Microeconomics Lecture Notes

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Part I
The Consumer Problem

1 Budget

1.1 Consumption Set X

Assumptions: (Universe of Choice Objects): X

Bundles: Elements of X. $x \in X$

Assumptions about X.
1. $\emptyset \neq X \subseteq \mathbb{R}^n_+$.
2. X is closed.
3. X is convex.
4. $0 \in X$.

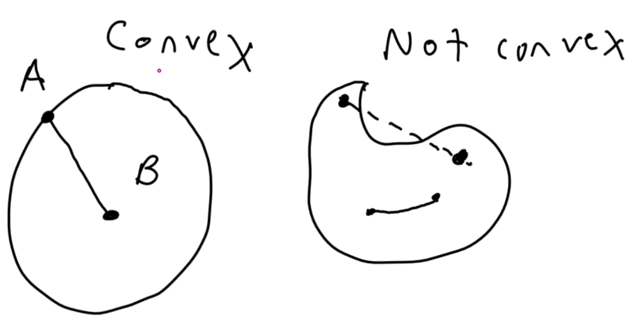

Figure 1.1: Examples of a Convex/Non-Convex Set.

1.2 Budget Set B

Budget Set: $B \subseteq X$

X defines the scope of the model. B is what an *individual consumer* chooses among.

Example. Budget Set with Prices and Income

$B = \{x | x \in X \,\&\, x_1 p_1 + x_2 p_2 \leq m\}$

Example. Ice Cream Bowls

Every ice cream bowl x has some non-negative number of scoops of Vanilla, Chocolate, Strawberry.

$$X = \mathbb{R}^3_+$$

Budget B is the set of bowls with *no more than one scoop of ice cream.*

$$B = \left\{ x | x \in R^3_+ \& \sum_{i=1}^{3} x_i \leq 1 \right\}$$

This is the unit-simplex in $\mathbb{R}_3$.

$(1, 0, 0) \in B$. (On the boundary.)

$(0.5, 0.5, 0) \in B$. (On the boundary.)

$(0.25, 0.25, 0.25) \in B$. (In the interior.)

$(2, 0, 0) \notin B$

2 Preference

2.1 The Preference Relation

Preference Relation is a **Binary Relation**.

Formally, a binary relation on set X is a subset of the Cartesian product X with itself.

$$\succeq \subseteq X \times X$$

Another way to denote an ordered pair is "in" the relation:

If $(x, y) \in \succeq$ we can also write $x \succeq y$.

Informally we say "x" is at least as good as "y", or "x" preferred "y".

Axioms of $\succeq$.

Axiom 0 *(reflexive):* $\forall x \in X$, $x \succeq x$. This is implied by *axiom 1*.

Axiom 1 *(complete):* $\forall x, x' \in X$, either $x \succeq x'$ or $x' \succeq x$ (or both).

The consumer has "some" preference over every pair of objects.

Axiom 2 *(transitive):* $\forall x, x', x'' \in X$ *if* $x \succeq x'$ *and* $x' \succeq x'' \Rightarrow x \succeq x''$.

$\succeq$ is a "weak order" if it is complete and transitive.

2.2 Relations and Sets Related to $\succeq$

Subrelations:

$\sim$ is the indifference relation. $x \succeq y$ and $y \succeq x \Leftrightarrow x \sim y$.

$\succ$ is the strict relation. $x \succeq y$ and not $y \succeq x \Leftrightarrow x \succ y$.

Related Sets:

$\succeq (x)$ "upper contour set", "no worse than set"

$\precsim (x)$ "lower contour set", "no better than set"

2.3 From Preferences to Choice

Choice Correspondence.

We will assume that from a budget set B a consumer "chooses" *choice set* C according to their preference $\succeq$. $C = \{x | x \in B \,\&\, \forall x' \in B, x \succeq x'\}$.

Informally, C is the set of objects that are at least as good as anything else in the set.

Example With Transitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, c \succeq b$.

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = c$$

$$C(\{a, b, c\}) = c$$

2.4 Cycles Lead to Empty Choice Sets

The Problem with Intransitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, b \succeq c$. *This is intransitive.*

The choice correspondence is:

$$C : P(X) / \emptyset \rightarrow X$$

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a,b\}) = a, C(\{a,c\}) = c, C(\{b,c\}) = b$$

$$C(\{a,b,c\}) = \emptyset$$

This consumer cannot make a choice from the set $\{a,b,c\}$.

2.5 Cycles and Empty Choices

Notice in the previous example, $a \succ b, a \succ c, c \succ a$. We have proved (essentially) that if there is a cycle, there is an empty choice set.

In fact, suppose, there is an empty choice set **and** X is finite. There must be a cycle.

$$\forall x \in B, \#(\succsim(x)) < \#(B)$$

By completeness, $\forall x \exists x' \in X : x' \succ x$. Choose an x_1, let x_2 be any element of $\succ(x_1)$. We have $x_2 \succ x_1$. If there is an $x_3 \in \succ(x_2)$ such that $x_1 \succ x_3$ we have identified a cycle. Otherwise, we continue with an inductive step. Suppose we have $x_n \succ ... \succ x_1$. $\succ(x_n)$ is non-empty. Either it contains an element x_{n+1} such that there is an $x_i \succ x_{n+1}$ in which case we have identified a cycle or it does not and we continue with another inductive step. Either we find a cycle or reach the N_{th} step with $x_N \succ x_{n-1} \succ ... \succ x_1$. $\succ(x_N)$ is non-empty.

So, the cycle condition is equivalence to a non-empty choice set. Transitivity of $\succsim$ implies transitivity of $\succ$ which implies no cycles (try this last step at home). But do no-cycles imply transitivity of $\succsim$? No. Here is a counter-example:

$$x \succ y, y \sim z, z \succ x$$

If no-cycles of the strict preference is equivalent to non-empty choice (in finite sets), and transitvity of $\succsim$ is not equivelent to no-cycles, why do we assume it?

2.6 Intransitivity: Empty Choices, Incoherent Choices: Pick One.

Finite non-emptyness: *For any B with $\#(B) \in \mathbb{I}$, $C(B) \neq \emptyset$*
Coherence: *For every x, y and B, B' such that $x, y \in B \cap B'$, $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$.*

Suppose there is an intransitive $\succsim$. There exists either a B where $C(B) = \emptyset$ or there exists a x, y, B, B' where the choice correspondence is incoherent.

If $\succsim$ is intransitive then there is some set of three bundles x, y, z such that one of the following 4 scenarios holds:

$$1)\, x \succ y, y \succ z, z \succ x$$

$$C(\{x, y, z\}) = \emptyset$$

$$2)\, x \sim y, y \sim z, z \succ x$$

$$3)\, x \sim y, y \succ z, z \succ x$$

$$x \notin C(\{x, y, z\})$$

$$y \in C(\{x, y, z\})$$

$$x \in C(\{x, y\})$$

$$4)\, x \succ y, y \sim z, z \succ x$$

Can you find the incoherent choice?

3 Indifference Sets

3.1 Indifference Maps

To understand preferences, we often draw sets of the form $\sim(x)$. Many times these are one dimension smaller than the space of bundles, in which case we often call them *indifference curves*, but they need not have any special structure, unless we make further assumptions about preferences. There is only one things we really know about these sets.

3.2 Complete, Transitive Preferences have Indifference Sets that Do Not Intersect

Result. *Indifference curves do not cross.* For two bundles $x \succ y$, $\sim(x) \cap \sim(y) = \emptyset$.

