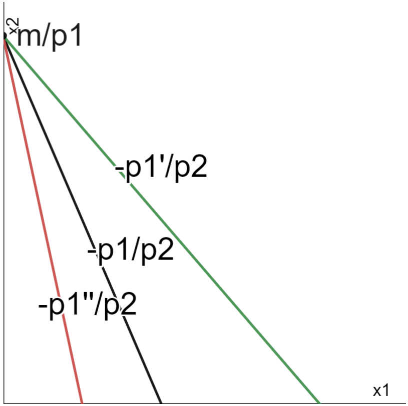


Figure 3.3: How Budget Changes with change in p_1.

In summary:

m **changes:**

Both endpoints change. If m increases, $\frac{m}{p_1}$ (the amount I can buy of good 1 changes) increases and $\frac{m}{p_2}$ (maximum affordable x_2) increases. The slope does not change. If m decreases, the opposite happens.

p_1 **changes:**

p_1. If p_1 goes up, the slope decreases (more negative). If p_1 goes down, the slope increases. The x_2 intercept stays the same.

p_2 **changes:**

p_2. If p_2 goes up, the slope increases and the x_2 intercept decreases. If p_2 goes down the slope decreases (becomes more negative) and the x_2 intercept increases. The x_1 intercept stays the same.

3.3 Taxes

Taxes represent a certain kind of price change. There are two kinds of taxes that are used frequently: *quantity* and *ad valorem* taxes.

A **quantity tax** is determined by **number of units** (x_i) purchased where an **ad valorem** tax is determined by the **value** of the good purchased $(x_i p_i)$.

With a quantity tax of t dollars on good i, the amount paid in tax is tx_i. With an ad valorem tax of percentage τ on good i, the amount pain in tax is $\tau(p_i x_i)$. The key difference is that as price of a good changes, the amount collected by the government does not change with a quantity tax (assuming the amount purchased does not change), but it does with an ad valorem tax. Most sales taxes are ad valorem. However, there are quantity taxes we encounter frequently. Pay close attention next time you are pumping gas, there is usually a sticker showing how much you pay in tax *per gallon*. That's a quantity tax.

Here's what happens to the budget line when we ad a quantity tax and ad valorem tax on good 1.

Quantity tax on good 1:

$$p_1x_1 + tx_1 + p_2x_2 = m$$

$$(p_1 + t)\,x_1 + p_2x_2 = m$$

Ad valorem Tax on good 1:

$$(p_1x_1) + \tau\,(p_1x_1) + p_2x_2 = m$$

$$[(1+\tau)\,p_1]x_1 + p_2x_2 = m$$

Notice that in both cases, the tax effectively just increases the price of the good. This makes taxes easy to plot, they have the same effect as a price increase. However, there are some complex scenarios you should think about. What if a quantity tax only kicked in after buying a certain amount of some good? What if instead of a tax, a subsidy (a decrease in price) was put on a good? What if that subsidy only held for the first k units of the good? We will talk about many of these scenarios in class and work with them in practice problems.

Part II

Preferences

4 The Preference Relation $\succsim$

4.1 Definitions

Now that we know how to model what a consumer can have, we should talk about what the prefer. We represent preferences with a mathematical tool called a **relation**.

Preference Relation

The preference relation denoted $\succsim$ is a set of statements about **pairs** of bundles. The statement *"bundle x is preferred to bundle x'"* is shortened to:

$$x \succsim x'$$

Example: Ice Cream
Suppose a consumer eats bowls of ice cream. The bundles (bowls) are written with the vanilla scoops first and chocolate second. For example: $(2,0)$ is two scoops of vanilla and zero of chocolate.
A consumer who likes vanilla more than chocolate might have these preferences:

$$(1,0) \succsim (0,1)\,,(2,0) \succsim (0,2)$$

A consumer who like more ice cream to less might have these preferences:

$$(2,0) \succsim (1,0)\,,(2,2) \succsim (1,1)$$

A consumer who gets sick of ice cream: (does anyone want to eat 100 scoops of ice cream?)

$$(1,0) \succsim (100,0)$$

A consumer who does not care about flavor might have:

$$(1,0) \succsim (0,1)\,,(0,1) \succsim (1,0)$$

In the case of the consumer who does not care about flavor above, notice that we have both $(1,0) \succsim (0,1)$ and $(0,1) \succsim (1,0)$. That is, a scoop of vanilla is just as good as a scoop of chocolate and a scoop of chocolate is just as good as a scoop of vanilla. When this is the case, we say the consumer is **indifferent.**

Indifference Relation: $\sim$
When $x \succsim y$ and $y \succsim x$ we say "x is indifferent to y" and write $x \sim y$.

When a consumer is not indifferent, we say they have strict preference for some bundle.

Strict Preference Relation: $\succ$
When $x \succsim y$ and **not** $y \succsim x$ we say "x is strictly preferred to y" and write $x \succ y$.

4.2 Assumptions on $\succsim$

In economics, we like to make as few assumptions about consumer's preferences as we can. There's a surprising amount we can say about consumer choice with just a few assumptions about the structure of preferences.

The first three assumptions or **axioms** we will look at ensure that for any budget set, consumers will have some favorite or set of favorite bundles. That is, given any set of bundles, they will actually be able to choose *something*. We will talk more about why these assumptions assure that fact in class.

Axiom 1. Reflexive.
For all bundles. The bundle is at least as good as itself.
In set notation:

$$\forall x \in X : x \succsim x$$

This is what we call a *technical* assumption. It does not carry a lot of content for us to talk about, but it helps ensure some minimal structure. After all, if a bundle was not "as least as good as itself", we'd have some trouble since that would imply that either it cannot be compared to itself or that it is both strictly better than itself and at the same time strictly worse than itself.

Axiom 2. Complete.
For every pair of distinct bundles. Either one is at least as good as the other or the consumer is indifferent.
In set notation:

$$\forall x, y \in X \& x \neq y : x \succsim y \, or \, y \succsim x \, or \, both$$

This axiom is a little more interesting. It says that for every pair of bundles, the consumer has *some* preference. The consumer can say "I'm indifferent." but not "I don't know". That is, everything is comparable.

Axiom 3. Transitivity.
If x is at least as good as y and y is at least as good as z then x is at least as good as z.

$$x \succsim y, y \succsim z \, \textbf{implies} \, x \succsim z$$

Transitivity lets us chain together preferences. It is really the **key** and most powerful assumption here. Transitivity ensures (along with the other assumptions) implies we can always put a set of objects into a **ranking** (possible with ties). Once we have a ranking, there's always going to be some things that are at the top of that ranking. Those are the things our consumers will choose.

