Intermediate Microeconomics*

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These notes are based on my Vanderbilt Economics Course 3012. **They are preliminary.** If you find any typos or errors in this text, please e-mail me at g.leo@vanderbilt.edu.

Contents

1 Bundles & Budget

1.1 Bundles

Bundles are the fundamental object of study in microeconomics. In our models, when a consumer makes a choice, they choose a **bundle** from the set of bundles available to them (the **budget set**). Bundles can be anything or combination of things you can think of. In this course, however, bundles are *usually* going to be amounts of some things we call **goods** and very often we will just look at two goods.

Bundle: $x = (x_1, x_2)$

Example. Ice Cream Bowls (the bundles) are made of up two goods: scoops of vanilla ice cream and scoops of chocolate ice cream. x_1 is the amount of vanilla. x_2 is the amount of chocolate. $(1, 1)$ represents one scoop of each flavor, $(2, 2)$ two scoops of each flavor, and $(0.28, 100)$ a lot of chocolate (100 scoops) and a little vanilla (0.28 scoops).

Since bundles with two goods are represented by ordered pairs, we can plot bundles on and x_1, x_2 axis. An example of this is shown below.

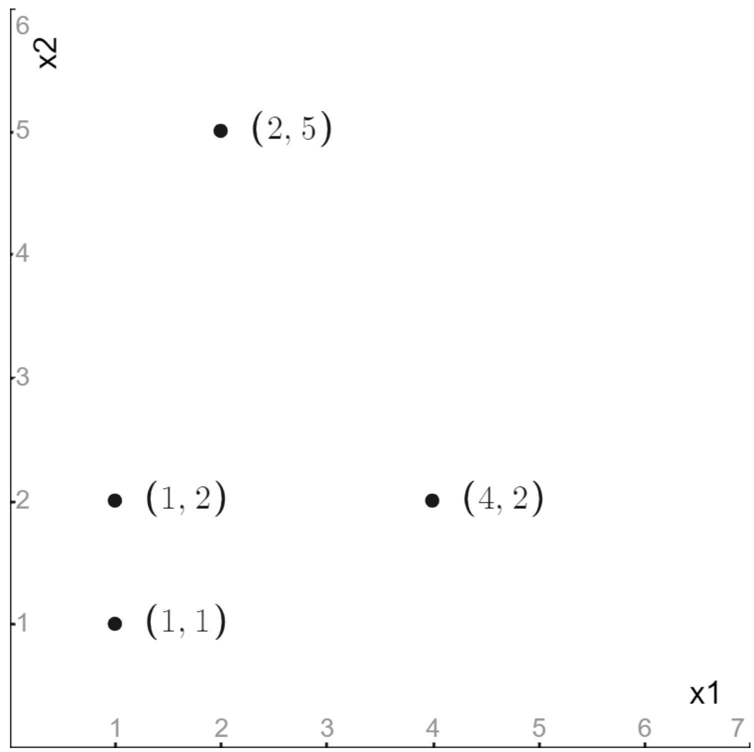

Figure 1.1: A Few Bundles on The Cartesian Plane.

1.2 Feasible Set

The set of all bundles relevant to a model is called the **Feasible Set**. The feasible set defines the scope of a model.

The Feasible Set: X is the "feasible" set of bundles.

Example. The feasible set for a model about choosing ice cream bowls is the set of all ordered pairs possible ice cream bowls: (x_1, x_2). Of course, it does not make sense to have a negative amount of ice cream, so in this case we might say $X = \mathbb{R}^2_+$. (This notation says that the feasible set is made up of 2 real numbers that are non-negative.)

2 Budget Set

Budget Set: B

The budget set is the set of bundles *available* to a particular consumer. The budget set must be a subset of the feasible set. In set notation we write: $B \subseteq X$

2.1 Budget Sets from Prices and Income

Not everything in the feasible set is going to be achievable for every consumer. Some bundles are affordable and others are not. The set of bundles that a consumer can *actually choose from* is called the **budget set**. Our budget sets will be constructed by assuming consumers have some income and that each good has a price.

Prices: p_1, p_2: Price units of good 1 and good 2.
Income: m.

With these, we can define the cost of a bundle:

Cost of a bundle: $p_1x_1 + p_2x_2$

The set of all bundles that a consumer can afford is called the **Budget Set.** We can define if formally this way:

Budget set: $B = \{x | x \in X \,\&\, x_1p_1 + x_2p_2 \leq m\}$.[a]

[a] In "normal" language, this says the budget set is the set of bundles such that the price of the bundle is less than income.

Since we are able to plot bundles, we can also plot the budget set. To do this, it is easiest to first, we draw the **Budget Line**. This is the set of bundles that are "just affordable".

Budget Line: $x_1p_1 + x_2p_2 = m$

Now we can plot this on an x_1, x_2 plane. Let's put x_2 of the vertical axis. In this case, it is useful to rewrite the budget line into a form we are more familiar with:

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$$

This is now clearly an equation for a line with intercept $\frac{m}{p_2}$ and slope $-\frac{p_1}{p_2}$. Before we plot it, let's interpret it a little. Notice that if $x_1 = 0$ we get $x_2 = \frac{m}{p_2}$. This says "If I were only to buy x_2, I could afford $\frac{m}{p_2}$ units of x_2. Furthermore, for every unit that we increase x_1 by, x_2 goes down by $-\frac{p_1}{p_2}$. This says "If I am spending all my money, if I want to buy one more unit of x_1, I have to give up $-\frac{p_1}{p_2}$ units of x_2. This is a very important thing to know about the slope of the budget line. **The slope of the budget line represents the trade-off between x_1 and x_2 at the market prices.** We are now ready to plot the budget set. It is the budget line and all of the bundles "below" the budget line.

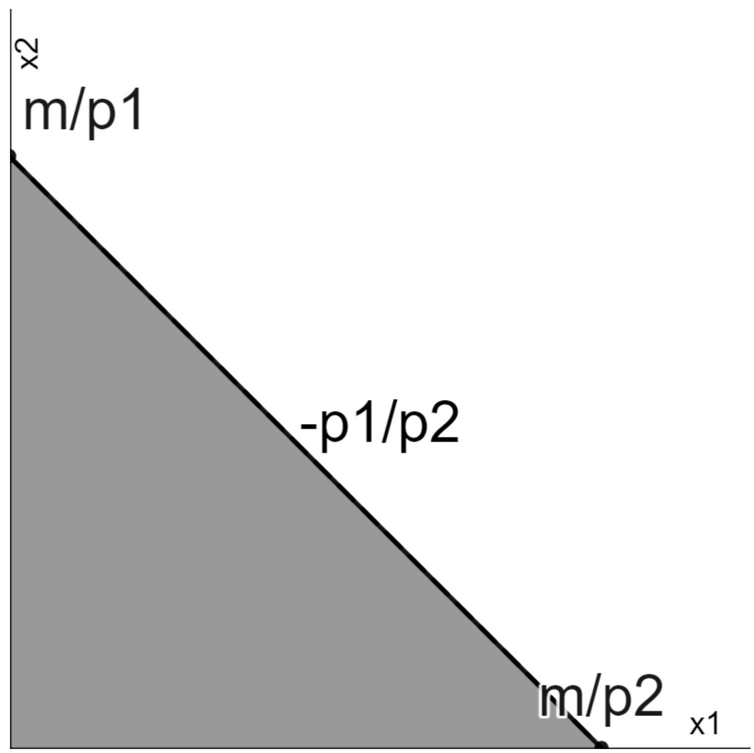

Figure 2.1: Graphical Representation of the Budget Set with slope $-\frac{p_1}{p_2}$ and intercepts $\frac{m}{p_1}$ and $\frac{m}{p_2}$.

2.2 Changing Prices and Income

We are often interested in looking at how the budget set changes when we change on of the parameters of the model: m, p_1, or p_2.

We can work out how the budget set changes by looking at changes in the budget line. There are three key elements to the budget line: the slope $-\frac{p_1}{p_2}$ and the intercepts $\frac{m}{p_1}$ and $\frac{m}{p_2}$.

When income changes, notice that only the intercepts change. If m increases, both intercepts increase. This should be intuitive. Since the intercepts represent how much of a good we can buy if we only buy that good, then if income increases, we can afford more. When income decreases, the opposite happens.

Importantly, when income changes, the slope of the budget line does not change. This is because the trade-off between the goods stays the same regardless of income (as long as the price remain the same).

When a price changes on the other hand, the slop of the budget line changes and **one** of the intercepts changes. For instance, if p_1 goes up, the slope of the budget line becomes steeper (because more x_2 has to be given up to get an extra unit of x_1). Furthermore, the x_1 intercept decreases because less x_1 can be afforded if we only buy x_1.

Some of the possible changes are demonstrated in the graphs below.

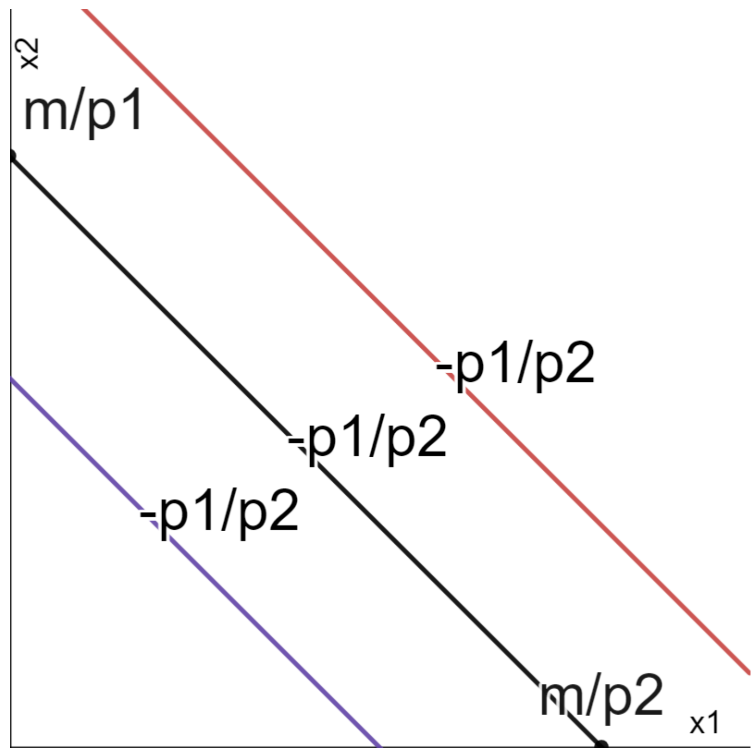

Figure 2.2: How Budget Changes with Income.

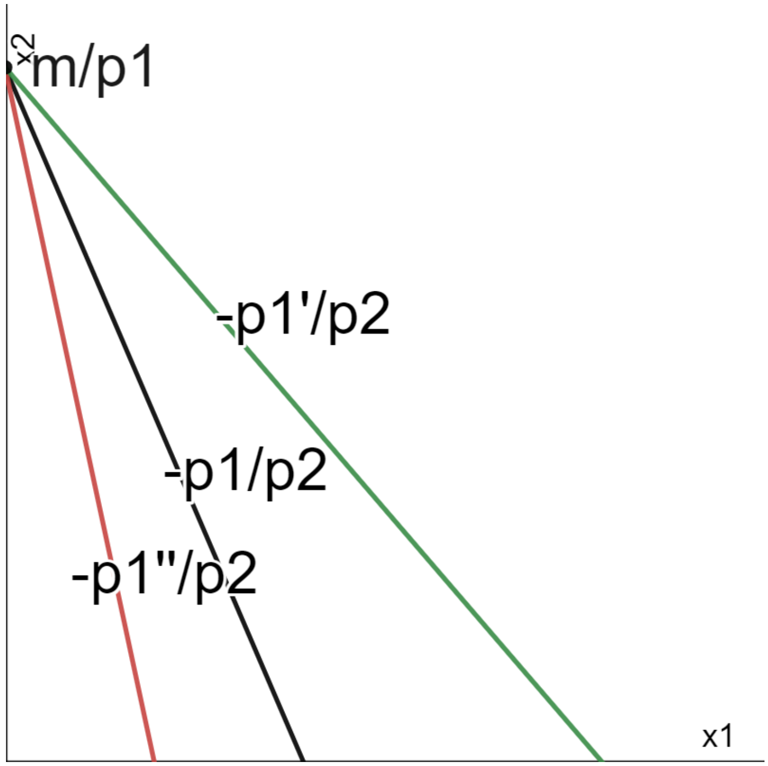

Figure 2.3: How Budget Changes with change in p_1.

In summary:

m changes:

Both endpoints change. If m increases, $\frac{m}{p_1}$ (the amount I can buy of good 1 changes) increases and $\frac{m}{p_2}$ (maximum affordable x_2) increases. The slope does not change. If m decreases, the opposite happens.

p_1 changes:

p_1. If p_1 goes up, the slope decreases (more negative). If p_1 goes down, the slope increases. The x_2 intercept stays the same.

p_2 changes:

p_2. If p_2 goes up, the slope increases and the x_2 intercept decreases. If p_2 goes down the slope decreases (becomes more negative) and the x_2 intercept increases. The x_1 intercept stays the same.

2.3 Taxes

Taxes represent a certain kind of price change. There are two kinds of taxes that are used frequently: *quantity* and *ad valorem* taxes.

A **quantity tax** is determined by **number of units** (x_i) purchased where an **ad valorem** tax is determined by the **value** of the good purchased ($x_i p_i$).

With a quantity tax of t dollars on good i, the amount paid in tax is tx_i. With an ad valorem tax of percentage τ on good i, the amount pain in tax is $\tau\,(p_i x_i)$. The key difference is that as price of a good changes, the amount collected by the government does not change with a quantity tax (assuming the amount purchased does not change), but it does with an ad valorem tax. Most sales taxes are ad valorem. However, there are quantity taxes we encounter frequently. Pay close attention next time you are pumping gas, there is usually a sticker showing how much you pay in tax *per gallon*. That's a quantity tax.

Here's what happens to the budget line when we ad a quantity tax and ad valorem tax on good 1.

Quantity tax on good 1:

$$p_1x_1 + tx_1 + p_2x_2 = m$$

$$(p_1 + t)\, x_1 + p_2x_2 = m$$

Ad valorem Tax on good 1:

$$(p_1x_1) + \tau\,(p_1x_1) + p_2x_2 = m$$

$$[(1+\tau)\, p_1]x_1 + p_2x_2 = m$$

Notice that in both cases, the tax effectively just increases the price of the good. This makes taxes easy to plot, they have the same effect as a price increase. However, there are some complex scenarios you should think about. What if a quantity tax only kicked in after buying a certain amount of some good? What if instead of a tax, a subsidy (a decrease in price) was put on a good? What if that subsidy only held for the first k units of the good? We will talk about many of these scenarios in class and work with them in practice problems.

3 The Preference Relation $\succsim$

3.1 Definitions

Now that we know how to model what a consumer can have, we should talk about what the prefer. We represent preferences with a mathematical tool called a **relation**.

Preference Relation
The preference relation denoted $\succsim$ is a set of statements about **pairs** of bundles. The statement *"bundle x is preferred to bundle x'"* is shortened to:

$$x \succsim x'$$

Example: Ice Cream
Suppose a consumer eats bowls of ice cream. The bundles (bowls) are written with the vanilla scoops first and chocolate second. For example: $(2, 0)$ is two scoops of vanilla and zero of chocolate.
A consumer who likes vanilla more than chocolate might have these preferences:

$$(1,0) \succsim (0,1)\,, (2,0) \succsim (0,2)$$

A consumer who like more ice cream to less might have these preferences:

$$(2,0) \succsim (1,0)\,, (2,2) \succsim (1,1)$$

A consumer who gets sick of ice cream: (does anyone want to eat 100 scoops of ice cream?)

$$(1,0) \succsim (100,0)$$

A consumer who does not care about flavor might have:

$$(1,0) \succsim (0,1)\,, (0,1) \succsim (1,0)$$

In the case of the consumer who does not care about flavor above, notice that we have both $(1,0) \succsim (0,1)$ and $(0,1) \succsim (1,0)$. That is, a scoop of vanilla is just as good as a scoop of chocolate and a scoop of chocolate is just as good as a scoop of vanilla. When this is the case, we say the consumer is **indifferent.**

Indifference Relation: $\sim$
When $x \succsim y$ and $y \succsim x$ we say "x is indifferent to y" and write $x \sim y$.

When a consumer is not indifferent, we say they have strict preference for some bundle.

Strict Preference Relation: $\succ$
When $x \succsim y$ and **not** $y \succsim x$ we say "x is strictly preferred to y" and write $x \succ y$.

3.2 Assumptions on $\succsim$

In economics, we like to make as few assumptions about consumer's preferences as we can. There's a surprising amount we can say about consumer choice with just a few assumptions about the structure of preferences.

The first three assumptions or **axioms** we will look at ensure that for any budget set, consumers will have some favorite or set of favorite bundles. That is, given any set of bundles, they will actually be able to choose *something.* We will talk more about why these assumptions assure that fact in class.

Axiom 1. Reflexive.
For all bundles. The bundle is at least as good as itself.
In set notation:

$$\forall x \in X : x \succsim x$$

This is what we call a *technical* assumption. It does not carry a lot of content for us to talk about, but it helps ensure some minimal structure. After all, if a bundle was not "as least as good as itself", we'd have some trouble since that would imply that either it cannot be compared to itself or that it is both strictly better than itself and at the same time strictly worse than itself.

Axiom 2. Complete.
For every pair of distinct bundles. Either one is at least as good as the other or the consumer is indifferent.
In set notation:

$$\forall x, y \in X \& x \neq y : x \succsim y \, or \, y \succsim x \, or \, both$$

This axiom is a little more interesting. It says that for every pair of bundles, the consumer has *some* preference. The consumer can say "I'm indifferent." but not "I don't know". That is, everything is comparable.

Axiom 3. Transitivity.
If x is at least as good as y and y is at least as good as z then x is at least as good as z.

$$x \succsim y, y \succsim z \, \textbf{implies} \, x \succsim z$$

Transitivity lets us chain together preferences. It is really the **key** and most powerful assumption here. Transitivity ensures (along with the other assumptions) implies we can always put a set of objects into a **ranking** (possible with ties). Once we have a ranking, there's always going to be some things that are at the top of that ranking. Those are the things our consumers will choose.

3.3 Example of Violating Transitivity

In many circumstances, transitivity is an uncontroversial assumption. However, it is possible to construct perfectly reasonable decision processes where transitivity fails. Here is one of those examples:

Suppose there are three people on a dating app:

Person 1. Rich, Very Intelligent, Average Looking

Person 2. Financially Constrained, Genius, Good Looking

Person 3. Moderately Well Off, Average Intelligence, Best Looking

Now let's compare every pair of people. Person 2 is both more intelligent and better looking than person 1. Person 3 is wealthier and better looking than person 2. Person 1 is wealthier and more intelligent than person 3.

From this, we can construct a preference ordering: $2 \succ 1, 3 \succ 2, 1 \succ 2$. Notice, this is intransitive. It is clear who is better in any pair, but who would is is best from the set of all three? This kind of multi-dimensional comparison can easily cause intransitivity.

3.4 From Preference to Choice

So far, we have a pretty satisfying model of preferences, but economics is about *choice.* How do we model choice? Intuitively, we want to write down formal that, from any budget set, the consumer will choose the best thing (according to their preferences). To do this, let's define a **Choice Function.** We can write:

$$C : B \to B$$

This says that C is a function that maps the set B (a budget set) into itself. That is, from the set B, the function C returns some objects from the set B. This statement ensures that the set of "choices" will always be a subset of the budget set. In set notation, that would be expressed as: $C(B) \subseteq B$.

That's good, but there's no structure here involving the preference relation. What we really want is that $C(B)$ (the potential choices from the set budget set B) is the set of all bundles in B that are at **least as good as everything else in B.** We can express that formal as follows.

$$C(B) = \{x | x \in B : \forall x' \in B, x \succsim x'\}$$

This says, $C(B)$ is defined to be all the bundles (x) in the budget set (B) such that (:) for all ($\forall$) other bundles (x') in ($\in$) the budget set, we have that x is at least as good as x'. This is not the easiest statement to read if you are not familiar with this kind of formal expression, but I hope that you will agree that it is a rather elegant, and efficient way of expressing an otherwise rather complicated idea.

Notice that in the example in the last section of choosing a partner on a dating app, there is no partner that is at least as good as all the other partners. In that case, **the choice set is empty!** Having empty choice sets is potentially problematic for a mathematical model of choice. So, when can we be sure that there is always some bundle that a consumer will choose from any budget set in our models.

Fortunately for us, our three assumptions: reflexivity, completeness, and transitivity are enough to ensure that the consumer will always have some favorite things in any budget and will be able to make a choice. As an aside, transitivity is even a little stronger than we need for this, as it also ensures a form of consistency of choice called "coherence". We will talk a little about that in class.

3.5 Indifference Curves and the Weakly Preferred Set

At this point, we have spent a good amount of time looking at how to formally express preferences. In practice, it is hard to work with these formal statements. Like anything else, it is nice to be able to visualize preferences. We can achieve this through **indifference curves.**

Indifference curve: an indifference curve is a set bundles such that a consumer is indifferent between every pair of bundles in the set.

In mathematics terms, an indifference curve is called an *equivalence class.* That is, it is some set that are "equivalent" in terms of preferences. This term is not necessary to know, but it may come up in future courses.

Note: There are many indifference curves. We only sketch a few to get an idea of the "shape" of preferences. Every bundle has an indifference curve passing through it.

Let's look at an example. Suppose we have a consumer who likes apples just as much as oranges. They are indifferent between the bundle "two apple" $(2, 0)$ and the bundle "two orange" $(0, 2)$. These two bundles are on the same indifference curve in the graph below. The consumer is also indifferent between $(4, 0), (2, 2)$ and $(0, 4)$ they are on the same indifference curve in the graph below.

It is very useful to interpret the slope of an indifference curve at a particular point. Pick a bundle (x_1, x_2) and imagine adding a unit of x_1 to get a new bundle $(x_1 + 1, x_2)$. If the consumer wants more x_1, then the resulting bundle must be better, that is $(x_1 + 1, x_2) \succ (x_1, x_2)$. We now much ask, how much would we have to decrease x_2 by to get a new bundle $(x_1 + 1, x_2 - b)$ that the consumer is indifferent to the original? That is, what is the b such that $(x_1 + 1, x_2 - b) \sim (x_1, x_2)$. In a sense, we are asking, how much x_2 is a consumer willing to give up to get an extra unit of x_1. Since $(x_1 + 1, x_2 - b) \sim (x_1, x_2)$ they must be on the same indifference curve. So, we are also asking, if we start on some point of an indifference curve, and move one unit right, how far down do we need to move to bump into that same indifference curve. That amount is approximately the slope of the indifference curve.[1]

[1]Technically the slope at a particular point is defined as the limit of the ratio of how far we have to move down to how far we move to the right as that distance we move to the right shrinks to zero. You know, calculus stuff.

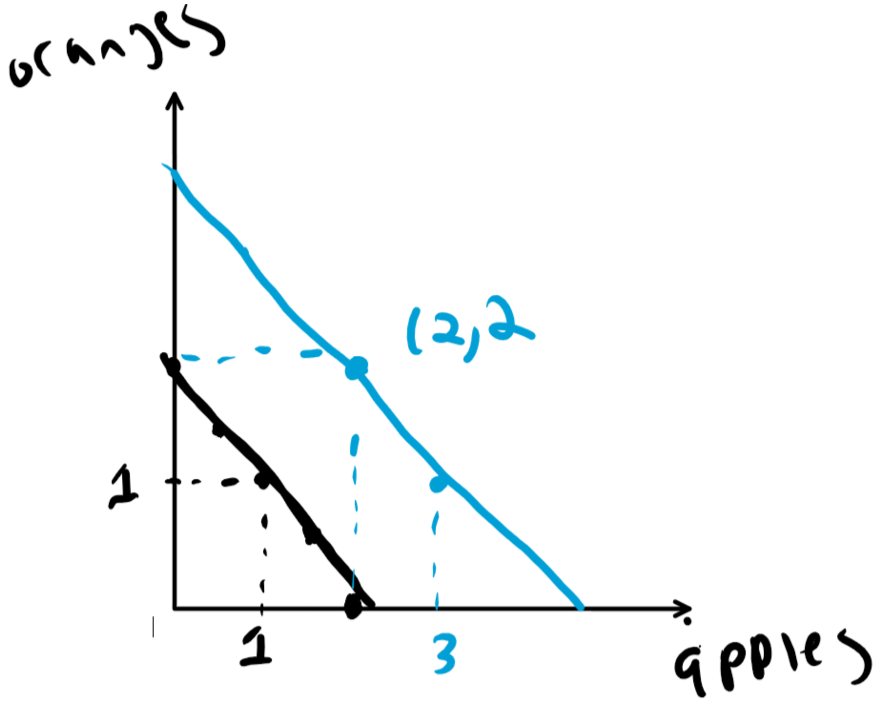

Figure 3.1: Indifference curves when apples are just as good as oranges.

3.6 Indifference Curves Cannot Cross

What can we say about the shape of indifference curves? It turns out, with only the assumptions of reflexivity, completeness, and transitivity: not much. Indifference curves could have some wild shapes. But under these assumptions there is one thing we know: **two distinct indifference curves cannot cross.**

Below is a proof of this claim. You are not responsible for knowing this proof, but you may be interested to see the logic. Understanding the logic might help you understand the way our axioms are used in proving formal statements about preferences.

