Economics 8100

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Part I
Budget

1 Consumption Set X

Assumptions: (Universe of Choice Objects): X
Bundles: Elements of X. $x \in X$

Assumptions about X.
1. $\emptyset \neq X \subseteq \mathbb{R}^n_+$.
2. X is closed.
3. X is convex.
4. $0 \in X$.

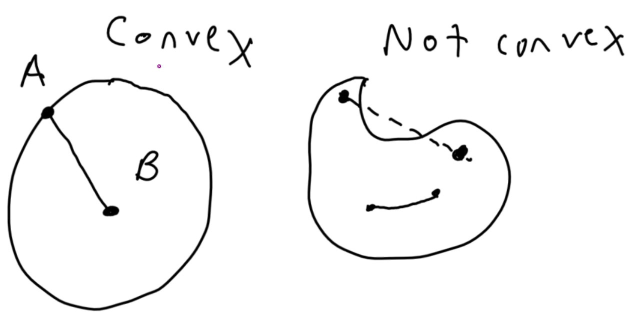

Figure 1.1: Examples of a Convex/Non-Convex Set.

2 Budget Set B

Budget Set: $B \subseteq X$

X defines the scope of the model. B is what an *individual consumer* chooses among.

Example. Budget Set with Prices and Income
$B = \{x | x \in X \,\&\, x_1 p_1 + x_2 p_2 \leq m\}$

Example. Ice Cream Bowls
Every ice cream bowl x has some non-negative number of scoops of Vanilla, Chocolate, Strawberry.

$$X = \mathbb{R}^3_+$$

Budget B is the set of bowls with *no more than one scoop of ice cream.*

$$B = \left\{ x | x \in R^3_+ \& \sum_{i=1}^{3} x_i \leq 1 \right\}$$

This is the unit-simplex in $\mathbb{R}_3$.
$(1, 0, 0) \in B$. (On the boundary.)
$(0.5, 0.5, 0) \in B$. (On the boundary.)
$(0.25, 0.25, 0.25) \in B$. (In the interior.)
$(2, 0, 0) \notin B$

Part II

Preference

3 The Preference Relation

Preference Relation is a **Binary Relation**.
Formally, a binary relation on set X is a subset of the Cartesian product X with itself.

$$\succeq \subseteq X \times X$$

Another way to denote an ordered pair is "in" the relation:
If $(x, y) \in \succeq$ we can also write $x \succeq y$.
Informally we say "x" is at least as good as "y", or "x" preferred "y".

Axioms of $\succeq$.
Axiom 0 *(reflexive):* $\forall x \in X$, $x \succeq x$. This is implied by *axiom 1.*

Axiom 1 *(complete):* $\forall x, x' \in X$, either $x \succeq x'$ or $x' \succeq x$ (or both).

The consumer has "some" preference over every pair of objects.

Axiom 2 *(transitive):* $\forall x, x', x'' \in X$ *if* $x \succeq x'$ *and* $x' \succeq x'' \Rightarrow x \succeq x''$.

$\succeq$ is a "weak order" if it is complete and transitive.

4 Relations and Sets Related to $\succeq$

Subrelations:

$\sim$ is the indifference relation. $x \succeq y$ and $y \succeq x \Leftrightarrow x \sim y$.

$\succ$ is the strict relation. $x \succeq y$ and not $y \succeq x \Leftrightarrow x \succ y$.

Related Sets:

$\succeq (x)$ "upper contour set", "no worse than set"

$\precsim (x)$ "lower contour set", "no better than set"

5 From Preferences to Choice

Choice Correspondence.

We will assume that from a budget set B a consumer "chooses" *choice set* C according to their preference $\succeq$. $C = \{x | x \in B \,\&\, \forall x' \in B, x \succeq x'\}$.

Informally, C is the set of objects that are at least as good as anything else in the set.

Example With Transitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, c \succeq b$.

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = c$$

$$C(\{a, b, c\}) = c$$

6 Cycles Lead to Empty Choice Sets

6.1 The Problem with Intransitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, b \succeq c$. *This is intransitive!*

Choice correspondence:

$$C : P(X)/\emptyset \to X$$

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a,b\}) = a, C(\{a,c\}) = c, C(\{b,c\}) = b$$

$$C(\{a,b,c\}) = \emptyset$$

This consumer cannot make a choice from the set $\{a,b,c\}$.

6.2 Cycles and Empty Choices

Notice in the previous example, $a \succ b, a \succ c, c \succ a$. We have proved (essentially) that if there is a cycle, there is an empty choice set.

In fact, suppose, there is an empty choice set **and** X is finite. There must be a cycle.

$$\forall x \in B, \#\left(\succsim(x)\right) < \#(B)$$

By completeness, $\forall x \exists x' \in X : x' \succ x$. Choose an x_1, let x_2 be any element of $\succ(x_1)$. We have $x_2 \succ x_1$. If there is an $x_3 \in \succ(x_2)$ such that $x_1 \succ x_3$ we have identified a cycle. Otherwise, we continue with an inductive step. Suppose we have $x_n \succ ... \succ x_1$. $\succ(x_n)$ is non-empty. Either it contains an element x_{n+1} such that there is an $x_i \succ x_{n+1}$ in which case we have identified a cycle or it does not and we continue with another inductive step. Either we find a cycle or reach the N_{th} step with $x_N \succ x_{n-1} \succ ... \succ x_1$. $\succ(x_N)$ is non-empty.

So, the cycle condition is equivalence to a non-empty choice set. Transitivity of $\succsim$ implies transitivity of $\succ$ which implies no cycles (try this last step at home). But do no-cycles imply transitivity of $\succsim$? No. Here is a counter-example:

$$x \succ y, y \sim z, z \succ x$$

7 Intransitivity: Empty Choices, Incoherent Choices: Pick One.

So if no-cycles of the strict preference is equivalent to non-empty choice (in finite sets), and transitvity of $\succsim$ is not equivelent to no-cycles, why do we assume it?

Finite non-emptyness: *For any B with $\#(B) \in \mathbb{I}$, $C(B) \neq \emptyset$*

Coherence: *For every x, y and B, B' such that $x, y \in B \cap B'$, $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$.*

Suppose there is an intransitive $\succsim$. There exists either a B where $C(B) = \emptyset$ or there exists a x, y, B, B' where the choice correspondence is incoherent.

By intransitivity:

$$1)\, x \succ y, y \succ z, z \succ x$$

$$C(\{x, y, z\}) = \emptyset$$

$$2)\, x \sim y, y \sim z, z \succ x$$

$$3)\, x \sim y, y \succ z, z \succ x$$

$$x \notin C(\{x, y, z\})$$

$$y \in C(\{x, y, z\})$$

$$x \in C(\{x, y\})$$

$$4)\, x \succ y, y \sim z, z \succ x$$

Can you find the incoherent choice?

8 Indifference Sets

8.1 Indifference Maps

To understand preferences, we often draw sets of the form $\sim(x)$. Many times these are one dimension smaller than the space of bundles, in which case we often call them *indifference curves*, but they need not have any special structure, unless we make further assumptions about preferences. There is only one things we really know about these sets.

8.2 Complete, Transitive Preferences have Indifference Sets that Do Not Intersect

Result. *Indifference curves do not cross.* For two bundles $x \succ y$, $\sim(x) \cap \sim(y) = \emptyset$.

Proof is given visually below:

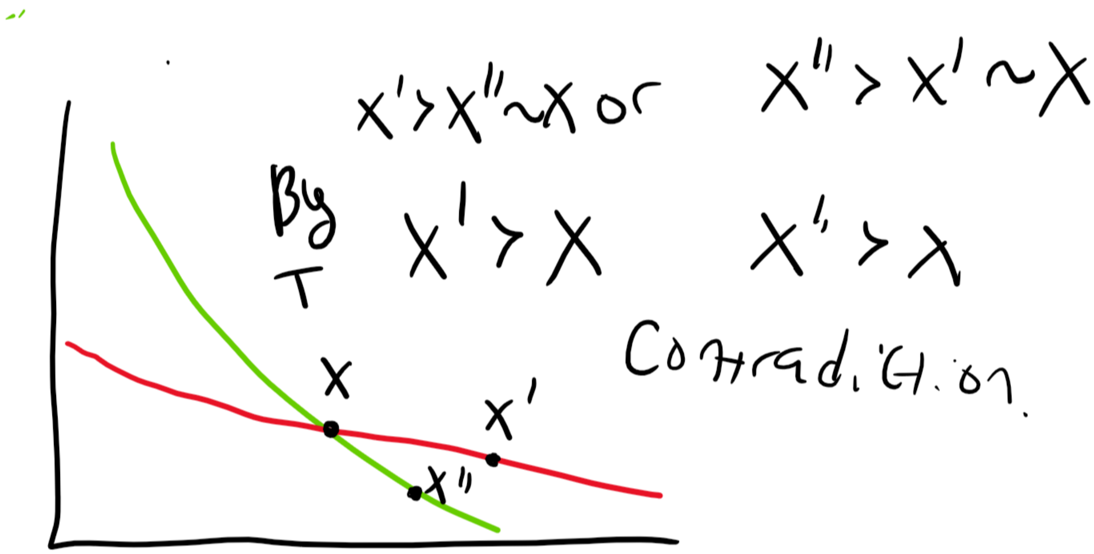

Figure 8.1: Distinct Indifference Sets do not Intersect

Part III

From Preference to Utility

9 Utility Represents Preferences

Suppose there is some $U : X \to \mathbb{R}$ such that $U(x) \geq U(x') \Leftrightarrow x \succsim x'$ then we say $u()$ represents preference relation $\succsim$. When does such a representation exist?