4.3 Example of Violating Transitivity

In many circumstances, transitivity is an uncontroversial assumption. However, it is possible to construct perfectly reasonable decision processes where transitivity fails. Here is one of those examples:

Suppose there are three people on a dating app:

Person 1. Rich, Very Intelligent, Average Looking

Person 2. Financially Constrained, Genius, Good Looking

Person 3. Moderately Well Off, Average Intelligence, Best Looking

Now let's compare every pair of people. Person 2 is both more intelligent and better looking than person 1. Person 3 is wealthier and better looking than person 2. Person 1 is wealthier and more intelligent than person 3.

From this, we can construct a preference ordering: $2 \succ 1, 3 \succ 2, 1 \succ 2$. Notice, this is intransitive. It is clear who is better in any pair, but who would is is best from the set of all three? This kind of multi-dimensional comparison can easily cause intransitivity.

4.4 From Preference to Choice

So far, we have a pretty satisfying model of preferences, but economics is about *choice*. How do we model choice? Intuitively, we want to write down formal that, from any budget set, the consumer will choose the best thing (according to their preferences). To do this, let's define a **Choice Function.** We can write:

$$C : B \to B$$

This says that C is a function that maps the set B (a budget set) into itself. That is, from the set B, the function C returns some objects from the set B. This statement ensures that the set of “choices” will always be a subset of the budget set. In set notation, that would be expressed as: $C(B) \subseteq B$.

That’s good, but there’s no structure here involving the preference relation. What we really want is that $C(B)$ (the potential choices from the set budget set B) is the set of all bundles in B that are at **least as good as everything else in B.** We can express that formal as follows.

$$C(B) = \{x | x \in B : \forall x' \in B, x \succsim x'\}$$

This says, $C(B)$ is defined to be all the bundles (x) in the budget set (B) such that (:) for all ($\forall$) other bundles (x') in ($\in$) the budget set, we have that x is at least as good as x'. This is not the easiest statement to read if you are not familiar with this kind of formal expression, but I hope that you will agree that it is a rather elegant, and efficient way of expressing an otherwise rather complicated idea.

Notice that in the example in the last section of choosing a partner on a dating app, there is no partner that is at least as good as all the other partners. In that case, **the choice set is empty!** Having empty choice sets is potentially problematic for a mathematical model of choice. So, when can we be sure that there is always some bundle that a consumer will choose from any budget set in our models.

Fortunately for us, our three assumptions: reflexivity, completeness, and transitivity are enough to ensure that the consumer will always have some favorite things in any budget and will be able to make a choice. As an aside, transitivity is even a little stronger than we need for this, as it also ensures a form of consistency of choice called “coherence”. We will talk a little about that in class.

4.5 Indifference Curves and the Weakly Preferred Set

At this point, we have spent a good amount of time looking at how to formally express preferences. In practice, it is hard to work with these formal statements. Like anything else, it is nice to be able to visualize preferences. We can achieve this through **indifference curves.**

Indifference curve: an indifference curve is a set bundles such that a consumer is indifferent between every pair of bundles in the set.

In mathematics terms, an indifference curve is called an *equivalence class.* That is, it is some set that are “equivalent” in terms of preferences. This term is not necessary to know, but it may come up in future courses.

Note: There are many indifference curves. We only sketch a few to get an idea of the “shape” of preferences. Every bundle has an indifference curve passing through it.

Let’s look at an example. Suppose we have a consumer who likes apples just as much as oranges. They are indifferent between the bundle “two apple” $(2, 0)$ and the bundle “two orange” $(0, 2)$. These two bundles are on the same indifference curve in the graph below. The consumer is also indifferent between $(4, 0), (2, 2)$ and $(0, 4)$ they are on the same indifference curve in the graph below.

It is very useful to interpret the slope of an indifference curve at a particular point. Pick a bundle (x_1, x_2) and imagine adding a unit of x_1 to get a new bundle $(x_1 + 1, x_2)$. If the consumer wants more x_1, then the resulting bundle must be better, that is $(x_1 + 1, x_2) \succ (x_1, x_2)$. We now much ask, how much would we have to decrease x_2 by to get a new bundle $(x_1 + 1, x_2 - b)$ that the consumer is indifferent to the original? That is, what is the b such that $(x_1 + 1, x_2 - b) \sim (x_1, x_2)$. In a sense, we are asking, how much x_2 is a consumer willing to give up to get an extra unit of x_1. Since $(x_1 + 1, x_2 - b) \sim (x_1, x_2)$ they must be on the same indifference curve. So, we are also asking, if we start on some point of an indifference curve, and move one unit right, how far down do we need to move to bump into that same indifference curve. That amount is approximately the slope of the indifference curve.[1]

[1] Technically the slope at a particular point is defined as the limit of the ratio of how far we have to move down to how far we move to the right as that distance we move to the right shrinks to zero. You know, calculus stuff.

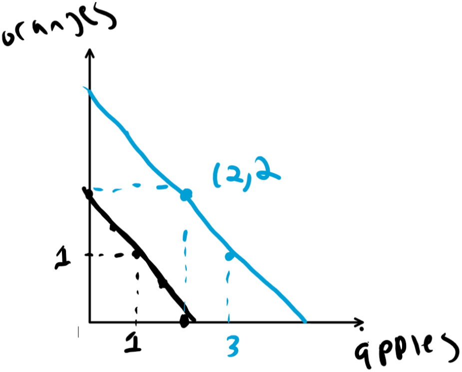


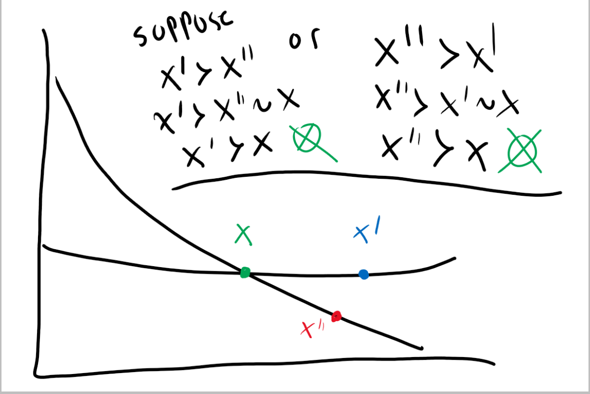
Figure 4.1: Indifference curves when apples are just as good as oranges.