Proof the two indifference curves cannot cross.
Look at the graph below. Here I have drawn two distinct indifference curves that cross each other. Notice that if two curves cross, they have to cross *somewhere*. I have labeled that somewhere x in the graph. This is a bundle that is on **both** indifference curves. However, since these are distinct indifference curves, there must be some bundle x' and x'' that are respectively on the different indifference curves and thus not indifferent to each other. However, since x is on both indifference curves, we must have $x' \sim x$ and $x'' \sim x$. Let's derive a contradiction to prove this scenario can never happen. Since it is not the case that $x' \sim x''$ if preferences are **complete**, it must be that either $x' \succ x''$ or $x'' \succ x'$. If we take the first possibility $x' \succ x''$ we have $x' \succ x''$ and $x'' \sim x$. By **transitivity,** it must be that $x' \succ x$ but we already know that $x' \sim x$. If we take the second possibility $x'' \succ x'$ we have $x'' \succ x'$ and $x' \sim x$. By **transitivity,** it must be that $x'' \succ x$ but we already know that $x'' \sim x$. Thus, no matter what, we have found a contradiction.

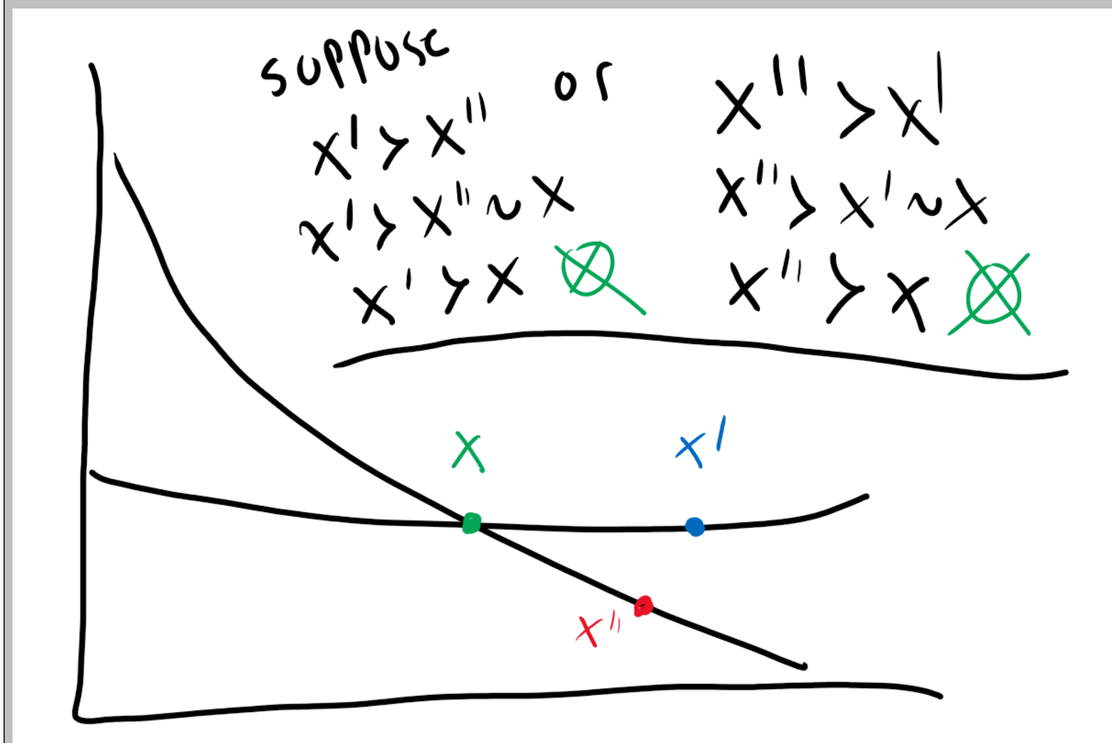

Figure 3.2: Indifference curves cannot cross if preferences are transitive.

3.7 Common Types of Preferences

There are a few "families" of preferences you should know about. These different families represent different types of trade-offs consumers are willing to make between two goods.

3.7.1 Perfect Substitutes

Perfect Substitutes preferences are such that a consumer's willingness to trade-off between the goods is the **same everywhere**.

The indifference curves are always downward sloping lines with the same slope. Recall The slope measures the amount of x_2 the consumer is willing to give up to get 1 more unit of x_1.

Steep slope: stronger preference for x_1.

Shallow slope: stronger preference for x_2.

Figure 3.3: Indifference curves for perfect substitutes preferences. This consumer would be willing to give up 2 units of x_2 in exchange for 1 unit of x_1.

3.7.2 Perfect Complements

Perfect Complements preferences are such that a consumer must consume the goods in a **fixed proportion**.

An example of this is left and right shoes. You always consume left and right shoes in a 1-to-1 proportion. That is, you want one left shoe for every right shoe. If you have the same number of left and right shoes, you are not willing to give up any left shoes to get more right shoes, because that would reduce the number of usable pairs you have.

Another example is ingredients in a recipe. Suppose you bake pies and a pie always needs two apples and one crust. If you have two apples and one crust, or four apples and two crusts, or six apples and three crusts, you would not be willing to give up apples to get more crusts or give up crusts to get more apples, it would reduce the number of pies you can make.

The indifference curves for these preferences are **L-shaped**. The kinks of these L-shaped curves pass along a line through the origin where the points on that line are the points where the goods are consumed in the "correct" proportion. That is, where there is not too much of either good. For left and right shoes, if left shoes are x_1 and right shoes are x_2, that the line $x_2 = x_1$ (the 45-degree line). For pies, if apples are x_1 and crusts are x_2 then the line through the kink points is where $2x_2 = x_1$. *I have plotted these below.*

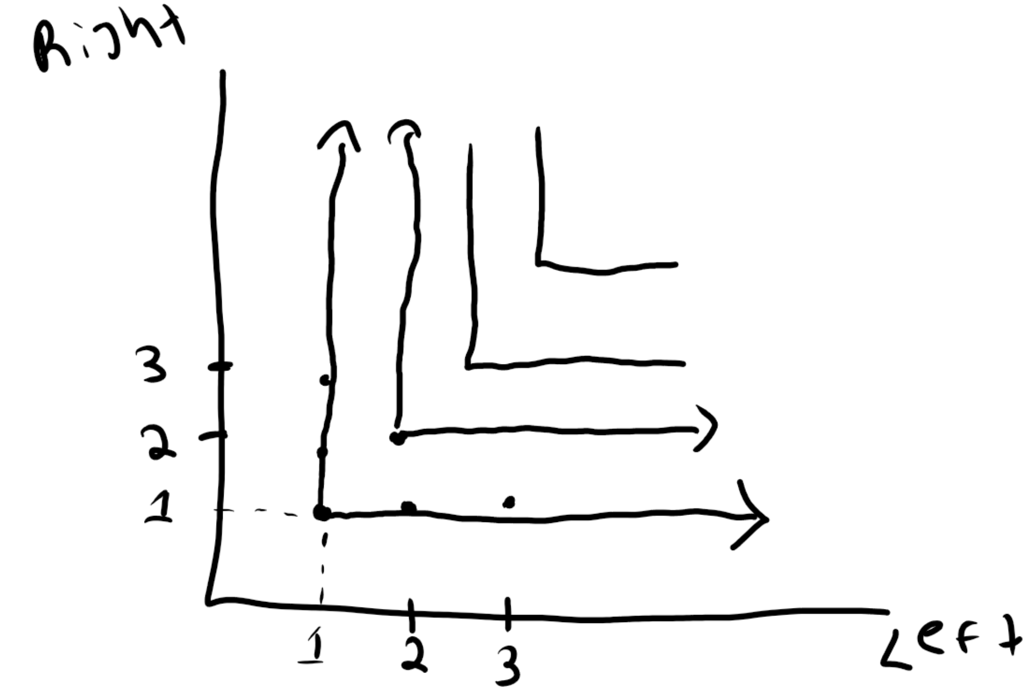

Figure 3.4: Indifference curves for perfect complements preferences where Left/Right shoes must be consumed in a 1-to-1 one ratio.

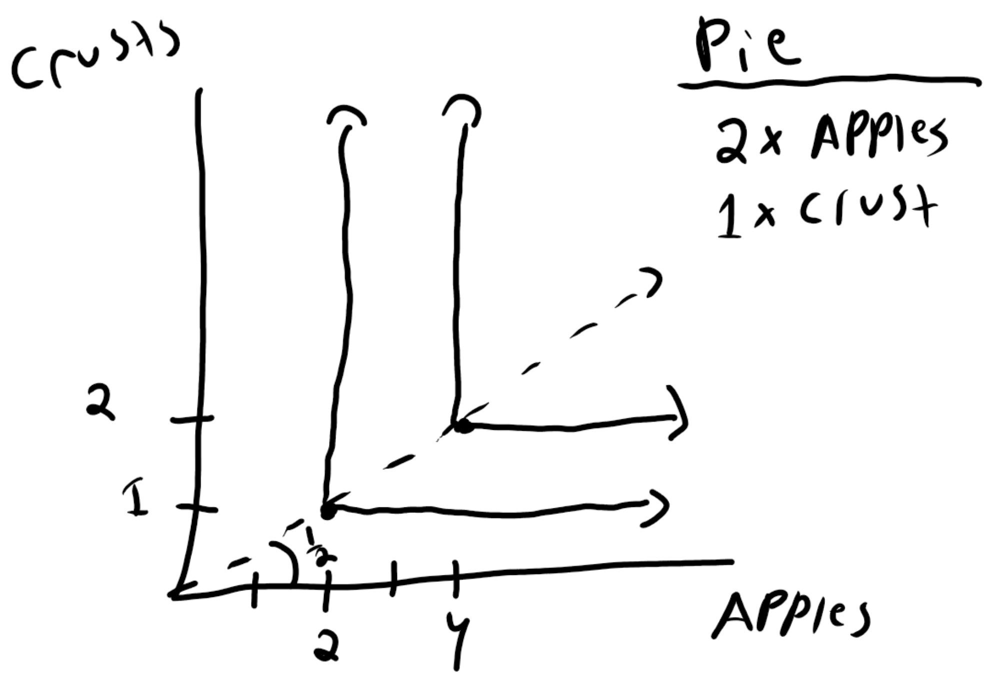

Figure 3.5: Indifference curves for perfect complements preferences where the goods are consumed in a 2-to-1 ratio. In this case, 2 apple and 1 crust make a pie.

3.7.3 Bads

So far, our examples have involved situations where both goods are in fact "good". That is, the consumer wants more (or at least does not want less) of either good. It is possible to model situations where that is not true. When a consumer wants less of something, we call that thing a **bad**.

When both x_1 and x_2 are goods, indifference curves are downward sloping. It is worth pausing to think about the intuition for this. The slope represents the tradeoff a consumer is willing to make between x_1 and x_2. Approximately, it is how much x_2 a consumer will give up to get one more unit of x_1. But now suppose x_2 is a bad. If a consumer gets one more unit of x_1, they will be happier. If we take away from x_2 they will be even happier than that! We could not possibly end up on the same indifference curve by adding some x_1 and taking away some x_2. We have to **add** x_2 to bring them back to indifference. Thus, the indifference curve is actually **upward sloping!**

The indifference curve will also be upward sloping if x_1 is a bad and x_2 is a good. Try to convince yourself of that using the same logic as above. However, if both goods are bad, the indifference curve is again downward sloping. However, unlike when both x_1 and x_2 are goods, preference increases as we move towards the origin: the bundle $(0, 0)$. These are demonstrated in the graphs below.

Figure 3.6: When one good is a "bad", indifference curves slope upward!

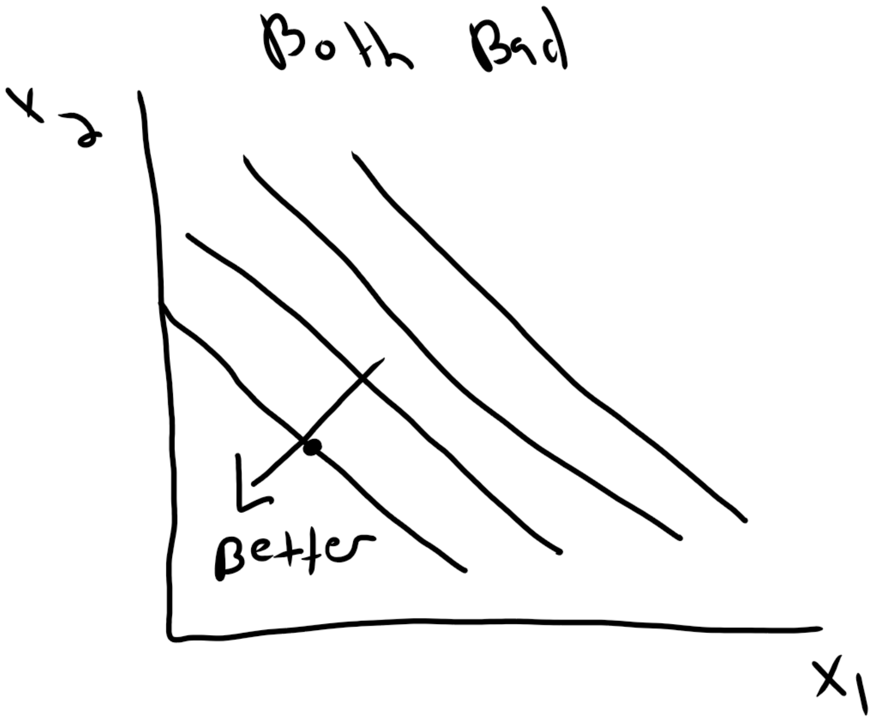

Figure 3.7: When both goods are bad, indifference curves slope down, but preference "increases" towards the origin (to the south west).

3.8 Well Behaved Preferences

As outlined in section 3.4the assumptions of reflexivity, completeness, and transitivity are sufficient to model consumers that are able to make choices from any budget set. That effectively enough to let us do some economics. However, preferences meeting just those conditions can be a little wild. We know that distinct indifference curves cannot cross but that's about it. Sometimes, we want to make some assumptions that will ensure that preferences are a little more well behaved. The following assumptions will prove convenient.

3.8.1 Monotonicity

Monotoncitiy: The assumption that everything is a "good".

There are two forms of this assumption. *Strict* and *Weak.* Strict monotonicity says that more of any good makes a consumer strictly better off. Weak monotonicity says that more of *every* good makes a consumer strictly better off, but more of any particular good might not. For instance, perfect substitutes are strictly monotonic. Perfect complements are weakly monotonic. We sometimes just call weakly monotonic "monotonic" (I know it must makes things more confusing- that's the problem with natural language. So, let's be formal:

Strict Monotonicity: For two bundles (x_1, x_2) and (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. $(x_1, x_2) \succ (y_1, y_2)$ if either $x_1 > y_1$ or $x_2 > y_2$

Weak Monotonicity. (AKA "Monotonic"): For two bundles (x_1, x_2) and (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1 \geq y_1$ and $x_2 \geq y_2$. $(x_1, x_2) \succ (y_1, y_2)$ if **both** $x_1 > y_1$ **and** $x_2 > y_2$

Weak monotonicity implies indifference curves are downward sloping (they have negative slope or zero slope) that is, they cannot be strictly upward sloping. Furthermore, it implies that preference increases to the north east. That is, as we move out, away from the origin, the bundles get better. Strict monotonicity additionally implies indifference curves are always *strictly* downward sloping. **It is worth thinking about why these assumptions imply these facts about the slope of the indifference curves and what those facts translates to in terms of trade-offs.**

Note that, it is often possible to convert non-monotonic preference to monotonic preference by thinking of a "bad" as the "lack of a bad". For instance, if we were writing a model of preferences over candy and Brussels sprouts we might have the bundle $(2, 2)$ which is two candies and two Brussels sprouts. If Brussels sprouts are a bad, then we might have: $(2, 2) \succ (2, 3)$. These preferences are non-monotonic. However, suppose we rewrite the number of sprouts as "how many less than 10 sprouts do I have?". Then the two bundles are $(2, 8)$ and $(2, 7)$. They are the same physical bundle so we still have $(2, 8) \succ (2, 7)$, but notice now we have patched up monotonicity.

Monotonicity ensures that, as long as there is no "savings" in the model, consumers will always spend all of their money. Why not? If more is better, then spending less than their income must be sub-optimal- they could get more of everything. This is helpful, since it tells us we can look for optimal bundles **on the budget line**.

Technically, we do not even need monotonicity for this to be true, a far weaker condition called local nonsatiation will ensure the same thing. Local nonsatiation says that for any bundle, there is another bundle "nearby" that is strictly better. That bundle might involve less stuff, it might involve more stuff. Effectively it ensures that if the consumer were not spending all of their money, they could change their bundle by a "little bit" and make themselves better off. Because of this, they could not possibly spend less than their income, because (by local nonsatiation) there will always be some other affordable bundle nearby that is strictly better. You do not need to know about this, but I think it is kind of interesting.

3.8.2 Convexity and Strict Convexity

Convexity: The assumption that mixtures are better than extremes.

Monotonicity tells us we can look for optimal bundles on the budget line. But where? This assumption can help tell us where to look. It is not a requirement for doing economics by any means. It is a convenience. There are two forms of this assumption. Both of them essentially say that if we take two bundles that are indifferent and mix them together, we will get a better bundle. Strict convexity says that bundle is strictly better and weak convexity (or just convexity) just tells us that it is weakly better. Here are the formal statements.

Strictly Convex: For two indifferent bundles $(x_1, x_2) \sim (y_1, y_2)$, for any $t \in (0, 1)$, the mixture given by $(tx_1 + (1-t)\, y_1, tx_2 + (1-t$
(x_1, x_2) and $(tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2) \succ (y_1, y_2)$.

Weakly Convex: For two indifferent bundles $(x_1, x_2) \sim (y_1, y_2)$, for any $t \in [0, 1]$, the mixture given by $(tx_1 + (1-t)\, y_1, tx_2 + (1-t)$
(x_1, x_2) and $(tx_1 + (1-t)\, y_1, tx_2 + (1-t)\, y_2) \succsim (y_1, y_2)$.

Notice how we are mixing the bundles together. We take a t portion of the bundle (x_1, x_2) and mix it with a $(1-t)$ portion of (y_1, y_2). To demonstrate this, let's mix together $(2, 1)$ and $(1, 2)$. If we take $t = 0.5$, we are taking half of $(2, 1)$ which is $(1, 0.5)$ and adding half of $(1, 2)$ which is $(0.5, 1)$. The result is the bundle $(1.5, 1.5)$. If instead we take (0.25) we get a quarter of $(2, 1)$ which is $(0.5, 0.25)$ and three-quarters of $(1, 2)$ which is $(0.75, 1.5)$. Adding these together we get the bundle $(1.25, 1.75)$.

The mixtures are also referred to as "convex combinations". If we were to plot all of the **convex combinations of two points, the convex combinations would simply be the straight line through the two points.**

Using this, we can talk about the geometry of indifference curves meeting these conditions. Under the assumption of monotonicity:

If preferences are **strictly convex**, then the **indifference curve always lies strictly below a line between any two points on that indifference curve**.

If preferences are **weakly convex**, then the **indifference curve always lies weakly below a line between any two points on that indifference curve**.

An example of an indifference curve for strictly convex preferences is shown below.

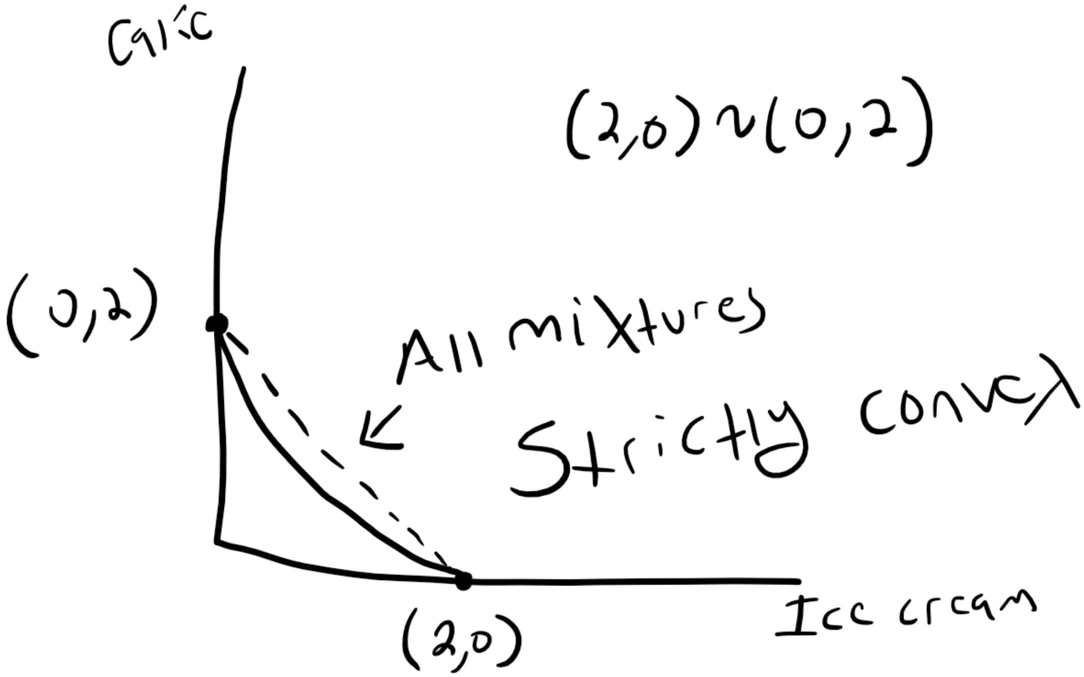

Figure 3.8: Example of Convex Indifference Curves

3.9 Marginal Rates of Substitution and Slope of the Indifference Curve

The marginal rate of substitution is defined as the rate at which a consumer will give up x_2 in order to get more x_1. We have already seen that this rate of trade-off is captured by the slope of the indifference curve at a point. Approximately, we can think of the MRS as how much x_2 a consumer would give up to get one more unit of x_1. The MRS and equivalently the slope of the indifference curves will play a critical role in finding optimal bundles. We will see that in the next two sections.

4 Utility

4.1 Definition

A utility function is a way of assigning "scores" to bundles, such that better bundles according to $\succsim$ get a higher score. For example, suppose a consumer's preferences are:

$$A \succ B \succ C \sim D$$

Some utility functions that represent these preferences:

$U(A) = 10$, $U(B) = 5$, $U(C) = U(D) = 0$

$U(A) = 12$, $U(B) = 1$, $U(C) = U(D) = -100$

Utility function: $U(x)$ represents preferences $\succsim$ when for every pair of bundles x and y, $U(x) \geq U(y)$ if and only if $x \succsim y$.

That is, if x is better than y according to $\succsim$ it gets a higher utility according to $U()$. To reiterate, a utility function is a convenient mathematical representation of the fundamental preference relation $\succsim$. We do not need to believe utility functions actually exist to use them, since they are just how we represent preferences.

We say that u**tility is ordinal** since the magnitude of the numbers are meaningless, and only the relationships matter. There is no sense in which two times higher utility means that the preference is two times stronger. If we could say something like that, we could call utility a **cardinal** measure. Since we can only infer the ranking of bundles, but not say anything about how *strong* the preferences are from the relation $\succsim$, the utility function that represents $\succsim$ also has no such content.

4.2 Monotonic Transformations

Because utility is ordinal, we are free transform one utility function into another, as long as it maintains the same preferences. **Any strictly increasing function of a utility function represents the same preferences as the original utility function.** For example, suppose:

$$U(x_1, x_2) = x_1 + x_2$$

This represents the preferences of someone who only cares about the total amount of stuff, but not the composition. In fact, this utility function represents *perfect substitutes preferences*. Here are some other utility functions that represent the same preferences:

$$\tilde{U}(x_1, x_2) = x_1 + x_2 + 100 = U(x_1, x_2) + 100$$

$$\tilde{U}(x_1, x_2) = (x_1 + x_2)^2 = (U(x_1, x_2))^2$$

Since the functions $f(u) = u + 100$ and $f(u) = u^2$ are strictly increasing for $u \geq 0$ (which is always true for the original utility function), these are monotonic transformations of the original utility function. It is often useful to use monotonic transformations to modify a utility function that is hard to work with into one that is more convenient.

For instance, suppose we had the utility function: $u = 38(x_1 + x_2)^2 + 100$. We could transform this into the utility function $u = x_1 + x_2$ which is much simpler. The two utility functions represent the exactly same preferences.

4.3 MRS from Utility Function

As we have discussed above, the **Marginal Rate of Substitution** (MRS) is the slope of the indifference curve. We can get the MRS from a utility function by taking the ratio of partial derivatives of the utility function. Let's first define those partial derivatives:

Marginal Utility of good i is $mu_i = \frac{\partial u(x_1,x_2)}{\partial x_i}$.

With this, we can define the MRS in terms of the marginal utilities:

The Marginal Rate of Substitution (MRS) is given by: $MRS = -\frac{mu_1}{mu_2} = -\frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}}$

Note that MRS is an **ordinal** property since it represents the slope of indifference curves. Because two preferences that are the same have the same indifference curves, they will also have the same MRS. This is actually a convenient way to check whether two utility functions represent the same preferences.

Same MRS, same preferences.