Proof is given visually below:

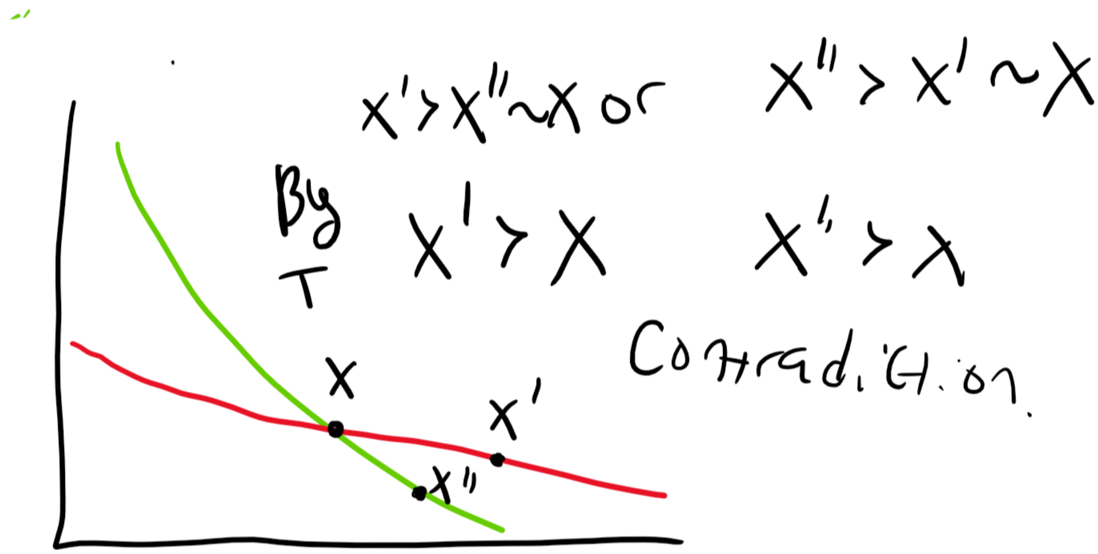

Figure 3.1: Distinct Indifference Sets do not Intersect

4 From Preference to Utility

4.1 Utility Represents Preferences

Suppose there is some $U : X \to \mathbb{R}$ such that $U(x) \geq U(x') \Leftrightarrow x \succsim x'$ then we say $u()$ represents preference relation $\succsim$. When does such a representation exist?

4.2 Constructing U under Finite X

Proposition 1. *A $U()$ exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.*

Proof. Let's start with $\Rightarrow$.

Because $\geq$ is complete on the real numbers, for every $x, y \in X$ either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ thus because $u()$ represents $\succsim$, it is complete.

By similar argument, $\succsim$ is transitive. For every three $x, y, z \in X$. If $u(x) \geq u(y)$ and $u(y) \geq u(z)$ then $u(x) \geq u(z)$ because $\geq$ is transitive on the real numbers.

Now we prove $\Leftarrow$:

Define $U(x) \equiv \#(\precsim(x))$

Example: $a \succ b, b \succ c$. $\precsim(a) = \{a, b, c\}.U(a) = 3$.

Lemma: For $x \succsim y$, $\precsim(y) \subseteq \precsim(x)$ (proved in PS1).

By this lemma, for $x \succsim y$, $\precsim(y) \subseteq \precsim(x)$ and thus $\# \precsim(y) \leq \# \precsim(x)$ and $u(x) \geq u(y)$. $\square$

4.3 Constructing U under Countably infinite X

Pick any arbitrary order on the bundles: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim(x)} w(y)$$

Example: "π shows up unexpectedly when eating ice cream."

An even number of scoops of ice cream are better than an odd number of scoops and otherwise more is better than less.

$$u(2) = \sum_{i=1}^{\infty} \left(\frac{1}{(2i-1)^2} \right) = \frac{\pi^2}{8}$$

$$u(4) = \frac{1}{4} + \frac{\pi^2}{8}$$

4.4 Failures under Uncountable X

The *Lexicographic* preferences have no utility representation:

$$X = \mathbb{R}_+^2$$

$(x_1, x_2) \succ (y_1, y_2)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$.

$\succsim$ is complete, and transitive. [*Prove this for practice*].

Pick two real numbers $v_2 > v_1$ and construct four bundles $(v_1, 1), (v_2, 1), (v_1, 2), (v_2, 2)$.

$$(v_2, 2) \succ (v_2, 1) \succ (v_1, 2) \succ (v_1, 1)$$

Suppose there is a utility function representing these preferences, then we have two disjoint intervals:

$$[u(v_2, 1), u(v_2, 2)]$$

$$[u(v_1, 1), u(v_1, 2)]$$

For every real number, we can construct an interval like this. Because the rationals are dense in the reals, there is a rational number in each of these intervals. Thus, for every real, we can find a unique rational number. That is, we have a mapping from the reals into the rationals which implies that the cardinality of the rationals is at least as large as that of the reals. $\#\mathbb{Q} \geq \#\mathbb{R}$. This contradicts that the cardinality of the rationals is strictly smaller than the reals.

4.5 An example of a lexicographic preference relation with a utility representation.

Cars have horse power in $[0, 999]$ and cup holders in $\mathbb{Z}_+$ (integers).

Suppose preferences are lexicographic and more cup holders are more important than more horsepower.

$u(c_i, h_i) = c_i + \frac{h_i}{1000}$ represents these preferences.

See problem set 2 for example where we do not bound the horse power.

4.6 What ensures a utility representation in an uncountable universe?

A preference relation is representable by a utility function $U(x)$ iff $\forall x, y \in X\, s.t.\, x \succ y$, $\exists x^* \in X^* \subset X\, s.t.\, x \succsim x^* \succ y$ and the set X^* is countable.

To construct the utility function, $U(x)$, Pick any arbitrary order on the bundles in X^*: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim(x) \cap X^*} w(y)$$

4.7 Continuous $\succsim$.

Preference relation $\succsim$ is continuous if $\forall x \in X$, $\succsim(x)$ and $\precsim(x)$ are closed in X.

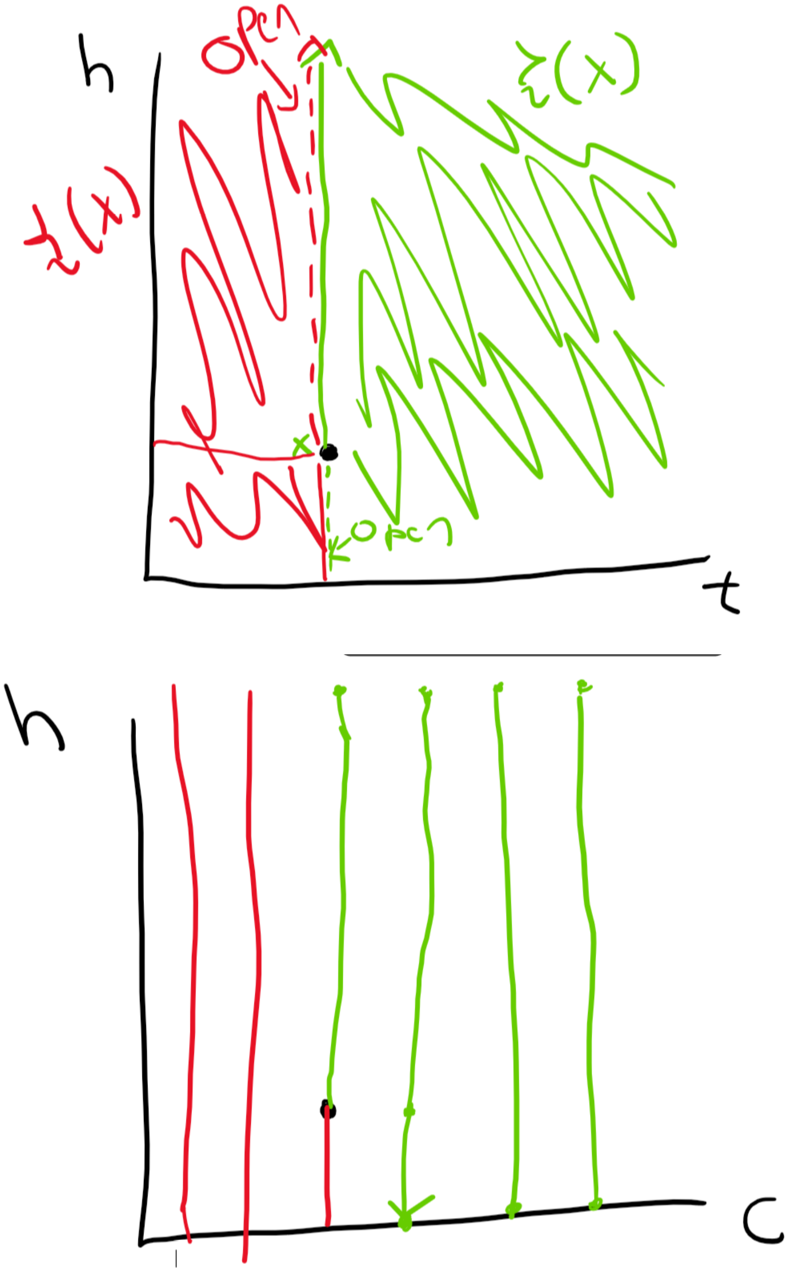

Figure 4.1: Not Continuous/Continuous Lexicographic Preferences.