4.6 Indifference Curves Cannot Cross

What can we say about the shape of indifference curves? It turns out, with only the assumptions of reflexivity, completeness, and transitivity: not much. Indifference curves could have some wild shapes. But under these assumptions there is one thing we know: **two distinct indifference curves cannot cross.**

Below is a proof of this claim. You are not responsible for knowing this proof, but you may be interested to see the logic. Understanding the logic might help you understand the way our axioms are used in proving formal statements about preferences.

Proof the two indifference curves cannot cross.
Look at the graph below. Here I have drawn two distinct indifference curves that cross each other. Notice that if two curves cross, they have to cross *somewhere.* I have labeled that somewhere x in the graph. This is a bundle that is on **both** indifference curves. However, since these are distinct indifference curves, there must be some bundle x' and x'' that are respectively on the different indifference curves and thus not indifferent to each other. However, since x is on both indifference curves, we must have $x' \sim x$ and $x'' \sim x$. Let's derive a contradiction to prove this scenario can never happen.
Since it is not the case that $x' \sim x''$ if preferences are **complete**, it must be that either $x' \succ x''$ or $x'' \succ x'$. If we take the first possibility $x' \succ x''$ we have $x' \succ x''$ and $x'' \sim x$. By **transitivity,** it must be that $x' \succ x$ but we already know that $x' \sim x$. If we take the second possibility $x'' \succ x'$ we have $x'' \succ x'$ and $x' \sim x$. By **transitivity,** it must be that $x'' \succ x$ but we already know that $x'' \sim x$. Thus, no matter what, we have found a contradiction.



4.7 Common Types of Preferences

There are a few "families" of preferences you should know about. These different families represent different types of trade-offs consumers are willing to make between two goods.

4.7.1 Perfect Substitutes

Perfect Substitutes preferences are such that a consumer's willingness to trade-off between the goods is the **same everywhere**.

The indifference curves are always downward sloping lines with the same slope. Recall The slope measures the amount of x_2 the consumer is willing to give up to get 1 more unit of x_1.

Steep slope: stronger preference for x_1.

Shallow slope: stronger preference for x_2.

Figure 4.3: Indifference curves for perfect substitutes preferences. This consumer would be willing to give up 2 units of x_2 in exchange for 1 unit of x_1.

4.7.2 Perfect Complements

Perfect Complements preferences are such that a consumer must consume the goods in a **fixed proportion**.

An example of this is left and right shoes. You always consume left and right shoes in a 1-to-1 proportion. That is, you want one left shoe for every right shoe. If you have the same number of left and right shoes, you are not willing to give up any left shoes to get more right shoes, because that would reduce the number of usable pairs you have.

Another example is ingredients in a recipe. Suppose you bake pies and a pie always needs two apples and one crust. If you have two apples and one crust, or four apples and two crusts, or six apples and three crusts, you would not be willing to give up apples to get more crusts or give up crusts to get more apples, it would reduce the number of pies you can make.

The indifference curves for these preferences are **L-shaped**. The kinks of these L-shaped curves pass along a line through the origin where the points on that line are the points where the goods are consumed in the "correct" proportion. That is, where there is not too much of either good. For left and right shoes, if left shoes are x_1 and right shoes are x_2, that the line $x_2 = x_1$ (the 45-degree line). For pies, if apples are x_1 and crusts are x_2 then the line through the kink points is where $2x_2 = x_1$. *I have plotted these below.*

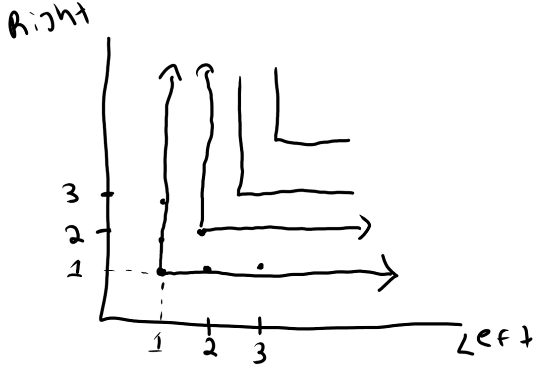


Figure 4.4: Indifference curves for perfect complements preferences where Left/Right shoes must be consumed in a 1-to-1 one ratio.

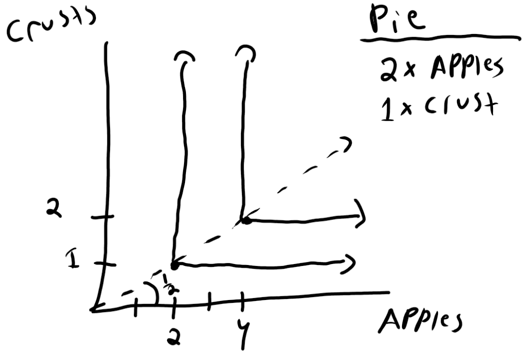


Figure 4.5: Indifference curves for perfect complements preferences where the goods are consumed in a 2-to-1 ratio. In this case, 2 apple and 1 crust make a pie.

4.7.3 Bads

So far, our examples have involved situations where both goods are in fact "good". That is, the consumer wants more (or at least does not want less) of either good. It is possible to model situations where that is not true. When a consumer wants less of something, we call that thing a **bad**.

When both x_1 and x_2 are goods, indifference curves are downward sloping. It is worth pausing to think about the intuition for this. The slope represents the tradeoff a consumer is willing to make between x_1 and x_2. Approximately, it is how much x_2 a consumer will give up to get one more unit of x_1. But now suppose x_2 is a bad. If a consumer gets one more unit of x_1, they will be happier. If we take away from x_2 they will be even happier than that! We could not possibly end up on the same indifference curve by adding some x_1 and taking away some x_2. We have to **add** x_2 to bring them back to indifference. Thus, the indifference curve is actually **upward sloping!**

The indifference curve will also be upward sloping if x_1 is a bad and x_2 is a good. Try to convince yourself of that using the same logic as above. However, if both goods are bad, the indifference curve is again downward sloping. However, unlike when both x_1 and x_2 are goods, preference increases as we move towards the origin: the bundle $(0, 0)$. These are demonstrated in the graphs below.

Figure 4.6: When one good is a "bad", indifference curves slope upward!

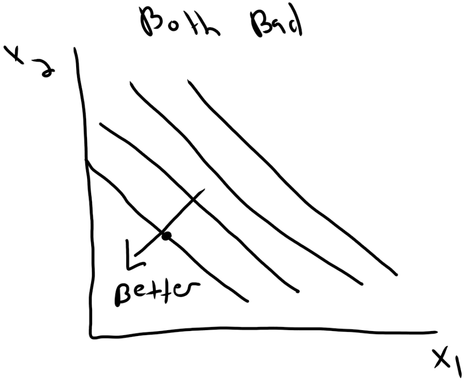


Figure 4.7: When both goods are bad, indifference curves slope down, but preference "increases" towards the origin (to the south west).