9.1 Finite X

Proposition 1. *A $U()$ exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.*

Proof. Let's start with $\Rightarrow$.

Because $\geq$ is complete on the real numbers, for every $x, y \in X$ either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ thus because $u()$ represents $\succsim$, it is complete.

By similar argument, $\succsim$ is transitive. For every three $x, y, z \in X$. If $u(x) \geq u(y)$ and $u(y) \geq u(z)$ then $u(x) \geq u(z)$ because $\geq$ is transitive on the real numbers.

Now we prove $\Leftarrow$:

Define $U(x) \equiv \#(\precsim(x))$

Example: $a \succ b, b \succ c$. $\precsim (a) = \{a, b, c\}.U(a) = 3$.
Lemma: For $x \succsim y$, $\precsim (y) \subseteq \precsim (x)$ (proved in PS1).
By this lemma, for $x \succsim y$, $\precsim (y) \subseteq \precsim (x)$ and thus $\# \precsim (y) \leq \# \precsim (x)$ and $u(x) \geq u(y)$. $\square$

9.2 Countably infinite X

Pick any arbitrary order on the bundles: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim(x)} w(y)$$

Example: "π shows up unexpectedly when eating ice cream."
An even number of scoops of ice cream are better than an odd number of scoops and otherwise more is better than less.

$$u(2) = \sum_{i=1}^{\infty} \left(\frac{1}{(2i-1)^2} \right) = \frac{\pi^2}{8}$$

$$u(4) = \frac{1}{4} + \frac{\pi^2}{8}$$

9.3 Uncountable X

The *Lexicographic* preferences have no utility representation:

$$X = \mathbb{R}_+^2$$

$(x_1, x_2) \succ (y_1, y_2)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$.
$\succsim$ is complete, and transitive. [*Prove this for practice*].
Pick two real numbers $v_2 > v_1$ and construct four bundles $(v_1, 1), (v_2, 1), (v_1, 2), (v_2, 2)$.

$$(v_2, 2) \succ (v_2, 1) \succ (v_1, 2) \succ (v_1, 1)$$

Suppose there is a utility function representing these preferences, then we have two disjoint intervals:

$$[u(v_2, 1), u(v_2, 2)]$$

$$[u(v_1, 1), u(v_1, 2)]$$

For every real number, we can construct an interval like this. Because the rationals are dense in the reals, there is a rational number in each of these intervals. Thus, for every real, we can find a unique rational number. That is, we have a mapping from the reals into the rationals which implies that the cardinality of the rationals is at least as large as that of the reals. $\#\mathbb{Q} \geq \#\mathbb{R}$. This contradicts that the cardinality of the rationals is strictly smaller than the reals.

9.4 An example of preference relation with a utility representation.

Cars have horse power in $[0, 999]$ and cup holders in $\mathbb{Z}_+$ (integers).

Suppose preferences are lexicographic and more cup holders are more important than more horse-power.

$u(c_i, h_i) = c_i + \frac{h_i}{1000}$ represents these preferences.

See problem set 2 for example where we do not bound the horse power.

9.5 What ensures a utility representation in an uncountable universe?

A preference relation is representable by a utility function $U(x)$ iff $\forall x, y \in X\, s.t.\, x \succ y$, $\exists x^* \in X^* \subset X\, s.t.\, x \succsim x^* \succ y$ and the set X^* is countable.

To construct the utility function, $U(x)$, Pick any arbitrary order on the bundles in X^*: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim(x) \cap X^*} w(y)$$

9.6 Continuous $\succsim$.

Preference relation $\succsim$ is continuous if $\forall x \in X$, $\succsim(x)$ and $\precsim(x)$ are closed in X.

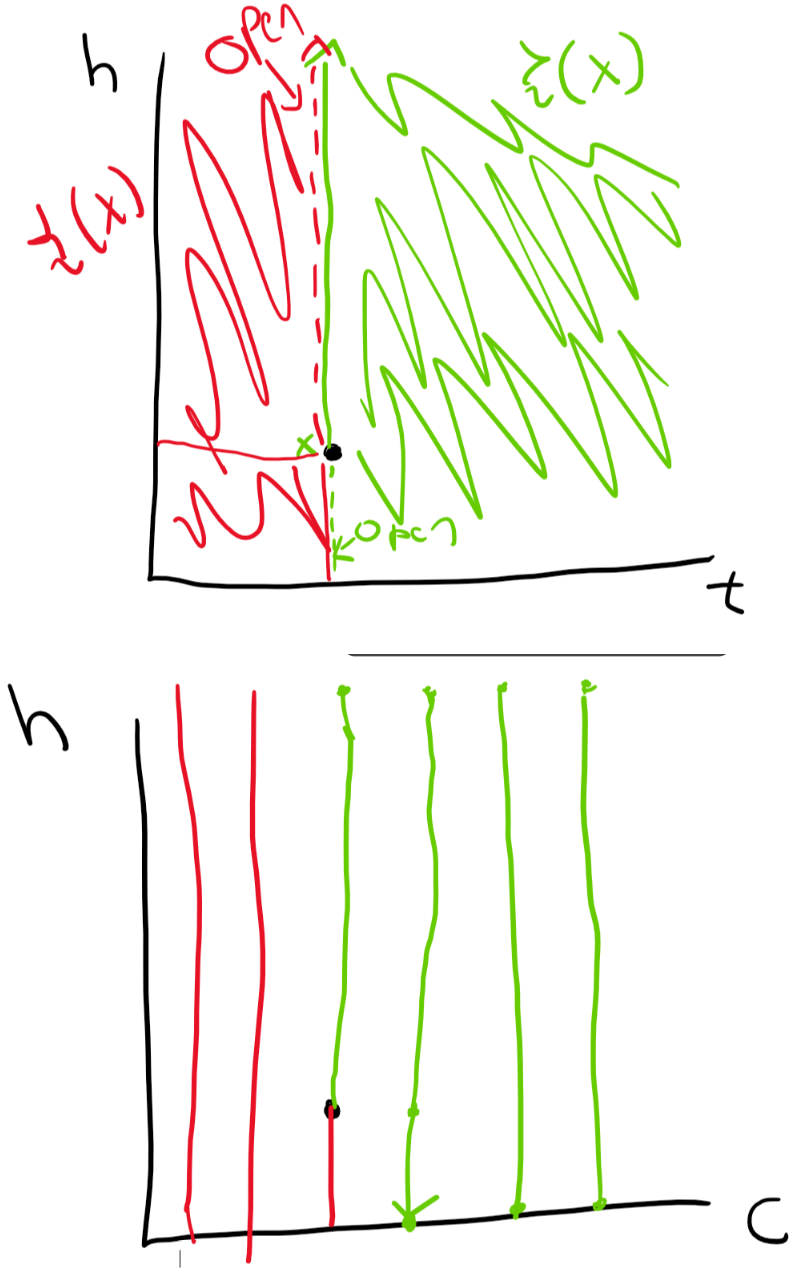

Figure 9.1: Not Continuous/Continuous Lexicographic Preferences.

9.7 What ensures a continuous utility representation?

A *complete*, *transitive,* and *continuous* preference relation $\succsim$ can be represented by a continuous utility function $U(x)$ and, a continuous utility function represented C,T,C preferences.

10 Other Properties of $\succsim$

10.1 Monotonicity

Ensure consumers consume on the boundary of the budget set.

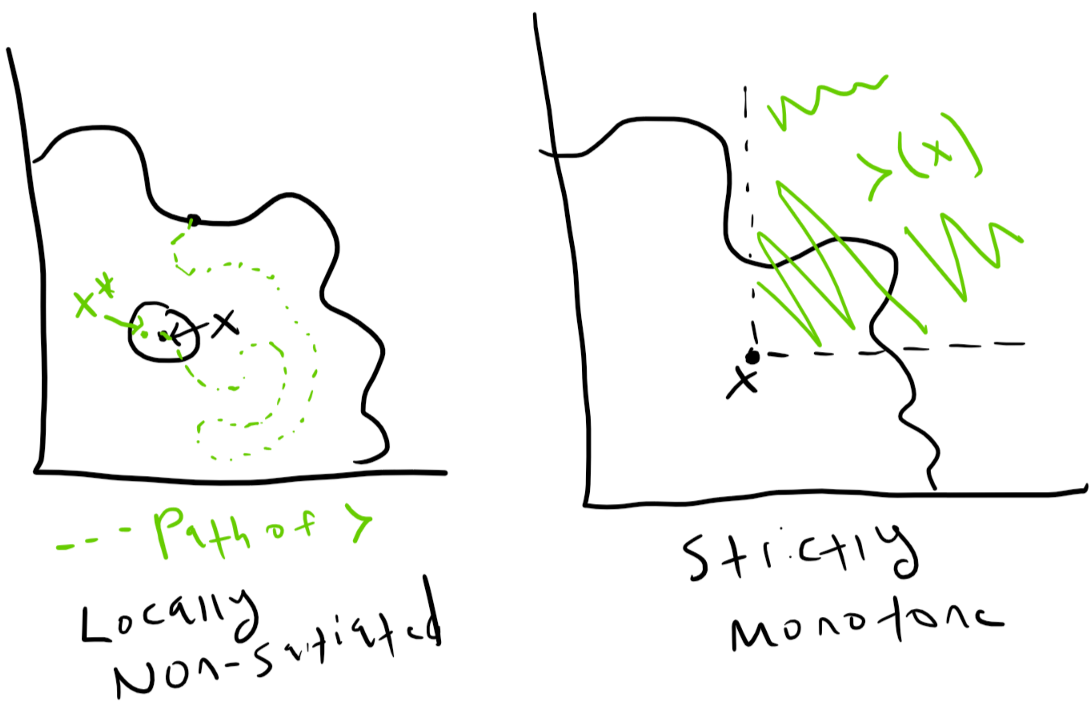

Figure 10.1: Locally Non-satiated vs. Strictly Monotone

Strict Monotonicity

More stuff is better.