4.8 What ensures a continuous utility representation?

A *complete*, *transitive,* and *continuous* preference relation $\succsim$ can be represented by a continuous utility function $U(x)$ and, a continuous utility function represented C,T,C preferences.

5 Other Properties of $\succsim$

5.1 Monotonicity

Ensure consumers consume on the boundary of the budget set.

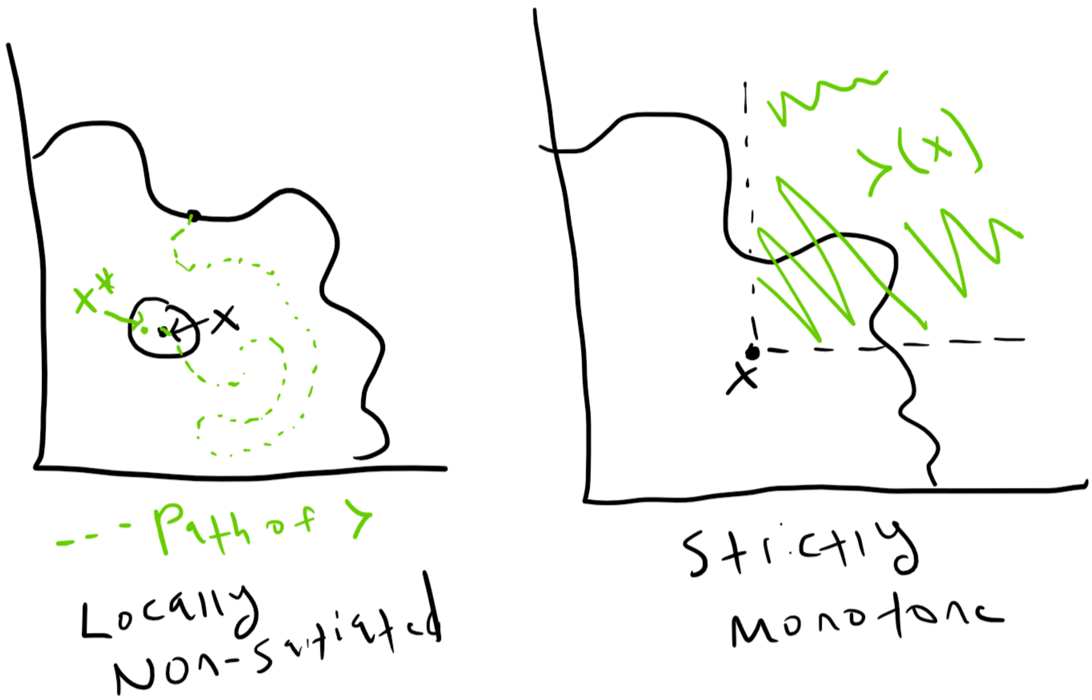

Figure 5.1: Locally Non-satiated vs. Strictly Monotone

Strict Monotonicity

More stuff is better.

First, some notation:

For $X \subseteq \mathbb{R}^n$

$x \geq x'$ iff $x_i \geq x'_i$ for all $i \in \{1, 2, ..., n\}$

$x \gg x'$ iff $x_i > x'_i$ for all $i \in \{1, 2, ..., n\}$

Fro example: $(2,2) >> (1,1), (2,1) \geq (1,1), (1,1) \geq (1,1)$

Definition. Strict Monotonicity. $x \geq x' \Rightarrow x \succsim x'$ and $x \gg x' \Rightarrow x \succ x'$

Local Nonsatiation

Definition. Local Nonsatiatin. $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x^* \in B_\varepsilon(x)$ such that $x^* \succ x$.

A consumer can always change the bundle a "little bit" no matter how small that little bit is, and find something strictly better.

5.2 Convex Sets, Convex/Concave Functions, Quasi-Convex/Concave Functions

Convex Sets

In a subset of euclidean space X, the line between $x \in X$ and $x' \in X$ is another point in the set X given by $tx + (1-t)x'$ where $t \in [0,1]$. We call points like this **Convex Combinations** of x and x'.

For example: $x = (1,0), x' = (0,1)$. If we take $t = 0.5$. The convex combination is $0.5(1,0) + 0.5(0,1) = (0.5, 0.5)$.

A **convex set** $S \subseteq X$ is a set of points that contains all of its convex combinations.

Formally, $\forall x, x' \in S$, $\forall t \in [0,1]$, $tx + (1-t)x' \in S$.

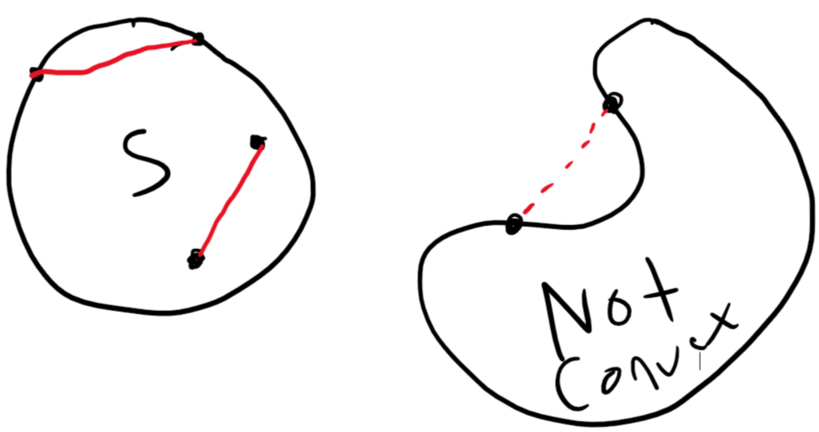

Figure 5.2: A Convex and Non-Convex Set

Convex Functions

A line between two points "on the function" lies above the function itself.
***Convex Function**:*

$$\forall x, x' \in X, t \in [0,1], tf(x) + (1-t) f(x') \geq f(tx + (1-t)x')$$

***Strictly Convex Function**:*

$$\forall x, x' \in X, t \in (0,1), tf(x) + (1-t) f(x') > f(tx + (1-t)x')$$

Contour Sets:
A convex function has **convex lower contour sets**.

Concave Functions

A line between two points "on the function" lies above the function itself.
***Concave Function**:*

$$\forall x, x' \in X, t \in [0,1], tf(x) + (1-t) f(x') \leq f(tx + (1-t)x')$$

***Strictly Concave Function**:*

$$\forall x, x' \in X, t \in (0,1), tf(x) + (1-t) f(x') < f(tx + (1-t)x')$$

A concave function has **convex upper contour sets**.

Quasi-Concave Functions

A function $f(x)$ is quasi-concave if **and only** it has convex upper contour sets.
A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a concave function.
A function $f(x)$ is quasi-concave if and only if $f(tx + (1-t)x') \geq min\{f(x), f(x')\}$ for $t \in [0,1]$.
A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t)x') > min\{f(x), f(x')\}$ for $t \in (0,1)$.

Notice that, for a strictly quasi-concave utility function, let $x' \succ x$, then the set $tx+(1-t)x' \in \succ(x)$ for $t \in (0,1)$. Thus, there is a small enough ball around that point $B_\epsilon(tx + (1-t)x') \in \succsim(x)$. Thus, these points are in the interior of $\succsim(x)$ and $\succsim$ is **strictly convex.**

Quasi-Convex Functions

A function $f(x)$ is quasi-convex if **and only** it has convex lower contour sets.
A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a convex function.
A function $f(x)$ is quasi-convex if and only if $f(tx + (1-t)x') \leq max\{f(x), f(x')\}$ for $t \in [0,1]$.
A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t)x') < max\{f(x), f(x')\}$ for $t \in (0,1)$.