4.8 Well Behaved Preferences

As outlined in section 4.4the assumptions of reflexivity, completeness, and transitivity are sufficient to model consumers that are able to make choices from any budget set. That effectively enough to let us do some economics. However, preferences meeting just those conditions can be a little wild. We know that distinct indifference curves cannot cross but that's about it. Sometimes, we want to make some assumptions that will ensure that preferences are a little more well behaved. The following assumptions will prove convenient.

4.8.1 Monotonicity

Monotoncitiy: The assumption that everything is a "good".

There are two forms of this assumption. *Strict* and *Weak.* Strict monotonicity says that more of any good makes a consumer strictly better off. Weak monotonicity says that more of *every* good makes a consumer strictly better off, but more of any particular good might not. For instance, perfect substitutes are strictly monotonic. Perfect complements are weakly monotonic. We sometimes just call weakly monotonic "monotonic" (I know it must makes things more confusing-that's the problem with natural language. So, let's be formal:

Strict Monotonicity: For two bundles (x_1, x_2) and (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. $(x_1, x_2) \succ (y_1, y_2)$ if either $x_1 > y_1$ or $x_2 > y_2$

Weak Monotonicity. (AKA "Monotonic"): For two bundles (x_1, x_2) and (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. $(x_1, x_2) \succ (y_1, y_2)$ if **both** $x_1 > y_1$ **and** $x_2 > y_2$

Weak monotonicity implies indifference curves are downward sloping (they have negative slope or zero slope) that is, they cannot be strictly upward sloping. Furthermore, it implies that preference increases to the north east. That is, as we move out, away from the origin, the bundles get better. Strict monotonicity additionally implies indifference curves are always *strictly* downward sloping. **It is worth thinking about why these assumptions imply these**

facts about the slope of the indifference curves and what those facts translates to in terms of trade-offs.

Note that, it is often possible to convert non-monotonic preference to monotonic preference by thinking of a "bad" as the "lack of a bad". For instance, if we were writing a model of preferences over candy and Brussels sprouts we might have the bundle $(2, 2)$ which is two candies and two Brussels sprouts. If Brussels sprouts are a bad, then we might have: $(2, 2) \succ (2, 3)$. These preferences are non-monotonic. However, suppose we rewrite the number of sprouts as "how many less than 10 sprouts do I have?". Then the two bundles are $(2, 8)$ and $(2, 7)$. They are the same physical bundle so we still have $(2, 8) \succ (2, 7)$, but notice now we have patched up monotonicity.

Monotonicity ensures that, as long as there is no "savings" in the model, consumers will always spend all of their money. Why not? If more is better, then spending less than their income must be sub-optimal- they could get more of everything. This is helpful, since it tells us we can look for optimal bundles **on the budget line**.

Technically, we do not even need monotonicity for this to be true, a far weaker condition called local nonsatiation will ensure the same thing. Local nonsatiation says that for any bundle, there is another bundle "nearby" that is strictly better. That bundle might involve less stuff, it might involve more stuff. Effectively it ensures that if the consumer were not spending all of their money, they could change their bundle by a "little bit" and make themselves better off. Because of this, they could not possibly spend less than their income, because (by local nonsatiation) there will always be some other affordable bundle nearby that is strictly better.

You do not need to know about this, but I think it is kind of interesting.

4.8.2 Convexity and Strict Convexity

Convexity: The assumption that mixtures are better than extremes.

Monotonicity tells us we can look for optimal bundles on the budget line. But where? This assumption can help tell us where to look. It is not a requirement for doing economics by any means. It is a convenience. There are two forms of this assumption. Both of them essentially say that if we take two bundles that are indifferent and mix them together, we will get a better bundle. Strict convexity says that bundle is strictly better and weak convexity (or just convexity) just tells us that it is weakly better. Here are the formal statements.

Strictly Convex: For two indifferent bundles $(x_1, x_2) \sim (y_1, y_2)$, for any $t \in (0, 1)$, the mixture given by $(tx_1 + (1-t)\, y_1, t$
(x_1, x_2) and $(tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2) \succ (y_1, y_2)$.

Weakly Convex: For two indifferent bundles $(x_1, x_2) \sim (y_1, y_2)$, for any $t \in [0, 1]$, the mixture given by $(tx_1 + (1-t)\, y_1, t$
(x_1, x_2) and $(tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2) \succsim (y_1, y_2)$.

Notice how we are mixing the bundles together. We take a t portion of the bundle (x_1, x_2) and mix it with a $(1-t)$ portion of (y_1, y_2). To demonstrate this, let's mix together $(2, 1)$ and $(1, 2)$. If we take $t = 0.5$, we are taking half of $(2, 1)$ which is $(1, 0.5)$ and adding half of $(1, 2)$ which is $(0.5, 1)$. The result is the bundle $(1.5, 1.5)$. If instead we take (0.25) we get a quarter of $(2, 1)$ which is $(0.5, 0.25)$ and three-quarters of $(1, 2)$ which is $(0.75, 1.5)$. Adding these together we get the bundle $(1.25, 1.75)$.

The mixtures are also referred to as "convex combinations". If we were to plot all of the **convex combinations of two points, the convex combinations would simply be the straight line through the two points.**

Using this, we can talk about the geometry of indifference curves meeting these conditions. Under the assumption of monotonicity:

If preferences are **strictly convex**, then the **indifference curve always lies strictly below a line between any two points on that indifference curve**.

If preferences are **weakly convex**, then the **indifference curve always lies weakly below a line between any two points on that indifference curve**.

An example of an indifference curve for strictly convex preferences is shown below.

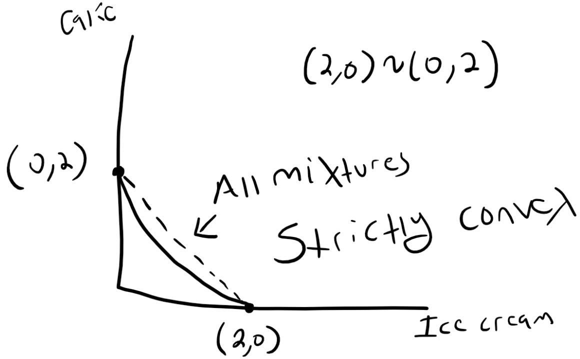


Figure 4.8: Example of Convex Indifference Curves

4.9 Marginal Rates of Substitution and Slope of the Indifference Curve

The marginal rate of substitution is defined as the rate at which a consumer will give up x_2 in order to get more x_1. We have already seen that this rate of trade-off is captured by the slope of the indifference curve at a point. Approximately, we can think of the MRS as how much x_2 a consumer would give up to get one more unit of x_1. The MRS and equivalently the slope of the indifference curves will play a critical role in finding optimal bundles. We will see that in the next two sections.