First, some notation:

For $X \subseteq \mathbb{R}^n$
$x \geq x'$ iff $x_i \geq x'_i$ for all $i \in \{1, 2, ..., n\}$
$x \gg x'$ iff $x_i > x'_i$ for all $i \in \{1, 2, ..., n\}$
Fro example: $(2,2) >> (1,1), (2,1) \geq (1,1), (1,1) \geq (1,1)$
Definition. Strict Monotonicity. $x \geq x' \Rightarrow x \succsim x'$ and $x \gg x' \Rightarrow x \succ x'$

Local Nonsatiation

Definition. Local Nonsatiatin. $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x^* \in B_\varepsilon(x)$ such that $x^* \succ x$.
A consumer can always change the bundle a "little bit" no matter how small that little bit is, and find something strictly better.

10.2 Convex Sets, Convex/Concave Functions, Quasi-Convex/Concave Functions

Convex Sets

In a subset of euclidean space X, the line between $x \in X$ and $x' \in X$ is another point in the set X given by $tx + (1-t)x'$ where $t \in [0,1]$. We call points like this **Convex Combinations** of x and x'.
For example: $x = (1,0), x' = (0,1)$. If we take $t = 0.5$. The convex combination is $0.5(1,0) + 0.5(0,1) = (0.5, 0.5)$.
A **convex set** $S \subseteq X$ is a set of points that contains all of its convex combinations.
Formally, $\forall x, x' \in S$, $\forall t \in [0,1]$, $tx + (1-t)x' \in S$.

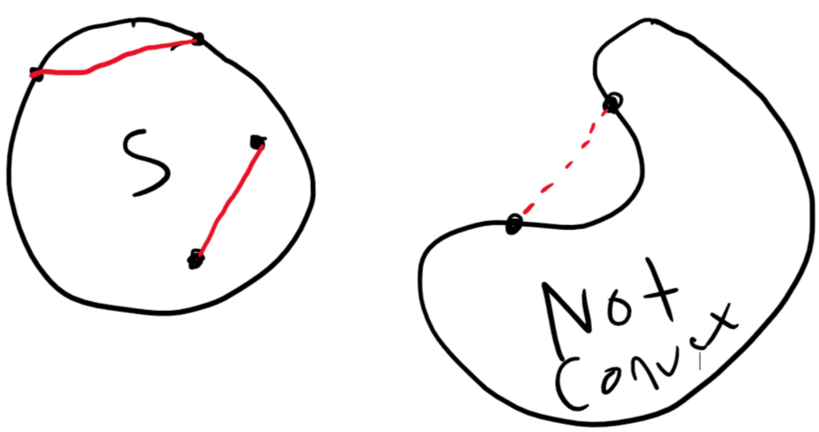

Figure 10.2: A Convex and Non-Convex Set

Convex Functions

A line between two points "on the function" lies above the function itself.
***Convex Function**:*

$$\forall x, x' \in X, t \in [0,1], tf(x) + (1-t) f(x') \geq f(tx + (1-t) x')$$

***Strictly Convex Function**:*

$$\forall x, x' \in X, t \in (0,1), tf(x) + (1-t) f(x') > f(tx + (1-t) x')$$

Contour Sets:
A convex function has **convex lower contour sets**.

Concave Functions

A line between two points "on the function" lies above the function itself.
***Concave Function**:*

$$\forall x, x' \in X, t \in [0,1], tf(x) + (1-t) f(x') \leq f(tx + (1-t) x')$$

***Strictly Concave Function**:*

$$\forall x, x' \in X, t \in (0,1), tf(x) + (1-t) f(x') < f(tx + (1-t) x')$$

A concave function has **convex upper contour sets**.

Quasi-Concave Functions

A function $f(x)$ is quasi-concave if **and only** it has convex upper contour sets.
A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a concave function.
A function $f(x)$ is quasi-concave if and only if $f(tx + (1-t) x') \geq min \{f(x), f(x')\}$ for $t \in [0,1]$.
A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t) x') > min \{f(x), f(x')\}$ for $t \in (0,1)$.

Notice that, for a strictly quasi-concave utility function, let $x' \succ x$, then the set $tx + (1-t) x' \in \succ (x)$ for $t \in (0,1)$. Thus, there is a small enough ball around that point $B_\epsilon (tx + (1-t) x') \in \succsim (x)$. Thus, these points are in the interior of $\succsim (x)$ and $\succsim$ is **strictly convex.**

Quasi-Convex Functions

A function $f(x)$ is quasi-convex if **and only** it has convex lower contour sets.
A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a convex function.
A function $f(x)$ is quasi-convex if and only if $f(tx + (1-t) x') \leq max \{f(x), f(x')\}$ for $t \in [0,1]$.
A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t) x') < max \{f(x), f(x')\}$ for $t \in (0,1)$.

10.3 Convexity of $\succsim$.

Convex Preferences: $x \succsim x' \Rightarrow t(x) + (1-t)x' \succsim x', t \in [0,1]$
$x \in \succsim (x') \Rightarrow t(x) + (1-t)x' \in \succsim (x')$
Thus, $\succsim (x)$ are convex if $\succsim$ is a convex preference relation.
Strictly Convex Preferences: $x \succsim x' \Rightarrow t(x) + (1-t)x' \succ x', t \in (0,1)$
The upper contour sets $\succsim (x)$ are *strictly* convex.

10.4 Utility and Preference Relationships

If U represents $\succsim$:
1) $\succsim$ (strictly) **convex** $\Leftrightarrow$ U is (strictly) **quasi-concave**.
2) $\succsim$ are **strictly monotonic** $\Leftrightarrow$ U is **strictly** increasing.
3) $\succsim$ are **strictly monotonic** $\Leftarrow$ U is **strongly** increasing.

11 The Consumer Problem

11.1 Choice

The set of all "best things" in the budget set. This is what we are looking for:
$C(B) = \{x | x \in B \wedge x \succsim x', \forall x' \in B\}$

Competitive Budgets:

$$B = \left\{x | x \in \mathbb{R}^n_+, p \cdot x \leq m\right\}$$

p is the vector of prices.
m is the "income".
Constrained problem:

$$Max_{x \in X} U(x) \, s.t. \, p \cdot x \leq m$$

11.2 The Lagrange Method- Some Intuition.

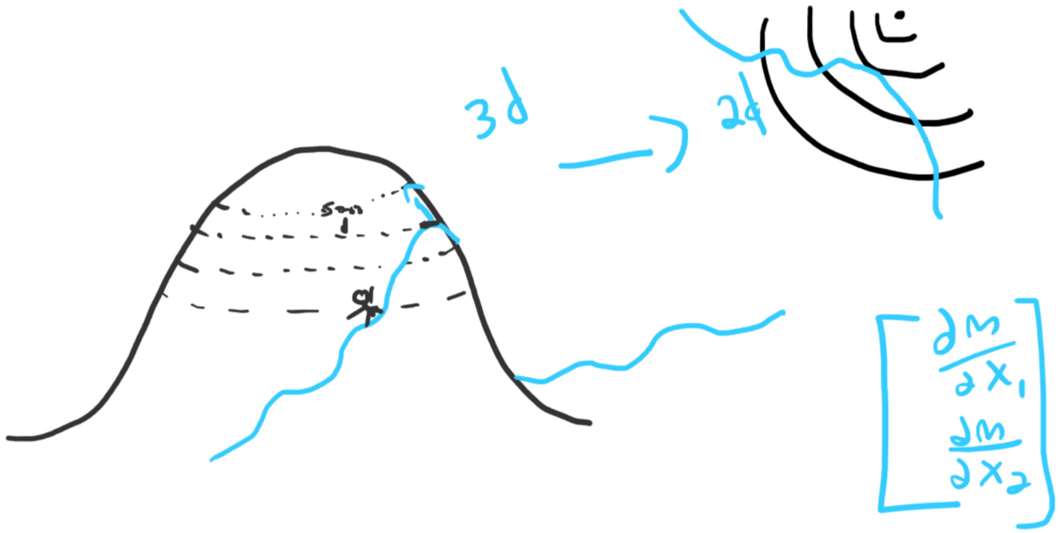

Figure 11.1: Finding the best spot for a selfie.

If both the objective and the constraint are smooth, at the optimal **the *direction* of the gradient of the objective has to be equal to the *direction* of the gradient of the constraint.** Otherwise, moving along the constraint boundary in *some* direction will yield a larger value of the objective! (Caveat: this assumes we *can* move in every direction along the constraint. That will only be true at non-boundary points.)