5.3 Convexity of $\succsim$.

Convex Preferences: $x \succsim x' \Rightarrow t(x) + (1-t)x' \succsim x', t \in [0,1]$
$x \in \succsim (x') \Rightarrow t(x) + (1-t)x' \in \succsim (x')$
Thus, $\succsim (x)$ are convex if $\succsim$ is a convex preference relation.
Strictly Convex Preferences: $x \succsim x' \Rightarrow t(x) + (1-t)x' \succ x', t \in (0,1)$
The upper contour sets $\succsim (x)$ are *strictly* convex.

5.4 Utility and Preference Relationships

If U represents $\succsim$:
1) $\succsim$ (strictly) **convex** $\Leftrightarrow$ U is (strictly) **quasi-concave**.
2) $\succsim$ are **strictly monotonic** $\Leftrightarrow$ U is **strictly** increasing.
3) $\succsim$ are **strictly monotonic** $\Leftarrow$ U is **strongly** increasing.

6 The Consumer Problem

6.1 Choice

The set of all "best things" in the budget set. This is what we are looking for:
$C(B) = \{x | x \in B \wedge x \succsim x', \forall x' \in B\}$

Competitive Budgets:

$$B = \left\{x | x \in \mathbb{R}^n_+, p \cdot x \leq m\right\}$$

p is the vector of prices.
m is the "income".
Constrained problem:

$$Max_{x \in X} U(x)\, s.t.\, p \cdot x \leq m$$

6.2 The Lagrange Method- Some Intuition.

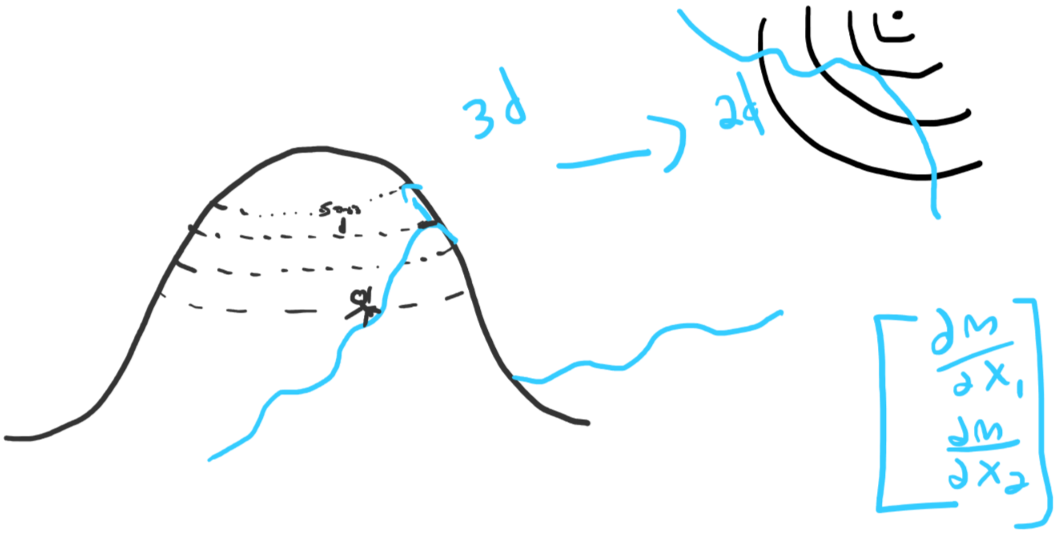

Figure 6.1: Finding the best spot for a selfie.

If both the objective and the constraint are smooth, at the optimal **the *direction* of the gradient of the objective has to be equal to the *direction* of the gradient of the constraint.** Otherwise, moving along the constraint boundary in *some* direction will yield a larger value of the objective! (Caveat: this assumes we *can* move in every direction along the constraint. That will only be true at non-boundary points.)

Thus, for smooth functions, the equality of the direction of the gradients of the objective and the constraint are **necessary** for an non-boundary optimum.

Since the direction of the gradient is just a scaling of the gradient, suppose U is our objective and G is the function for the boundary of the constraint. Then,

$$\nabla U(x) = \lambda \nabla G(x)$$

Can we write a function such that the first order condition will yield this gradient condition? *Sure*:

$$\mathcal{L} = U(x) - \lambda (G(x) - c)$$

Let's treat this as an unconstrained problem. The FOC. of this function is:

$$\nabla U(x) - \lambda \nabla G(x) = 0$$

$$\nabla U(x) = \lambda \nabla G(x)$$

This is precisely the necessary condition we need for the constrained problem.

Thus, FOC for unconstrained optimization of the Lagrangian is the necessary constrained optimization condition.

6.3 Example (Two Constraints)

$$Max_x\,(x_1x_2)$$

$$\left(x_1^2+x_2^2\right)^{\frac{1}{2}} \leq 10$$

$$2x_1+x_2 \leq m$$

One Binds. $m = 40$**.**

After plotting the two constraints, we can see that the distance constraint is entirely contained on the interior of the budget constraint. The only constraint that could possibly bind is the distance constraint:

$$x_1x_2-\lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)$$

$$\frac{\partial\left(x_1x_2-\lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)\right)}{\partial x_1} = x_2-\frac{\lambda x_1}{\sqrt{x_1^2+x_2^2}}$$

$$\frac{\partial\left(x_1x_2-\lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)\right)}{\partial x_2} = x_1-\frac{\lambda x_2}{\sqrt{x_1^2+x_2^2}}$$

$$x_2\frac{\sqrt{x_1^2+x_2^2}}{x_1}=\lambda$$

$$x_1\frac{\sqrt{x_1^2+x_2^2}}{x_2}=\lambda$$

$$x_2^2=x_1^2$$

$$x_1=x_2$$

$$x_1=x_2=\frac{10}{\sqrt{2}}$$

$m = 15$**.**

Now neither constraint is contained in the other. Let's set up the Lagrangian with both constraints:

$$x_1 x_2 - \lambda \left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10 \right) - \mu \left(2x_1 + x_2 - 15\right)$$

The FOCs:

$$\frac{\partial \left(x_1 x_2 - \lambda \left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10 \right) - \mu \left(2x_1 + x_2 - m\right) \right)}{\partial x_1} = -\frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} - 2\mu + x_2$$

$$\frac{\partial \left(x_1 x_2 - \lambda \left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10 \right) - \mu \left(2x_1 + x_2 - m\right) \right)}{\partial x_2} = -\frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} - \mu + x_1$$

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

Let suppose both constraints bind, then $\lambda \geq 0$, $\mu \geq 0$.
Since we are on the boundary of both constraints:

$$2x_1 + x_2 = 15$$

$$\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} = 10$$

The only point on the boundary of both constraints is:

$$x_1 \approx 2.68338, x_2 \approx 9.63325$$

Can the FOCs hold at this point? Let's see what λ and μ have to be. Both of these have to be true:

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

Plugging in x_1 and x_2 and solving we get:

$$\mu \approx 5.16181, \lambda \approx -2.57279$$

Notice the negative value of λ. This tells us, to get both to bind, we need to change the direction of the distance constraint from a $\leq$ to a $\geq$ constraint. But we can't do this. The reason both cannot bind is because that the point both bind, the slope of the boundary of both constraints is shallower than the slope of the indifference curve at that point. We cannot take a linear combination of the constraints can have the slope of that linear combination be equal to the slope of the indifference curve.

Let's suppose only the distance constraint binds:

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}}$$

We've already seen for this FOC, $x_1 = x_2$

$$x_1 = x_2 = \frac{10}{\sqrt{2}}$$

However, at this point, we are not on the interior of the budget constraint. This point costs $15\sqrt{2}$ but we only have $m = 15$. Thus, when $m = 15$, the budget constraint is violated on any point that could possible be optimal on the distance constraint.