In the section above, I claimed these two utility functions represent the same preferences: $u = 38(x_1 + x_2)^2 + 100$ and $u = x_1 + x_2$. Their marginal rates of substitution are identical:

$$-\frac{\frac{\partial\left(38(x_1+x_2)^2+100\right)}{\partial x_1}}{\frac{\partial\left(38(x_1+x_2)^2+100\right)}{\partial x_2}} = -\frac{76(x_1 + x_2)}{76(x_1 + x_2)} = -1$$

$$-\frac{\frac{\partial(x_1+x_2)}{\partial x_1}}{\frac{\partial(x_1+x_2)}{\partial x_2}} = -\frac{1}{1} = -1$$

4.4 Examples of Utility Functions

4.4.1 Perfect Substitutes

A constant MRS implies a constant willingness to trade off between the two goods. This is the case for perfect substitutes.

$$u(x_1, x_2) = ax_1 + bx_2$$

$$MRS = -\frac{a}{b}$$

4.4.2 Quasi-Linear

With quasi-linear preference, a consumer only gets tired of one of the two goods. For instance, if x_1 is ice cream and x_2 is money, we might want to represent preferences where the amount of money a consumer is willing to give up to get another unit of ice cream is decreasing in the amount of ice cream. This can be achieved with a quasi-linear utility function.

One common quasi-linear utility function is:

$$u(x_1, x_2) = ln(x_1) + x_2$$

Let's look at the MRS:

$$MRS = -\frac{\frac{\partial(ln(x_1)+x_2)}{\partial x_1}}{\frac{\partial(ln(x_1)+x_2)}{\partial x_2}} = -\frac{1}{x_1}$$

This says that as ice cream increases, (approximately) the amount of money a consumer is willing to give up to get another scoop of ice cream is one over the number of scoops they already have. With one scoop, they will give up a dollar to get another scoop. With two scoops, the would only give up 50 cents. And so on...

Another example of a quasi-linear utility function:

$$u(x_1, x_2) = \sqrt{x_1} + 10x_2$$

Practice taking the MRS of this function. Notice that it only depends on the amount of (ice cream) x_1!

4.4.3 Cobb-Douglass

Now suppose we want the consumer to get tired of both goods as they get more. We can use a **Cobb-Douglass** utility function:

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$$

Let's look at the MRS:

$$mu_1 = \frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta}$$

$$mu_2 = \frac{\partial\left(x_1^{\alpha} x_2^{\beta}\right)}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1}$$

$$MRS = -\frac{MU_1}{MU_2} = -\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}}$$

$$= -\frac{\alpha x_1^{\alpha} x_1^{-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta} x_2^{-1}} = -\frac{\alpha x_1^{-1}}{\beta x_2^{-1}} = -\frac{\alpha}{\beta}\frac{x_2}{x_1}$$

This says that the amount of x_2 a consumer is willing to give up to get another unit of x_1 is directly proportional to the ratio of x_2 to x_1. If they have a lot of x_2 relative to x_1 they will give up more x_2 to get another unit of x_1 and vise versa.

Let's compare two CD Functions:

Increasing the exponent on either good will increase the consumers desire for that good. They will still get tired of it, but between two consumers, one with a larger exponent on a good, that consumer will have a stronger desire for the good at the same bundle.

$$u(x_1, x_2) = x_1 x_2$$

$$MRS = -\frac{x_2}{x_1}$$

At the point $(1,1)$: $MRS = -1$. Now let's increase the exponent on x_1 to 10:

$$\tilde{u}(x_1, x_2) = x_1^{10} x_2$$

$$MRS = -10\frac{x_2}{x_1}$$

At the point $(1,1)$: $MRS = -10$

Notice that the consumer with $\tilde{u}(x_1, x_2) = x_1^{10} x_2$ would be willing to give up ten-times more x_2 to get the same amount of x_1 as the consumer with utility function $u(x_1, x_2) = x_1 x_2$.

5 Choice

Now that we have modeled budgets (what is available), preferences (what is desired) and know how to represent those preferences with utility functions, we are ready to talk about what consumers actually choose from the set of available bundles.

We have already modeled choice. Formally, we want to find the set of bundles that meet this condition

$$X^* = \{x : x \in B \,\&\, \forall x' \in B, x \succsim x'\}$$

This says that the set of optimal bundles X^* are bundles like x that are in the budget set B and are at least as good as any other x' that is also in the budget set.

There is one really powerful observation that makes the process of finding optimal bundles much simpler. In thinking about where an optimal bundle lives on the graph of preferences and budget, there are really only **three possibilities.** These come out of a very powerful observation about trade-offs when preferences are **complete**, **transitive** and **monotone**.

5.1 Three Possibilities

Assume $\succsim$ is **reflexive**, **complete**, **transitive** and $\succsim$ **monotonic**. A bundle cannot be optimal if it is on an indifference curve that crosses into the **interior** of the budget set.

The proof proceeds by contradiction. You are not responsible for this, but it might be nice to read through and try to understand. The proof is shown graphically below.

Suppose we found a bundle x we thought was optimal but was on an indifference curve that passed into the interior of the budget set. Then there is some bundle x' on the interiod of the duget set such that $x \sim x'$ (since it is on the same indifference curve). Since x' is on the interior of the budget set, there is some other bundle x'' such that x'' is in the budget set and has more of every good than x'. Since preferences are monotonic, $x'' \succ x'$. Since preferences are transitive, we have $x'' \succ x' \sim x$ and so $x'' \succ x$.

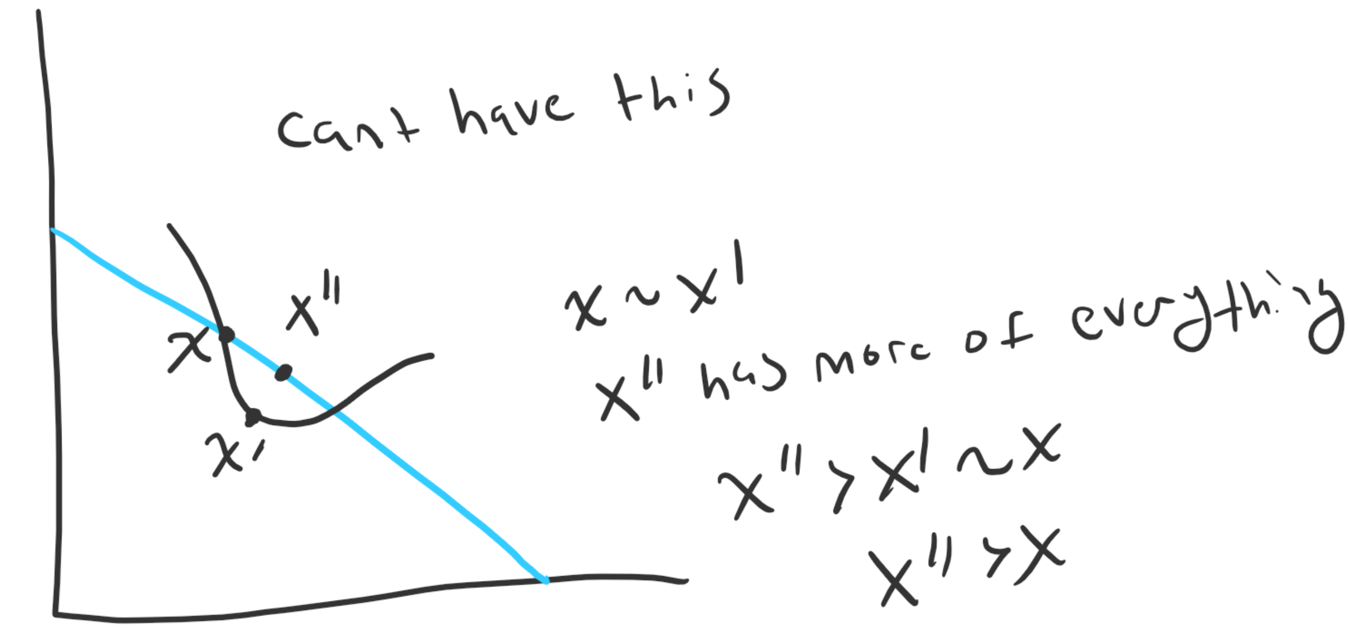

Figure 5.1: An optimal bundle cannot be on an indifference curve that passes "into" the budget set.

With this result in hand, there are only three ways a bundle can be on an indifference curve that is in the budget set and does not exist on an indifference curve that passes into the interior of the budget set. First of, it must be on the budget line. Then we have three possibilities:

1. (Tangent) It is at bundle where the indifference curve at that bundle had the same slope as the budget line.

2. (Touching but not tangent) The bundle is a "non-smooth" point on the indifference curve, but the that point just touches the budget line.

3. (Boundary) We are at one of the boundaries ($x_1 = 0$ or $x_2 = 0$) in this case the slope of the indifference curve and the slope of the budget need not be equal.

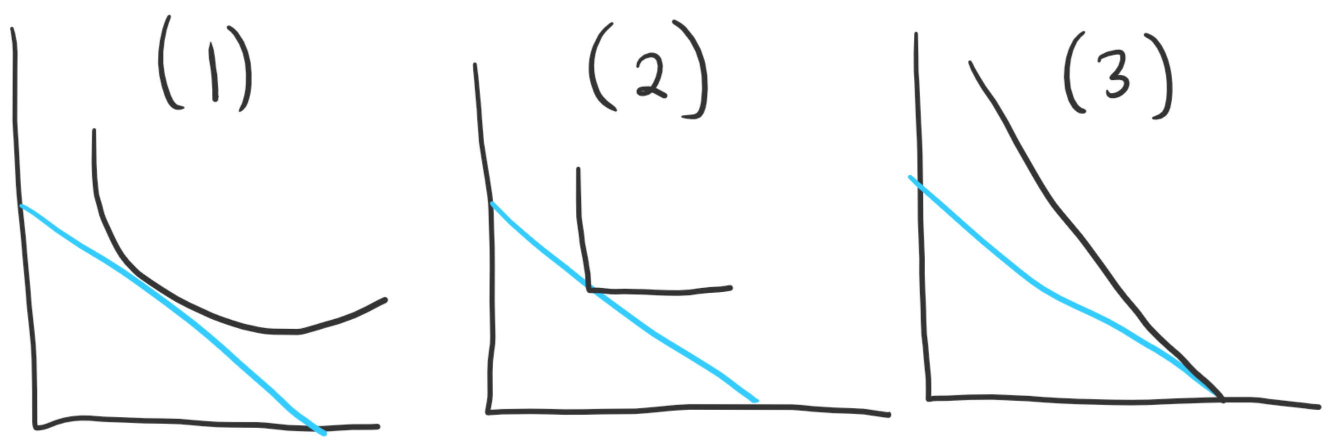

Figure 5.2: Graphical Examples of the Three Possibilities

Under some weak conditions (we can take derivatives of the utility function). The tangency condition is **necessary** for an *interior* optimum (involves consuming some of both things).

That is, if there is an optimal bundle that involves consuming some of both goods, it must have the property that the slope of the indifference curve at that optimal bundle is the same as the slope of the budget line. This is a very powerful result and also suggests why we are going to take a lot of derivatives in this class.

This condition is formalized by the familiar equation:

$$MRS = -\frac{p_1}{p_2}$$

Note, this is precisely that the slope of the indifference curve at a point is equal to the slope of the budget equation at that point. This also implies the trade-offs are the same.

The MRS is the way a consumer is **willing** to trade off between the goods and the slope of the budget equation is the rate a which the **must** in order to stay in their budget. If these are not equal either the consumer is willing to give up more x_2 than they have to in order to get more x_1 or they are willing to give up more x_1 than they have to in order to get more x_2. Neither situation can be optimal... unless they **can not get any more** x_1 **or** x_2. That would be the case at the boundary. That is, we can have an optimal bundle where this condition is not met, but it can only occur at a boundary or where this condition is not defined.

5.2 Examples

Let's look at a few examples of finding optimal bundles.

5.2.1 Cobb Douglass:

$$u(x_1, x_2) = x_1 x_2$$

$$p_1 x_1 + p_2 x_2 = m$$

This is a smooth utility function. We can find its MRS everywhere. Let's write down the tangency condition:

$$MRS = -\frac{\frac{\partial(x_1 x_2)}{\partial x_1}}{\frac{\partial(x_1 x_2)}{\partial x_2}} = -\frac{p_1}{p_2}$$

$$-\frac{x_2}{x_1} = -\frac{p_1}{p_2}$$

This simplifies to:

$$* \, x_1 p_1 = x_2 p_2$$

We also know that an optimal bundle occurs on the budget line. Let's write this down as a second condition. Budget Condition:

$$** \; x_1 p_1 + x_2 p_2 = m$$

We have two conditions and two unknowns. Plug the tangency condition into budget condition to get:

$$x_1 p_1 + x_1 p_1 = m$$

Simplify this:

$$x_1^* = \frac{1}{2}\frac{m}{p_1}$$

Plug this back into either of the two conditions gives us:

$$x_2^* = \frac{1}{2}\frac{m}{p_2}$$

The optimal bundle is:

$$\left(\frac{\frac{1}{2}m}{p_1}, \frac{\frac{1}{2}m}{p_2}\right)$$

Note the form of this bundle. $\frac{1}{2}m$ is half of the income. These optimal bundles have the consumer spending half of their income on both goods.

5.2.2 Perfect Substitutes

The utility function is:

$$u(x_1, x_2) = 2x_1 + x_2$$

Prices and income are: $p_1 = 1$, $p_2 = 1$, $m = 10$. This gives us the budget equation:

$$1x_1 + 1x_2 = 10$$

Finding the tangency condition:

$$-\frac{2}{1} = -\frac{1}{1}$$

$$-2 = -1$$

Ohh... But this is never true. The consumer would always be willing to give up 2 units of x_2 to get one unit of x_1. But they only have to give up 1. **There can't be an interior solution.** They will just buy as much x_1 as possible.

If you ever get lost doing a perfect substitutes problem you can use the following trick. With perfect substitutes, there has to be a boundary solution. Just check the utility of both intercepts (buying only x_1 and buying only x_2) see which is better. If they give the same utility, any bundle they can afford is optimal. In this problem we get:

Only consume x_1:

$$\left(\frac{m}{p_1}, 0\right) = (m, 0)$$

$$u(m, 0) = 2m = 20$$

Only consume x_2:

$$\left(0, \frac{m}{p_2}\right) = (0, m)$$

$$u(0, m) = m = 10$$

Since consuming only x_1 gives me more utility, that is the optimal bundle:

$$(m, 0)$$

5.2.3 Anything is Optimal

Here is an example where any affordable bundle is optimal.

$$u(x_1, x_2) = 2x_1 + x_2$$

$p_1 = 2$, $p_2 = 1$, $m = 10$. The budget equation is:

$$2x_1 + 1x_2 = 10$$

The tangency condition is:

$$-\frac{2}{1} = -\frac{2}{1}$$

$$-2 = -2$$

All of the bundles such that $2x_1 + 1x_2 = 10$ are optimal. Confirm this by checking the utility of some bundles on this line.

5.2.4 Perfect Complements

Suppose these are the utility function and budget equation:

$$u(x_1, x_2) = min\{x_1, x_2\}$$

$$2x_1 + x_2 = 15$$

We know the budget condition must be true at the optimum.

$$** \; 2x_1 + x_2 = 15$$

But, we can not take derivatives here. What is the other condition?

In this case, we have to use a little intuition. If the consumer ever consumed a bundle that was not on the kink of an indifference curve, than they could spend less on some good and use the leftover money to buy more of both goods, increasing utility.

"No Waste Condition". (Equation for the "kink" points).

$$* \, x_1 = x_2$$

Solving this equation together with the budget equation gives us:

$$x_1 = 5, x_2 = 5$$

5.2.5 Perfect Complements (2 Apples, 1 Crust)

Let's try another perfect complements problem. Suppose the utility function is:

$$u(x_1, x_2) = min\left\{\frac{1}{2}x_1, x_2\right\}$$

This represents the utility of someone who only eats pies and makes pies by using two apples and one crust per pie.

Suppose the budget equation is:

$$** \; 2x_1 + x_2 = 15$$

In this case, the "no waste condition" (equation for the "kink" points) is:

$$\frac{1}{2}x_1 = x_2$$

Notice we get this by setting the two terms in the $min\{\}$ function equal to each-other.

Combine the conditions and solve to get:

$$x_1 = 6, x_2 = 3$$

5.2.6 Max Preferences

The utility and the budget are:

$$u(x_1, x_2) = max\{x_1, x_2\}$$

$$2x_1 + x_2 = 15$$

Try this one at home: what is the optimal bundle?

6 Demand

In the previous chapter, we looked at how to find demand (the optimal bundle) given a utility function, prices, and income. In this chapter, we look at how demand changes when we change one of those parameters (p_1, p_2, m).

6.1 Marshallian Demand

The Mashallian demand is the optimal amount of a good, given prices and income. We denote these this way:

$$x_1^*(p_1, p_2, m), x_2^*(p_1, p_2, m)$$

For instance, the Marshallian for someone with utility function $u(x_1, x_2) = x_1 x_2$ is:

$$x_1^* = \frac{\frac{1}{2}m}{p_1}, x_2^* = \frac{\frac{1}{2}m}{p_2}$$

Marshallian for someone with utility function $u(x_1, x_2) = min\{x_1, x_2\}$ is:

$$x_1^* = \frac{m}{p_1 + p_2}, x_2^* = \frac{m}{p_1 + p_2}$$

We can now look at how these types of demands change.

6.2 Changes in Income

We first ask, "what happens to demand when we change income"? We can formalize this by thinking of this change as a derivative. We want to know what are the values of $\frac{\partial x_1^*(p_1,p_2,m)}{\partial m}$ and $\frac{\partial x_2^*(p_1,p_2,m)}{\partial m}$

6.2.1 Normal/Inferior

Depending on whether demand increases or decreases with income, we call goods **normal** or **inferior.**

If demand *increases* when income increases, we say the good is **"Normal".** If demand *decreases* when income increases, we say the good is **"Inferior".**

Examples:

We have seen that the demand for x_1 from the Cobb-Douglass utility function $u = x_1 x_2$ is $x_1^* = \frac{\frac{1}{2}m}{p_1}$. This is a normal good since this demand increase with income. Notice that $\frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial m} = \frac{1}{2p_1} > 0$.

Suppose we found demand for some good was $x_1 = \frac{10}{mp_1}$. This would be an inferior good since demand decreases with m. Noice $\frac{\partial\left(\frac{10}{mp_1}\right)}{\partial m} = -\frac{10}{m^2 p_1} < 0$.

6.2.2 Income Offer Curve

The income offer curve is a plot of optimal bundles (x_1^*, x_2^*) as income changes but prices remain fixed.

For example, suppose $u(x_1, x_2) = x_1 x_2$ and prices are $p_1 = 2, p_2 = 1$. We get demands: $x_1 = \frac{1}{4}m$, $x_2 = \frac{1}{2}m$. Let's pick a few points for m and plot the optimal bundles.

$$
\begin{array}{ll}
m = 1 & \left(\frac{1}{4}, \frac{1}{2}\right) \\
m = 2 & \left(\frac{1}{2}, 1\right) \\
m = 3 & \left(\frac{3}{4}, \frac{3}{2}\right) \\
m = 4 & (1, 2\}) \\
m = 5 & \left(\frac{5}{4}, \frac{5}{2}\right)
\end{array}
$$

Plotting these, we see quickly they live on a straight line.

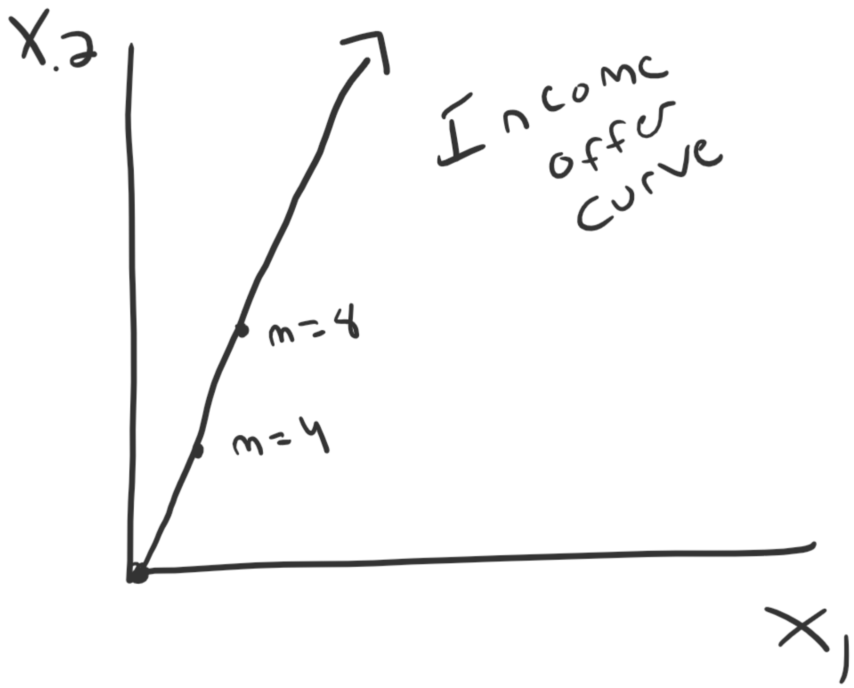

Figure 6.1: Income offer curve for Cobb Douglass preference example.

Notice how the income offer curve increases in both the x_1 and the x_2 direction as m increases. That is because both goods are normal. What if one good was inferior (both can not be inferior at the same time). We would get a graph like this:

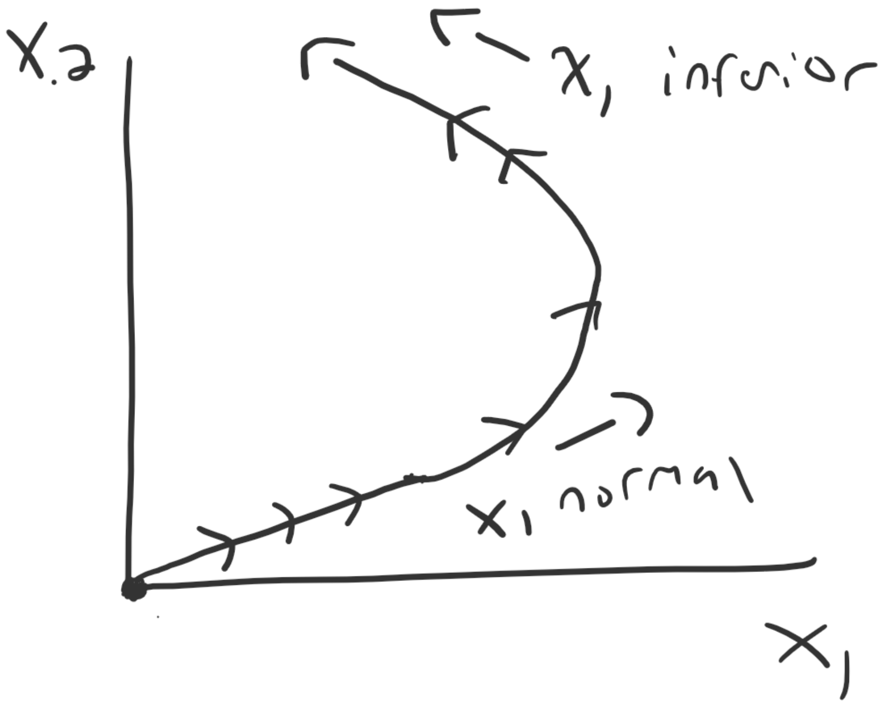

Figure 6.2: Example of an income offer curve where x_1 is initially normal but becomes inferior as m grows.

6.2.3 Engle Curve

The Engle curve is the relationship between income and a **single** good. Plotting m on the vertical axis against x_1 or x_2 on the horizontal axis. Suppose we had demand: $x_1 = \frac{1}{4}m$. To plot this with m on the vertical axis, it helps to isolate m. We get:

$$m = 4x_1$$

When we put m on the vertical axis, really what we are plotting is the amount of income a consumer would need to have to demand some amount x_1 of good 1.

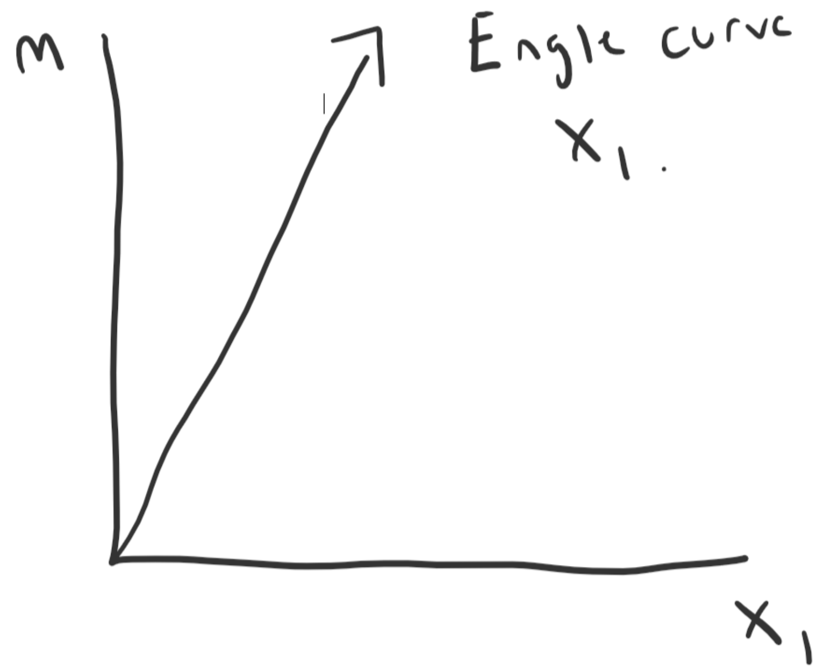

Figure 6.3: Engle curve for $x_1 = \frac{1}{4}m$.