Thus, for smooth functions, the equality of the direction of the gradients of the objective and the constraint are **necessary** for an non-boundary optimum.

Since the direction of the gradient is just a scaling of the gradient, suppose U is our objective and G is the function for the boundary of the constraint. Then,

$$\nabla U(x) = \lambda \nabla G(x)$$

Can we write a function such that the first order condition will yield this gradient condition? *Sure*:

$$\mathcal{L} = U(x) - \lambda(G(x) - c)$$

Let's treat this as an unconstrained problem. The FOC. of this function is:

$$\nabla U(x) - \lambda \nabla G(x) = 0$$

$$\nabla U(x) = \lambda \nabla G(x)$$

This is precisely the necessary condition we need for the constrained problem.

Thus, FOC for unconstrained optimization of the Lagrangian is the necessary constrained optimization condition.

11.3 Example (Two Constraints)

$$Max_x\,(x_1 x_2)$$

$$\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} \leq 10$$

$$2x_1 + x_2 \leq m$$

One Binds. $m = 40$**.**

After plotting the two constraints, we can see that the distance constraint is entirely contained on the interior of the budget constraint. The only constraint that could possibly bind is the distance constraint:

$$x_1 x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right)$$

$$\frac{\partial\left(x_1 x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right)\right)}{\partial x_1} \quad = \quad x_2 - \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial\left(x_1 x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right)\right)}{\partial x_2} \quad = \quad x_1 - \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}}$$

$$x_2 \frac{\sqrt{x_1^2 + x_2^2}}{x_1} = \lambda$$

$$x_1 \frac{\sqrt{x_1^2 + x_2^2}}{x_2} = \lambda$$

$$x_2^2 = x_1^2$$

$$x_1 = x_2$$

$$x_1 = x_2 = \frac{10}{\sqrt{2}}$$

$m = 15$.

Now neither constraint is contained in the other. Let's set up the Lagrangian with both constraints:

$$x_1x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right) - \mu\left(2x_1 + x_2 - 15\right)$$

The FOCs:

$$\frac{\partial\left(x_1x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right) - \mu\left(2x_1 + x_2 - m\right)\right)}{\partial x_1} = -\frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} - 2\mu + x_2$$

$$\frac{\partial\left(x_1x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right) - \mu\left(2x_1 + x_2 - m\right)\right)}{\partial x_2} = -\frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} - \mu + x_1$$

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

Let suppose both constraints bind, then $\lambda \geq 0$, $\mu \geq 0$.
Since we are on the boundary of both constraints:

$$2x_1 + x_2 = 15$$

$$\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} = 10$$

The only point on the boundary of both constraints is:

$$x_1 \approx 2.68338, x_2 \approx 9.63325$$

Can the FOCs hold at this point? Let's see what λ and μ have to be. Both of these have to be true:

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

Plugging in x_1 and x_2 and solving we get:

$$\mu \approx 5.16181, \lambda \approx -2.57279$$

Notice the negative value of λ. This tells us, to get both to bind, we need to change the direction of the distance constraint from a $\leq$ to a $\geq$ constraint. But we can't do this. The reason both cannot bind is because that the point both bind, the slope of the boundary of both constraints is shallower than the slope of the indifference curve at that point. We cannot take a linear combination of the constraints can have the slope of that linear combination be equal to the slope of the indifference curve.

Let's suppose only the distance constraint binds:

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}}$$

We've already seen for this FOC, $x_1 = x_2$

$$x_1 = x_2 = \frac{10}{\sqrt{2}}$$

However, at this point, we are not on the interior of the budget constraint. This point costs $15\sqrt{2}$ but we only have $m = 15$. Thus, when $m = 15$, the budget constraint is violated on any point that could possible be optimal on the distance constraint.

Suppose only the budget constaint binds:

$$\frac{x_2}{2} = \mu$$

$$x_1 = \mu$$

$$x_1 = \frac{x_2}{2}$$

$$4x_1 = 15$$

$$x_1 = \frac{15}{4}$$

$$x_2 = \frac{30}{4}$$

11.4 Example: Non-Negativity Constraints and Complementary Slackness

$$u = \log(x_1) + \sqrt{x_2} + x_3$$

$$x_1 + x_2 + x_3 \leq m$$

Let's set up the Lagrangian function while putting the non-negativity constraints in explicitly:

$$\log(x_1) + \sqrt{x_2} + x_3 - \lambda(x_1 + x_2 + x_3 - m) - \mu_1(-x_1) - \mu_2(-x_2) - \mu_3(-x_3)$$

The first order conditions are:

$$\mu_1 + \frac{1}{x_1} = \lambda$$

$$\mu_2 + \frac{1}{2\sqrt{x_2}} = \lambda$$

$$\mu_3 + 1 = \lambda$$

Suppose none of our non-negativity constraints bind. By complementary slackness: $\mu_1, \mu_2, \mu_3 = 0$. The first order conditions become:

$$\frac{1}{x_1} = \lambda$$

$$\frac{1}{2\sqrt{x_2}} = \lambda$$

$$1 = \lambda$$

Solving these for x_1, x_2, x_3 :

$$1 = x_1$$

$$\frac{1}{4} = x_2$$

$$x_3 = m - \frac{5}{4}$$

If $m \geq \frac{5}{4}$, this is a valid solution to the problem and since the utility function is concave, it is sufficient for the optimum. However, note that if $m < \frac{5}{4}$, this is not a feasible solution since it violates the non-negativity constraint for x_3. What is the optimal solution in that case?

Lets suppose $x_1 \geq 0$ is binding. Thus, $x_1 = 0$. The first order condition on x_1 requires $\frac{1}{x_1} = \lambda - \mu_1$. However, at $x_1 = 0$ this equation cannot hold. Thus, $x_1 > 0$ in any solution. Similarly, we can show that $x_2 \geq 0$ cannot bind since it's first order condition requires $\frac{1}{2\sqrt{x_2}} = \lambda - \mu_2$ which is not true at $x_2 = 0$. The only alternative is that $x_1 > 0$, $x_2 > 0$ and $x_3 = 0$. Since the non-negativity constraints on x_1 and x_2 do not bind, $\mu_1 = \mu_2 = 0$. The first order conditions are:

$$\frac{1}{x_1} = \lambda$$

$$\frac{1}{2\sqrt{x_2}} = \lambda$$

$$x_1 = 2\sqrt{x_2}$$

Solving these:

$$x_1 = 2\left(\sqrt{m+1} - 1\right), x_2 = m - 2\sqrt{m+1} + 2$$
$$x_3 = 0$$

Let's look at the multipliers:

$$\lambda = \frac{1}{2\left(\sqrt{m+1} - 1\right)}$$

$$\mu_3 = \frac{1}{2\left(\sqrt{m+1} - 1\right)} - 1$$

Let's suppose $m = 1$ which meets the condition:

$$\lambda \approx 1.20711$$

$$\mu_3 = 0.20711$$

The rate at which utility increases if you relax the budget constraint is about 1.20711 (it is 1 if $m > \frac{5}{4}$). Why is it that the utility increases at a rate one less than this if we relax the non-negativity constraint on x_3. If it is relaxed we can decrease x_3 which in-turn relaxes the budget constraint at a rate of one (the price of x_3). This allows us to increase utility at a rate of 1.20711 but decreasing x_3 decreases utility at a rate of 1 so the net effect is 0.20711.

11.5 Example: Two Budget Constraints

$$u = x_1 x_2$$

Suppose there are two budgets:

$$\frac{1}{2}x_1 + 2x_2 \leq 100$$

$$3x_1 + 2x_2 \leq 250$$

Would you rather add \$1 to budget 1 or \$1 to budget 2?

$$x_1 x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)$$

$$\frac{\partial\left(x_1x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)\right)}{\partial x_1} = -\frac{\lambda}{2} - 3\mu + x_2$$

$$\frac{\partial\left(x_1x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)\right)}{\partial x_2} = -2\lambda - 2\mu + x_1$$

$$x_2 = \frac{\lambda}{2} + 3\mu$$

$$x_1 = 2\lambda + 2\mu$$

$$x_1 = 60, x_2 = 35$$

$$\lambda = 22, \mu = 8$$

If budget one is relaxed, consumer can add utility at a rate of 22 while the rate is only 8 for budget 2. Budget one is more constraining. Think about why this is. Try to do the same problem with an OR constraint. That is, only one of the two constraints need to hold.

11.6 Example CD Utility

$$u\left(x_1, x_2\right) = x_1^{\alpha} x_2^{\beta}$$

$$p_1 x_1 + p_2 x_2 \leq m$$

$$x_1^{\alpha} x_2^{\beta} - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)$$

$$\frac{\partial\left(x_1^\alpha x_2^\beta - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1$$

$$\alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0$$

$$\frac{\partial\left(x_1^\alpha x_2^\beta - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_2} = \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2$$

$$\beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 = 0$$

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \lambda$$

$$\frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2} = \lambda$$

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

$$\frac{\alpha}{\beta} x_2 p_2 = x_1 p_1$$

Marshallian Demands:

$$x_1^* = \frac{\frac{\alpha}{\alpha+\beta} m}{p_1}$$

$$x_2^* = \frac{\frac{\beta}{\alpha+\beta} m}{p_2}$$

11.7 Indirect Utility

$$Max_x u\left(x\right)\ s.t.\, px \leq y$$

$$V\left(p, y\right) = u\left(x^*\left(p, y\right)\right)$$

For $x_1^\alpha x_2^\beta$ this is:

$$\left(\frac{\frac{\alpha}{\alpha+\beta}y}{p_1}\right)^{\alpha}\left(\frac{\frac{\beta}{\alpha+\beta}y}{p_2}\right)^{\beta}$$

If $\alpha=\beta=1$

$$\left(\frac{\frac{1}{2}y}{p_1}\right)\left(\frac{\frac{1}{2}y}{p_2}\right)=\frac{\frac{1}{4}y^2}{p_1p_2}$$

Homogeneous Functions:

$$f\left(t\boldsymbol{x}\right)=t^{\alpha}f\left(x\right)$$

f is homogeneous of degree α.