Suppose only the budget constaint binds:

$$\frac{x_2}{2} = \mu$$

$$x_1 = \mu$$

$$x_1 = \frac{x_2}{2}$$

$$4x_1 = 15$$

$$x_1 = \frac{15}{4}$$

$$x_2 = \frac{30}{4}$$

6.4 Example: Non-Negativity Constraints and Complementary Slackness

$$u = \log(x_1) + \sqrt{x_2} + x_3$$

$$x_1 + x_2 + x_3 \leq m$$

Let's set up the Lagrangian function while putting the non-negativity constraints in explicitly:

$$\log(x_1) + \sqrt{x_2} + x_3 - \lambda(x_1 + x_2 + x_3 - m) - \mu_1(-x_1) - \mu_2(-x_2) - \mu_3(-x_3)$$

The first order conditions are:

$$\mu_1 + \frac{1}{x_1} = \lambda$$

$$\mu_2 + \frac{1}{2\sqrt{x_2}} = \lambda$$

$$\mu_3 + 1 = \lambda$$

Suppose none of our non-negativity constraints bind. By complementary slackness: $\mu_1, \mu_2, \mu_3 = 0$. The first order conditions become:

$$\frac{1}{x_1} = \lambda$$

$$\frac{1}{2\sqrt{x_2}} = \lambda$$

$$1 = \lambda$$

Solving these for x_1, x_2, x_3 :

$$1 = x_1$$

$$\frac{1}{4} = x_2$$

$$x_3 = m - \frac{5}{4}$$

If $m \geq \frac{5}{4}$, this is a valid solution to the problem and since the utility function is concave, it is sufficient for the optimum. However, note that if $m < \frac{5}{4}$, this is not a feasible solution since it violates the non-negativity constraint for x_3. What is the optimal solution in that case?

Lets suppose $x_1 \geq 0$ is binding. Thus, $x_1 = 0$. The first order condition on x_1 requires $\frac{1}{x_1} = \lambda - \mu_1$. However, at $x_1 = 0$ this equation cannot hold. Thus, $x_1 > 0$ in any solution. Similarly, we can show that $x_2 \geq 0$ cannot bind since it's first order condition requires $\frac{1}{2\sqrt{x_2}} = \lambda - \mu_2$ which is not true at $x_2 = 0$. The only alternative is that $x_1 > 0$, $x_2 > 0$ and $x_3 = 0$. Since the non-negativity constraints on x_1 and x_2 do not bind, $\mu_1 = \mu_2 = 0$. The first order conditions are:

$$\frac{1}{x_1} = \lambda$$

$$\frac{1}{2\sqrt{x_2}} = \lambda$$

$$x_1 = 2\sqrt{x_2}$$

Solving these:

$$x_1 = 2\left(\sqrt{m+1} - 1\right), x_2 = m - 2\sqrt{m+1} + 2$$
$$x_3 = 0$$

Let's look at the multipliers:

$$\lambda = \frac{1}{2\left(\sqrt{m+1} - 1\right)}$$

$$\mu_3 = \frac{1}{2\left(\sqrt{m+1} - 1\right)} - 1$$

Let's suppose $m = 1$ which meets the condition:

$$\lambda \approx 1.20711$$

$$\mu_3 = 0.20711$$

The rate at which utility increases if you relax the budget constraint is about 1.20711 (it is 1 if $m > \frac{5}{4}$). Why is it that the utility increases at a rate one less than this if we relax the non-negativity constraint on x_3. If it is relaxed we can decrease x_3 which in-turn relaxes the budget constraint at a rate of one (the price of x_3). This allows us to increase utility at a rate of 1.20711 but decreasing x_3 decreases utility at a rate of 1 so the net effect is 0.20711.

6.5 Example: Two Budget Constraints

$$u = x_1 x_2$$

Suppose there are two budgets:

$$\frac{1}{2}x_1 + 2x_2 \leq 100$$

$$3x_1 + 2x_2 \leq 250$$

Would you rather add \$1 to budget 1 or \$1 to budget 2?

$$x_1 x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)$$

$$\frac{\partial\left(x_1x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)\right)}{\partial x_1} = -\frac{\lambda}{2} - 3\mu + x_2$$

$$\frac{\partial\left(x_1x_2 - \lambda\left(\frac{1}{2}x_1 + 2x_2 - 100\right) - \mu\left(3x_1 + 2x_2 - 250\right)\right)}{\partial x_2} = -2\lambda - 2\mu + x_1$$

$$x_2 = \frac{\lambda}{2} + 3\mu$$

$$x_1 = 2\lambda + 2\mu$$

$$x_1 = 60, x_2 = 35$$

$$\lambda = 22, \mu = 8$$

If budget one is relaxed, consumer can add utility at a rate of 22 while the rate is only 8 for budget 2. Budget one is more constraining. Think about why this is. Try to do the same problem with an OR constraint. That is, only one of the two constraints need to hold.

6.6 Example CD Utility

$$u\left(x_1, x_2\right) = x_1^{\alpha} x_2^{\beta}$$

$$p_1 x_1 + p_2 x_2 \leq m$$

$$x_1^{\alpha} x_2^{\beta} - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)$$

$$\frac{\partial\left(x_1^{\alpha}x_2^{\beta}-\lambda\left(p_1x_1+p_2x_2-m\right)\right)}{\partial x_1} = \alpha x_1^{\alpha-1}x_2^{\beta}-\lambda p_1$$

$$\alpha x_1^{\alpha-1}x_2^{\beta}-\lambda p_1=0$$

$$\frac{\partial\left(x_1^{\alpha}x_2^{\beta}-\lambda\left(p_1x_1+p_2x_2-m\right)\right)}{\partial x_2} = \beta x_1^{\alpha}x_2^{\beta-1}-\lambda p_2$$

$$\beta x_1^{\alpha}x_2^{\beta-1}-\lambda p_2=0$$

$$\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{p_1}=\lambda$$

$$\frac{\beta x_1^{\alpha}x_2^{\beta-1}}{p_2}=\lambda$$

$$\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{p_1}=\frac{\beta x_1^{\alpha}x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta}\frac{x_2}{x_1}=\frac{p_1}{p_2}$$

$$\frac{\alpha}{\beta}x_2p_2=x_1p_1$$

Marshallian Demands:

$$x_1^*=\frac{\frac{\alpha}{\alpha+\beta}m}{p_1}$$

$$x_2^*=\frac{\frac{\beta}{\alpha+\beta}m}{p_2}$$

6.7 Indirect Utility

$$Max_x u\left(x\right)\ s.t.\, px\leq y$$

$$V\left(p,y\right)=u\left(x^*\left(p,y\right)\right)$$

For $x_1^{\alpha}x_2^{\beta}$ this is:

$$\left(\frac{\frac{\alpha}{\alpha+\beta}y}{p_1}\right)^{\alpha}\left(\frac{\frac{\beta}{\alpha+\beta}y}{p_2}\right)^{\beta}$$

If $\alpha=\beta=1$

$$\left(\frac{\frac{1}{2}y}{p_1}\right)\left(\frac{\frac{1}{2}y}{p_2}\right)=\frac{\frac{1}{4}y^2}{p_1p_2}$$

Homogeneous Functions:

$$f\left(t\boldsymbol{x}\right)=t^{\alpha}f\left(x\right)$$

f is homogeneous of degree α.

Properties.