This is a normal good because x_1 increases as m increases. What about an inferior good? This one is subtle. We might think we could just make a graph where x_1 decreases *everywhere* as m increases. But, this is impossible. For x_1 to decreases, it has to have increased at some point. So, this shows us that good cannot possibly be "always inferior". The normal/inferior nature of a good can depend on income.

Here is an example where a good in normal for low income and inferior for larger income:

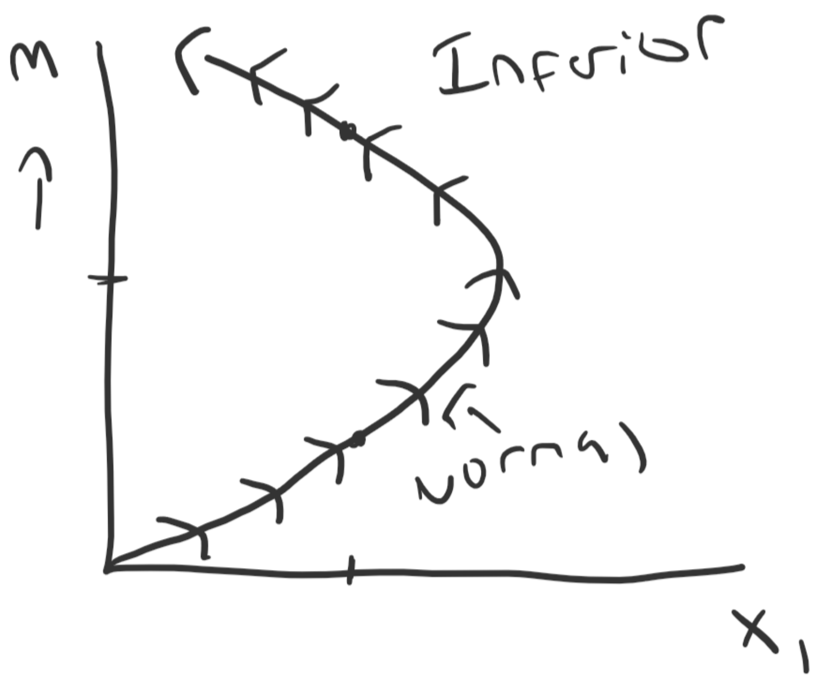

Figure 6.4: Engle curve for a good that is normal for low income and inferior for high income.

6.2.4 Example: Perfect Complements

Let's work an example with perfect complements. Suppose we have: $U(x_1, x_2) = min\{x_1, x_2\}$. $p_1 = 2$, $p_2 = 1$.

At the optimum, we have $x_1 = x_2$ (the no waste condition) and $2x_1 + 1x_2 = m$ (the budget condition). Solving these together gives us:

$$x_1 = \frac{m}{3}, x_2 = \frac{m}{3}$$

Plotting the income offer curve:

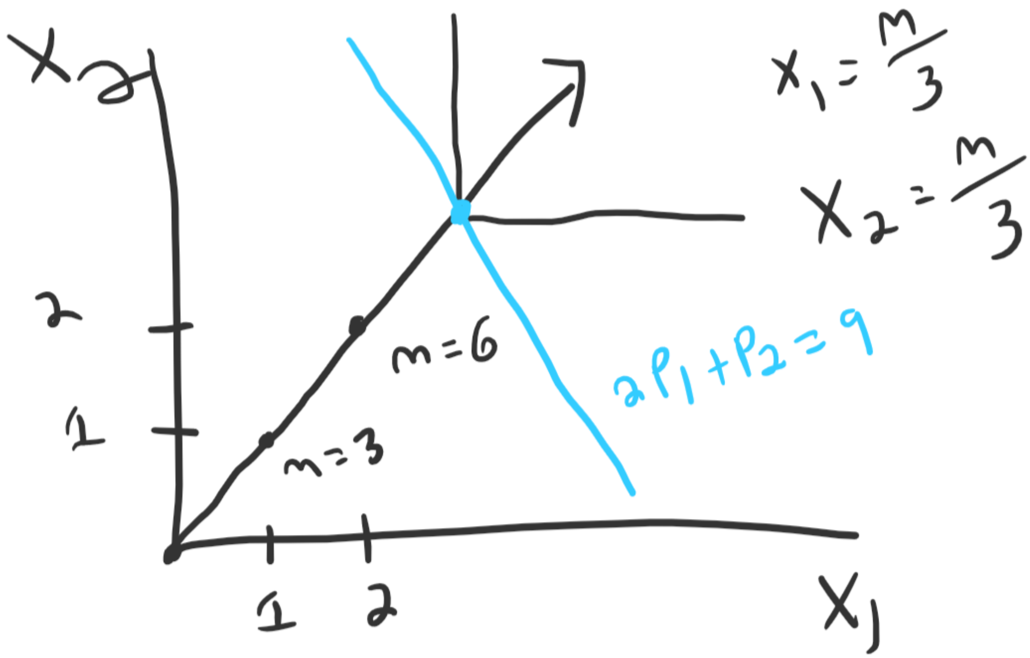

Figure 6.5: Income offer curve for $min\{x_1, x_2\}$ with $p_1 = 2$ and $p_2 = 1$

Plotting the Engle curve for x_1:

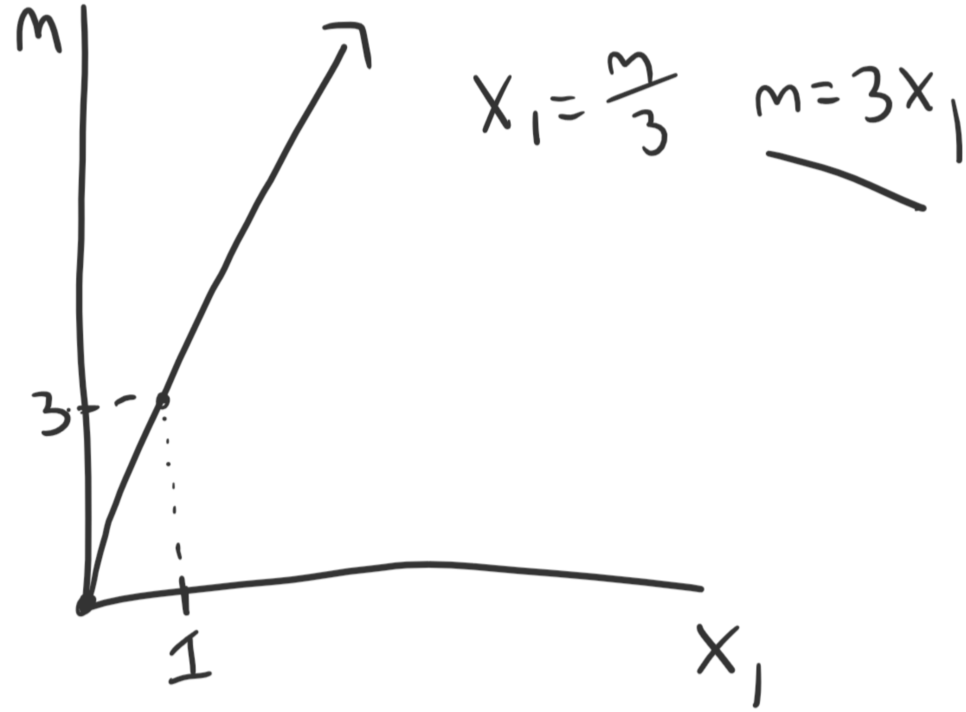

Figure 6.6: Engle Curve of x_1 for $min\{x_1, x_2\}$ with $p_1 = 2$ and $p_2 = 1$

6.3 Changes in "Own" Price

The analysis of how demand for a good changes when the price of that good changes is a common task in economics.

Most of the time, we expect demand to decrease when price increases. We call such goods *ordinary.* Perhaps surprisingly, nothing about our assumptions so far rule out the opposite. Demand can increase when price increases. We call such goods *giffen.*

Classifying goods:
When the price of a good goes up, and demand goes *down*, we say the good is **ordinary.**
When the price of a good goes up, and demand goes *up*, we say the good is **giffen.**

Here is a diagram showing how a good might be giffen. Notice that the price of good 1 increases from the green to the blue budget lines. Yet, the demand on the greed budget line is smaller than on the blue budget line.

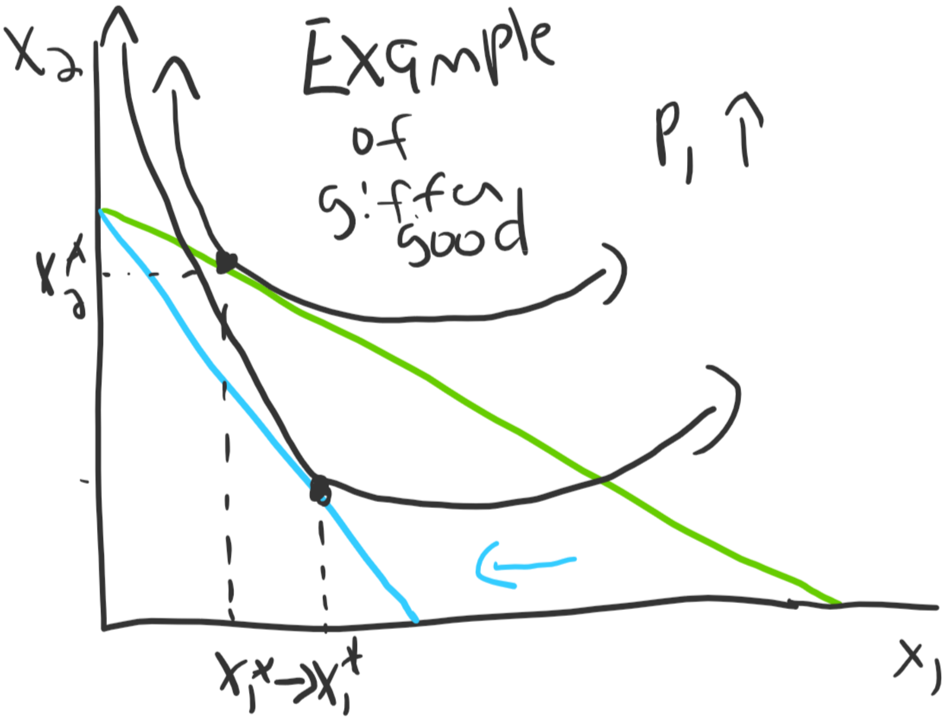

Figure 6.7: Indifference curves and budget for a "giffen" good. Note that as the price p_1 increases from the green to blue budget, the optimal amount of x_1 increases.

As we will see later on, a **giffen good must be inferior.** In fact, the was giffen goods arise is the following way: As the price of the good increases, it lowers the consumers effective income. This decrease in effective income can pressure demand up enough that the total effect on demand positive. *Yeah... it's weird*, but that's the beauty of math. Counter-intuitive outcomes occur all the time in formal models. This is not a bad thing. Investigating the formal mechanisms that create these strange outcomes can be just as informative as studying the more reasonable phenomenon we can produce.

6.3.1 Price Offer Curve

Like with changes in income, we have several ways of graphing the way demand changes when a price changes. The analogous graph to the income offer curve is the price offer curve. To produce this graph, hold income and one of the prices fixed, the price offer curve is the set of bundles x_1^*, x_2^* that are optimal at each level of the other price.

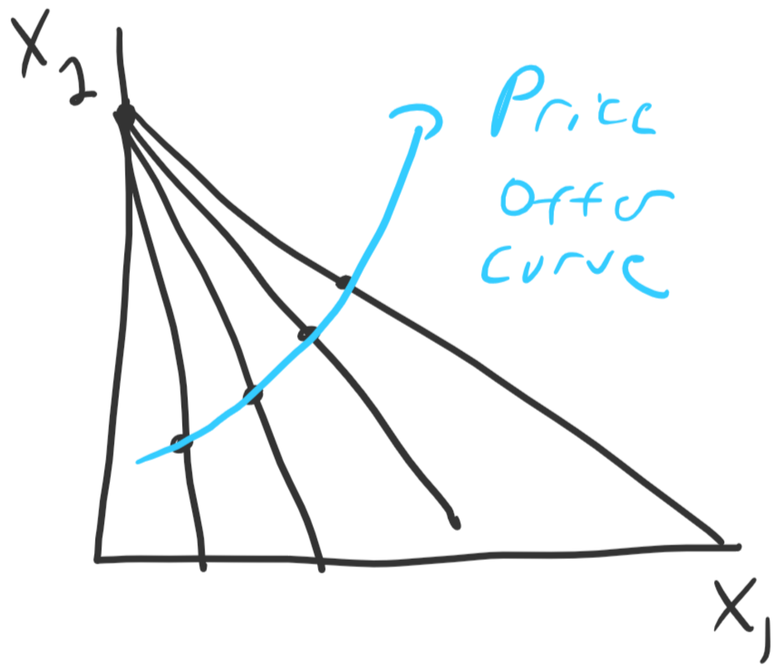

Figure 6.8: Example of a price offer curve (blue). The price offer curve plots the bundles that are optimal along each of the budget lines (black) as p_1 changes.

6.3.2 Plotting the Demand Curve

The analogous graph to the engle curve is the demand curve. As we have seen, the **demand** for a good is $x_i\left(p_1, p_2, m\right)$ that is, the optimal amount that a consumer chooses given the prices and income. However, when we talk about "plotting" the demand curve of x_1 we usually mean holding p_2 and m fixed and plotting how the demand for x_1 changes as p_1 changes. For this, we put p_1 on the vertical axis and x_1 on the horizontal axis. When we do this, we are actually plotting something called the *inverse demand.*

For example, suppose demand for x_1 is:

$$x_1 = \frac{\frac{1}{2}\left(10\right)}{p_1}$$

To plot this with p_1 on the vertical axis, we first solve for p_1. When we do this we get $p_1 = \frac{5}{x_1}$ this is the **inverse demand.** It is the price that would be responsible for the consumer buying some amount x_1 of the good. Plotting this we get the "demand plot".

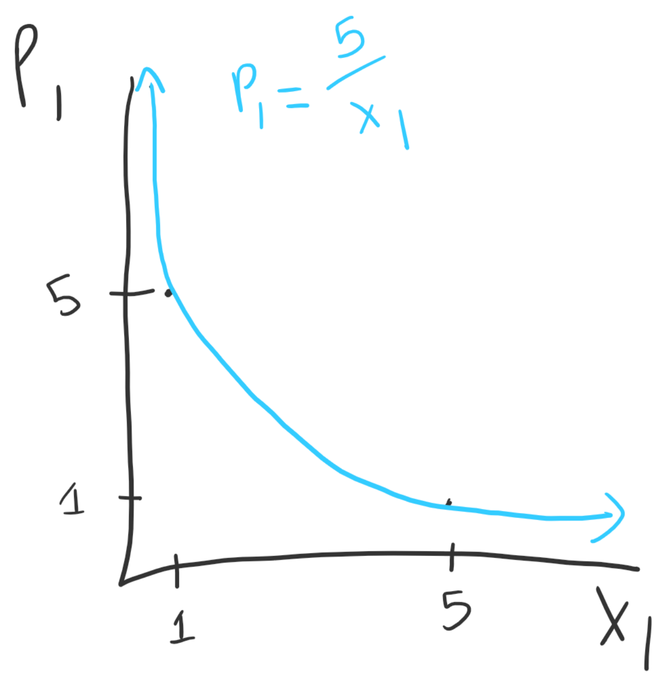

Figure 6.9: Plotting demand for $x_1 = \frac{5}{p_1}$.

6.4 Changes in "Other" Price

So far we have looked at what happens to a good when we change income and it's own price. We might also be interested in how demand changes for a good when the change the price of another good. Like the other parameters, we have two possibilities for classifying goods:

Classifying Goods
If the demand for a good goes **down** when the price of the other good goes up, we say the goods are **complements**.
If the demand for a good goes **up** when the price of the other good goes up, we say the goods are **substitutes**.
If demand for a good does not change when the price of the other good goes up, we say the goods are **neither complements nor substitutes**.

6.4.1 Example: Perfect Complements

Suppose we have

$$u = min\{x_1, x_2\}$$

The marshallian demand is: $x_1 = \frac{m}{p_1+p_2}$ and $x_2 = \frac{m}{p_1+p_2}$. For both goods, as you increase the price of the other good, the demand goes down. They are **complements** (*hopefully this is not a surprise*). One way to see this is just to note that both prices are in the denominator. Thus, as either increases, demand goes down (this shows both goods are also ordinary). Another way is to take a derivative. For instance, to see that x_1 is decreasing in p_2:

$$\frac{\partial\left(\frac{m}{p_1+p_2}\right)}{\partial\left(p_2\right)} = -\frac{m}{\left(p_1+p_2\right)^2}$$

Since m and $(p_1+p_2)^2$ are always positive, $-\frac{m}{(p_1+p_2)^2} < 0$. Thus, the derivative is negative, showing that x_1 decreases in p_2. A similar analysis will show that x_2 is decreasing in p_1.

6.4.2 Examples Perfect Substitutes

Suppose we have utility:

$$u = x_1 + x_2$$

Demand is $x_1 = \frac{m}{p_1}$ $x_2 = 0$ if $p_1 < p_2$ and $x_1 = 0$ $x_2 = \frac{m}{p_2}$ if $p_1 > p_2$.

This one is a litter trickier. Let's look at how x_2 changes in p_1. If $p_1 < p_2$, $x_2 = 0$. If p_1 increases enough so that $p_1 > p_2$ the demand for x_2 increases to $\frac{m}{p_2}$. So, as long as the change in price p_1 has any effect on the demand for p_2 (it might not if it does not change which price is higher in this example) then the goods are **substitutes.**

6.4.3 Examples Cobb Douglass

Suppose

$$u = x_1 x_2$$

Demand is $x_1 = \frac{\frac{1}{2}m}{p_1}$ and $x_2 = \frac{\frac{1}{2}m}{p_2}$. *Neither good's demand depends on the price of the other good.* **They are neither complements nor substitutes.**

7 Slutsky Decomposition

In the previous section, we looked at how demand changes when price changes. In this section, we dig a little deeper into *why* demand changes when price changes. Intuitively, there are two reason that a change in price can change demand. Let's look at them in the context of an increase in price.

Substitution Effect: *When the price goes up, you might substitute into buying alternatives, lowering demand.*
Income Effect: *When the price goes up, what you continue to buy is now more expensive. Effective income is lower and demand decreases if the good is normal, and increases if the good is inferior.*

Notice that, the income effect could be either positive or negative, depending on whether the good is normal or inferior. On the other hand, the substitution effect is always negative for an increase in price. We call this the law of demand:

Law of Demand:
For a change in price of good i the substitution effect (on good i) will always lead to a decrease or no change in demand x_i.

This leads to the first result for this section. I telegraphed it earlier, but here it is. If a good is **Giffen**, a price increase increases demand. However, a price increase will always decrease demand due to substitution. So, in order to get demand to decrease overall, the income effect *must increase demand enough to overcome the substitution effect.* That is, the good **must be inferior.**

This leads to three total possibilities for how a change in price can change demand through these two effects:

Ordinary/Normal- *Both effects decrease demand.*
Ordinary/Inferior- *Substitution decreases demand (it always does) and income effect increases demand, but not enough to overcome the decrease due to substitution.*
Giffen/Inferior- *Substitution decreases demand (it always does) and income effect increases demand so much that it overcomes the decrease due to substitution and increases demand overall.*

In the next section, we work through how to *quantitatively measure* these effects.

7.1 The Slutsky Decomposition.

The Slutsky decomposition is best thought of as a thought experiment. This thought experiment goes like this:

Suppose price of a good increases, we go from the budget:

$$p_1 x_1 + p_2 x_2 = m$$

To a new budget:

$$p_1^{'}x_1 + p_2x_2 = m$$

The **total effect** is:

$$x_1^*\left(p_1, p_2, m\right) - x_1^*\left(p_1^{'}, p_2, m\right)$$

To study substitution effect only, *we need to know what the consumer would choose if the price had changed, but their demand could not change due to income.* We need a way to to correct for the effect the price change has on effective income. Let's try this. Ask: *how much income would they need at the new prices to afford the old bundle?* If we were to give the consumer this extra income and then see what they buy at the new prices whatever changes about demand *couldn't be due to income.* We corrected their income. Any change can only be due to substitution.

To do this formally we calculate:

Compensating income: cost of the original bundle under the new prices.

If we are analyzing a change in p_1 this would be:

$$\tilde{m} = p_1^{'}x_1^*\left(p_1, p_2, m\right) + p_2x_2^*\left(p_1, p_2, m\right)$$

Now construct a new budget to asses the substitution effect. Use the new prices, but give the consumer this compensated income. The budget equation is: $p_1^{'}x_1 + p_2x_2 = \tilde{m}$. What does the consumer choose on this budget. This is denoted: $x_1^*\left(p_1^{'}, p_2, \tilde{m}\right)$

The **substitution effect** is the difference between what they chose under the original budget and what they choose under this budget with the new prices and compensated income.

$$x_1^*\left(p_1, p_2, m\right) - x_1^*\left(p_1^{'}, p_2, \tilde{m}\right)$$

Of course, since this difference takes care of the substitution effect. The **income effect** is the remainder:

$$x_1^*\left(p_1^{'}, p_2, \tilde{m}\right) - x_1^*\left(p_1^{'}, p_2, m\right)$$

That is, the difference between what they choose on the thought experiment budget (new prices, extra income) and what they choose under the new prices with their actual income.

7.2 Graphically Decomposing Demand

There are three budgets involved in the slusky decomposition. The original budget, shown in blue. The budget after the price change, shown in green. The budget we use to determine the substitution effect (shown in orange) which has the new prices and an extra amount of income so that the consumer can afford the old bundle at the new prices. Notice how the orange budget intersects the blue budget the the bundle the consumer demands under the prices before the price change. This is graphically showing that the income has been compensated so the consumer can afford that old bundle.

On each budget I have drawn the indifference curve through the optimal bundle on that budget. Notice in each case the indifference curve just touches the budget but does not pass through it. I have also marked the demand for x_1 from each of those budgets on the x_1 axis. x_1 is the demand under the original budget. $x_1^{'}$ is the demand under the new prices. $x_1^{''}$ is the demand under the new prices with compensated income. The total effect is $x_1 - x_1^{'}$. The substitution effect is $x_1 - x_1^{''}$. The total effect is the difference: $\left(x_1 - x_1^{'}\right) - \left(x_1 - x_1^{''}\right) = x_1^{''} - x_1^{'}$. This is shown to the right of the plot.

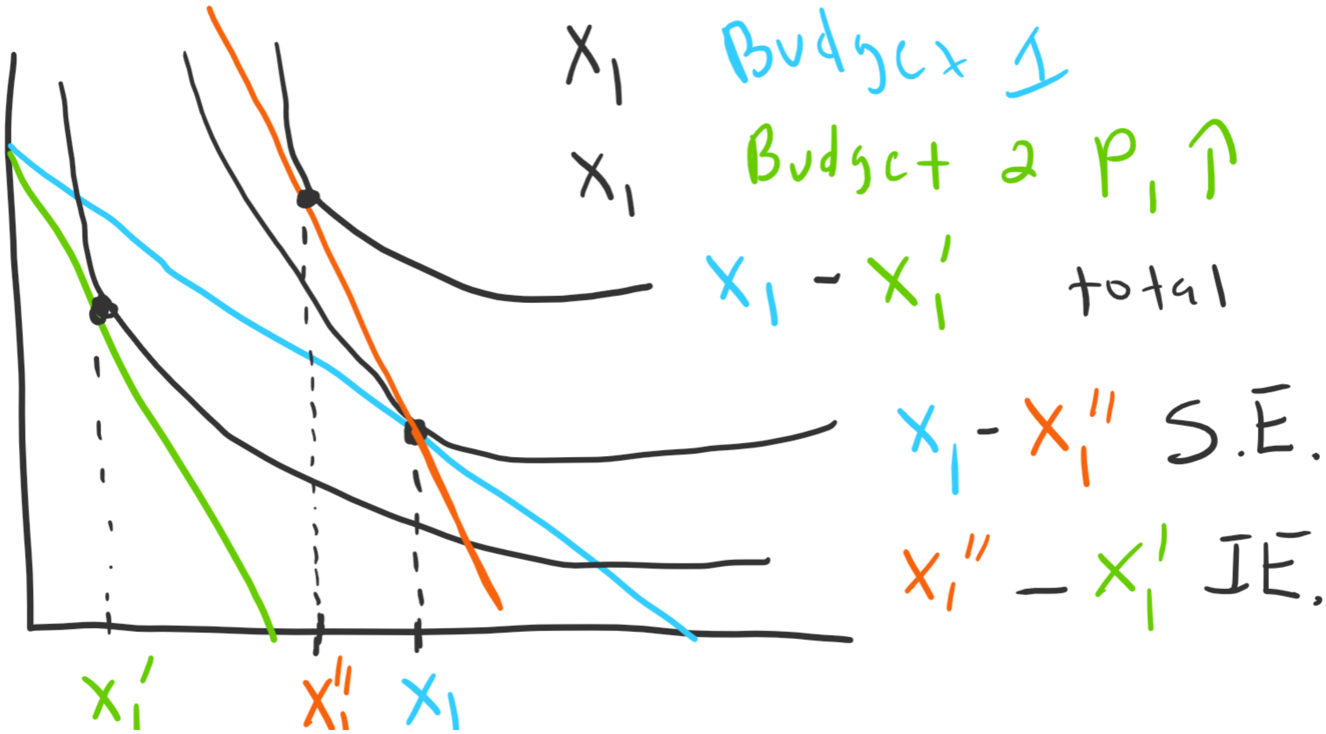

Figure 7.1: The Slutsky Decomposition for an Ordinary/Normal Good.