Properties.

1. Continuous.

Berge's Maximum Theorem

2. Homogeneous of degree zero in prices and income.

$V\left(t\boldsymbol{p},ty\right)=t^0V\left(\boldsymbol{p},y\right)=V\left(p,y\right)$

3. Strictly increasing in income.

Due to local non-satiation.

$$V=u\left(x_1^*,x_2^*\right)-\lambda\left(p_1x_1^*+p_2x_2^*-y\right)$$

$$V=u\left(x_1^*,x_2^*\right)-\lambda^*\left(p_1x_1^*+p_2x_2^*-y\right)$$

$$\frac{\partial V}{\partial y}=\lambda^*$$

$$\frac{MU_i}{p_i}=\lambda$$

4. Decreasing (weakly) in prices.

$$V=u\left(x_1^*,x_2^*\right)-\lambda^*\left(p_1x_1^*+p_2x_2^*-y\right)$$

$$\frac{\partial\left(u\left(x_1^*,x_2^*\right)-\lambda^*\left(p_1x_1^*+p_2x_2^*-y\right)\right)}{\partial p_1}=\left(-x_1^*\right)\lambda^*$$

5. Quasi-convex in (p,y).

$\left(p,y\right),\left(p',y'\right)\ \left(tp+\left(1-t\right)p',ty+\left(1-t\right)y'\right)$

$$(tp + (1-t)\,p', ty + (1-t)\,y')$$

Because (by homework) anything achieveable in budget $(tp + (1-t)\,p', ty + (1-t)\,y')$ is achievable in one of the two other budgets, then either $max\,\{V\,((p,y))\,, V\,((p',y'))\} \geq V\,((tp + (1-t)\,p', ty + (1-t)\,y'))$

6. Roy's Identity. (An envelope condition)

The ratio of the way utility changes with price i to the way it changes with income is proportional to the amount of i consumed. This is because as price i changes, it changes effective income by $(\Delta p_i)\,x_i$ and locally, there is no need to worry about changes in consumption level.

$$-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial y}} = -\frac{-x_i^*\lambda}{\lambda} = x_i^*$$

12 Cost Minimization

This is the dual to utility maximization.

12.1 The Dual Consumer Problem

$$min\,(px)\;\; s.t.\, u\,(x) \geq \bar{u}$$

12.2 Example CD Utility

$$min.\, p_1x_1 + p_2x_2$$

$$s.t.\, u\,(x_1, x_2) = x_1^\alpha x_2^\beta \geq \bar{u}$$

$$Min._{\boldsymbol{x}\in\mathbf{X}}\,(p_1x_1 + p_2x_2) - \lambda\left(x_1^\alpha x_2^\beta - \bar{u}\right)$$

$$\frac{\partial\left((p_1x_1 + p_2x_2) - \lambda\left(x_1^\alpha x_2^\beta - u\right)\right)}{\partial\,(x_1)} = p_1 - \lambda\alpha x_1^{\alpha-1}x_2^\beta$$

$$\frac{\partial\left((p_1x_1 + p_2x_2) - \lambda\left(x_1^\alpha x_2^\beta - u\right)\right)}{\partial\,(x_2)} = p_2 - \beta\lambda x_1^\alpha x_2^{\beta-1}$$

$$\frac{\alpha x_1^{\alpha-1}x_2^\beta}{p_1} = \frac{1}{\lambda}$$

$$\frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2} = \frac{1}{\lambda}$$

λ is the relative cost of increasing utility.

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

Tangency Condition (same as for utility maximization):

$$\frac{\alpha}{\beta} x_2 p_2 = x_1 p_1$$

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

Utility Condition:

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

$$\left(x_2 \frac{\alpha p_2}{\beta p_1} \right)^{\alpha} x_2^{\beta} = u$$

$$x_2 = \left(\frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

$$x_1 = \left(\frac{\alpha p_2}{\beta p_1} \right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

12.3 Marshallian / Hicksian Demand.

Marshallian Demand (Demand):
Amount of good i that is optimal given prices and income.

$$x_i^* \left(\boldsymbol{p}, y \right)$$

Hicksian Demand:
The amount of good i you choose to achieve utiliy u in the cheapest way:

$$x_i^h \left(\boldsymbol{p}, u \right)$$

12.4 Expenditure Function

$$e(\boldsymbol{p}, u) = \sum_{i=1}^{n} p_i x_i^h(\boldsymbol{p}, u)$$

This is the "optimized" value of the cost minimization objective subject to the utility constraint. It is analogous to the indirect utility function for utility maximization.

Properties.

1. Continuous.
Berge's Maximum Principle.
2. For $p \gg 0$, strictly increasing and unbounded above in u.
3. Increasing in p.
4. Homogeneous of degree 1 in p.
5. Concave in p.
The meaning of this in terms of economics:
If x^* is optimal at p, u and prices change. x^* still achieves the utility u. The cost of x^* thus represents an upper bound on the expenditure I need to achieve u at the new prices.
Let's talk about this one.
6. Shephard's lemma.

$$p_1 x_1^h + p_2 x_2^h - \lambda\left(u\left(x_1^h, x_2^h\right) - \bar{u}\right)$$

$$\frac{\partial\left(p_1 x_1^h + p_2 x_2^h - \lambda\left(u\left(x_1^h, x_2^h\right) - \bar{u}\right)\right)}{p_i} = x_i^h$$

$$\frac{\partial e(u, \boldsymbol{p})}{\partial p_i} = x_i^h$$

Another envelope condition that has no name:

$$\frac{\partial e(\bar{u}, \boldsymbol{p})}{\partial \bar{u}} = \lambda$$

12.5 Duality of Indirect Utility/Expenditure

In general:
$e(p, v(p, y)) \leq y$
$v(p, e(p, \bar{u})) \leq \bar{u}$
But with continuous, **strictly monotonic** utility:
$v(p, e(p, \bar{u})) = \bar{u}$
$e(p, v(p, y)) = y$
$x_i(p, y) = x_i^h(p, v(p, y))$

12.6 Example. Cobb-Douglas

Suppose $u = x_1 x_2$

Marshallian demands:

$$L = x_1 x_2 - \lambda \left(x_1 + x_2 - m \right)$$

$$\frac{\partial \left(x_1 x_2 - \lambda \left(p_1 x_1 + p_2 x_2 - m \right) \right)}{\partial x_1} = x_2 - \lambda p_1$$

$$\frac{\partial \left(x_1 x_2 - \lambda \left(p_1 x_1 + p_2 x_2 - m \right) \right)}{\partial x_2} = x_1 - \lambda p_2$$

$$\frac{x_2}{p_1} = \lambda$$

$$\frac{x_1}{p_2} = \lambda$$

Notice, the Lagrange Multiplier is exactly equal to the amount utility increases when the consumer spends marginally more on either good. Setting these equal,

$$x_1 = x_2$$

Plugging this into the budget equation: $x_1 + x_2 = m$

$$x_1 = x_2 = \frac{1}{2} \frac{m}{p_i}$$

Indirect Utility:

Plug the Marshallian demands into the utility function:

$$u = \frac{1}{2} \frac{m}{p_1} \frac{1}{2} \frac{m}{p_2}$$

$$v \left(p_1, p_2, m \right) = \frac{1}{4} \frac{m^2}{p_1 p_2}$$

Expenditure:

Invert the indirect utility function, solve for m:

$$e = \sqrt{4 u p_1 p_2}$$

Hicksian Demands:

Leverage the envelope condition, take derivative of the expenditure function to get Hicksian demands:

$$\frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_1} = x_1^h$$

$$x_1^h = \sqrt{u}\sqrt{\frac{p_2}{p_1}}$$

$$x_2^h = \sqrt{u}\sqrt{\frac{p_1}{p_2}}$$

13 Decomposition

$$T.E. = S.E. + I.E.$$

13.1 Two Types.

Hicksian

Hick's formalized the Substitution effect in the following way:

The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old utility level:**

$$\boldsymbol{SE:}\, x_i\left(p,y\right) - x_i\left(p', e\left(p', v\left(p,y\right)\right)\right)$$

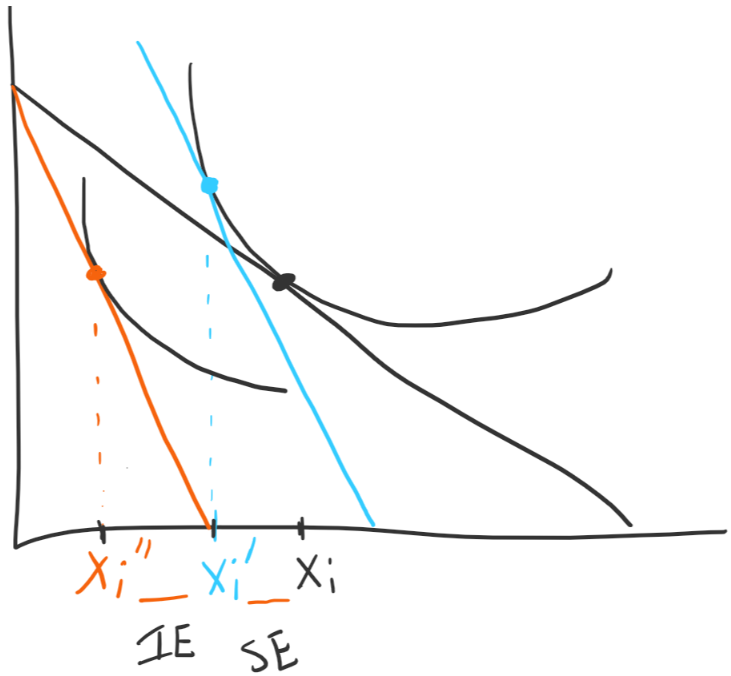

Figure 13.1: An example of the Hicksian decomposition.