5 Utility (4.1-4.5)

5.1 Definition

A utility function is a way of assigning "scores" to bundles, such that better bundles according to $\succsim$ get a higher score. For example, suppose a consumer's preferences are:

$$A \succ B \succ C \sim D$$

Some utility functions that represent these preferences:

$U(A) = 10$, $U(B) = 5$, $U(C) = U(D) = 0$

$U(A) = 12$, $U(B) = 1$, $U(C) = U(D) = -100$

Utility function: $U(x)$ represents preferences $\succsim$ when for every pair of bundles x and y, $U(x) \geq U(y)$ if and only if $x \succsim y$.

That is, if x is better than y according to $\succsim$ it gets a higher utility according to $U()$. To reiterate, a utility function is a convenient mathematical representation of the fundamental preference relation $\succsim$. We do not need to believe utility functions actually exist to use them, since they are just how we represent preferences.

We say that **utility is ordinal** since the magnitude of the numbers are meaningless, and only the relationships matter. There is no sense in which two times higher utility means that the preference is two times stronger. If we could say something like that, we could call utility a **cardinal** measure. Since we can only infer the ranking of bundles, but not say anything about how *strong* the preferences are from the relation $\succsim$, the utility function that represents $\succsim$ also has no such content.

5.2 Monotonic Transformations

Because utility is ordinal, we are free transform one utility function into another, as long as it maintains the same preferences. **Any strictly increasing function of a utility function represents the same preferences as the original utility function.** For example, suppose:

$$U(x_1, x_2) = x_1 + x_2$$

This represents the preferences of someone who only cares about the total amount of stuff, but not the composition. In fact, this utility function represents *perfect substitutes preferences.* Here are some other utility functions that represent the same preferences:

$$\tilde{U}(x_1, x_2) = x_1 + x_2 + 100 = U(x_1, x_2) + 100$$

$$\tilde{U}(x_1, x_2) = (x_1 + x_2)^2 = (U(x_1, x_2))^2$$

Since the functions $f(u) = u + 100$ and $f(u) = u^2$ are strictly increasing for $u \geq 0$ (which is always true for the original utility function), these are monotonic transformations of the original utility function. It is often useful to use monotonic transformations to modify a utility function that is hard to work with into one that is more convenient.

For instance, suppose we had the utility function: $u = 38(x_1 + x_2)^2 + 100$. We could transform this into the utility function $u = x_1 + x_2$ which is much simpler. The two utility functions represent the exactly same preferences.

5.3 MRS from Utility Function

As we have discussed above, the **Marginal Rate of Substitution** (MRS) is the slope of the indifference curve. We can get the MRS from a utility function by taking the ratio of partial derivatives of the utility function. Let's first define those partial derivatives:

Marginal Utility of good i is $mu_i = \frac{\partial u(x_1,x_2)}{\partial x_i}$.

With this, we can define the MRS in terms of the marginal utilities:

The Marginal Rate of Substitution (MRS) is given by: $MRS = -\frac{mu_1}{mu_2} = -\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}}$

Note that MRS is an **ordinal** property since it represents the slope of indifference curves. Because two preferences that are the same have the same indifference curves, they will also have the same MRS. This is actually a convenient way to check whether two utility functions represent the same preferences.

Same MRS, same preferences.

In the section above, I claimed these two utility functions represent the same preferences: $u = 38\left(x_1 + x_2\right)^2 + 100$ and $u = x_1 + x_2$. Their marginal rates of substitution are identical:

$$-\frac{\frac{\partial\left(38(x_1+x_2)^2+100\right)}{\partial x_1}}{\frac{\partial\left(38(x_1+x_2)^2+100\right)}{\partial x_2}} = -\frac{76\left(x_1 + x_2\right)}{76\left(x_1 + x_2\right)} = -1$$

$$-\frac{\frac{\partial(x_1+x_2)}{\partial x_1}}{\frac{\partial(x_1+x_2)}{\partial x_2}} = -\frac{1}{1} = -1$$

5.4 Examples of Utility Functions

5.4.1 Perfect Substitutes

A constant MRS implies a constant willingness to trade off between the two goods. This is the case for perfect substitutes.

$$u\left(x_1, x_2\right) = ax_1 + bx_2$$

$$MRS = -\frac{a}{b}$$

5.4.2 Quasi-Linear

With quasi-linear preference, a consumer only gets tired of one of the two goods. For instance, if x_1 is ice cream and x_2 is money, we might want to represent preferences where the amount of money a consumer is willing to give up to get another unit of ice cream is decreasing in the amount of ice cream. This can be achieved with a quasi-linear utility function.

One common quasi-linear utility function is:

$$u\left(x_1, x_2\right) = ln\left(x_1\right) + x_2$$

Let's look at the MRS:

$$MRS = -\frac{\frac{\partial(ln(x_1)+x_2)}{\partial x_1}}{\frac{\partial(ln(x_1)+x_2)}{\partial x_2}} = -\frac{1}{x_1}$$

This says that as ice cream increases, (approximately) the amount of money a consumer is willing to give up to get another scoop of ice cream is one over the number of scoops they already have. With one scoop, they will give up a dollar to get another scoop. With two scoops, the would only give up 50 cents. And so on...

Another example of a quasi-linear utility function:

$$u\left(x_1, x_2\right) = \sqrt{x_1} + 10x_2$$

Practice taking the MRS of this function. Notice that it only depends on the amount of (ice cream) x_1!