Here is an example of decomposing changes in demand for a consumer with perfect complements preferences. As you might expect, with perfect complements, there is no substitution effect. Have a look how the demand on the blue and orange budgets below are identical! Thus, the substitution effect is zero. The total effect is completely explained by the income effect.

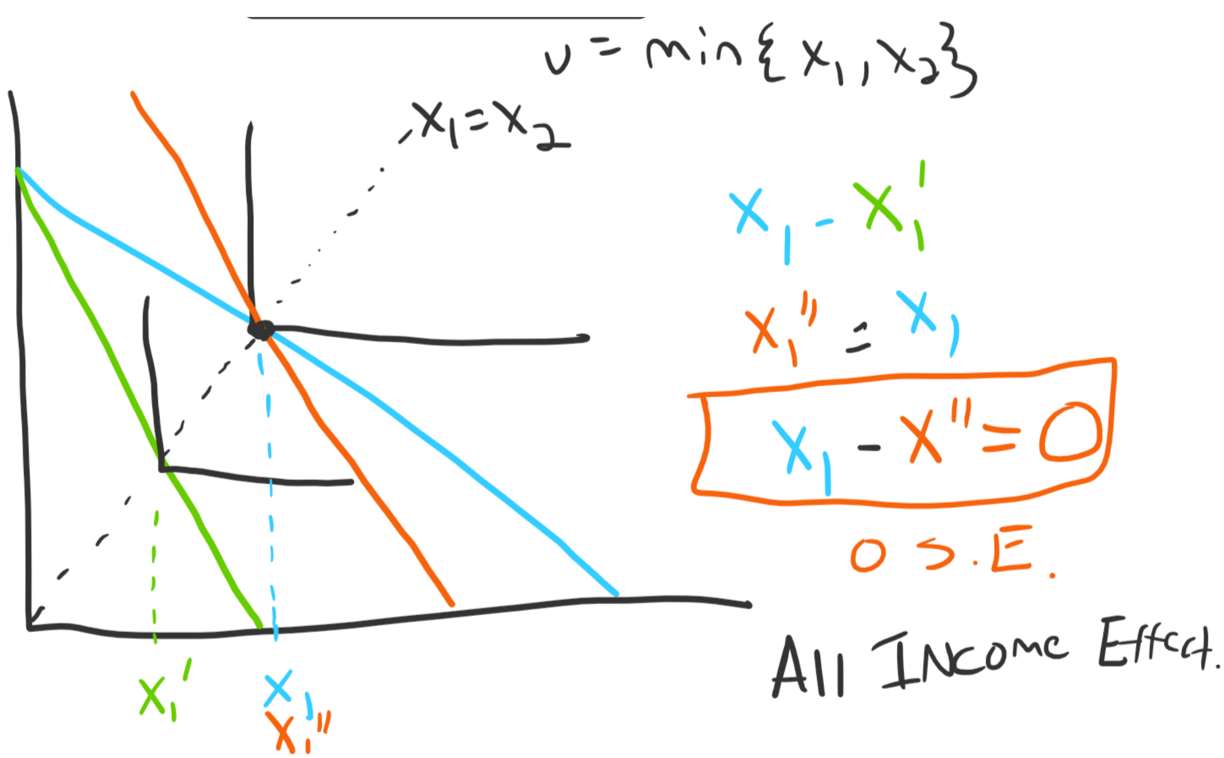

Figure 7.2: The Slutsky decomposition for perfect complements. There is **only** substitution effect.

7.3 Why is the substitution effect always negative?

I mentioned above that the substitution effect is always negative without much commentary. It is worth a reading through a short proof to convince yourself of this. You are not responsible for this proof, but as usual, understanding it might give you better insight into how demand works.

Below, I have drawn a diagram with the same color coding as the previous two examples. The original budget is blue, then price p_1 increases to create the green budget. The orange budget is the compensated budget we use to determine the substitution effect.

The fact that substitution effect must be negative is equivalent to the fact that, on the compensated budget, the consumer's optimal bundle will always lie to the left of the original bundle. Suppose it didn't. Suppose the consumer instead demanded *more* x_1 on the compensated budget. The fact that the price p_1 is higher, and the consumer has enough income to afford the old bundle means the whatever bundle they might demand to the right of the original, it will have to have been strictly affordable under the old budget. Inspect the graph to convince yourself of this. But if that bundle was *strictly affordable* under the old budget and wasn't chosen, why would it be chosen now?

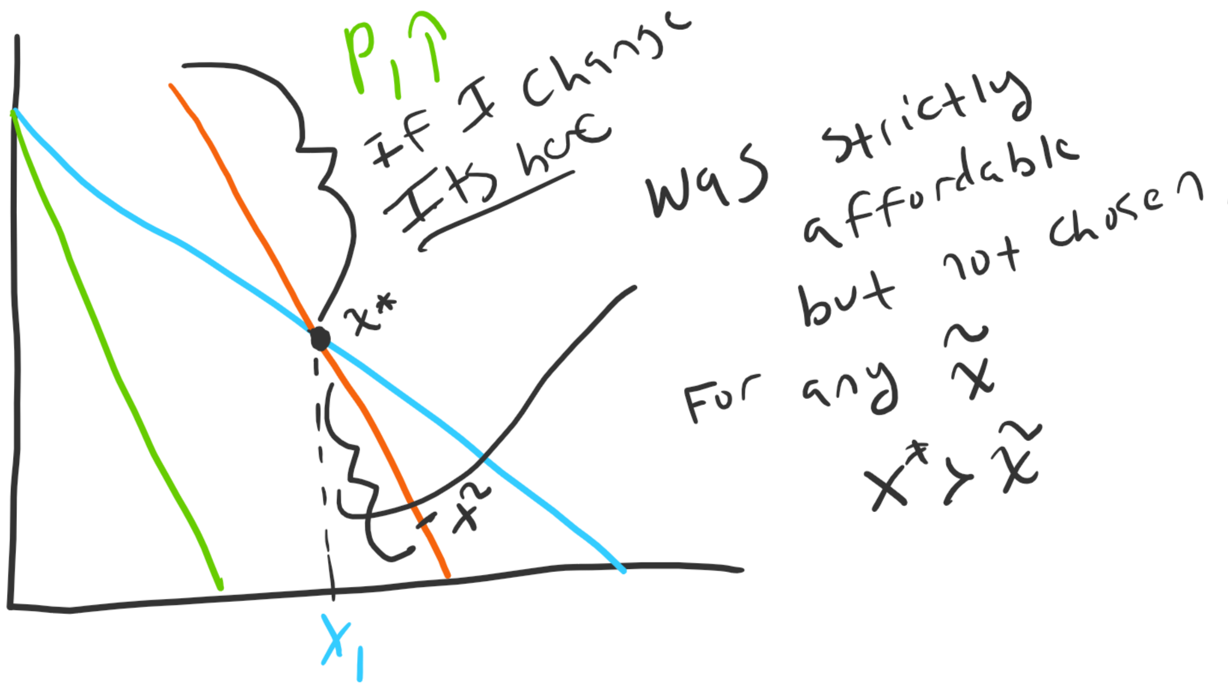

Figure 7.3: **The substitution effect must be negative.** Everything on the lower portion of the orange budget (which determines the substituion effect) was available under the original (blue) budget and was **strictly affordable**. Thus, the point chosen on the blue budget x^* must be strictly better than any of those points. Thus, no point on the lower point of the orange budget can be chosen. This implies demand must decrease due to substitution.

7.4 Example Problem

Let's work through a slutsky decomposition problem. Suppose utility is:

$$u = x_1 x_2$$

Demand for this consumer is: $x_1^* = \frac{\frac{1}{2}m}{p_1}, x_2^* = \frac{\frac{1}{2}m}{p_2}$.

Let's pick some income and prices. Suppose prices are $p_1 = 1$, $p_2 = 1$, $m = 20$ and the price of good one increases to $p_1 = 2$. Demand before the price change is the bundle $(10, 10)$. After the price change, demand changes to $(5, 10)$. The demand for x_1 has decrease by 5 units due to this change in prices. How much is due to substitution? We need to first calculate the compensating income. How much would this consumer need to afford the bundle $(10, 10)$ after the price change. The cost of that bundle under the prices $p_1 = 2$, $p_2 = 1$ is: $2\,(10) + (10) = 30$.

We now construct the new budget equation: $2x_1 + x_2 = 30$. What would the consumer demand under this budget? Using the demands above, we get the bundle: $\left(\frac{15}{2}, 15\right)$. Notice, the new demand for x_1 is $\frac{15}{2}$. This is $\frac{5}{2}$ higher than the original bundle. This difference cannot be due to income effect since we have given the consumer extra income to afford the old bundle. Thus this $\frac{5}{2}$ is due to the substitution effect. The remainder of the 5 unit change in demand must be due to income. That is $5 - \frac{5}{2} = \frac{5}{2}$. Thus, for this example, the income and substitution effect are both $\frac{5}{2}$.

8 From Income to Endowments

Until this point our consumers had income in terms of money. $m = \$10$ for instance. We did not model where this income comes from. In economics, we call such a parameter *exogenous*. That means, it is determined outside the model. This is fine for a lot of analysis. But it will not work if we want to model a more cohesive economy where income *has to come from somewhere in the economy.* We would want income to be *endogenous.* That is, determined within the model.

To take the first step towards endogenizing income, we will now think of the consumers as having an **endowment** of goods to start with. We denote the bundle a consumer starts with as (ω_1, ω_2).

Those little letters are actually each a lowercase greek omega rather than a w. But if you want to think of them as w, that's fine. The whole thing is made up anyway. In fact, I'll probably waver between ω and w in these notes since I often forget to type "\omega".

w_1 is the endowment of good 1. w_2 is the endowment of good 2.

Let's look at an example. Suppose we have an apple farmer who grows apples but consumes apples and crusts., x_1 is apples. x_2 is crusts. Endowment might be $\omega_1 = 10$, $\omega_2 = 0$. This says, the apple farmer stats with 10 apples and zero crusts.

We now construct a new budget equation for this consumer using the following intuitive constraint: the cost of their chosen bundles must equal the value of their endowment. Formally:

$$p_1 x_1 + p_2 x_2 = p_1 w_1 + p_2 w_2$$

For our apple farmer:

$$p_1 x_1 + p_2 x_2 = p_1 10$$

Notice how income now reacts to changes in prices. If the price of p_1 goes up, income increases as well! Income is endogenous to the model. The endowments are exogenous.

8.1 Gross Demand vs. Net Demand

We can re-write the above budget equation into another useful form.

$$p_1 (x_1 - w_1) + p_2 (x_2 - w_2) = 0$$

Notice here we use $x_1 - w_1$ and $x_2 - w_2$. These are called the **net demands.** In contrast, we call x_1 and x_2 the **gross demands**. This form of the budget says the cost of the net demand must be zero.

When $x_i - w_i > 0$ we say the consumer is a **net demander (buyer)** of that good. When $x_i - w_i < 0$ we say they are a **net supplier (seller)**. Notice that the fact that the cost of net demand must be zero implies that if a consumer is a net demander of one good, they are a net supplier of the other.

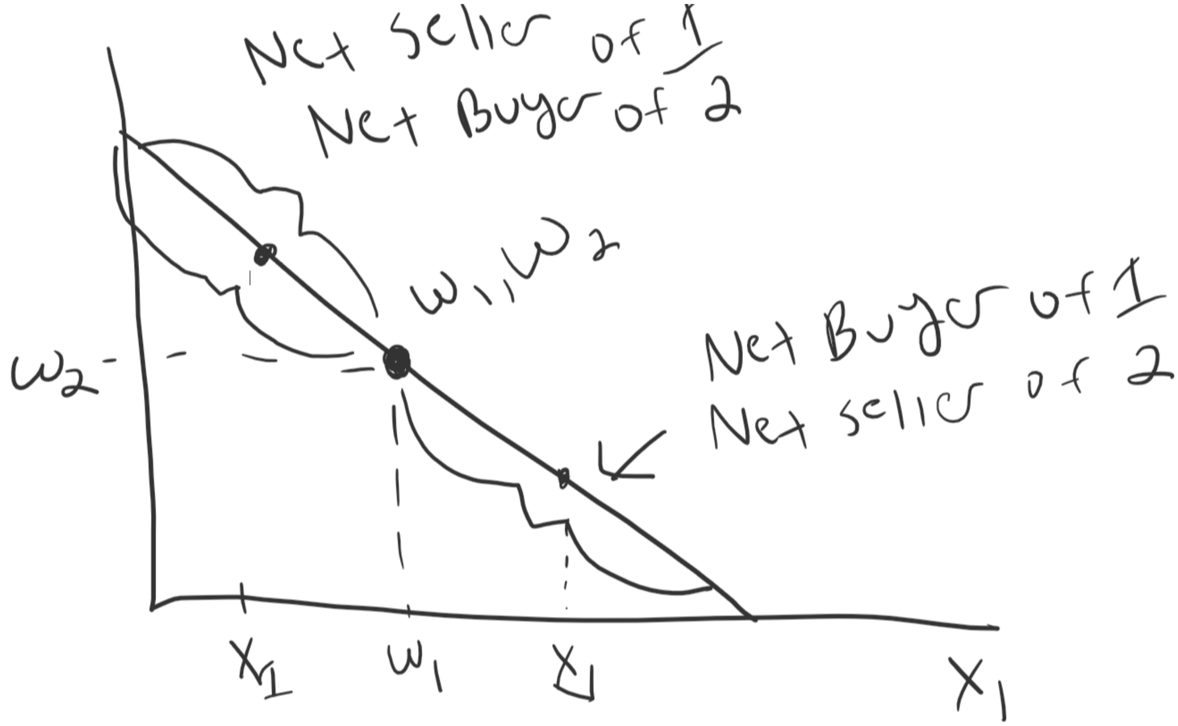

Figure 8.1: Regions where a consumer is a net buyer/seller.

8.2 Drawing the Budget Line and Changes to Price

Before, when income was exogenous, income did no react to changes in price. Now however, a change in price changes both the slope of the budget equation but also the income. This changes how the budget line reacts to changes in price.

An easy way to get the changes right is to make sure that the budget line *always passes through the endowment point*. It has to. The endowment is always available to the consumer. Look what happens when the price p_1 changes in the diagram below. Suppose the black line is the original budget. When p_1 increase, the slope must become steeper, but the line must still pass through the endowment. We end up with the blue budget. On the other hand, if p_1 decreases, we get the orange budget.

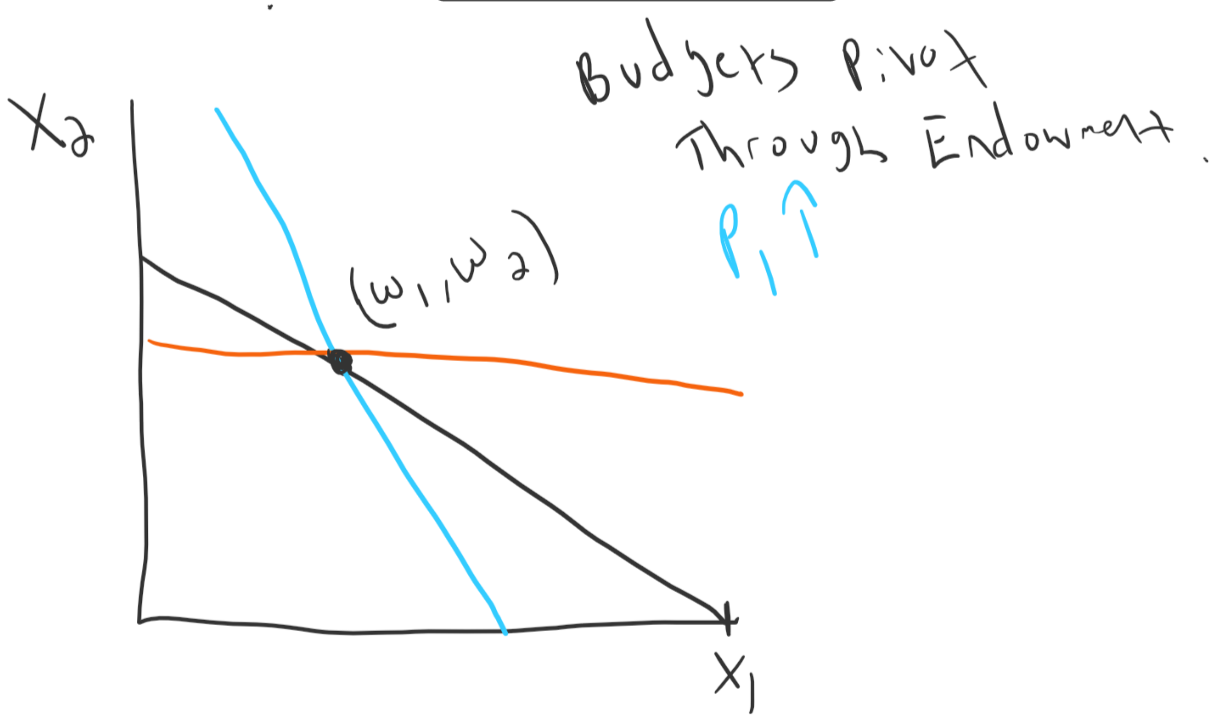

Figure 8.2: The budget line always passes through the endowment (w_1, w_2). If prices change, the slope changes, and the budget pivots through this point. p_1 increase (or p_2 decrease) is shown in blue. p_1 decrease (or p_2 increase) is shown in orange.

It is a good exercise to calculate the intercepts of the new budget equation. The x_1 intercept measure the amount of x_1 a consumer can afford if they only buy x_1. Plug in 0 for x_2 into the budget and solve for x_1:

$$p_1 x_1 + p_2 (0) = p_1 w_1 + p_2 w_2$$

$$x_1 = \frac{p_1 w_1 + p_2 w_2}{p_1} = w_1 + \frac{p_2 w_2}{p_1}$$

The x_2 intercept can be found analogously;

$$x_2 = \frac{p_1 w_1 + p_2 w_2}{p_2} = \frac{p_1 w_1}{p_2} + w_2$$

8.3 Price Changes and Net Buyers/Sellers

The way the budget equation reacts to changes in prices means we can often say something interesting about what happens to demand when prices change.

For a consumer who is a **net buyer of a good**, if the price of that good **decreases**, they will remain a **net buyer** and be **strictly better off**.
For a consumer who is a **net seller of a good**, if the price of that good **increases**, they will remain a **net seller** and be **strictly better off**.

I discussed the proof of these in class. *The proof is closely related to the proof of the fact that the substitution effect must be negative.* See if you can figure how why.

8.4 Example Problem

Let's work an example of an apple farmer, who only consumes apple pies and is endowed only with apples. Endowment is $\omega_1 = 10$, $\omega_2 = 0$. Prices are $p_1 = 1, p_2 = 1$. Preferences are $u = min\left\{\frac{1}{2}x_1, x_2\right\}$.

Let's set up the budget:

$$1x_1 + 1x_2 = 1\,(10) + 1\,(0)$$

$$x_1 + x_2 = 10$$

Now fund the no waste condition:

$$\frac{1}{2}x_1 = x_2$$

Combining this with the budget equation and solving for x_1 :

$$x_1 + \frac{1}{2}x_1 = 10$$

$$x_1 = \frac{20}{3}$$

Now plugging this back into the no waste condition gives:

$$x_2 = \frac{1}{2}\,(x_1) = \frac{10}{3}$$

So the optimal bundle is $\left(\frac{20}{3}, \frac{10}{3}\right)$.

9 Intertemporal Choice (Chapter)

In the last chapter, we learned about how we can endogenize income by assuming that a consumer has an *endowment* of goods. The goal of this chapter is to show that our model so far is more flexible than it might first seem. We can use what we have learned so far to build a very acceptable model of borrowing and saving behavior. We call this *intertemporal* (between times) choice.

9.1 Bundles (Consumption Today, Consumption Tomorrow)

To move from an endowment of goods to a model about borrowing and saving, we first need to define the goods we will use. Here, we will construct a two period model. Our **goods** will be c_1, c_2- consumption on in period 1 and consumption in period 2 respectively. A bundle is (c_1, c_2).

Endowments in this model will represent income in period 1- m_1, and income in period 2- m_2. As usual, the endowment is always affordable.

9.2 Prices (Interest Rate)

We also need to define prices. Price in this model will always be measured in terms of consumption in some period. The price of consumption in one period relative to another will depend on r the interest rate the consumer can borrow or save at.

If the consumer wants to borrow \$1000 in period 1, they pay back $1000\,(1+r)$ in period 2: $1000 + 1000\,(r)$. The cost of consumption in period 1 is $(1+r)$ times the cost of consumption in period 2. Similarly if they save \$1000 in period 1, they get back: $1000\,(1+r)$ in period two.

9.3 Budget Constraint (Future Value Version)

We can use the interest to builds the budget constraint. Suppose the consumer saved mone in period 1. Consumption in period 2 is their income in period 2 (m_2) plus how much they saved in period 1: $(m_1 - c_1)$ multiplied my $1 + r$. This gives us the amount they can consume in period 2:

$$c_2 = m_2 + (1 + r)(m_1 - c_1)$$

Suppose instead they borrowed money in period 1. Consumption in period 2 is income in period 2: m_2 minus the amount they have to pay back to cover their loan from period 1. This is given by:

$$c_2 = m_2 - (1 + r)(c_1 - m_1) = m_2 + (1 + r)(m_1 - c_1)$$

These are exactly the same equation. So we can use the single equation $c_2 = m_2 + (1 + r)(m_1 - c_1)$ to represent all the bundles available to them regardless of whether they borrow or save. We can also transform this equation to make it look a little more like one we are used to:

$$c_2 + (1 + r)c_1 = m_2 + (1 + r)(m_1)$$

This looks a lot like the budget equations we used in the last chapter. In this case, the prices are measured in terms of consumption in period 2 since the consumption in period 2 has the price of 1. If the price of consumption in period one is 1 then the price of consumption in period two is $(1 + r)$. Since prices are measured in terms of period two, we call this the "**future value**" version of the budget equation.

We use "future value" here the same way it is used in accounting. For instance, with this budget equation, we can easily calculate the **future value of income**. This tells us how much c_2 the consumer can consume if they only consume c_2. That is, it tells us the value of their stream of income measured totally in terms of consumption in period 2 . Plugging in $c_1 = 0$ we get:

$$c_2 = m_2 + (1 + r)m_1$$

We can also divide both sides of the budget equation by $(1 + r)$ to transform the budget equation in to a present value version where the price of c_1 is 1 and so everything is measured in terms of period one consumption. This is:

$$c_1 + \frac{c_2}{1 + r} = m_1 + \frac{m_2}{1 + r}$$

We can use this to easily find the **present value of income.** Plug in $c_2 = 0$ and we get:

$$c_1 = m_1 + \frac{m_2}{(1 + r)}$$

For some intuition about this, notice if the consumer takes out a loan of $\frac{m_2}{(1+r)}$ they will owe the bank m_2 in the next period which is exactly their income income. This is the biggest loan they can take out in period 1.

9.4 Plotting the Budget Equation.

We are used to seeing budget equations like this:

$$p_1x_1 + p_2x_2 = p_1w_1 + p_2w_2$$

Our now look like (either):

$$c_1 + \frac{c_2}{1 + r} = m_1 + \frac{m_2}{1 + r}$$

$$(1 + r)c_1 + c_2 = (1 + r)m_1 + m_2$$

For both of these, the ratio of prices. In either case that ratio is $(1+r)$, so the slope of the budget equation in $-(1+r)$. This should make sense. Consumption in period one is more expensive. To get one more dollar of consumption in period 1 you have to give up $(1+r)$ dollars of consumption in period 2.

So we have the slope, now we just need the intercepts. However, we already found these. The intercepts are always "how much of _____ can I have if I only have ____". In this case the c_1 intercept is the present value of income and the c_2 intercept is the future value of income.

On the budget equation, we should also plot the endowment. Like in the previous chapter, any bundle to the right of the endowment on the budget line is a bundle where the consumer is "net buyer" of consumption in period one. In this case, we call that person a **borrower.** If the consumer chooses a point to the left of the endowment, they are a net buyer of consumption in period two. We call them a **saver** or **lender.** I have plotted an example budget equation below.