The substitution effect has to be negative because a**ny demand on the compensated budget that lies below the old budget equation was previously strictly affordable.** That is, *if the price of a good increases, a consumer will demand less of it due to the substitution effect.*

Slutsky

The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old bundle:**

$$\boldsymbol{SE:} \; x_i(p, y) - x_i(p', p \cdot x^*(p, y))$$

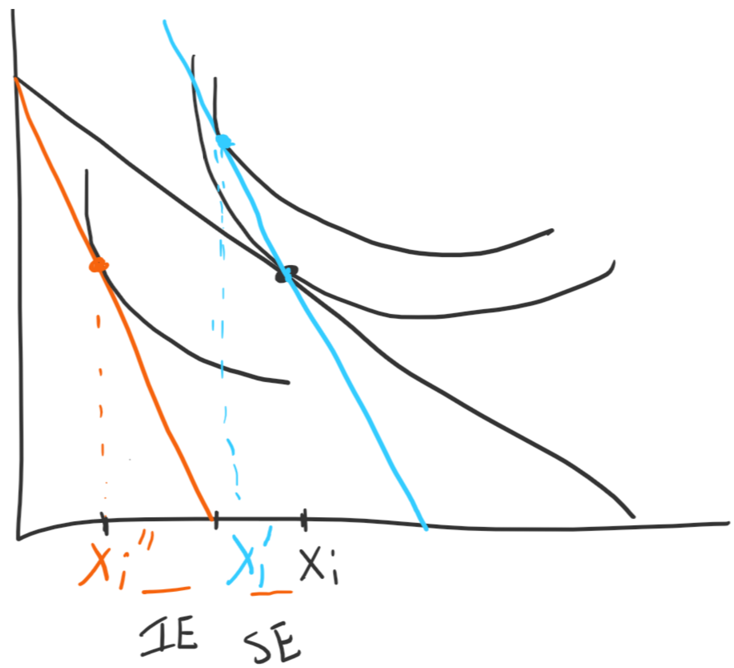

Figure 13.2: Example of the Slutsky decomposition.

The substitution effect has to be (non-positive) because any bundle with more x_i (when p_i is the price that is increasing) was previously strictly affordable.

13.2 Slutsky Equation

At the margin, the amount of money I need to maintain the same utility level is the same as the money I need to maintain the same bundle. Why? (Think first-order condition). Thus, at the margin, both methods are the same.

Let's look at the Hicksian decomposition at the margin.

$$S.E. = T.E. - I.E.$$

The substitution effect is:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_j}$$

By duality:

$$x_i(p, e(p, \bar{u})) = x_i^h(p, \bar{u})$$

Thus, T.E.-I.E. must be:

$$\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j}$$

Set them equal and expand this:

$$\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + \frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y}\frac{\partial e\left(p, \bar{u}\right)}{\partial p_j} = \frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Let's label the various effects to keep them straight.

$$(T.E.)\,\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + (-I.E.)\,\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y}\frac{\partial e\left(p, \bar{u}\right)}{\partial p_j} = (S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Use envelope condition to replace $\frac{\partial e(p,\bar{u})}{\partial p_j}$ with x_j^h.

$$(T.E.)\,\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + (-I.E.)\left(\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y}x_j^h\right) = (S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Let $e\left(p, \bar{u}\right) = y$.

$$(T.E.)\,\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j} + (-I.E.)\,\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y}x_j^h = (S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Rearranging this, we get the Slutsky Equation:

$$\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j} = \frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j} - \frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y}x_j^h$$

$$(T.E.)\,\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j}$$

$$(S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

$$(I.E.)\left(-\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y}x_j^h\right)$$

13.3 Example Cobb Douglass

Let's decompose demand:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - \frac{\partial x_i}{\partial m}\frac{\partial e}{\partial p_i}$$

Cobb Douglass utility: $x_1 x_2$

$$x_1^* = \frac{\frac{1}{2}m}{p_1}$$

$$x_2^* = \frac{\frac{1}{2}m}{p_2}$$

Indirect Utility:

$$V(p,m) = \frac{\frac{1}{2}m}{p_1}\frac{\frac{1}{2}m}{p_2} = \frac{\frac{1}{4}m^2}{p_1 p_2}$$

Leverage Duality:

$$\bar{u} = \frac{\frac{1}{4}e(p,\bar{u})^2}{p_1 p_2}$$

$$\sqrt{4\bar{u}p_1p_2} = e(p,\bar{u})$$

Use Shephards Lemma:

$$x_1^h = \frac{\partial e(p,\bar{u})}{\partial p_j} = \frac{\partial\left(\sqrt{4\bar{u}p_1p_2}\right)}{\partial p_1}$$

$$x_1^h = \frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_1} \quad = \quad \frac{p_2\bar{u}}{\sqrt{p_1p_2\bar{u}}}$$

$$x_2^h = \frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_2} = \frac{p_1\bar{u}}{\sqrt{p_1p_2\bar{u}}}$$

Substitution Effect for good 1 when $p_1$changes:

$$\frac{\partial\left(\frac{p_2u}{\sqrt{p_1p_2u}}\right)}{\partial p_1} \quad = \quad -\frac{p_2^2u^2}{2\left(p_1p_2u\right)^{3/2}}$$

Income Effect:

$$-\frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial m}\frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_1} \quad = \quad -\frac{p_2u}{2p_1\sqrt{p_1p_2u}}$$

$$-\frac{p_2u}{2p_1\sqrt{p_1p_2u}} = -\frac{p_2^2u^2}{2\left(p_1p_2u\right)^{3/2}}$$

$$-\frac{p_2u}{2p_1\sqrt{p_1p_2u}} = -\frac{\sqrt{p_2u}}{2p_1\left(p_1\right)^{1/2}}$$

Income and Substitution effect are the same for the consumer at all levels of income and prices.

13.4 Neagtive Own-Substitution Effect

$$\frac{\partial x_i^h}{\partial p_i} \leq 0$$

$$\frac{\partial^2 (e)}{(\partial p_i)^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0$$

Because e is concave, it has to be concave in any direction. Thus, the second derivative must be non-positive.

13.5 Elasticities

Suppose the price of a good changes from 1 to 2. Consumer 1's demand changes from 100 to 50 and consumer 2's changes from 10 to 5. Their behavior in terms of absolute changes in demand $\frac{\Delta x_i}{\Delta p_i}$ is wildly different, but their behavior in terms of percentage terms $\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}}$ is identical. Elasticity is simply a way of quantifying comparative statics in unit-free percentage terms.

$$\eta_i = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial y}{y}} = \frac{\partial x_i}{\partial y}\frac{y}{x_i}$$

Example: $x_1^\alpha x_2^\alpha$

$$\frac{\partial \left(\frac{A*y}{p_i}\right)}{\partial y}\frac{y}{\left(\frac{A*y}{p_i}\right)} \quad = \quad 1$$

Price and Cross-Price Elasticity.

$$\epsilon_{ij} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$$

Example: $x_1^\alpha x_2^\alpha$

$$\frac{\partial \left(\frac{A*y}{p_i}\right)}{\partial p_i}\frac{p_i}{\left(\frac{A*y}{p_i}\right)} \quad = \quad -1$$

$$\frac{\partial \left(\frac{A*y}{p_i^2}\right)}{\partial p_i}\frac{p_i}{\left(\frac{A*y}{p_i^2}\right)} \quad = \quad -2$$