5.4.3 Cobb-Douglass

Now suppose we want the consumer to get tired of both goods as they get more. We can use a **Cobb-Douglass** utility function:

$$u\left(x_1, x_2\right) = x_1^{\alpha} x_2^{\beta}$$

Let's look at the MRS:

$$mu_1 = \frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta}$$

$$mu_2 = \frac{\partial \left(x_1^{\alpha} x_2^{\beta} \right)}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta - 1}$$

$$MRS = -\frac{MU_1}{MU_2} = -\frac{\alpha x_1^{\alpha - 1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta - 1}}$$

$$= -\frac{\alpha x_1^{\alpha} x_1^{-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta} x_2^{-1}} = -\frac{\alpha x_1^{-1}}{\beta x_2^{-1}} = -\frac{\alpha}{\beta} \frac{x_2}{x_1}$$

This says that the amount of x_2 a consumer is willing to give up to get another unit of x_1 is directly proportional to the ratio of x_2 to x_1. If they have a lot of x_2 relative to x_1 they will give up more x_2 to get another unit of x_1 and vise versa.

Let's compare two CD Functions:

Increasing the exponent on either good will increase the consumers desire for that good. They will still get tired of it, but between two consumers, one with a larger exponent on a good, that consumer will have a stronger desire for the good at the same bundle.

$$u(x_1, x_2) = x_1 x_2$$

$$MRS = -\frac{x_2}{x_1}$$

At the point $(1, 1)$: $MRS = -1$. Now let's increase the exponent on x_1 to 10:

$$\tilde{u}(x_1, x_2) = x_1^{10} x_2$$

$$MRS = -10\frac{x_2}{x_1}$$

At the point $(1, 1)$: $MRS = -10$

Notice that the consumer with $\tilde{u}(x_1, x_2) = x_1^{10} x_2$ would be willing to give up ten-times more x_2 to get the same amount of x_1 as the consumer with utility function $u(x_1, x_2) = x_1 x_2$.

6 Choice

Now that we have modeled budgets (what is available), preferences (what is desired) and know how to represent those preferences with utility functions, we are ready to talk about what consumers actually choose from the set of available bundles.

We have already modeled choice. Formally, we want to find the set of bundles that meet this condition

$$X^* = \{x : x \in B \,\&\, \forall x' \in B, x \succsim x'\}$$

This says that the set of optimal bundles X^* are bundles like x that are in the budget set B and are at least as good as any other x' that is also in the budget set.

There is one really powerful observation that makes the process of finding optimal bundles much simpler. In thinking about where an optimal bundle lives on the graph of preferences and budget, there are really only **three possibilities.** These come out of a very powerful observation about trade-offs when preferences are **complete**, **transitive** and **monotone**.

6.1 Three Possibilities

Assume $\succsim$ is **reflexive**, **complete**, **transitive** and $\succsim$ **monotonic**. A bundle cannot be optimal if it is on an indifference curve that crosses into the **interior** of the budget set.

The proof proceeds by contradiction. You are not responsible for this, but it might be nice to read through and try to understand. The proof is shown graphically below.

Suppose we found a bundle x we thought was optimal but was on an indifference curve that passed into the interior of the budget set. Then there is some bundle x' on the interiod of the duget set such that $x \sim x'$ (since it is on the same indifference curve). Since x' is on the interior of the budget set, there is some other bundle x'' such that x'' is in the budget set and has more of every good than x'. Since preferences are monotonic, $x'' \succ x'$. Since preferences are transitive, we have $x'' \succ x' \sim x$ and so $x'' \succ x$.

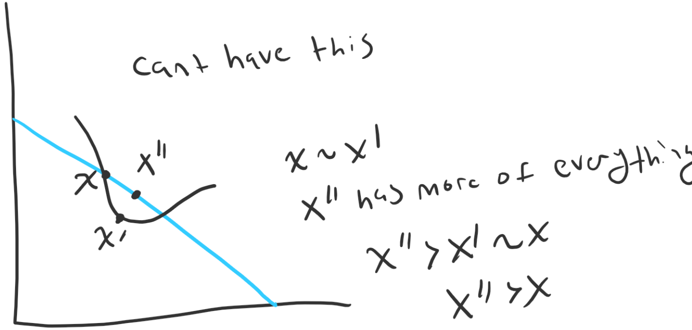


Figure 6.1: An optimal bundle cannot be on an indifference curve that passes "into" the budget set.

With this result in hand, there are only three ways a bundle can be on an indifference curve that is in the budget set and does not exist on an indifference curve that passes into the interior of the budget set. First of, it must be on the budget line. Then we have three possibilities:

1. (Tangent) It is at bundle where the indifference curve at that bundle had the same slope as the budget line.

2. (Touching but not tangent) The bundle is a "non-smooth" point on the indifference curve, but the that point just touches the budget line.

3. (Boundary) We are at one of the boundaries ($x_1 = 0$ or $x_2 = 0$) in this case the slope of the indifference curve and the slope of the budget need not be equal.

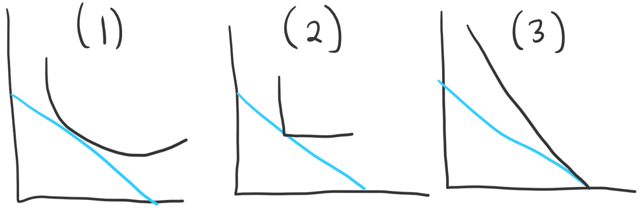


Figure 6.2: Graphical Examples of the Three Possibilities

Under some weak conditions (we can take derivatives of the utility function). The tangency condition is **necessary** for an *interior* optimum (involves consuming some of both things).

That is, if there is an optimal bundle that involves consuming some of both goods, it must have the property that the slope of the indifference curve at that optimal bundle is the same as the slope of the budget line. This is a very powerful result and also suggests why we are going to take a lot of derivatives in this class.

This condition is formalized by the familiar equation:

$$MRS = -\frac{p_1}{p_2}$$

Note, this is precisely that the slope of the indifference curve at a point is equal to the slope of the budget equation at that point. This also implies the trade-offs are the same.

The MRS is the way a consumer is **willing** to trade off between the goods and the slope of the budget equation is the rate a which the **must** in order to stay in their budget. If these are not equal either the consumer is willing to give up more x_2 than they have to in order to get more x_1 or they are willing to give up more x_1 than they have to in order to get more x_2. Neither situation can be optimal... unless they **can not get any more x_1 or x_2**. That would be the case at the boundary. That is, we can have an optimal bundle where this condition is not met, but it can only occur at a boundary or where this condition is not defined.

6.2 Examples

Let's look at a few examples of finding optimal bundles.