1. Continuous.

Berge's Maximum Theorem

2. Homogeneous of degree zero in prices and income.

$V\left(t\boldsymbol{p},ty\right)=t^0V\left(\boldsymbol{p},y\right)=V\left(p,y\right)$

3. Strictly increasing in income.

Due to local non-satiation.

$$V=u\left(x_1^*,x_2^*\right)-\lambda\left(p_1x_1^*+p_2x_2^*-y\right)$$

$$V=u\left(x_1^*,x_2^*\right)-\lambda^*\left(p_1x_1^*+p_2x_2^*-y\right)$$

$$\frac{\partial V}{\partial y}=\lambda^*$$

$$\frac{MU_i}{p_i}=\lambda$$

4. Decreasing (weakly) in prices.

$$V=u\left(x_1^*,x_2^*\right)-\lambda^*\left(p_1x_1^*+p_2x_2^*-y\right)$$

$$\frac{\partial\left(u\left(x_1^*,x_2^*\right)-\lambda^*\left(p_1x_1^*+p_2x_2^*-y\right)\right)}{\partial p_1}=\left(-x_1^*\right)\lambda^*$$

5. Quasi-convex in (p,y).

$\left(p,y\right),\left(p',y'\right)\ \left(tp+\left(1-t\right)p',ty+\left(1-t\right)y'\right)$

$$(tp + (1-t)\,p', ty + (1-t)\,y')$$

Because (by homework) anything achieveable in budget $(tp + (1-t)\,p', ty + (1-t)\,y')$ is achievable in one of the two other budgets, then either $max\,\{V\left((p,y)\right), V\left((p',y')\right)\} \geq V\left((tp + (1-t)\,p', ty + (1-t)\,y')\right)$

6. Roy's Identity. (An envelope condition)

The ratio of the way utility changes with price i to the way it changes with income is proportional to the amount of i consumed. This is because as price i changes, it changes effective income by $(\Delta p_i)\,x_i$ and locally, there is no need to worry about changes in consumption level.

$$-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial y}} = -\frac{-x_i^*\lambda}{\lambda} = x_i^*$$

7 Cost Minimization

This is the dual to utility maximization.

7.1 The Dual Consumer Problem

$$min\,(px)\;\; s.t.\, u\,(x) \geq \bar{u}$$

7.2 Example CD Utility

$$min.\, p_1x_1 + p_2x_2$$

$$s.t.\, u\,(x_1, x_2) = x_1^\alpha x_2^\beta \geq \bar{u}$$

$$Min._{\boldsymbol{x}\in\boldsymbol{X}}\,(p_1x_1 + p_2x_2) - \lambda\left(x_1^\alpha x_2^\beta - \bar{u}\right)$$

$$\frac{\partial\left((p_1x_1 + p_2x_2) - \lambda\left(x_1^\alpha x_2^\beta - u\right)\right)}{\partial\,(x_1)} = p_1 - \lambda\alpha x_1^{\alpha-1}x_2^\beta$$

$$\frac{\partial\left((p_1x_1 + p_2x_2) - \lambda\left(x_1^\alpha x_2^\beta - u\right)\right)}{\partial\,(x_2)} = p_2 - \beta\lambda x_1^\alpha x_2^{\beta-1}$$

$$\frac{\alpha x_1^{\alpha-1}x_2^\beta}{p_1} = \frac{1}{\lambda}$$

$$\frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2} = \frac{1}{\lambda}$$

λ is the relative cost of increasing utility.

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

Tangency Condition (same as for utility maximization):

$$\frac{\alpha}{\beta} x_2 p_2 = x_1 p_1$$

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

Utility Condition:

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

$$\left(x_2 \frac{\alpha p_2}{\beta p_1} \right)^{\alpha} x_2^{\beta} = u$$

$$x_2 = \left(\frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

$$x_1 = \left(\frac{\alpha p_2}{\beta p_1} \right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

7.3 Marshallian / Hicksian Demand.

Marshallian Demand (Demand):
Amount of good i that is optimal given prices and income.

$$x_i^* (\boldsymbol{p}, y)$$

Hicksian Demand:
The amount of good i you choose to achieve utiliy u in the cheapest way:

$$x_i^h (\boldsymbol{p}, u)$$

7.4 Expenditure Function

$$e(\boldsymbol{p}, u) = \sum_{i=1}^{n} p_i x_i^h(\boldsymbol{p}, u)$$

This is the "optimized" value of the cost minimization objective subject to the utility constraint. It is analogous to the indirect utility function for utility maximization.

Properties.

1. Continuous.
Berge's Maximum Principle.
2. For $p \gg 0$, strictly increasing and unbounded above in u.
3. Increasing in p.
4. Homogeneous of degree 1 in p.
5. Concave in p.
The meaning of this in terms of economics:
If x^* is optimal at p, u and prices change. x^* still achieves the utility u. The cost of x^* thus represents an upper bound on the expenditure I need to achieve u at the new prices.
Let's talk about this one.
6. Shephard's lemma.

$$p_1 x_1^h + p_2 x_2^h - \lambda\left(u\left(x_1^h, x_2^h\right) - \bar{u}\right)$$

$$\frac{\partial\left(p_1 x_1^h + p_2 x_2^h - \lambda\left(u\left(x_1^h, x_2^h\right) - \bar{u}\right)\right)}{p_i} = x_i^h$$

$$\frac{\partial e(u, \boldsymbol{p})}{\partial p_i} = x_i^h$$

Another envelope condition that has no name:

$$\frac{\partial e(\bar{u}, \boldsymbol{p})}{\partial \bar{u}} = \lambda$$

7.5 Duality of Indirect Utility/Expenditure

In general:
$e(p, v(p, y)) \leq y$
$v(p, e(p, \bar{u})) \leq \bar{u}$
But with continuous, **strictly monotonic** utility:
$v(p, e(p, \bar{u})) = \bar{u}$
$e(p, v(p, y)) = y$
$x_i(p, y) = x_i^h(p, v(p, y))$

7.6 Example. Cobb-Douglas

Suppose $u = x_1 x_2$

Marshallian demands:

$$L = x_1 x_2 - \lambda\left(x_1 + x_2 - m\right)$$

$$\frac{\partial\left(x_1 x_2 - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_1} = x_2 - \lambda p_1$$

$$\frac{\partial\left(x_1 x_2 - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_2} = x_1 - \lambda p_2$$

$$\frac{x_2}{p_1} = \lambda$$

$$\frac{x_1}{p_2} = \lambda$$

Notice, the Lagrange Multiplier is exactly equal to the amount utility increases when the consumer spends marginally more on either good. Setting these equal,

$$x_1 = x_2$$

Plugging this into the budget equation: $x_1 + x_2 = m$

$$x_1 = x_2 = \frac{1}{2}\frac{m}{p_i}$$

Indirect Utility:

Plug the Marshallian demands into the utility function:

$$u = \frac{1}{2}\frac{m}{p_1}\frac{1}{2}\frac{m}{p_2}$$

$$v\left(p_1, p_2, m\right) = \frac{1}{4}\frac{m^2}{p_1 p_2}$$

Expenditure:

Invert the indirect utility function, solve for m:

$$e = \sqrt{4up_1p_2}$$

Hicksian Demands:

Leverage the envelope condition, take derivative of the expenditure function to get Hicksian demands:

$$\frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_1} = x_1^h$$

$$x_1^h = \sqrt{u}\sqrt{\frac{p_2}{p_1}}$$

$$x_2^h = \sqrt{u}\sqrt{\frac{p_1}{p_2}}$$

8 Decomposition

When price increases, there are two intuitive effects. The substitution effect captures the decrease into demand due to a consumer's willingness to trade-off to goods which are now relatively cheaper. The income effect captures the change in demand due to the fact that whatever the consumer's remaining demand after substitution, the good is not relatively more expensive. This can either increase or decrease demand, depending on how the consumer's preferences react to changes in relative income.

$$T.E. = S.E. + I.E.$$

8.1 Two Types

There are two ways to decompose demand. These correspond to two closely related thought experiments that attempt to capture these effects in an objective way.