13.6 Elasticity Relations

$$y = \sum_{j \in I} p_j x_j (p, y)$$

Derivative w.r.t. Price:

$$\frac{\partial y}{\partial p_i} = \frac{\partial \sum_{j \in I} p_j x_j (p, y)}{\partial p_i}$$

$$0 = \frac{\partial \sum_{j \in I} p_j x_j (p, y)}{\partial p_i}$$

$$0 = \sum_{j \neq i} p_j \frac{\partial x_j}{\partial p_i} + p_i \frac{\partial x_i}{\partial p_i} + x_i$$

$$0 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i} + x_i$$

$$-x_i = \sum_{i=1}^{n} p_j \frac{\partial x_j}{\partial p_i} \left(\frac{p_i}{x_j} \frac{x_j}{p_i} \right)$$

$$-x_i = \sum_{j=1}^{n} p_j \left(\frac{\partial x_j}{\partial p_i} \frac{p_i}{x_j} \right) \left(\frac{x_j}{p_i} \right)$$

$$-x_i = \sum_{j=1}^{n} p_j \left(\frac{x_j}{p_i} \right) \varepsilon_{j,i}$$

$$-x_i = \sum_{j=1}^{n} \left(\frac{p_j x_j}{p_i} \right) \varepsilon_{j,i}$$

$$-x_i = \frac{1}{p_i} \sum_{j=1}^{n} \left(\frac{p_j x_j}{1} \right) \varepsilon_{j,i}$$

$$-\left(\frac{1}{y} \right) \frac{x_i p_i}{1} = \left(\frac{1}{y} \right) \sum_{j=1}^{n} \left(\frac{p_j x_j}{1} \right) \varepsilon_{j,i}$$

$$-s_i = \sum_{j=1}^{n} \left(\frac{p_j x_j}{y} \right) \varepsilon_{j,i}$$

$$-s_i = \sum_{j=1}^{n} s_j \varepsilon_{j,i}$$

Derivative w.r.t. Income:

$$y = \sum_{j \in I} p_j x_j (p, y)$$

$$\frac{\partial y}{\partial y} = \frac{\partial \sum_{j \in I} p_j x_j (p, y)}{\partial y}$$

$$1 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial y}$$

Multiply by a "one"

$$1 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial y} \left(\frac{y}{x_j} \frac{x_j}{y} \right)$$

$$1 = \sum_{j=1}^{n} \frac{p_j x_j}{y} \eta_j$$

$$1 = \sum_{j \in I} s_j \eta_j$$

14 Decisions Under Uncertainty

Outcomes:

$$A \equiv \{a_1, ..., a_n\}$$

Simple gambles:

Probability distribution over outcomes.

$$\mathcal{G}_s \equiv \left\{ (p_1 \circ a_1, p_2 \circ a_2, ..., p_n \circ a_n) \,|\, \sum p_i = 1, p_i \geq 0 \right\}$$

Let $g_1, ..., g_n \in \mathcal{G}_s$

Example: $A = \{\$10, \$5, \$0\}$

$g_1 = \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0 \right)$

$g_2 = (1 \circ \$5)$ (degenerate gamble)

First-order Compound Gambles:

for all i, $g_i \in \mathcal{G}_s$

$$\mathcal{G}_{c_1} \equiv \left\{ (p_1 \circ g_1, p_2 \circ g_2, ..., p_n \circ g_n) \,|\, \sum p_i = 1 \right\}$$

Example:

$$\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ (1 \circ \$5)\right)$$

Nth-order Compound Gambles:

Let $g_i \in \mathcal{G}_{c_j}$ for $j \in \mathbb{Z}_+$.

Compound gambles:

$$\mathcal{G} \equiv \left\{(p_1 \circ g_1, p_2 \circ g_2, ..., p_m \circ g_m) \mid \sum p_i = 1\right\}$$

Example:

$$\left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2} \circ \$7\right)\right)$$

14.1 Expected Utility

$$g = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2} \circ \$7\right)\right)$$

Calculate the probability of each outcome under this gamble and construct a simple gamble that gives that outcome with it's probability. This is the *induced* simple gamble for g.

$$g_s = \frac{1}{8} \circ \$10, \frac{1}{8} \circ \$0, \frac{1}{4} \circ \$5, \frac{1}{2} \circ \$7$$

We'd like to represent their preferences over that compound gamble by:

$$\frac{1}{8} u\left(\$10\right) + \frac{1}{8} u\left(\$0\right) + \frac{1}{4} u\left(\$5\right) + \frac{1}{2} u\left(\$7\right)$$

This is the **simple gamble induced** by compound gamble g.

Suppose we have $u\left(a_i\right)$:

$$U\left(g\right) = E_g\left(u\left(a_i\right)\right)$$

Suppose gamble g induced simple gamble g_s. Let $p_i^{g_s}$ be the probability of outcome i in g_s. This utility function is of the expected value form.

$$U\left(g\right) = \sum_{i=1}^{n} p_i^{g_s} u\left(a_i\right)$$

$p_i^{g_s}$ is the probability outcome a_i occurs under gamble g. It is also the probability it occurs in the induced simple lottery g_s.

When can we represent preferences over $\mathcal{G}$ with an expected utility function?

14.2 Example

$$g = \left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2} \circ \$7\right)\right)$$

Utility is linear in money: $u(x) = x$. Under this utility function, your utility is the expected amount of money from the gamble.

$$u(g) = \frac{1}{8}(10) + \frac{1}{8}0 + \frac{1}{4}5 + \frac{1}{2}7$$

$$u(g) = 6$$

Suppose utility of money is $u(x) = log(x+1)$

$$u(g) = \frac{1}{8}(\log(10+1)) + \frac{1}{8}(\log(0+1)) + \frac{1}{4}(\log(5+1)) + \frac{1}{4}(\log(7+1))$$

$$u(g) = 1.26754$$

Utility of the expected outcome:

$$u(6) = \log(6+1)$$

$$1.94591$$

This demonstrates that a preference to reduce risk is the same as having concave utility function over prices.

14.3 Expected Utility Theorem

When can we represent $\succsim$ with an expected utility function? We really need two key things. First, this utility function is continuous, so we need to at least represent the preferences with a continuous utility function. This will require the preferences to be complete, transitive, and most importantly ***continuous.*** Second, preferences over compound gambles need to be the same as the preferences over their induced simple gambles. This relies on the preferences to have a structure that is ***linear*** in probabilities.

14.4 Expanded Proof- Continuity

Let $\succsim$ be the preference relation on $\mathcal{G}$:

Axiom 1. ***Complete:*** *$\succsim$ is complete.*

Axiom 2. ***Transitive:*** *$\succsim$ is transitive.*

Assume $a_1 \succsim a_2 ... \succsim a_n$.

Axiom 3. ***Monotonic:*** For all $(\alpha \circ a_1, (1-\alpha) \circ a_n) \succsim (\beta \circ a_1, (1-\beta) \circ a_n)$ iff $\alpha \geq \beta$, Axiom 4. ***Continuous/Archemedian:*** For all g $\exists p \in [0,1]$ such that $g \sim (p \circ a_1, (1-p) \circ a_n)$

Under assumptions $1-4$, we can represent with a continuous utility function. Let $u(g)$ be defined this way. Find the simple gamble over the best and worst outcome that g is indifferent to. Let p be the probability of the best outcome in that simple gamble. Then $u(g) = p$. For instance, if $g \sim \left(\frac{1}{4} \circ a_1, \frac{3}{4} \circ a_n\right)$ then $u(g) = \frac{1}{4}$. Implicit, we can define $u(g)$ as:

$$u(g) : g \sim (u(g) \circ a_1, (1-u(g)) \circ a_n)$$

Let's see that:

$$u(g) \geq u(g') \Leftrightarrow g \succsim g'$$

Start with:

$$g \succsim g'$$

By continuity, these are indifferent to gambles over the best and worst outcome:

$$(p \circ a_1, (1-p) \circ a_n) \sim g \succsim g' \sim (p' \circ a_1, (1-p') \circ a_n)$$

By transitivity:

$$g \succsim g' \Leftrightarrow (p \circ a_1, (1-p) \circ a_n) \succsim (p' \circ a_1, (1-p') \circ a_n)$$

By monotonicity the right side if true if and only if $p \geq p'$. We now have:

$$g \succsim g' \Leftrightarrow (p \circ a_1, (1-p) \circ a_n) \succsim (p' \circ a_1, (1-p') \circ a_n) \Leftrightarrow p \geq p'$$

By construction, $u(g) = p$. Thus:

$$g \succsim g' \Leftrightarrow (p \circ a_1, (1-p) \circ a_n) \succsim (p' \circ a_1, (1-p') \circ a_n) \Leftrightarrow p \geq p' \Leftrightarrow u(g) \geq u(g')$$

14.5 Expanded Proof- Linearity

In the above utility construction we **measure utility through the probabilities of the indifferent gambles** over the best and worst outcome. But, under the above definition, the simple gamble $(p \circ a_1, (1-p) \circ a_n)$ that is indifferent to g *can be anything.* However, for a utility function with the expected utility property, $u(g) = u(g_s)$ since all that matters are the utility of the outcomes and the probabilities of those outcome. However, if we have enough structure to ensure that simple lottery over the best and worst outcomes that is indifferent to any gamble g is the same simple lottery over the best and worst outcomes that is indifferent to the simple lottery induced by g, then we can extend the above construction and show that it is linear.

We need two additional assumptions:

Axiom 5. ***Substitution:*** If $g = (p_1 \circ g_1, ..., p_k \circ g_k)$ and $h = (p_1 \circ h_1, ..., p_k \circ h_k)$ and if $g_i \sim h_i$ for all $i \in \{1, ..., k\}$ then $g \sim h$.

Axiom 6. ***Reduction:*** For any gamble g and the simple gamble it induces g_s, $g \sim g_s$.