6.2.1 Cobb Douglass:

$$u(x_1, x_2) = x_1 x_2$$

$$p_1 x_1 + p_2 x_2 = m$$

This is a smooth utility function. We can find its MRS everywhere. Let's write down the tangency condition:

$$MRS = -\frac{\frac{\partial (x_1 x_2)}{\partial x_1}}{\frac{\partial (x_1 x_2)}{\partial x_2}} = -\frac{p_1}{p_2}$$

$$-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$$

This simplifies to:

$$* \, x_1 p_1 = x_2 p_2$$

We also know that an optimal bundle occurs on the budget line. Let's write this down as a second condition. Budget Condition:

$$** \; x_1 p_1 + x_2 p_2 = m$$

We have two conditions and two unknowns. Plug the tangency condition into budget condition to get:

$$x_1 p_1 + x_1 p_1 = m$$

Simplify this:

$$x_1^* = \frac{1}{2}\frac{m}{p_1}$$

Plug this back into either of the two conditions gives us:

$$x_2^* = \frac{1}{2}\frac{m}{p_2}$$

The optimal bundle is:

$$\left(\frac{\frac{1}{2}m}{p_1}, \frac{\frac{1}{2}m}{p_2}\right)$$

Note the form of this bundle. $\frac{1}{2}m$ is half of the income. These optimal bundles have the consumer spending half of their income on both goods.

6.2.2 Perfect Substitutes

The utility function is:

$$u(x_1, x_2) = 2x_1 + x_2$$

Prices and income are: $p_1 = 1$, $p_2 = 1$, $m = 10$. This gives us the budget equation:

$$1x_1 + 1x_2 = 10$$

Finding the tangency condition:

$$-\frac{2}{1} = -\frac{1}{1}$$

$$-2 = -1$$

Ohh... But this is never true. The consumer would always be willing to give up 2 units of x_2 to get one unit of x_1. But they only have to give up 1. **There can't be an interior solution.** They will just buy as much x_1 as possible.

If you ever get lost doing a perfect substitutes problem you can use the following trick. With perfect substitutes, there has to be a boundary solution. Just check the utility of both intercepts (buying only x_1 and buying only x_2) see which is better. If they give the same utility, any bundle they can afford is optimal. In this problem we get:

Only consume x_1:

$$\left(\frac{m}{p_1}, 0\right) = (m, 0)$$

$$u(m, 0) = 2m = 20$$

Only consume x_2:

$$\left(0, \frac{m}{p_2}\right) = (0, m)$$

$$u(0, m) = m = 10$$

Since consuming only x_1 gives me more utility, that is the optimal bundle:

$$(m, 0)$$

6.2.3 Anything is Optimal

Here is an example where any affordable bundle is optimal.

$$u(x_1, x_2) = 2x_1 + x_2$$

$p_1 = 2$, $p_2 = 1$, $m = 10$. The budget equation is:

$$2x_1 + 1x_2 = 10$$

The tangency condition is:

$$-\frac{2}{1}=-\frac{2}{1}$$

$$-2=-2$$

All of the bundles such that $2x_1 + 1x_2 = 10$ are optimal. Confirm this by checking the utility of some bundles on this line.

6.2.4 Perfect Complements

Suppose these are the utility function and budget equation:

$$u\left(x_1, x_2\right) = min\left\{x_1, x_2\right\}$$

$$2x_1 + x_2 = 15$$

We know the budget condition must be true at the optimum.

$$**\ 2x_1 + x_2 = 15$$

But, we can not take derivatives here. What is the other condition?

In this case, we have to use a little intuition. If the consumer ever consumed a bundle that was not on the kink of an indifference curve, than they could spend less on some good and use the leftover money to buy more of both goods, increasing utility.

"No Waste Condition". (Equation for the "kink" points).

$$* \, x_1 = x_2$$

Solving this equation together with the budget equation gives us:

$$x_1 = 5, x_2 = 5$$

6.2.5 Perfect Complements (2 Apples, 1 Crust)

Let's try another perfect complements problem. Suppose the utility function is:

$$u\left(x_1, x_2\right) = min\left\{\frac{1}{2}x_1, x_2\right\}$$

This represents the utility of someone who only eats pies and makes pies by using two apples and one crust per pie.

Suppose the budget equation is:

$$** \; 2x_1 + x_2 = 15$$

In this case, the "no waste condition" (equation for the "kink" points) is:

$$\frac{1}{2}x_1 = x_2$$

Notice we get this by setting the two terms in the $min\{\}$ function equal to each-other.

Combine the conditions and solve to get:

$$x_1 = 6, x_2 = 3$$

6.2.6 Max Preferences

The utility and the budget are:

$$u(x_1, x_2) = max\{x_1, x_2\}$$

$$2x_1 + x_2 = 15$$

Try this one at home: what is the optimal bundle?

7 Demand

In the previous chapter, we looked at how to find demand (the optimal bundle) given a utility function, prices, and income. In this chapter, we look at how demand changes when we change one of those parameters (p_1, p_2, m).

7.1 Marshallian Demand

The Mashallian demand is the optimal amount of a good, given prices and income. We denote these this way:

$$x_1^*(p_1, p_2, m), x_2^*(p_1, p_2, m)$$

For instance, the Marshallian for someone with utility function $u(x_1, x_2) = x_1 x_2$ is:

$$x_1^* = \frac{\frac{1}{2}m}{p_1}, x_2^* = \frac{\frac{1}{2}m}{p_2}$$

Marshallian for someone with utility function $u(x_1, x_2) = min\{x_1, x_2\}$ is:

$$x_1^* = \frac{m}{p_1 + p_2}, x_2^* = \frac{m}{p_1 + p_2}$$

We can now look at how these types of demands change.

7.2 Changes in Income

We first ask, "what happens to demand when we change income"? We can formalize this by thinking of this change as a derivative. We want to know what are the values of $\frac{\partial x_1^*(p_1,p_2,m)}{\partial m}$ and $\frac{\partial x_2^*(p_1,p_2,m)}{\partial m}$

7.2.1 Normal/Inferior

Depending on whether demand increases or decreases with income, we call goods **normal** or **inferior.**

If demand *increases* when income increases, we say the good is **"Normal".**
If demand *decreases* when income increases, we say the good is **"Inferior".**

Examples:

We have seen that the demand for x_1 from the Cobb-Douglass utility function $u = x_1 x_2$ is $x_1^* = \frac{\frac{1}{2}m}{p_1}$. This is a normal good since this demand increase with income. Notice that $\frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial m} = \frac{1}{2p_1} > 0$.

Suppose we found demand for some good was $x_1 = \frac{10}{mp_1}$. This would be an inferior good since demand decreases with m. Noice $\frac{\partial\left(\frac{10}{mp_1}\right)}{\partial m} = -\frac{10}{m^2 p_1} < 0$.

7.2.2 Income Offer Curve

The income offer curve is a plot of optimal bundles (x_1^*, x_2^*) as income changes but prices remain fixed.