Hicksian

Hick's formalized the Substitution effect in the following way:

The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old utility level:**

$$\boldsymbol{SE:}\, x_i\left(p, y\right) - x_i\left(p', e\left(p', v\left(p, y\right)\right)\right)$$

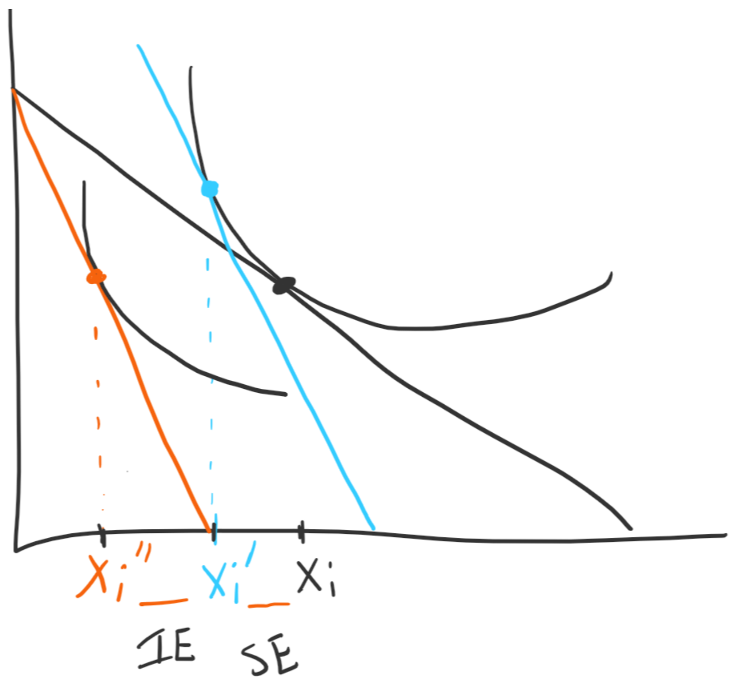

Figure 8.1: An example of the Hicksian decomposition.

The substitution effect has to be negative because a**ny demand on the compensated budget that lies below the old budget equation was previously strictly affordable.** That is, *if the price of a good increases, a consumer will demand less of it due to the substitution effect.*

Slutsky

The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old bundle:**

$$\boldsymbol{SE:}\ x_i\left(p,y\right)-x_i\left(p',p\cdot x^*\left(p,y\right)\right)$$

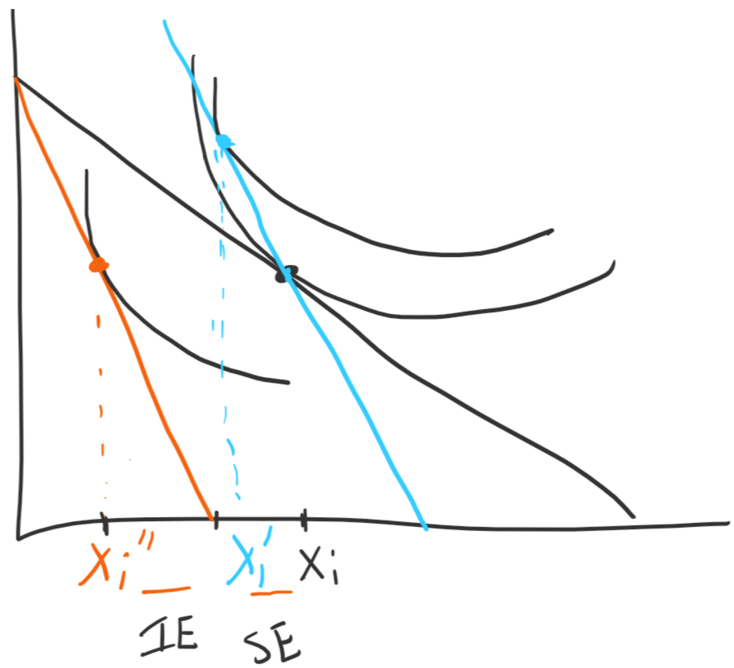

Figure 8.2: Example of the Slutsky decomposition.

The substitution effect has to be (non-positive) because any bundle with more x_i (when p_i is the price that is increasing) was previously strictly affordable.

8.2 Slutsky Equation

At the margin, the amount of money a consumers needs to maintain the same utility level is the same as the money the consumer needs to maintain the same bundle. Why? At the optimum, the consumers utility changes in the same way regardless of what they spend (a marginal amount) of money on. Thus, at the margin, both methods are the same.

Let's look at the Hicksian decomposition at the margin.

$$S.E. = T.E. - I.E.$$

The substitution effect is:

$$\frac{\partial x_i^h\left(p, \bar{u}\right)}{\partial p_j}$$

By duality:

$$x_i\left(p, e\left(p, \bar{u}\right)\right) = x_i^h\left(p, \bar{u}\right)$$

Thus, T.E.-I.E. must be:

$$\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j}$$

Set them equal and expand this:

$$\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + \frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y} \frac{\partial e\left(p, \bar{u}\right)}{\partial p_j} = \frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Let's label the various effects to keep them straight.

$$(T.E.)\,\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + (-I.E.)\,\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y} \frac{\partial e\left(p, \bar{u}\right)}{\partial p_j} = (S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Use envelope condition to replace $\frac{\partial e(p,\bar{u})}{\partial p_j}$ with x_j^h.

$$(T.E.)\,\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial p_j} + (-I.E.)\left(\frac{\partial\left(x_i\left(p, e\left(p, \bar{u}\right)\right)\right)}{\partial y} x_j^h\right) = (S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Let $e\left(p, \bar{u}\right) = y$.

$$(T.E.)\,\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j} + (-I.E.)\,\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y} x_j^h = (S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

Rearranging this, we get the Slutsky Equation:

$$\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j} = \frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j} - \frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y} x_j^h$$

$$(T.E.)\,\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial p_j}$$

$$(S.E.)\,\frac{\partial\left(x_i^h\left(p, \bar{u}\right)\right)}{\partial p_j}$$

$$(I.E.)\left(-\frac{\partial\left(x_i\left(p, y\right)\right)}{\partial y} x_j^h\right)$$

8.3 Example Cobb Douglass

Let's decompose demand:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - \frac{\partial x_i}{\partial m} \frac{\partial e}{\partial p_i}$$

Cobb Douglass utility: $x_1 x_2$

$$x_1^* = \frac{\frac{1}{2}m}{p_1}$$

$$x_2^* = \frac{\frac{1}{2}m}{p_2}$$

Indirect Utility:

$$V(p,m) = \frac{\frac{1}{2}m}{p_1}\frac{\frac{1}{2}m}{p_2} = \frac{\frac{1}{4}m^2}{p_1 p_2}$$

Leverage Duality:

$$\bar{u} = \frac{\frac{1}{4}e(p,\bar{u})^2}{p_1 p_2}$$

$$\sqrt{4\bar{u}p_1p_2} = e(p,\bar{u})$$

Use Shephards Lemma:

$$x_1^h = \frac{\partial e(p,\bar{u})}{\partial p_j} = \frac{\partial\left(\sqrt{4\bar{u}p_1p_2}\right)}{\partial p_1}$$

$$x_1^h = \frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_1} \quad = \quad \frac{p_2\bar{u}}{\sqrt{p_1p_2\bar{u}}}$$

$$x_2^h = \frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_2} = \frac{p_1\bar{u}}{\sqrt{p_1p_2\bar{u}}}$$

Substitution Effect for good 1 when $p_1$changes:

$$\frac{\partial\left(\frac{p_2 u}{\sqrt{p_1p_2u}}\right)}{\partial p_1} \quad = \quad -\frac{p_2^2u^2}{2\left(p_1p_2u\right)^{3/2}}$$

Income Effect:

$$-\frac{\partial\left(\frac{\frac{1}{2}m}{p_1}\right)}{\partial m}\frac{\partial\left(\sqrt{4up_1p_2}\right)}{\partial p_1} \quad = \quad -\frac{p_2u}{2p_1\sqrt{p_1p_2u}}$$

$$-\frac{p_2u}{2p_1\sqrt{p_1p_2u}} = -\frac{p_2^2u^2}{2\left(p_1p_2u\right)^{3/2}}$$

$$-\frac{p_2u}{2p_1\sqrt{p_1p_2u}} = -\frac{\sqrt{p_2u}}{2p_1\left(p_1\right)^{1/2}}$$

Income and Substitution effect are the same for the consumer at all levels of income and prices.