Now, construct utility this way: let $u(a_i)$ be defined as above:

$$u(a_i) : a_i \sim (u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n)$$

Note, instead of finding the simple gamble over the best and worst outcome **for every g.** Here, it is enough to know only those indifferent lotteries *for outcomes.* Then we extend the utility function to all gambles through expectation. Let p_i^g be the probability of outcome a_i in the simple gamble induced by g.

$$u(g) = \sum_{i=1}^{n} p_i^g u(a_i)$$

We have the following result, **under axioms 1,2,3,4,5,6:**

$$u(g) \geq u(g) \Leftrightarrow g \succsim g$$

Lets Prove it:

*By **Reduction**:*

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n,) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n,\right)$$

*By **Continuity** we know every outcome is indifferent to some simple gamble over the best and worst outcome:*

$$a_i \sim (\alpha \circ a_1, (1 - \alpha) \circ a_n)$$

*By Substitution we can glue these in to the above expression. The gamble on the right of the expression above is a **simple gamble over multiple outcomes**. By replacing every outcome by it's simple gamble over the best and worst outcome, we turn this into a **compound gamble over only the best and worst outcome**.*

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n\right)$$
$$\Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_i^{g_s} \circ (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_i^{g_s^{'}} \circ (\alpha_i a_1, (1 - \alpha_i) a_n), ..., p_n^{g_s^{'}} \circ a_n\right)$$

We can now make this a simple lottery again by applying reduction a second time:

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n\right)$$
$$\Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_i^{g_s} \circ (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_i^{g_s^{'}} \circ (\alpha_i a_1, (1 - \alpha_i) a_n), ..., p_n^{g_s^{'}} \circ a_n\right)$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} \alpha_i p_i^{g_s} \circ a_1, \sum_{i=1}^{n} (1-\alpha_i)\, p_i^{g_s} \circ a_n\right) \succsim \left(\sum_{i=1}^{n} \alpha_i p_i^{g_s^{'}} \circ a_1, \sum_{i=1}^{n} (1-\alpha_i)\, p_i^{g_s^{'}} \circ a_n\right)$$

We now have a simple gamble over the best and worst outcome again. By Monotonicity:

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow \left(p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n\right) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n\right)$$

$$\Leftrightarrow \left(p_1^{g_s} \circ a_1, ..., p_i^{g_s} \circ (\alpha_i \circ a_1, (1-\alpha_i) \circ a_n), ..., p_n^{g_s} \circ a_n\right) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_i^{g_s^{'}} \circ (\alpha_i a_1, (1-\alpha_i)\, a_n), ..., p_n^{g_s^{'}} \circ a_n\right)$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} \alpha_i p_i^{g_s} \circ a_1, \sum_{i=1}^{n} (1-\alpha_i)\, p_i^{g_s} \circ a_n\right) \succsim \left(\sum_{i=1}^{n} \alpha_i p_i^{g_s^{'}} \circ a_1, \sum_{i=1}^{n} (1-\alpha_i)\, p_i^{g_s^{'}} \circ a_n\right)$$

$$\Leftrightarrow \sum_{i=1}^{n} \alpha_i p_i^{g_s} \geq \sum_{i=1}^{n} \alpha_i p_i^{g_s^{'}}$$

By *Construction of* $u(g)$:

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow \left(p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n\right) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n\right)$$

$$\Leftrightarrow \left(p_1^{g_s} \circ a_1, ..., p_i^{g_s} \circ (\alpha_i \circ a_1, (1-\alpha_i) \circ a_n), ..., p_n^{g_s} \circ a_n\right) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_i^{g_s^{'}} \circ (\alpha_i a_1, (1-\alpha_i)\, a_n), ..., p_n^{g_s^{'}} \circ a_n\right)$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} \alpha_i p_i^{g_s} \circ a_1, \sum_{i=1}^{n} (1-\alpha_i)\, p_i^{g_s} \circ a_n\right) \succsim \left(\sum_{i=1}^{n} \alpha_i p_i^{g_s^{'}} \circ a_1, \sum_{i=1}^{n} (1-\alpha_i)\, p_i^{g_s^{'}} \circ a_n\right)$$

$$\Leftrightarrow \sum_{i=1}^{n} \alpha_i p_i^{g_s} \geq \sum_{i=1}^{n} \alpha_i p_i^{g_s^{'}} \Leftrightarrow \sum_{i=1}^{n} u(a_i)\, p_i^{g_s} \geq \sum_{i=1}^{n} u(a_i)\, p_i^{g_s^{'}} \Leftrightarrow u(g) \geq u(g')$$

14.6 Another Representation

There are other combinations of axioms that give the same result, many make use of the following axiom:

Axiom. ***Independence:*** For any gambles $g, g', g''$and any $\alpha \in (0,1]$, $\alpha g + (1-\alpha)\, g'' \succsim \alpha g + (1-\alpha)\, g' + (1-\alpha)\, g'' \Leftrightarrow g \succsim g'$.

A common form of the representation theorem is the following:

Complete and transitive preference admit an EU representation IFF they are Independent and Monotone.

The same proof we used above can be applied after showing that all of the assumption 1-6 are implied by *Completeness, Transitivity, Independence, and Monotonicity.*

14.7 Risk Preferences

We look at gambles over wealth. If a consumer is an expected utility maximizer, we can represent their preferences with the following utility function:

$$u(g) = E_g(v(w)) = \sum p(w_i)\, v(w_i)$$

Suppose $v(w) = \log(w)$ and consider the gamble $g = \left(\frac{1}{2} \circ 10, \frac{1}{2} \circ 20\right)$

$$u(g) = \frac{1}{2}(v(10)) + \frac{1}{2}(v(20))$$

$$u(g) = \frac{1}{2}(\log(10)) + \frac{1}{2}(\log(20)) = 2.64916$$

Let's compare this to the **utility of the expected wealth from the gamble**:

$$E_g(w) = \frac{1}{2}10 + \frac{1}{2}20 = 15$$

$$v(E_g(w)) = \log(15.0) \quad = \quad 2.70805$$

The expected wealth of the gamble is preferred to the gamble itself. We say this person is **risk averse.** Formally:

Risk Averse:

$$E_g(v(w)) < v(E_g(w))$$

Risk Loving:

$$E_g(v(w)) > v(E_g(w))$$

Risk Neutral:

$$E_g(v(w)) = v(E_g(w))$$

By **Jensen's inequality**: $E_g(v(w)) \leq (\geq)\, v(E_g(w))$ is true for all "random variables" (in this gambles g) if and only if v is concave (convex). If v is differentiable: $v''(w) \leq (\geq)\, 0$.

Note that this also gives: $E_g(v(w)) = v(E_g(w))$ if and only if v is both convex and concave. The only such functions are linear.

14.8 Certainty Equivalent / Risk Premium

The **certainty equivalent** is an amount of money such that having that money for sure is indifferent to gamble g. Formally:

$$CE(g) = v^{-1}(u(g))$$

Here, v^{-1} is the inverse of the utility function for wealth. If v is assumed strictly increasing, this is well defined. We can define risk averse in terms of the certainty equivalent:

A person is risk averse iff for all g:

$$CE(g) < E_g(w)$$

A person is risk loving iff for all g:

$$CE(g) > E_g(w)$$

From our previous example:

$$\frac{1}{2}(\log(10)) + \frac{1}{2}(\log(20.0)) = 2.64916$$

What is the certain wealth that gives this utility?

$$\log(CE) = 2.64916$$

$$CE = 14.1422$$

The expected wealth is 15 but the consumer would accept less than 15 in place of the gamble. Again this shows risk aversion.

The **risk premium** is the difference between the expected wealth and the certainty equivelent: $RP_g = E_g(w) - CE(g)$.

$$RP = 15 - 14.1422 = 0.8578$$

14.9 Two Measures of Risk Preferences and Differential Equations

Absolute Risk Aversion: $-\frac{v''(w)}{v'(w)}$

Relative Risk Aversion:$-\frac{wv''(w)}{v'(w)}$

Both of these are reasonable measures of the curvature of a utility function and have some nice properties. I bring them up to demonstrate a useful technique for finding functions that have properties given in terms of derivatives:

Suppose we want a constant relative risk aversion for some model. What utility functions can we use? We want to find all utility functions such that:

$$-\frac{wv''(w)}{v'(w)} = c$$

This is a differential equation. We can solve these easily with mathematica even if we are rusty on the various forms:

$$DSolve[-\frac{w * v''[w]}{v'[w]} == c, v[w], w]$$

$$\left\{\left\{v(w) \to \frac{c_1 w^{1-c}}{1-c} + c_2\right\}\right\}$$

This tells us every utility function for wealth with constant relative risk aversion is an affine transformation of:

$$v(w) = \frac{w^{1-c}}{1-c}$$

Part IV
The Firm

15 The Firm's Problem

15.1 Technology

The most general way to represent the technology available to a firm is through a production possibilities set.

$$Y \subset \mathbb{R}^m$$

For instance the vector $(-1,-1,1)$ says:l take one input of goods 1 and 2 and create one output of good 3.

Example. Two apples and one crust make a pie.

$$Y = (-2,-1,1),(-4,-2,2),(-6,-3,3)...$$

If only one of the goods is ever an output and the rest are always inputs, we can represent this with a function. For instance, the production function for pies:

$$f(\boldsymbol{x}) = min\left\{\frac{1}{2}x_1, x_2\right\}$$

We will work with production functions rather than possibilities set through the rest of the class, but it is nice to be aware more flexible language exists.

15.2 Assumptions on $f(\boldsymbol{x})$

1. **Continuous**
2. **Strictly Increasing**
3. **Strictly Quasi-Concave**
4. $f(0) = 0$

15.3 Cost Minimization

15.4 MRTS and Separability

15.5 Homogeneous/Homothetic Production