For example, suppose $u(x_1, x_2) = x_1 x_2$ and prices are $p_1 = 2, p_2 = 1$. We get demands: $x_1 = \frac{1}{4}m$, $x_2 = \frac{1}{2}m$. Let's pick a few points for m and plot the optimal bundles.

$$\begin{array}{ll}
m = 1 & \left(\frac{1}{4}, \frac{1}{2}\right) \\
m = 2 & \left(\frac{1}{2}, 1\right) \\
m = 3 & \left(\frac{3}{4}, \frac{3}{2}\right) \\
m = 4 & (1, 2\}) \\
m = 5 & \left(\frac{5}{4}, \frac{5}{2}\right)
\end{array}$$

Plotting these, we see quickly they live on a straight line.

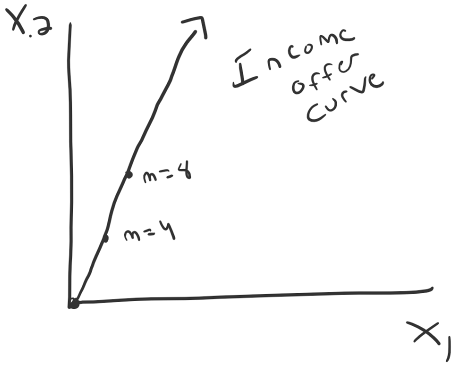


Figure 7.1: Income offer curve for Cobb Douglass preference example.

Notice how the income offer curve increases in both the x_1 and the x_2 direction as m increases. That is because both goods are normal. What if one good was inferior (both can not be inferior at the same time). We would get a graph like this:

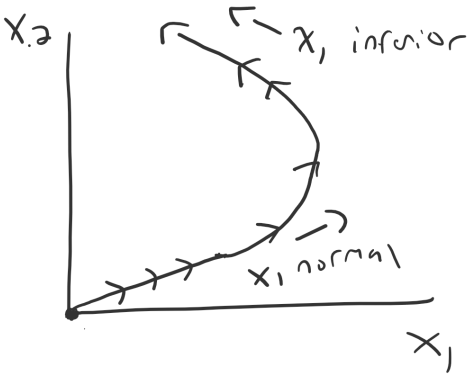


Figure 7.2: Example of an income offer curve where x_1 is initially normal but becomes inferior as m grows.

7.2.3 Engle Curve

The Engle curve is the relationship between income and a **single** good. Plotting m on the vertical axis against x_1 or x_2 on the horizontal axis. Suppose we had demand: $x_1 = \frac{1}{4}m$. To plot this with m on the vertical axis, it helps to isolate m. We get:

$$m = 4x_1$$

When we put m on the vertical axis, really what we are plotting is the amount of income a consumer would need to have to demand some amount x_1 of good 1.

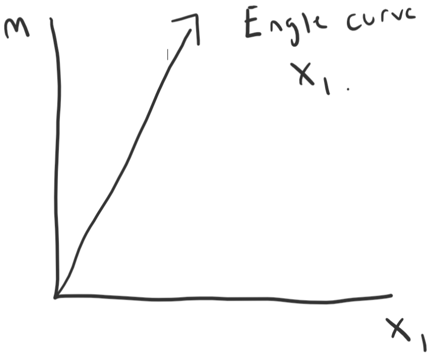


Figure 7.3: Engle curve for $x_1 = \frac{1}{4}m$.

This is a normal good because x_1 increases as m increases. What about an inferior good? This one is subtle. We might think we could just make a graph where x_1 decreases *everywhere* as m increases. But, this is impossible. For x_1 to decreases, it has to have increased at some point. So, this shows us that good cannot possibly be "always inferior". The normal/inferior nature of a good can depend on income.

Here is an example where a good in normal for low income and inferior for larger income:

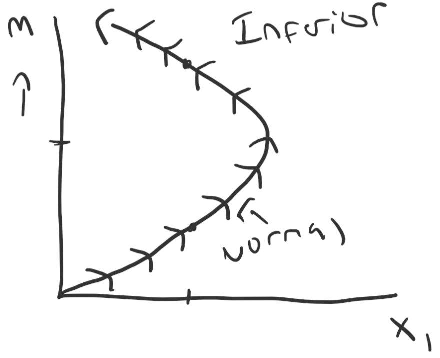


Figure 7.4: Engle curve for a good that is normal for low income and inferior for high income.

7.2.4 Example: Perfect Complements

Let's work an example with perfect complements. Suppose we have: $U(x_1, x_2) = min\{x_1, x_2\}$. $p_1 = 2$, $p_2 = 1$.

At the optimum, we have $x_1 = x_2$ (the no waste condition) and $2x_1 + 1x_2 = m$ (the budget condition). Solving these together gives us:

$$x_1 = \frac{m}{3}, x_2 = \frac{m}{3}$$

Plotting the income offer curve:

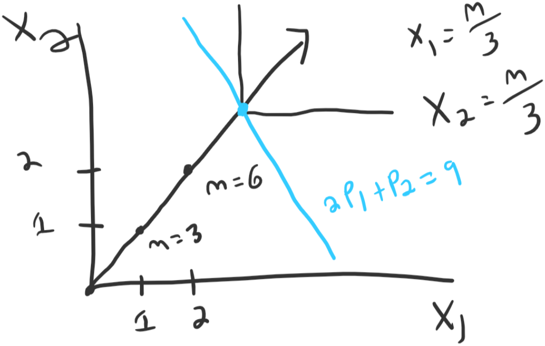


Figure 7.5: Income offer curve for $min\{x_1, x_2\}$ with $p_1 = 2$ and $p_2 = 1$

Plotting the Engle curve for x_1:

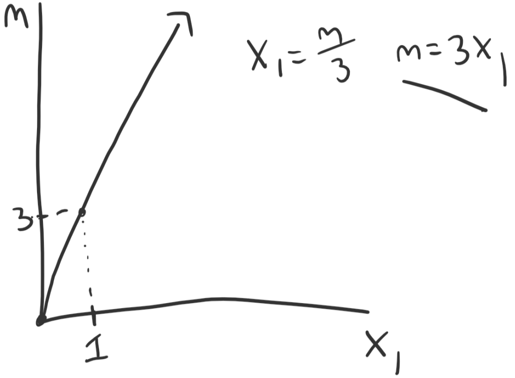


Figure 7.6: Engle Curve of x_1 for $min\{x_1, x_2\}$ with $p_1 = 2$ and $p_2 = 1$