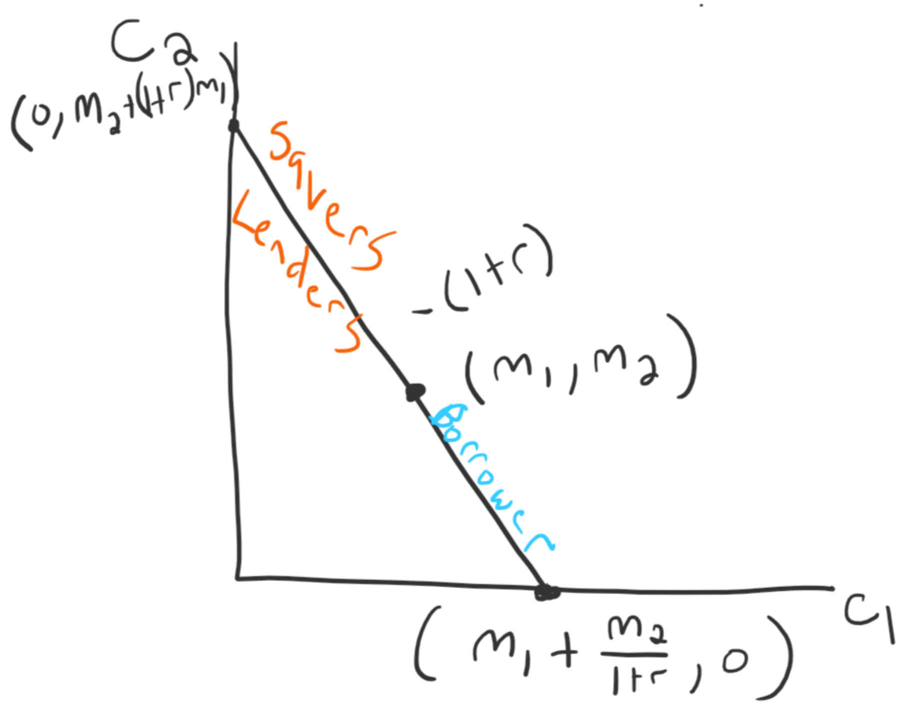

Figure 9.1: The budget equation for an intertemporal choice problem.

9.5 Comparative Statics

As in the last chapter, there are some instances where we know how a consumer will react to changes in prices.

A borrow, when the interest rate goes down, remains a borrower and is strictly better off.
Because they are a "net buyer" of "good 1" and an interest rate decrease is really a decrease in the price of consumption in period 1, we can use the result of the last chapter to show this.

A saver, when the interest rate goes up, remains a saver. And must be strictly better off.
A saver is really a net seller of c_1. If the price of c_1 goes up because the interest rate increased, they will remain a seller.

9.6 Example Problem

We can now work through an example problem. Once we set up the budget equation, there really is not much difference between solving these problems and solving the ones in the last chapter.

Suppose income is: $m_1 = 200.m_2 = 600$. The interest rate is a (very high) $r = \frac{1}{2}$. Utility is $U(c_1, c_2) = c_1 c_2$.

Write down the budget equation:

$$(1+r)\,c_1 + c_2 = (1+r)\,m_1 + m_2$$

Plugging in incomes and interest rate:

$$(1.5)\, c_1 + c_2 = (1.5)\,(200) + 600$$

Let's calculate the present value of income. Set $c_2 = 0$:

$$(1.5)\, c_1 = (1.5)\,(200) + 600$$

$$c_1 = 600$$

Let's calculate the future value of income. Set $c_1 = 0$:

$$c_2 = (1.5)\,(200) + 600$$

$$c_2 = 900$$

Now let's find the optimal consumption. The equal slope condition is: $MRS = -(1 + r)$. Plugging in for these:

$$-\frac{c_2}{c_1} = -1.5$$

Rearranged, we get:

$$c_2 = (1.5)\, c_1$$

Plug this back into the budget equation to solve for c_1:

$$1.5c_1 + 1.5c_1 = (1.5)\,200 + 600$$

$$c_1 = \frac{900}{3} = 300$$

Plugging $c_1 = 300$ back into either the equal slope of budget condition will give us c_2:

$$c_2 = 450$$

At this interest rate the consumer is a borrower since $c_1 = 300 > 200 = m_1$. We know that if the interest rate were to decrease to $\frac{1}{4}$, the consumer will remain a borrower. Try to confirm this by solving for the actual optimal bundle of c_1 and c_2 when $r = \frac{1}{4}$.

10 Market Demand

We have now spent a lot of time studying individual consumers. Since this is a microeconomics course, that should not be too surprising. However, microeconomics is not only about the choices of individual consumers, one area often studied in microeconomics are choices of individual firms and the small-scale interaction of firms.

Importantly, firms do not serve just one individual. Firms sell to many individuals. So we cannot really start to study firms until we have at least talked a little bit about how we go from individual demands to the aggregated market demands that firms actually care about. In the end a firm really wants to know the total amount of stuff they can sell at certain prices. That will depend on demand of *many* consumers. In this chapter, we move from the demand of an individual to the total demand in a market.

10.1 Adding Demand Curves

Suppose we have n consumers, each with some demand for good 1 and some demand for good 2. When we denote a demand we now have to indicate not only what good it is for but also *who's* demand it is. We do it this way:

Demand of consumer i for good j is written $x_i^j(p_1, p_2, m_i)$.

With this notation, we can **sum over consumers** to get the market or aggregate demand for a particular **good**. For instance, the market demand for good 1 is:

$$X^1(p_1, p_2, m_1, ..., m_n) = \sum_{i=1}^{n} x_i^1(p_1, p_2, m_i)$$

And the market demand for good 2 is:

$$X^2(p_1, p_2, m_1, ..., m_n) = \sum_{i=1}^{n} x_i^2(p_1, p_2, m_i)$$

Note how we are denoting market demands with capital letters to keep them separate from individual demand which use a lower case x.

10.2 Example Cobb Douglass Demand

Let's look at an example of aggregating demand. Suppose we have Cobb Douglass consumers all with the utility function:

$$u_i\left(x_i^1, x_i^2\right) = \left(x_i^1\right)\left(x_i^2\right)$$

Note: the 1 and 2 superscripts are no exponents, but rather the label for the good. The individual demand for consumer i for each good is:

$$x_i^1 = \frac{\frac{1}{2}m_i}{p_1}, x_i^2 = \frac{\frac{1}{2}m_i}{p_2}$$

Market demand for good 1 is the sum of the individual demands:

$$X^1 = \sum_{i=1}^{n}\left(x_i^1\right) = \sum_{i=1}^{n}\left(\frac{\frac{1}{2}m_i}{p_1}\right)$$

Suppose $p_1 = 1$ and $m_1 = 10$, $m_2 = 20$, $m_3 = 30$.

$$X^1 = \left(\frac{\frac{1}{2}10}{1}\right) + \left(\frac{\frac{1}{2}20}{1}\right) + \left(\frac{\frac{1}{2}30}{1}\right) \quad = \quad 30$$

Notice that if we denote M as the sum of the individual incomes so that $M = \sum_{i=1}^{n} m_i$, then M is the total amount of income in the economy. We could re-write the total demand above in terms of M like this:

$$\sum_{i=1}^{n}\left(\frac{\frac{1}{2}m_i}{p_1}\right) = \frac{1}{2}\frac{1}{p_1}\sum_{i=1}^{n} m_i = \frac{\frac{1}{2}M}{p_1}$$

So in this case, the market demand for the good only depends on the aggregate income. It does not matter how income is distributed. Let's look at another example.

Suppose $p_1 = 1$ and $m_1 = 20$, $m_2 = 20$, $m_3 = 20$. The aggregate income is 60, just as it was in the other example above.

$$X^1 = \left(\frac{\frac{1}{2}20}{1}\right) + \left(\frac{\frac{1}{2}20}{1}\right) + \left(\frac{\frac{1}{2}20}{1}\right) \quad = \quad 30$$

The demand is the same. This is a pretty nice feature. In fact, if we just gave all the income to one person, we'd have something like $m_1 = 60, m_2 = 0, m_3 = 0$. The demand would still be 30. To determine market demand all we have to do in this example is figure out what the aggregate income is, then imagine giving that whole income to one consumer and asking what that *representative consumer* would choose. Whatever they would choose is the same as the market demand with that aggregate income under any distribution of incomes. This use of a *representative agent* is very common in some macroeconomic models because it vastly simplifies some calculations. But when does this condition hold?

It turns out, it always holds if consumers always have the same **homothetic** preferences.

10.3 Homothetic Preferences.

for any $t \geq 0$:
$\succsim$ is **homothetic if** $x \succsim y$ implies $tx \succsim ty$.
u is **homothetic if** $u(x) \geq u(y)$ **implies** $u(tx) \geq u(ty)$**.**
If u represents $\succsim$ then u is homothetic if and only if $\succsim$ is homothetic.

For example, suppose $(1,2) \succsim (2,1)$ and preferences are homothetic. Then we know $(2,4) \succsim (4,2)$ since these bundles are respective $t = 2$ times $(1,2)$ and $(2,1)$.

In the example in the last section, we saw that consumers with the same cobb dougalss utility function have the representative consumer property, and I told you that this is because the utility function is homothetic. Let's see that it is true here:

Suppose utility is:

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$$

Pick two bundles $(x_1, x_2), (\tilde{x}_1, \tilde{x}_2)$ such that $u(x_1, x_2) \geq u(\tilde{x}_1, \tilde{x}_2)$. Then:

$$x_1^{\alpha} x_2^{\beta} > \tilde{x}_1^{\alpha} \tilde{x}_2^{\beta}$$

Multiply (x_1, x_2) and $(\tilde{x}_1, \tilde{x}_2)$ by t. To show preferences are homothetic, we need to show that:

$$(tx_1)^{\alpha} (tx_2)^{\beta} > (t\tilde{x}_1)^{\alpha} (t\tilde{x}_2)^{\beta}$$

Factor out the t from each variable:

$$t^{\alpha} t^{\beta} \left(x_1^{\alpha} x_2^{\beta} \right) > t^{\alpha} t^{\beta} \left(\tilde{x}_1^{\alpha} \tilde{x}_2^{\beta} \right)$$

Notice that $t^{\alpha} t^{\beta}$ cancels from each side. We get:

$$\left(x_1^{\alpha} x_2^{\beta} \right) > \left(\tilde{x}_1^{\alpha} \tilde{x}_2^{\beta} \right)$$

This is true by our initial asusmption. Thus, Cobb Douglass is homothetic.

10.4 Homotheticity via MRS

Showing a utility function is homothetic as we did in the previous example can be tedious, but there is an easier way to test if preferences are homothetic.

For a utility function of two goods where the MRS is well defined, preferences are homothetic if and only if the MRS depends only on the ratio of goods but not the amount of either good.

Let's try this for the Cobb Douglas example above. The MRS is:

$$-\frac{\frac{\partial \left(x_1^{\alpha} x_2^{\beta} \right)}{\partial x_1}}{\frac{\partial \left(x_1^{\alpha} x_2^{\beta} \right)}{\partial x_2}} \quad = \quad -\frac{\alpha x_2}{\beta x_1}$$

This only depends on the ratio of good (and the parameters α and β). For instance, the MRS for $(1, 1)$ is $-\frac{\alpha}{\beta}$ and the MRS for $(2, 2)$ is $-\frac{\alpha}{\beta}$.

Here are some non-homothetic preferences:

$$u = x_1 + \sqrt{x_2}$$

Let's check the MRS:

$$-\frac{\frac{\partial(x_1+\sqrt{x_2})}{\partial x_1}}{\frac{\partial(x_1+\sqrt{x_2})}{\partial x_2}} = - \quad 2\sqrt{x_2}$$

This depends on the absolute level x_2, not the ratio of x_1 and x_2. It is not homothetic.

The fact that the MRS is the same for the same ratio of goods in a homothetic utility function implies something nice about the indifference curves:

The indifference curves of a homothetic utility function are parallel along a ray through the origin.

That is, draw any line emanating from the point $(0, 0)$. At every point on that ray, the slope of the indifference curve at that point is exactly the same.

Another nice fact about homothetic prefrences is that:

Consumers with homothetic preferences will always have linear Engle curves.

I'll prove that to you in class.

10.5 Elasticity

Because we are now working with many different consumers, who may have wildly different levels of demand for a good, it is nice to have a way of comparing consumer demand that does not depend on the levels of demand.

For example, suppose the price of a good changes from 1 to 2. Consumer 1's demand changes from 100 to 50 and consumer 2's changes from 10 to 5. Their behavior in terms of absolute changes in demand $\frac{\Delta x_i}{\Delta p_i}$ is wildly different, but their behavior in terms of percentage terms $\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}}$ is identical. Elasticity is simply a way of quantifying comparative statics in **unit-free percentage terms.**

Let's look at this for both consumers above:

$$\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}} = \frac{\frac{100-50}{100}}{\frac{1-2}{1}} = -\frac{1}{2}$$

$$\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}} = \frac{\frac{10-5}{10}}{\frac{1-2}{1}} = -\frac{1}{2}$$

10.6 Price Elasticity

In the example above, we used absolute percent changes to compare the consumer demands: $\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}}$. For very small absolute changes, we can measure the same thing through derivatives, change the Δ to ∂ and we have it.

Price elasticity of demand is given by:

$$\epsilon_{i,i} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_i}{p_i}} = \frac{\partial x_i}{\partial p_i}\frac{p_i}{x_i}$$

Price elasticity measures (roughly) the percent change in demand of a good for a one percent increase in price of that good.

Cross-price elasticity is:

$$\epsilon_{i,j} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$$

Cross price elasticity measures (roughly) the percent change in demand of a good for a one percent increase in price of some other good.

Here is an example. Suppose utility is $u = x_1 x_2$. Demand for good 1 is $x_1 = \frac{\frac{1}{2}m}{p_1}$.

Price Elasticity is:

$$\epsilon_{1,1} = \frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial p_1}\frac{p_1}{\frac{\frac{1}{2}m}{p_1}}$$

$$= -\left(\frac{\frac{1}{2}m}{p_1^2}\right)\frac{p_1}{\frac{\frac{1}{2}m}{p_1}}$$

$$= -\left(\frac{\frac{1}{2}m}{p_1^2}\right)\frac{p_1^2}{\frac{1}{2}m}$$

$$= -1$$

This is what we call **constant unit-elastic demand** because the elasticity is always the same (constant) and is always -1 (unit). In this case, we would say "a 1% increase in price always leads to a 1% decrease in demand".

If elasticity is less than -1 (For instance -2.) we say demand is "**Elastic**". A 1% increase in price leads to a more than 1% decrease in demand.
If elasticity is more than -1 (For instance $-\frac{1}{2}$.) we say demand is "**Inelastic**". A 1% increase in price leads to a less than 1% decrease in demand.

Can you think of some goods that have elastic demand? Inelastic demand?

10.7 Cross-Price Elasticity:

We defined cross-price elasticity above, but let's look at some examples. Recall that the cross price-elasticity for good 1 with respect to price 2 is:

$$\epsilon_{1,2} = \frac{\partial\left(x_1\right)}{\partial p_2}\frac{p_2}{x_1}$$

For the cobb-Douglass utility function we looked at above, demand for good 1 is: $\frac{\frac{1}{2}m}{p_1}$. Notice, this does not depend on p_2 at all. In fact:

$$\epsilon_{1,2} = \frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial p_2}\frac{p_2}{x_1} = 0$$

Because demand for one good does not depend on the price of another, the cross-price elasticities are zero for **Cobb Douglass.**

Let's try a less trivial one now. S*uppose utility is:* $u = min\left\{x_1, x_2\right\}$. Optimal demand for good 1: $x_1 = \frac{m}{p_1+p_2}$. This now depends on p_2. The cross price elasticity is:

$$\epsilon_{1,2} = \frac{\partial\left(\frac{m}{p_1+p_2}\right)}{\partial p_2}\frac{p_2}{\frac{m}{p_1+p_2}}$$

$$\epsilon_{1,2} = -\frac{p_2}{p_1+p_2}$$

To interpret this, when price p_2 increases by 1%, demand for x_1 goes down by $\frac{p_2}{p_1+p_2}$%. For example, suppose $p_1 = p_2 = 1$. Then $\epsilon_{1,2} = -\frac{1}{2}$. If price of p_2 goes up by 1%, demand for x_1 goes down by $\frac{1}{2}$%. Think about why that would be for this case of perfect complements at these prices.

10.8 Income Elasticity

There is another type of elasticity we can talk about as well. **Income elasticity.** Roughly,

> Income elasticity measures: *the percent that demand changes when we increase income by 1%.*
> The formula is: $\eta_i = \frac{\partial x_i}{\partial m}\frac{m}{x_i}$

Let's have a look at our usual cobb douglass demand example. Demand for good 1 is $x_1 = \frac{\frac{1}{2}m}{p_1}$. The income elasticity is:

$$\eta_i = \frac{\partial\left(\frac{\frac{1}{2}m}{p_i}\right)}{\partial m}\frac{m}{\frac{\frac{1}{2}m}{p_i}}$$

$$= \frac{1}{2p_i}\frac{m}{\frac{\frac{1}{2}m}{p_i}}$$

$$= \frac{1}{2p_i}\frac{2p_i}{1}$$

$$\eta_i = 1$$

When income goes up by 1%, demand will go up by 1%. In fact, this will hold for *any* cobb douglass utility function and for either good:

Suppose $u = x_1^\alpha x_2^\beta$:

$$x_1 = \frac{\frac{\alpha}{\alpha+\beta} * m}{p_1}, x_2 = \frac{\frac{\beta}{\alpha+\beta} * m}{p_2}$$

$$\eta_1 = \frac{\partial\left(\frac{\frac{\alpha}{\alpha+\beta}*m}{p_1}\right)}{\partial m}\frac{m}{\frac{\frac{\alpha}{\alpha+\beta}*m}{p_1}} = 1$$

$$\eta_2 = \frac{\partial\left(\frac{\frac{\beta}{\alpha+\beta}*m}{p_2}\right)}{\partial m}\frac{m}{\frac{\frac{\beta}{\alpha+\beta}*m}{p_2}} = 1$$

Let's look at one where the income elasticity is not 1. Suppose demand is $\frac{\frac{1}{2}m^2}{p_1}$. Here we get:

$$\eta = \frac{\partial\left(\frac{\frac{1}{2}m^2}{p_1}\right)}{\partial m}\frac{m}{\left(\frac{\frac{1}{2}m^2}{p_1}\right)} \quad = \quad 2$$

In this case, we have a **constant income elasticity** of 2 which means that if income increases by 1%, demand increases by 2%.

Finally, let's look at an example where the elasticity is not constant. Suppose demand is $\frac{\log(m)}{p_1}$:

$$\eta = \frac{\partial\left(\frac{\log(m)}{p_1}\right)}{\partial m}\frac{m}{\left(\frac{\log(m)}{p_1}\right)} \quad = \quad \frac{1}{\log(m)}$$

In this case, if income increases by 1%, demand increases by $\frac{1}{log(m)}$%, which depends on the income level itself.

11 Equilibrium

At this point in our study of microconomics, we have focused only on consumer choice behavior- working through this topic from a very fundamental level. If you have followed along, you can now start with a description of consumers' preferences and a budgets and work out those consumers demands, then aggregate those demands into a market or aggregate demands. You might think the next step is to study the producers and work through their choices from the same fundamental level. We will do this, but first, it is nice to take a step back and look at markets again, using a little of what we have learned so far.

In studying two consumer choices, we have tended to look at models with two goods. We need two goods to model trade-offs and actually derive demands. However, when we are ready to study markets themselves, as long as we have those demands in-hand, we do not need to deal with two markets at a time, we can zoom in on a market for a single good. Studying one market at a time is called: *Partial Equilibrium.* It is partial, because we know that changes in the price of good in whatever market we study will demand in other markets (as long as goods are either substitutes or complements). When we study multiple markets at the same time and consider these spillover effects, we call that studying *General Equilibrium.* Let's save that for the end of the semester.

11.1 Market Demand/Supply

A market has two sides. The consumers demand the good, the producers supply the good. The description of *how much* each side respectively demands or supplies are given by the demand/supply function:

Definitions:

Market demand $Q_d(p)$ (what is the total amount demanded at price p).

Market supply $Q_s(p)$ (what is the total amount supplied at price p).

Inverse market demand: $p_d(Q)$ (at what price are Q units demanded?)

Inverse market supply: $p_s(Q)$ (at what price are Q units supplied?)

Example. Linear Demand. Suppose market Demand is $Q_d = 1000 - p$. Then, inverse market demand is $p = 1000 - Q_d$.

Example. Cobb Douglass. Suppose the consumers in a market all have utility $x_1 x_2$. They all demand $\frac{\frac{1}{2}m_i}{p_1}$ units of x_1. In total they demand $Q_d = \frac{\frac{1}{2}M}{p}$ where $M = \sum_{i=1}^{n} m_i$ (the aggregate income). The inverse demand is: $p = \frac{\frac{1}{2}M}{Q_d}$.

11.2 What is an equilibrium?

An equilibrium in a market is defined as a price p^* such that supply equals demand. Formally, it is a p^* such that $Q_d(p^*) = Q_s(p^*)$.

We tend to analyze markets at their equilibrium (in our case there will only be one possible equilibrium). This is because a market is at rest when the price reaches equilibrium. There is no pressure on prices to either increase or decrease. This is a convenient way to study markets— the same way that it's easier for a dentist to clean your teeth if you aren't jogging.

To see why an economy is at rest at equilibrium, suppose at some price p, supply exceeds demand:

$$Q_s(p) > Q_d(p)$$

There are **surplus** units of the good, some firms did not get to sell. A firm would rather sell their excess unit rather than have it go to waste, even if that means selling at a lower price than p. There are no consumers left who are willing to buy at a price of p, but there are consumers willing to buy at a price lower than p. Thus, the firm can offer their unit for sale at a lower price, this will make them better off and some consumer better off. However, this creates **downward pressure** on prices as firms start to list their units of the good at these lower prices.

Suppose demand exceeds supply:

$$Q_d(p) > Q_s(p)$$

In this case, there is a **shortage.** Some consumers are willing to buy at a higher price and there are some firms willing to sell at that higher price. There is **upward pressure** on prices.

In either case, there is pressure for prices to move. At the equilibrium price however, there is no consumer willing to buy at a higher price, or no firm willing to sell at a lower price.

11.3 Solving for an Equilibrium.

Solving for equilibrium prices is straight-forward. Simply use the condition $Q_d(p^*) = Q_s(p^*)$ and solve for the resulting price.

Example. Suppose demand is $Q_d = \frac{500}{p}$ and supply is $Q_s = 100p$. Let's set these equal and solve for p.

$$\frac{500}{p} = 100p$$

$$p^* = \sqrt{5}$$

This is the equilibrium price. To get equilibrium quantity, plug into either supply or demand. We should get the same thing:

$$Q_s = 100\left(\sqrt{5}\right)$$

$$Q_d = \frac{\frac{1}{2}1000}{\sqrt{5}} = \frac{500}{\sqrt{5}} = 100\frac{5}{\sqrt{5}} = 100\sqrt{5}$$

This is the equilibrium quantity:

$$Q^* = 100\sqrt{5}$$

11.4 Fixed Supply

We have not talked at all about how we would actually derive market supply. In the problem above, we had $Q_s = 100p$. Where would a market supply like this actually come from? We will see that in the next few chapters.

However, there are some types of supply that are easy to understand without doing anything formal. The simplest of these is **fixed** supply**.** With fixed supply, the Q_s is constant for any price. The inverse supply curve (the thing we plot on the "equilibrium graph" is a vertical line). This would be the case, for instance, with concert tickets. The size of the venue is fixed regardless of the price of tickets.

As an example, suppose there was a fixed supply of 1000 and demand is $Q_d = \frac{500}{p}$. The equilibrium price is whatever price will get consumers to demand 1000. In this case, that $p^* = \frac{1}{2}$.

11.5 Effect of a Tax

Once we can solve for equilibrium, we might want to analyze how different types of policies affect an market. The key one we will study in this course is a tax.

Suppose the government imposes a tax of t per unit of good. In this case, we should think of the price in the market p as the "sticker price" of the good. p is the price firms receive when the good is purchased, but consumers have to pay $p+t$, and the t goes to the government. In this sense, while p is the relevant price for the firms supply function, $p+t$ is now the relevant price for consumers. That is what they actually have to pay[2]. This gives us a new equilibrium condition:

$$Q_s(p) = Q_d(p+t)$$

Example.

Suppose demand is $Q_d = 300 - 50p$ and supply is $Q_s = 100p$. Let's first solve for the equilibrium without a tax. Set $Q_s = Q_d$:

$$100p = 300 - 50p$$

[2]We could alternatively think of p as the price consumers actually pay (this would be the case when tax is included in the posted price). Then the firm gets $p - t$. It turns out, these will be exactly the same. Since the "sticker price" is sort of what we are used to when we go to the store, I will use that for the examples.

This gives us:

$$p^* = 2$$

$$q^* = 200$$

Now, suppose the government adds a tax of $t = 1$. Consumers will pay $p+1$ since they pay the "sticker price" p plus the tax $t = 1$. In equilibrium, demand still needs to equal supply. Let's set these equal keeping in mind $p+1$ is now the relevant price for consumers.

$$300 - 50(p+1) = 100p$$

$$p^* = \frac{5}{3}$$

In this case, in equilibrium, suppliers get $p = \frac{5}{3}$ per unit and consumers pay $p + t = \frac{5}{3} + 1 = \frac{8}{3}$ per unit. To get the market quantity, plug $p = \frac{5}{3}$ back into the supply function:

$$Q_s = 100\left(\frac{5}{3}\right) = \frac{500}{3}$$

Let's check that consumers actually demand $\frac{500}{3}$. Plugging $\frac{8}{3}$ into the demand function:

$$Q_d = 300 - 50\left(\frac{8}{3}\right) = 300 - \frac{400}{3} = \frac{900-400}{3} = \frac{500}{3}$$

Notice, the effect of the tax is that the new equilibrium has a lower quantity. Consumers pay more then they used to and suppliers receive less than they used to. Both are worse off. How should we quantify "worse off" though? To do this, we use a concept called *surplus*.

11.6 Surplus

Consumer surplus is a measure of welfare that tells us how much "better-off" consumers are because the market sells them some quantity q of a good at price p.

To motivate how we measure surplus, think of the height of the inverse demand function at a point as being the price *some* consumer is willing to pay for a unit of that good. The difference between that height and the price is the difference between what they would pay and what they have to pay. That difference is a measure of that consumers surplus- how happy the are to pay less than they are willing to.

"Summing" over all the consumers who actually buy the good gives that area below the inverse demand curve and above price. This is effectively adding up all the surplus of the consumers who buy the good. The same argument motivates the area above the inverse demand and below price as being the producer surplus. The inverse supply represents the price some firm would be willing to take to sell a unit of the good. The difference between price and that willingness is their surplus.

In summary, **consumer surplus** is the area below the inverse demand curve but above price from 0 to the equilibrium quantity q^*. Analogously, the **producer surplus** is the area above the inverse supply curve but below price from 0 to q^*.

It is useful to draw the inverse supply and demand curves to calculate these areas. I have plotted this below. If demand and supply are linear, we can calculate them without using an integral since the area is just a triangle. In the case of the example above, we had demand is $Q_d = 300 - 50p$ and supply is $Q_s = 100p$. Thus inverse demand is $p = 6 - \frac{q}{50}$ and inverse supply is $p = \frac{q}{100}$. Equilibrium (without a tax) price was $p^* = 2$ and quantity was $q^* = 200$. Notice that the inverse demand has a intercept at 6. Thus, the area of consumer surplus is a triangle with points $(0,2), (0,6), (2,200)$. This is a triangle with base of 200 and a height of 4. The area is:

$$\frac{1}{2}(4 * 200) = 400$$

The producer surplus is a triangle with points $(0,0),(2,0),(2,200)$. This is a triangle with base of 200 and height 2. The area of consumer surplus is:

$$\frac{1}{2}(2*200) = 200$$

Total welfare is the sum of consumer and producer surplus. In this case it is 600.

z

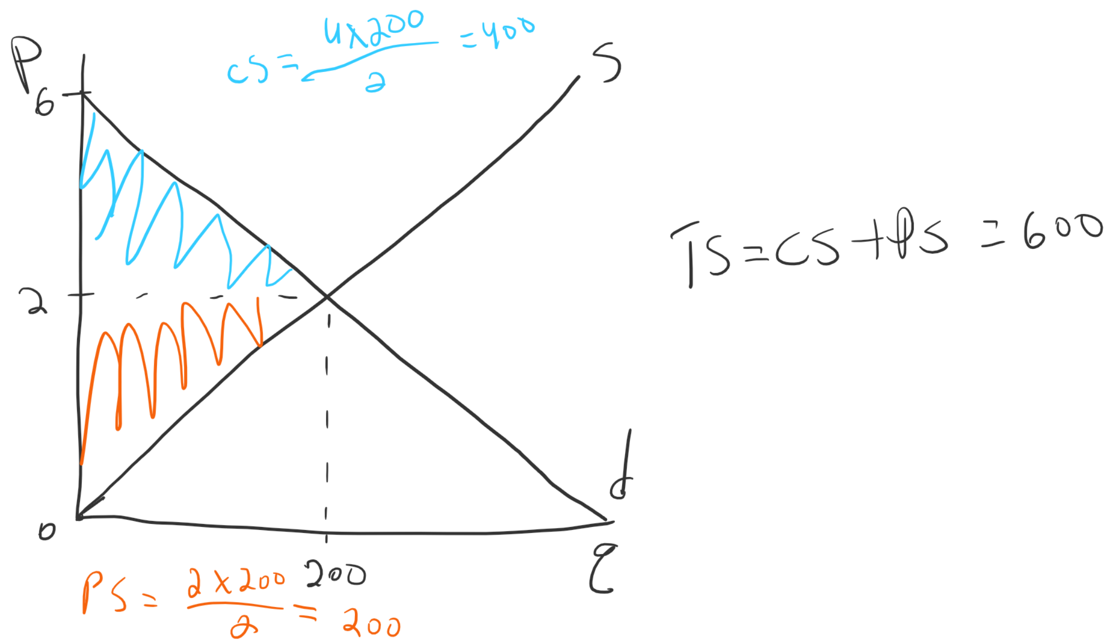

Figure 11.1: Calculating Consumer and Producer Surplus

11.7 Pareto Efficiency

Look at the equilibrium graph for the last problem. Start at equilibrium. Could we make any firm and consumer happier at the same time? Imagine we were to lower quantity. In this case, we move to the left of the equilibrium point. The inverse demand is above the inverse supply. There was previously some consumer buying the good at a price below their willingness to pay and some firm selling at a price higher than their willingness to pay. Both this consumer and this firm were happy having transacted, but now we removed that transaction. Thus, reducing quantity from equilibrium does not make anyone better off.

Let's try moving to the right of equilibrium. We have to increase quantity above the equilibrium level. But, to the right of equilibrium, inverse demand is below inverse supply. Thus, to sell additional units, it has to be at some price that is either above what a consumer is willing to spend or below the price some firm is willing to accept. To sell more than the equilibrium quantity, we have to make *someone* worse off.

In fact, equilibrium is the only point where we **can not make someone better off without making someone else worse off.** This condition is called **pareto efficiency**. The equilibrium is the only pareto efficient point.

11.8 Deadweight Loss

Above, we discussed the notion of Pareto efficiency. **Deadweight loss** is a measure related to the loss of efficiency due to a market being out-of-equilibrium. It is the measured by the difference in total surplus from some out-of-equilibrium point to the equilibrium point. For instance, from a previous problem, we calculated the equilibrium under a tax of size 1. In equilibrium, the total surplus was 600. Let's calculate it under the tax. I have plotted the equilibrium with tax $t = 1$ below.

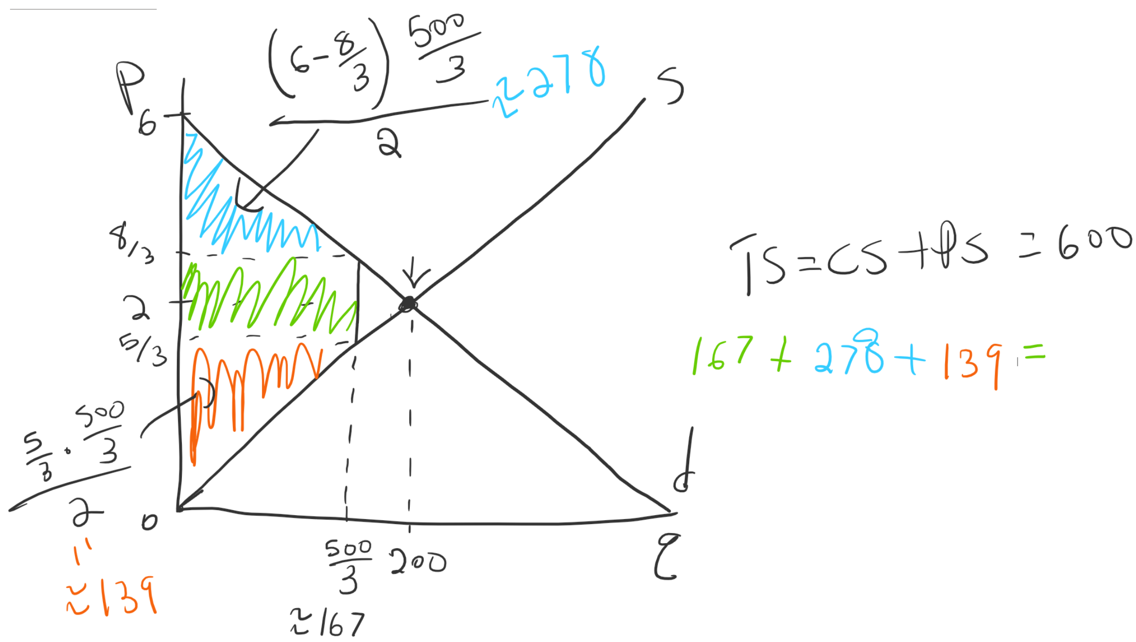

Figure 11.2: Calculating Surplus For Previous Example

The consumer surplus under the tax:

$$\frac{\left(6-\frac{8}{3}\right)\frac{500}{3}}{2} = \frac{2500}{9} = 277.778$$

The producer surplus under the tax:

$$\frac{\left(\frac{5}{3}\right)\frac{500}{3}}{2} = \frac{1250}{9} = 138.889$$

When there is no tax, the total surplus is the sum of producer and consumer surplus. But with a tax, the tax revenue can be considered surplus as well. It is sort of the governments surplus. It will always be the tax t times the quantity: $t\left(q\right)$. In this case, the government revenue under the tax is:

$$1\left(\frac{500}{3}\right) = 166.667$$

The total surplus under the tax is:

$$\frac{2500}{9}+\frac{1250}{9}+\frac{500}{3.0} = 583.333$$

Compare this to the original surplus which was 600. The dead-weight loss is the difference.

$$600 - 583.333 = 16.667$$

We can also find this by the area of the missing triangle in the chart above:

$$\frac{\left(\frac{8}{3}-\frac{5}{3}\right)\left(200-\frac{500}{3}\right)}{2} = 16.6667$$

11.9 Tax Burden

Notice how the tax in our problem above lowered the surplus of consumers and producers. After a tax is imposed, consumers will pay more and producers will receive less than they did without the tax. Sometimes, a tax has a bigger impact on consumer welfare and sometimes on producer welfare. Calculating **tax burden** or **tax incidence** is the process of calculating who ends up "paying" for the tax.

From the previous problem, after the tax is imposed, consumers pay $\frac{8}{3}$. They paid only 2 before the tax. We say that **the burden on consumers** is $\frac{8}{3} - 2 = \frac{2}{3}$.

Producers used to get 2 per unit, but after the tax they only get $\frac{5}{3}$. The burden of the tax on producers is $\frac{1}{3}$.

Notice that the burden on consumers plus the burden on producers adds to 1, the size of the tax. This will always be the case. This also allows us to calculate the proportion of burden. Just divide the burden on each size by the size of the tax.

In this case the consumers bear 66.7% of the tax. Producers bear 33.3% of the tax.

In this case, the consumers bear a little more of the tax than producers. This relative ratio of burden is governed by the relative elasticities of supply and demand. If demand is relative elastic and supply is relatively inelastic, then most of burden will be on producers. This is because, if demand is elastic, suppliers can't "pass on" much of the tax since it will lower quantity demanded too much. When demand is relatively inelastic compared to supply, most of the burden of the tax will be on the consumers. This is because the suppliers can pass on most of the tax to consumers without having a significant decrease in quantity demanded.

12 Technology

In this chapter, we start our study of the production side of an economy. Once we build up a little foundation, producers are going to behave a lot like consumers– just sort of in reverse. Consumers maximize utility given a fixed amount they can spend, producers minimize the amount they spend to produce some fixed amount of output. In a few chapters, we will see how these two problems are ultimately quite closely related.

To start, we need to represent what a producer can do. We do this by defining their **technology.** At very general level we could allow producers to use different kids of inputs, produce different kinds of outputs, maybe produce some kind of intermediate output that they then use to produce other output, and so on. By as with consumers, we can get pretty far using the simplest *interesting* model we can come up with. In this case, that will consist of a technology made up of two kinds of inputs x_1, x_2 and and a single output y. For instance, x_1 and x_2 might be apples and crusts and y would be pies.

Defining the ***technology*** is about describing how the inputs turn into output. We do this through production functions.

12.1 Production Functions

A production function is a mapping from amounts of inputs, in our case an ordered pairs of amounts of both inputs (x_1, x_2), into an amount of output y. Formally, we would define it this way. $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ that is, f is a function that maps every pair of two non-negative real numbers (the input amounts) into a non-negative real number representing the output amount.

Here's an example. Suppose we have a pie producer that uses 1 crust and 2 apples to make each pie. Some points on the production function are $f(1,2) = 1, f(2,4) = 2, f(3,6) = 3$... We can represent the production function more generally as: $f(x_1, x_2) = min\left\{x_1, \frac{1}{2}x_2\right\}$.

Notice that this looks a lot like a utility function. In fact, we used a similar example to motivate the perfect complements utility function. One difference between a production function and a utility function is that the numbers matter. We can't take transformations of this function, because our production function represents something real. A utility function is just a way of representing ordinal preferences. For instance, if we were to multiply the production function above by 2, we would get a technology that only needs $\frac{1}{2}$ crust and 1 apple to make a pie. That's not the same technology.

When we are given a production function, that's the one we are stuck with. No transformations allowed.

12.2 Isoquants

Like utility functions, we can use the "contours" of the production function to understand its shape. For consumers we had indifference curves. For producers we have: **Isoquants**– combinations of input that give you the same amount of output.

For instance, in our example above the input bundles $(1,2), (1,3), (1,4), (2,2), (3,2)$ ... all give 2 pie. They are on the same isoquant.

The isoquants can be graphs with all the same techniques you used to graph indifference curves for consumers. Here are some examples of some common technologies and their associated isoquants.

12.3 Example - Fixed Proportions/Perfect Complements

Fixed proportions / perfect complements production works just like perfect complements utility. The inputs are used in a certain proportion and extra of one of the two inputs does not contribute to additional production. These have the functional form $min\{ax_1, bx_2\}$. The isoquants are L-shapes with the kinks following the line $bx_2 = ax_1$ or more familiarly $x_2 = \frac{a}{b}x_1$ that is, a line through the origin with slope $\frac{b}{a}$. For the case looked at above $f(x_1, x_2) = min\left\{x_1, \frac{1}{2}x_2\right\}$, here are the isoquants.

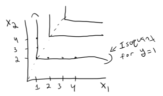

Figure 12.1: Isoquants for fixed-proportion technology.

12.4 Example - Perfect Substitutes

Perfect substitutes represent a technology where the inputs are interchangeable. For instance, maybe there are two types of tools you can use to make the output. The production functions have the functional form $f(x_1, x_2) = ax_1 + bx_2$. The isoquants are straight lines with slope $-\frac{a}{b}$. To see this, take the production function and ask what bundle of inputs produce output y? We get $y = ax_1 + bx_2$ now solve for x_2 to get $x_2 = \frac{y}{b} - \frac{a}{b}x_1-$ a line with slope $-\frac{a}{b}$.

For instance, let's look $f(x_1, x_2) = 2x_1 + x_2$. I have drawn some of the isoquants below:

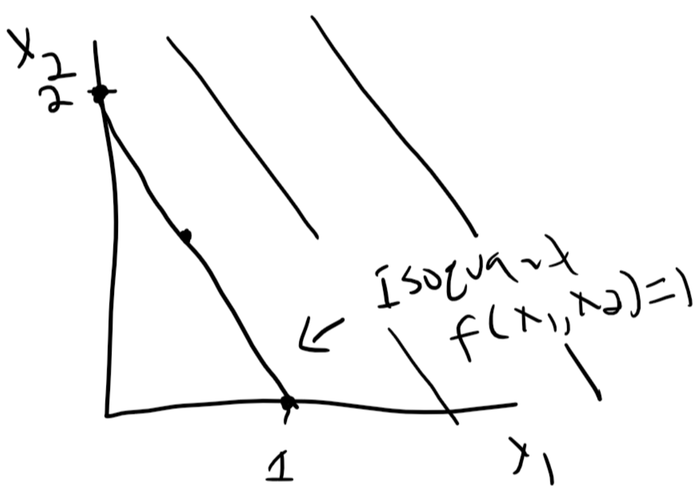

Figure 12.2: Isoquants for perfect substitutes technology.

12.5 Marginal Products

For consumers, we talked a lot about the ratio of partial derivatives, but did not make much of the partial derivatives themselves which are called the *marginal utilities* for consumers. This is because, the actually of the marginal utilities are meaningless for consumers since the actual value of the utility function does not matter. For producers however, the actual value of the production function is meaningful– it is measured in units of output. So here, the partial derivatives are meaningful and we call these the **marginal products.** The marginal product of an input measures how much production will increase if that input is increased a small amount.

Marginal product for good i is This is the partial derivative of the production function with respect to input x_i.

$$MP_i = \frac{\partial f(x_1, x_2)}{\partial x_i}$$

Example $f(x_1, x_2) = 2x_1 + x_2$

$$MP_1 = 2, MP_2 = 1$$

Example $f(x_1, x_2) = (x_1 + x_2)^{\frac{1}{2}}$. (By the way, this production is called the CES production function. CES stands for Constant Elasticity of Substitution. Don't worry too much about what elasticity of substitution is just yet.

$$MP_1 = \frac{\partial \left((x_1 + x_2)^{\frac{1}{2}} \right)}{\partial x_1} = \frac{1}{2} (x_1 + x_2)^{-\frac{1}{2}}$$

$$MP_2 = \frac{1}{2} (x_1 + x_2)^{-\frac{1}{2}}$$

Example $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ *(Cobb Douglass Production)*

$$MP_1 = \frac{\partial \left(x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} \right)}{\partial x_1} = \frac{1}{2} x_1^{\frac{1}{2}-1} x_2^{\frac{1}{2}} = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} = \frac{1}{2} \frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} = \frac{\sqrt{x_2}}{2\sqrt{x_1}}$$

$$MP_2 = \frac{\sqrt{x_1}}{2\sqrt{x_2}}$$

12.6 Diminishing Marginal Product

Diminishing marginal product is the idea that if you increase one of the inputs while holding the other input fixed, the extra output you become smaller and smaller. That is, each input becomes less productive as you increase only that input.

Diminishing marginal product requires that the derivative of the marginal product is negative. Since the marginal product is also a derivative, this is equivalent to the second derivative of the production function being negative.

$$\frac{\partial (MP_i)}{\partial x_i} = \frac{\partial f(x_1, x_2)}{\partial x_i \partial x_i} < 0$$

Let's take a look at the Cobb Douglass example. It has negative marginal products. Let's check it for input 1.

$$MP_1 = \frac{\sqrt{x_2}}{2\sqrt{x_1}}$$

$$\frac{\partial \left(\frac{\sqrt{x_2}}{2\sqrt{x_1}} \right)}{\partial x_1} = -\frac{\sqrt{x_2}}{4x_1^{3/2}} < 0$$

The CES production function $f(x_1, x_2) = (x_1 + x_2)^{\frac{1}{2}}$ also has decreasing marginal product. The marginal product for input 1 is:

$$MP_1 = \frac{\partial \left((x_1 + x_2)^{\frac{1}{2}} \right)}{\partial x_1} = \frac{1}{2} (x_1 + x_2)^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{\sqrt{x_1 + x_2}}$$

By inspection, this decreases as x_1 increases since x_1 only appears in the denominator. We can also check the derivative formally:

$$\frac{\partial \left(\frac{1}{2} \frac{1}{\sqrt{x_1 + x_2}} \right)}{\partial x_1} = -\frac{1}{4 (x_1 + x_2)^{3/2}}$$

As one final example, let's try the: $f(x_1, x_2) = x_1^2 x_2^2$. Marginal product is:

$$\frac{\partial\left(x_1^2x_2^2\right)}{\partial x_1} = 2x_1x_2^2$$

The derivative of marginal product is:

$$\frac{\partial\left(2x_1x_2^2\right)}{\partial x_1} = 2x_2^2 > 0$$

This has increasing marginal product since the marginal product of x_1 is increasing with x_1.

12.7 Technical Rate of Substitution

The slope of the isoquants will be important. The interpretation of will be that the slope of the isoquant measures how much of input x_2 you can give up if you add 1 unit of x_1 (so that you continue producing the same amount of output).

This slope is given by the **Technical Rate of Substitution** which is the ratio of the marginal products. It is analogous to the MRS for consumers.

$$TRS = -\frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}} = -\frac{MP_1}{MP_2}$$

Notice how, like the MRS, this is measuring tradeoffs. In this case the tradeoff a firm is *willing* to make between inputs. Eventually it will play a key role in finding optimal input bundles.

12.8 Returns to Scale

Another type of analysis we can do on the shape of the production function besides looking at the marginal products is to look at the returns to scale. Notice that marginal product was about the shape of the production function as a single input is increased. The idea of returns to sale is about asking what happens to output when we increase both inputs. Specifically, we might ask questions like "if we doubles both inputs would output double? less than double? more than double?"

Let's look at the pie baking production function in the example above with $f\left(x_1,x_2\right) = min\left\{x_1, \frac{1}{2}x_2\right\}$.

1 crust, 2 apples make 1 pie:

$$f\left(1,2\right) = 1$$

Double the input now using 2 crusts, 4 apples. This makes 2 pies.

$$f\left(2,4\right) = 2$$

Doubling the inputs, exactly doubled the output. In fact, this is true generally for the function. We can show it formally this way:

$$f\left(x_1,x_2\right) = min\left\{x_1, \frac{1}{2}x_2\right\}$$

$$f\left(2x_1,2x_2\right) = min\left\{\left(2\right)x_1, \left(2\right)\frac{1}{2}x_2\right\} = 2min\left\{x_1, \frac{1}{2}x_2\right\}$$

We call this **linear returns to scale**. More abstractly, a production function has linear returns to scale if for any $t > 1$

$$f\left(tx_1,tx_2\right) = tf\left(x_1,x_1\right)$$

If on the other hand, doubling the inputs less than doubled the outputs, we would say the production function has **decreasing returns to scale.** Formally a production function will have this property if for any $t > 1$:

$$f(tx_1, tx_2) < tf(x_1, x_1)$$

Finally, **increasing returns to scale** requires that doubling the inputs more than doubles the outputs or more generally that for any $t > 1$:

$$f(tx_1, tx_2) > tf(x_1, x_1)$$

Take for example the CES production function: $f(x_1, x_2) = (x_1 + x_2)^{\frac{1}{2}}$. Multiply both inputs by t:

$$f(tx_1, tx_2) = (tx_1 + tx_2)^{\frac{1}{2}}$$

How does this compare to multiplying the output by t? That is $t(x_1 + x_2)^{\frac{1}{2}}$?

$$f(tx_1, tx_2) = \sqrt{t}(x_1 + x_2)^{\frac{1}{2}}$$

Notice that for any $t > 1$:

$$\sqrt{t}(x_1 + x_2)^{\frac{1}{2}} < t(x_1 + x_2)^{\frac{1}{2}}$$

Thus, multiplying inputs by t less than increases output by t. **This has decreasing returns to scale.**

13 Profit Maximization / Cost Minimization

Now that we have established a way or representing the technology available to a form, we can look at the optimization problem facing a firm. Of course, an optimization problem needs a well defined objective. For consumers, that is the maximization of utility (or more formally, finding a "best" bundle according to $\succsim$ from the budget set). For firms, we will assume that they attempt to maximize profit.

13.1 Profit Maximization

Let p the price of output. Since the amount produced is given by the production function $f(x_1, x_2)$, we can write the firms **revenue** as $pf(x_1, x_2)$.

Cost is simply the cost of the chosen inputs. Let w_i be the input cost of input i. Then, cost can be written $w_1x_1 + w_2x_2$.

Putting these together gives up the profit function as a function of the input bundle (x_1, x_2):

$$\pi(x_1, x_2) = pf(x_1, x_2) - (w_1x_1 + w_2x_2)$$

Caveat: The assumption that p is fixed no matter how much output the firm produces is not realistic in most markets. This is called the *price taking* assumption. We will relax this later.

13.2 Short-Run Profit Maximization

In economics we often distinguish between the "short-run" and "long-run" profit maximization. The distinction is that in the short run, some inputs are fixed. For instance, in the short run, the firm might be stuck using a certain size of factory. In the long run, they can move to a different factory, but for now, it is fixed.

Suppose x_2 is fixed at level $\bar{x_2}$, then profit is only a function of x_1.

$$\pi(x_1, \bar{x_2}) = pf(x_1, \bar{x_2}) - w_1x_1 - w_2\bar{x_2}$$

In this case profit maximization only involves choosing the optimal level of the variable input x_1.

13.3 Example

Suppose $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ but x_2 is fixed at $x_2 = 10$. Price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$.

Plugging in $x_2 = 10$, the short-run production function is given by:

$$f(x_1, 10) = x_1^{\frac{1}{2}} (10)^{\frac{1}{2}} = \sqrt{10} x_1^{\frac{1}{2}}$$

The short-run profit function is then:

$$\pi(x_1, 10) = 100\sqrt{10} x_1^{\frac{1}{2}} - (x_1 + 10)$$

Maximizing this is a simple one-dimensional optimization problem. We need to look for a point where the slope is zero. Since this is a nice smooth function, the slope will have to be zero at the maximum.

$$\frac{\partial \left(100\sqrt{10} x_1^{\frac{1}{2}} - (x_1 + 10)\right)}{\partial x_1} = \frac{50\sqrt{10}}{\sqrt{x_1}} - 1$$

Setting this to zero gives the first-order condition:

$$\frac{50\sqrt{10}}{\sqrt{x_1}} - 1 = 0$$

Solving this for x_1,we get the profit maximizing level of x_1:

$$x_1^* = 25000$$

What is the maximum profit the firm can earn in the short run? Plug the optimal x_1 into the short run profit function:

$$\pi(x_1, 10) = 100\sqrt{10.0}\sqrt{25000} - (25000 + 10) = 24990$$

13.4 Long-Run Profit Maximization Example

In the long run, every input can be changed. We have to find the optimal level of both x_1 and x_2. Suppose $f(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$. Price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$.

As we discussed in our class on technology, when the production function has increasing returns or constant returns to scale, there might not be a profit maximizing level of inputs. Fortunately, this one is decreasing returns to scale. For any $t > 1$ we have:

$$f(tx_1, tx_2) = \left((tx_1)^{\frac{1}{3}} (tx_2)^{\frac{1}{3}}\right) = \left(t^{\frac{1}{3}} x_1^{\frac{1}{3}} t^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) =$$

$$= t^{\frac{2}{3}} \left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}\right) = t^{\frac{2}{3}} f(x_1, x_2) < t f(x_1, x_2)$$

So, let's try the long-run profit maximization. We want to maximize:

$$\pi(x_1, x_2) = p x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (w_1 x_1 + w_2 x_2)$$

Imagine the profit function being a mountain. You can move in two directions. North-south and east-west. Imagine you were at a point where the mountains slope was non-zero in one of these directions. Then you could move in that direction and get to a higher point on the mountain. Both of these slopes have to be zero for you to possible be at the peak of the mountain. The same goes for maximizing any function, the partial derivatives of that function have to **both** be zero at the optimal value. For profit, we have:

$$\frac{\partial \pi (x_1, x_2)}{\partial x_1} = 0, \frac{\partial \pi (x_1, x_2)}{\partial x_2} = 0$$

To reinforce this. Suppose either of these is non-zero. First, suppose $\frac{\partial \pi(x_1,x_2)}{\partial x_i} > 0$. If you increase x_i profit will go up. If on the other hand we have $\frac{\partial \pi(x_1,x_2)}{\partial x_i} < 0$, decreasing x_i profit will go up.

Let's suppose now the price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$. Plugging these into our example above, we get:

$$Max_{x_1 x_2} 100 x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)$$

Let's fund the partial derivatives in both variables:

$$\frac{\partial \left(100 x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)\right)}{\partial x_1} = \frac{\sqrt[3]{x_2}}{3 x_1^{2/3}} - 1$$

$$\frac{\partial \left(x_1^{\frac{1}{3}} x_2^{\frac{1}{3}} - (x_1 + x_2)\right)}{\partial x_2} = \frac{\sqrt[3]{x_1}}{3 x_2^{2/3}} - 1$$

This gives us these two first order conditions:

$$100 \frac{\sqrt[3]{x_2}}{3 x_1^{2/3}} - 1 = 0, \; 100 \frac{\sqrt[3]{x_1}}{3 x_2^{2/3}} - 1 = 0$$

To solve for the profit max, we need to solve these two equations for the two unknowns x_1, x_2. That's not so easy, so we might want to use a calculator for help. I'll use mathematica. Here's the command you would run for that:

$$Solve[\{100 \frac{\sqrt[3]{x_2}}{3 x_1^{2/3}} - 1 == 0, 100 \frac{\sqrt[3]{x_1}}{3 x_2^{2/3}} - 1 == 0\}, \{x_1, x_2\}]$$

There is only one solution:

$$x_1 = \frac{1000000}{27}, x_2 = \frac{1000000}{27}$$

13.5 Example- Constant Returns to Scale

Suppose for this example we have the production function $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$. Profit is given by: $\pi(x_1, x_2) = 100 x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - (x_1 + x_2)$. Let's find the first order conditions:

$$\frac{\partial \left(100 x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - (x_1 + x_2)\right)}{\partial x_1} = 100 \frac{\sqrt{x_2}}{2\sqrt{x_1}} - 1$$

$$\frac{\partial \left(100 x_1^{\frac{1}{2}} x_2^{\frac{1}{2}} - (x_1 + x_2)\right)}{\partial x_2} = 100 \frac{\sqrt{x_1}}{2\sqrt{x_2}} - 1$$

Set these equal to zero and rearrange:

$$50\sqrt{x_1} = \sqrt{x_2}, 50\sqrt{x_2} = \sqrt{x_1}$$

Square both sides of both equations:

$$2500x_2 = x_1, 2500x_1 = x_2$$

The only solution to this is $x_1 = x_2 = 0$. Something weird is going on here. Why would the optimal level be zero? It isn't. $x_1 = x_2 = 0$ is actually a profit **minimizing** input choice. It turns out that there is no profit maximizing solution. The firm can always increase profit by increasing production. This will be much more clear when we take a different approach to the problem: cost minimization.

13.6 Cost Minimization

Profit maximization really involves solving two problems at once: finding the optimal amount of output to produce and finding the cheapest way of producing that output. By doing this all at once, we can lose some of the the intuition about what we are doing. However, it is possible to break this problem up into two steps.

1. Calculate the cheapest way to produce any level of output y.
2. Calculate the most profitable y.

To formalize step 1, we really want to solve the problem:

$Min_{x_1,x_2} w_1x_1 + w_2x_2$ subject to $f(x_1, x_2) = y$.

In words, the constraint $f(x_1, x_2) = y$ is: "among all the pairs of x_1 and x_2 that produce y" and the $Min_{x_1,x_2} w_1x_1 + w_2x_2$ is "find the cheapest (x_1, x_2) pair". Notice that constraint is identical to saying "from the pairs (x_1, x_2) on the y isoquant. That is, our constraint is an isoquant. We want to find the cheapest bundle. One way to represent the cost of bundles is to draw **isocost curves**. These are sets of (x_1, x_2) that all cost the same to use. Effectively, they look like a bunch of budget lines. In the figure below, I have drawn the isoquant constraint in black and a bunch of isocost curves in red. Notice that the bundle that is only the lowest isocost curve that is also on the isoquant is the bundle marked in red. It occurs at a point where the isocost is tangent to the isoquant. This is almost like the consumer problem in reverse.

Of course, since the slope of the isoquant measures the tradeoff the firm can make between x_1 and x_2 to keep producing the same amount of output and the slope of the isocost represents how the firm can tradeoff between x_1 and x_2 to continue incurring the same cost, if the slopes aren't equal, then the firm can find some way of trading off between x_1 and x_2 such that they will produce the same amount of output at the same cost. The intuition for this is identical to the intuition for why the slopes of the indifference curve and budget need to be the same for a consumer (at least on the *interior* where some of both goods are being consumed).

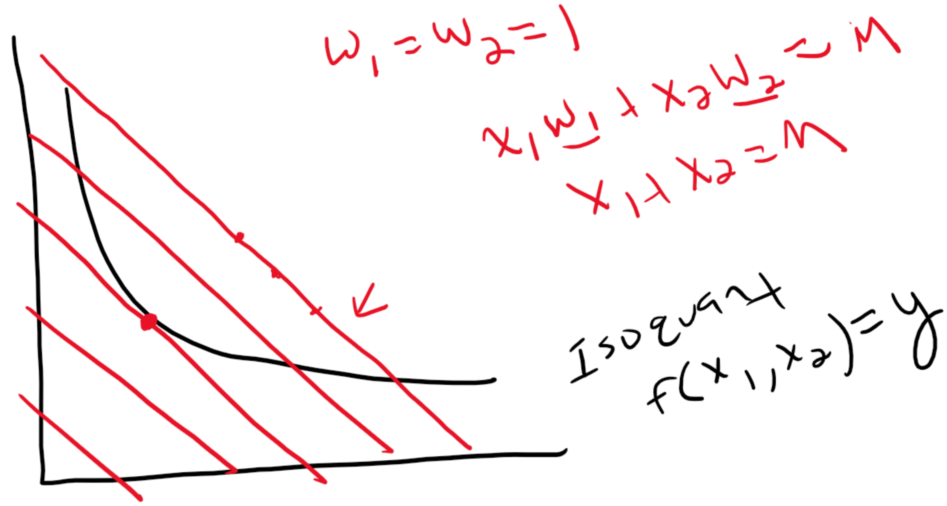

Figure 13.1: Cost minimization requires finding the lowest isocost curve on the fixed isoquant.

Since we need to look for points where the curves are tangent, we can use an optimal condition (equal slopes condition) just like we used for consumers. Here, it looks like this:

$$TRS = -\frac{w_1}{w_2}$$

This condition (if it exists and can be satisfied), combined with the constraint $f(x_1, x_2) = y$, will provide a cost minimizing bundle or more specifically, what we call the **conditional factor demands** $x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)$**.** These tell us how much x_1 and x_2 to use to produce y at the prices w_1 and w_2. Plugging the conditional factor demands back into the function $w_1 x_1 + w_2 x_2$ provides the **cost function** $c(y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$ which gives the cheapest way of producing y at the prices w_1 and w_2. Note that while the cost function $c(y)$ is a function of the input prices w_1 and w_2 as well, I tend to drop them from the functional notation and just write $c(y)$ instead of $c(w_1, w_2, y)$.

13.7 Example- Minimizing Cost for a Cobb Douglass Production Function

Minimize the cost of producing y units of output with production function $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$.

The TRS is:

$$TRS = -\frac{\frac{\partial\left(x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}\right)}{\partial x_1}}{\frac{\partial\left(x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}\right)}{\partial x_2}} = -\frac{x_2}{x_1}$$

Equal-slope condition:

$$-\frac{x_2}{x_1} = -\frac{w_1}{w_2}$$

Solving this condition for x_1:

$$x_1 = \frac{x_2 w_2}{w_1}$$

Instead of plugging this into a budget equation like we would for the consumer utility maximization, we need to plug it into the producer's constraint– the production constraint: $x_1^{\frac{1}{4}} x_2^{\frac{1}{4}} = y$. Plugging this is gives us:

$$\left(\frac{x_2 w_2}{w_1}\right)^{\frac{1}{4}} x_2^{\frac{1}{4}} = y$$

Solve for x_2 to get **conditional factor demand for x_2:**

$$x_2 = y^2 \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}$$

Plug this back into the equal slope condition above to get x_1:

$$x_1 = y^2 \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}$$

To calculate the cost function (what is the cheapest way to produce y): plug these conditional factor demands into the cost equation:

$$c(y) = w_1 \left(y^2 \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}\right) + w_2 \left(y^2 \left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}\right) = 2 w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$$

13.8 Profit Maximization Through Cost Minimization

Once we have the cost function, we know the cheapest way of producing any amount of output y. Now, if we want to maximize profit, we have already done the harder of the two parts (figuring out the cheapest way to produce). We are left with a relatively easy profit maximization problem. To see this, replace the cost $w_1 x_1 + w_2 x_2$ with $c(y)$. Now we have written the profit function completely in terms of y. And, we know whatever y we choose, $c(y)$ will already represent the cheapest way of producing that amount:

$$\pi(y) = py - c(y)$$

This is very easy to maximize since it is just one-dimensional. *It only depends on y.*

13.9 Example

Maximize profit of $f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$. We have already found $c(y)$ for this firm in the example above.

$$c(y) = 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$$

Let's write the profit function in terms of only y:

$$\pi(y) = py - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2$$

For an interior maximum (y is some number other than 0), the slope of this will have to be zero. Otherwise, the firm could increase or decrease output and increase profit. Taking the derivative:

$$\frac{\partial \left(py - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} y^2\right)}{\partial y} = p - 4\sqrt{w_1}\sqrt{w_2}y$$

Setting this equal to zero we have:

$$p = 4\sqrt{w_1}\sqrt{w_2}y$$

Notice, that in this form we have written it as marginal revenue (p) equal to marginal cost $\left(4\sqrt{w_1}\sqrt{w_2}y\right)$. In general, it is always true that the first order condition when we maximize profit in terms of only y can be written as $MR = MC$. Under the price taking assumption (that price p does not depend on y) the marginal revenue is just p and we have $p = MC$. Returning to the example, we can solve y to get the optimal y for any set of prices. This the optimal (profit maximizing) level of output for any price.

$$y^* = \frac{p}{4\sqrt{w_1}\sqrt{w_2}}$$

We can also determine the profit a firm can earn under any output and inpout prices by plugging this optimal y back into the profit function written in terms of y. In this case, it takes a little work to simplify it down to something satisfying:

$$\pi(y^*) = p\left(\frac{p}{4\sqrt{w_1}\sqrt{w_2}}\right) - 2w_1^{\frac{1}{2}} w_2^{\frac{1}{2}} \left(\frac{p}{4\sqrt{w_1}\sqrt{w_2}}\right)^2 =$$

$$\frac{p^2}{4\sqrt{w_1}\sqrt{w_2}} - \frac{p^2}{8\sqrt{w_1}\sqrt{w_2}} = \frac{p^2}{8\sqrt{w_1}\sqrt{w_2}}$$

Suppose $p = 10$ and $w_1 = w_2 = 1$ the maximum profit the firm can earn is (plug prices into the profit function and optimal y^* above):

$$\pi^* = \frac{100}{8} = \frac{25}{2}$$

$$y^*(p, w_1, w_2) = \frac{p}{4\sqrt{w_1}\sqrt{w_2}}$$

13.10 What can go wrong?

If returns to scale are linear or increasing then if we can find any output level y where the firm earns positive profit **then there is no profit maximizing level of y.** The firm wants to produce as much as possible. This is because with linear or increasing returns to scale, doubling inputs will double cost and **at least** double output- so profit will at least double. Thus, if we can find a point were profit is positive, we can always use of all inputs and increase profit.

Recall this example from our profit maximization example above. Suppose $f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, price of output is $p = 100$. $w_1 = 1$, $w_2 = 1$. Let's see how easy it is to see here that there will be no profit maximizing level of output.

In this case, the cost minimizing level of inputs are (try this yourself using cost minimization):

$$x_1^* = x_2^* = y$$

The cost function is:

$$c(y) = 2y$$

The profit function in terms of y is:

$$\pi(y) = 100y - 2y = 98y$$

Now it is very clear that this profit function is increasing in y. there is not profit maximizing solution!

13.11 Linear Returns to Scale Example 2

Maximize profit of with $f(x_1, x_2) = min\left\{\frac{1}{2}x_1, x_2\right\}$ using cost minimization and then profit maximization.

To minimize costs, the firm should use $\frac{1}{2}x_1 = x_2$ since otherwise there will be wasted inputs. Plugging this back into the production function to get the conditional factor demands, we have:

$$x_2^* = y, x_1^* = 2y$$

This gives us the cost function:

$$c(y) = w_1 2y + w_2 y = (2w_1 + w_2)\, y$$

Now we can write the profit function in terms of y:

$$\pi(y) = py - (2w_1 + w_2)\, y = (p - 2w_1 - w_2)\, y$$

If $p > 2w_1 + w_2$, this is a line with positive slope. There is no profit maximizing level.

If $p < 2w_1 + w_2$ this is a line with negative slope. The optimal level is $y = 0$ and profit is 0.

If $p = 2w_1 + w_2$ this is $\pi(y) = 0$. Profit will always be zero and the firm can choose whatever they want.

14 Firm Supply

In this chapter, we focus on the firm's supply functions. We are still focusing on firms that are price takers. The is, they assume price is fixed and not a function of their output. In this case, their profit functions can be written as: $\pi(y) = py - c(y)$ where the function $c(y)$ is their cost function which can be found my cost minimimization using the methods in the previous section.

14.1 Inverse supply is equal to the marginal cost function.

If a firm is maximizing profit at any point except $y = 0$, it must be that the slop of the profit function is zero. That is, for a firm to be producing *something* and also be maximizing profit, we need that: $\frac{\partial \pi(y)}{\partial y} = 0$. For price taking firms, this is equivalent to:

$$p = mc(y)$$

Let's suppose the firm's cost function is $c(y) = 2y^{\frac{3}{2}}$. Let's suppose that the firm wants to determine how much to supply any **any** price. Profit in terms of p and y is:

$$\pi(y) = py - 2y^{\frac{3}{2}}$$

The firm's marginal cost is $mc(y) = 3\sqrt{y}$. At the optimum, they want to find the output y^* that sets the $mc(y^*) = p$. We have:

$$p = 3\sqrt{y}$$

Solving for y gives us the supply function:

$$y = \left(\frac{p}{3}\right)^2$$

For instance, what will they produce at price of 30? We can plug in $p = 30$ and we get $y = 100$.

We can also get the inverse supply function which is useful in solving equilibrium problems. It tells us at what price a firm would produce some y output of output. Isolating p, this is given by:

$$p = 3\sqrt{y}$$

Notice, this is exactly the condition we wrote down above where $mc = p$. The firm's inverse supply function comes right out of this profit maximization condition. A firm's inverse supply **is it's marginal cost function.**

14.2 Profit at The Maximum

Once we have a firm's supply function, we can write down how much the firm can earn at any price. In the previous problem, we had the profit function:

$$\pi(y) = py - 2y^{\frac{3}{2}}$$

At $p = 30$ for instance, we saw the firm produces $y = 100$. Plugging in $p = 30$ and $y = 100$, we get that the firm earns 1000 at this price.

$$\pi = 30(100) - 2(100)^{\frac{3}{2}} = 1000$$

We can also write down a profit function that gives us the firm's maximum profit at **any price.** This is very useful for trying to find, for example, prices for which the firm can earn positive profit. To write down this function, we plug the firm's supply function in for y in the profit function to get everything in terms of p. In the example above, supply was $y = \left(\frac{p}{3}\right)^2$. Plugging this in, we get:

$$\pi^* = p\left(\left(\frac{p}{3}\right)^2\right) - 2\left(\left(\frac{p}{3}\right)^2\right)^{\frac{3}{2}} = \frac{1}{9}p^3 - \frac{2}{27}p^3 = \frac{1}{27}p^3$$

Notice that this firm will earn positive profit at *any* price. However, we can ask questions like "what does the price need to be so the firm earns at least 10000 in profit?" To do this, we write the inequality $\pi^* \geq 10000$. This is:

$$\frac{1}{27}p^3 \geq 10000$$

Solving for p we get a solution of about:

14.3 Fixed Costs Don't Affect Firm Behavior

Let's suppose the firm in the previous problem had a fixed cost of 1000 so that cost was $c(y) = 2y^{\frac{3}{2}} + 1000$. The profit function is:

$$\pi(y) = py - 2y^{\frac{3}{2}} - 1000$$

Maximizing this profit function, we look for the point where the slope is zero. We get:

$$p = 3\sqrt{y}$$

... and the firms supply is:

$$y = \left(\frac{p}{3}\right)^2$$

This is the exact same supply function. The firm's behavior is the same. This is no coincidence, firms always ignore fixed costs in choosing output. This is because fixed costs simply translates the the profit function downwards but do not change it's shape. If some output y^* maximizes some profit function it will also maximize another profit function that is just a translated version of that function as well. In fact, we we were to work out the profit function for this firm at any price, we would get:

$$\pi^* = \frac{1}{27}p^3 - 1000$$

It's just 1000 lower than last time.

When we have fixed costs, it's often interesting to ask when a firm will have positive profit. Notice that is price was, $p = 10$, profit would be: -962.963. This is perfectly fine. It's still **the most** the firm can earn at the price $p = 10$. If they did not produce anything, they would get -1000 (their fixed cost). Sometimes profit is negative even when the firm is acting optimally. Would would price need to be so that price is positive in this case?

$$\frac{1}{27}p^3 - 1000 \geq 0$$

$$p \geq 30$$

14.4 What Can go Wrong With $p = mc$?

There are two things that can go wrong. When the output y we get from setting $p = mc$ is not actually the optimal output.

Notice above we said the $p = mc$ is **necessary** for a profit maximizing y when $y > 0$. So there are two things that can happen.

1. $p = mc$ gives us a negative y

2. $y = 0$ gives a higher profit than the y where $p = mc$

Let's look at these in turn.

First 1. $p = mc$ gives us a negative y. This is an easy condition to check. Here's an example:

Suppose $c(y) = y^2 + 10y + 100$. Let's find the supply function. Set price to marginal cost:

$$p = 2y + 10$$

Solving this for y gives the output where price is equal to marginal cost:

$$y = \frac{p - 10}{2}$$

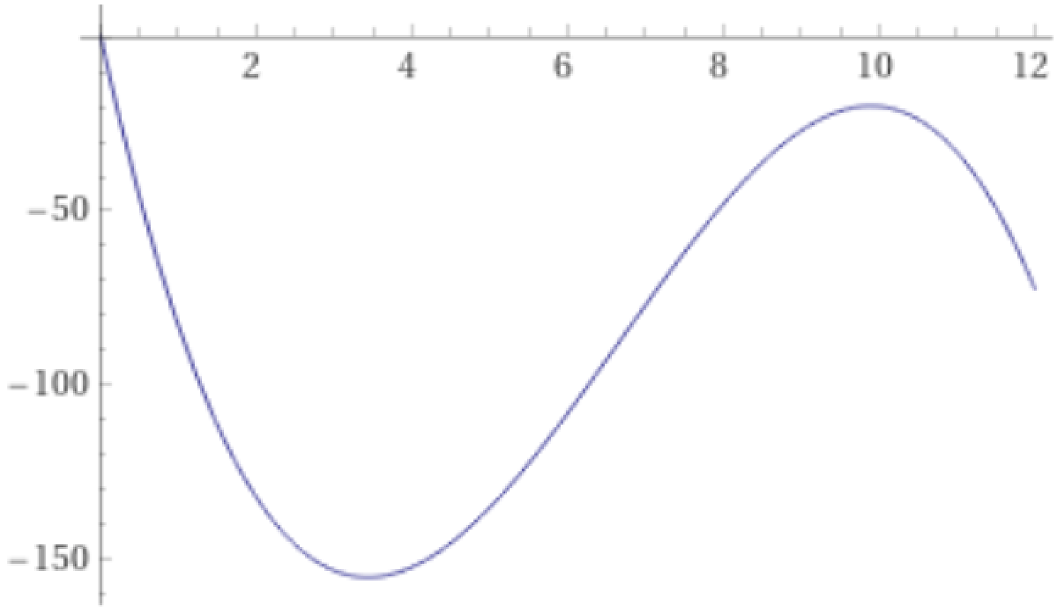

Figure 14.1: A plot of $\pi(y) = 8y - y(y-10)^2 - 10y$

If $p < 10$, this is a negative number. The firm cannot produce negative. So for any price $p < 10$, there is no $y > 0$ where $p = mc$, so no $y > 0$ could possibly maximize profit. The firm want's to produce $y = 0$ when $p < 10$. So the true supply function is:

$$y = \frac{p-10}{2}, p \geq 10$$

$$y = 0, p < 10$$

2. $y = 0$ gives a higher profit than the y where $p = mc$

Notice, above, I wrote that $p = mc$ is only necessary for a profit maximizing point where $y > 0$. It may well be that $y = 0$ gives more profit that any point where $p = mc$. We will not worry too much about this condition since it requires a bit of a weird cost function, but it is good to know that this **can happen**.

Suppose we have $c(y) = y(y-10)^2 + 10y$. Profit is:

$$\pi(y) = py - y(y-10)^2 - 10y$$

Suppose $p = 8$:

$$\pi(y) = 8y - y(y-10)^2 - 10y$$

Let's have a look at this function:

Let's set $p = mc$. The derivative is a little messy, but we can simplify it. We get:

$$8 = 3y^2 - 40y + 110$$

This has two solutions for y:

$$y \approx 3.43, y \approx 9.90$$

Looking at the plot above, $y \approx 3.43$ is actually a local **minimum.** $y \approx 9.90$ is a local maximum. In fact, it is the most profit the firm can earn if they produce $y > 0$ and earns them about $\pi \approx -19.899$. However, if the firm just chose to produce zero, profit would be 0!

The point of this was to show you what **can** go wrong by looking for profit maximizing points where $p = mc$. They can give you local minima instead of local maxima, and furthermore, the profit of producing $y = 0$ might be higher than the the most firm firm could earn by producing some $y > 0$. This won't affect most of our problems, but again, it is useful to know the limitations of our methods.

14.5 Example

Suppose $c(y) = 4y^2 + 50$. Suppose this firm is a price taker. Find the firms supply function and how much they can earn at at price p. What does the price need to be so they earn positive profit?

First, we set $p = mc$ to find the optimal output:

$$p = 8y$$

Solve for y gives us the supply function:

$$y = \frac{1}{8}p$$

This is never negative so we don't have to worry about that. Let's find the profit the firm gets by using this production for any p. Plug $y = \frac{1}{8}p$ into the profit function:

$$\pi^* = p\left(\frac{1}{8}p\right) - 4\left(\left(\frac{1}{8}p\right)^2\right) - 50$$

$$= \frac{1}{16}p^2 - 50$$

Let's just double check it's never better to produce $y = 0$. If the firm did choose $y = 0$ instead of $y = \frac{1}{8}p$, they would get profit -50. However, π^* above is strictly greater than -50 for any p. All good.

Now let's find when profit is positive:

$$\frac{1}{16}p^2 - 50 > 0$$

$$\frac{1}{16}p^2 > 50$$

$$p^2 > 800$$

$$p > \sqrt{800}$$

$$p > 28.285$$

As long as price is above 28.285, profit will be positive.