8.4 Neagtive Own-Substitution Effect

The derivative of the Hicksian demand must be (weakly) negative. Thus, the substitution effect for a good with respect to it's own price must be negative. That is: $\frac{\partial x_i^h}{\partial p_i} \leq 0$. This is due to the concavity of the expenditure function. Because e is concave, it has to be concave in any direction. Thus, the second derivative must be non-positive.

$$\frac{\partial^2 (e)}{(\partial p_i)^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0$$

8.5 Elasticities

Suppose the price of a good changes from 1 to 2. Consumer 1's demand changes from 100 to 50 and consumer 2's changes from 10 to 5. Their behavior in terms of absolute changes in demand $\frac{\Delta x_i}{\Delta p_i}$ is wildly different, but their behavior in terms of percentage terms $\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}}$ is identical. Elasticity is simply a way of quantifying comparative statics in unit-free percentage terms.

$$\eta_i = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial y}{y}} = \frac{\partial x_i}{\partial y}\frac{y}{x_i}$$

Example: $x_1^\alpha x_2^\alpha$

$$\frac{\partial\left(\frac{A*y}{p_i}\right)}{\partial y}\frac{y}{\left(\frac{A*y}{p_i}\right)} \quad = \quad 1$$

Price and Cross-Price Elasticity.

$$\epsilon_{ij} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$$

Example: $x_1^\alpha x_2^\alpha$

$$\frac{\partial\left(\frac{A*y}{p_i}\right)}{\partial p_i}\frac{p_i}{\left(\frac{A*y}{p_i}\right)} \quad = \quad -1$$

$$\frac{\partial\left(\frac{A*y}{p_i^2}\right)}{\partial p_i}\frac{p_i}{\left(\frac{A*y}{p_i^2}\right)} \quad = \quad -2$$

8.6 Elasticity Relations

There are some interesting relationships between elasticities that hold when the budget constraint binds. To derive either, start with a statement of the budget equation:

$$y = \sum_{j \in I} p_j x_j (p, y)$$

To derive the first, take the derivative w.r.t. a price:

$$\frac{\partial y}{\partial p_i} = \frac{\partial \sum_{j \in I} p_j x_j (p, y)}{\partial p_i}$$

$$0 = \frac{\partial \sum_{j \in I} p_j x_j (p, y)}{\partial p_i}$$

$$0 = \sum_{j \neq i} p_j \frac{\partial x_j}{\partial p_i} + p_i \frac{\partial x_i}{\partial p_i} + x_i$$

$$0 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i} + x_i$$

$$-x_i = \sum_{i=1}^{n} p_j \frac{\partial x_j}{\partial p_i} \left(\frac{p_i}{x_j} \frac{x_j}{p_i} \right)$$

$$-x_i = \sum_{j=1}^{n} p_j \left(\frac{\partial x_j}{\partial p_i} \frac{p_i}{x_j} \right) \left(\frac{x_j}{p_i} \right)$$

$$-x_i = \sum_{j=1}^{n} p_j \left(\frac{x_j}{p_i} \right) \varepsilon_{j,i}$$

$$-x_i = \sum_{j=1}^{n} \left(\frac{p_j x_j}{p_i} \right) \varepsilon_{j,i}$$

$$-x_i = \frac{1}{p_i} \sum_{j=1}^{n} \left(\frac{p_j x_j}{1} \right) \varepsilon_{j,i}$$

$$-\left(\frac{1}{y} \right) \frac{x_i p_i}{1} = \left(\frac{1}{y} \right) \sum_{j=1}^{n} \left(\frac{p_j x_j}{1} \right) \varepsilon_{j,i}$$

Denote $\frac{x_i p_i}{y} = s_i$ (the share of income spend on i):

$$-s_i = \sum_{j=1}^{n} \left(\frac{p_j x_j}{y} \right) \varepsilon_{j,i}$$

$$-s_i = \sum_{j=1}^{n} s_j \varepsilon_{j,i}$$

If instead we take the derivative of the budget constraint w.r.t. Income:

$$y = \sum_{j \in I} p_j x_j (p, y)$$

$$\frac{\partial y}{\partial y} = \frac{\partial \sum_{j \in I} p_j x_j (p, y)}{\partial y}$$

$$1 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial y}$$

$$1 = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial y} \left(\frac{y}{x_j} \frac{x_j}{y} \right)$$

$$1 = \sum_{j=1}^{n} \frac{p_j x_j}{y} \eta_j$$

$$1 = \sum_{j \in I} s_j \eta_j$$

9 Decisions Under Uncertainty

Outcomes:

$$A \equiv \{a_1, ..., a_n\}$$

Simple gambles:
Probability distribution over outcomes.

$$\mathcal{G}_s \equiv \left\{ (p_1 \circ a_1, p_2 \circ a_2, ..., p_n \circ a_n) \,|\, \sum p_i = 1, p_i \geq 0 \right\}$$

Let $g_1, ..., g_n \in \mathcal{G}_s$
Example: $A = \{\$10, \$5, \$0\}$
$g_1 = \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0 \right)$
$g_2 = (1 \circ \$5)$ (degenerate gamble)
First-order Compound Gambles:
for all i, $g_i \in \mathcal{G}_s$

$$\mathcal{G}_{c_1} \equiv \left\{ (p_1 \circ g_1, p_2 \circ g_2, ..., p_n \circ g_n) \,|\, \sum p_i = 1 \right\}$$

Example:

$$\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ(1\circ\$5)\right)$$

Nth-order Compound Gambles:
Let $g_i \in \mathcal{G}_{c_j}$ for $j \in \mathbb{Z}_+$.
Compound gambles:

$$\mathcal{G} \equiv \left\{(p_1 \circ g_1, p_2 \circ g_2, ..., p_m \circ g_m) \,|\, \sum p_i = 1\right\}$$

Example:

$$\left(\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ\$5\right)\right),\left(\frac{1}{2}\circ\$7\right)\right)$$

9.1 Expected Utility

$$g = \left(\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ\$5\right)\right),\left(\frac{1}{2}\circ\$7\right)\right)$$

Calculate the probability of each outcome under this gamble and construct a simple gamble that gives that outcome with it's probability. This is the *induced* simple gamble for g.

$$g_s = \frac{1}{8}\circ\$10,\frac{1}{8}\circ\$0,\frac{1}{4}\circ\$5,\frac{1}{2}\circ\$7$$

We'd like to represent their preferences over that compound gamble by:

$$\frac{1}{8}u(\$10)+\frac{1}{8}u(\$0)+\frac{1}{4}u(\$5)+\frac{1}{2}u(\$7)$$

This is the **simple gamble induced** by compound gamble g.
Suppose we have $u(a_i)$:

$$U(g) = E_g(u(a_i))$$

Suppose gamble g induced simple gamble g_s. Let $p_i^{g_s}$ be the probability of outcome i in g_s. This utility function is of the expected value form.

$$U(g) = \sum_{i=1}^{n} p_i^{g_s} u(a_i)$$

$p_i^{g_s}$ is the probability outcome a_i occurs under gamble g. It is also the probability it occurs in the induced simple lottery g_s.
When can we represent preferences over $\mathcal{G}$ with an expected utility function?

9.2 Example

$$g = \left(\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\left(\frac{1}{2}\circ\$10,\frac{1}{2}\circ\$0\right),\frac{1}{2}\circ\$5\right)\right),\left(\frac{1}{2}\circ\$7\right)\right)$$

Utility is linear in money: $u(x) = x$. Under this utility function, your utility is the expected amount of money from the gamble.

$$u(g) = \frac{1}{8}(10) + \frac{1}{8}0 + \frac{1}{4}5 + \frac{1}{2}7$$

$$u(g) = 6$$

Suppose utility of money is $u(x) = log(x+1)$

$$u(g) = \frac{1}{8}(\log(10+1)) + \frac{1}{8}(\log(0+1)) + \frac{1}{4}(\log(5+1)) + \frac{1}{4}(\log(7+1))$$

$$u(g) = 1.26754$$

Utility of the expected outcome:

$$u(6) = \log(6+1)$$

$$1.94591$$

This demonstrates that a preference to reduce risk is the same as having concave utility function over prices.

9.3 Expected Utility Theorem

When can we represent $\succsim$ with an expected utility function? We really need two key things. First, this utility function is continuous, so we need to at least represent the preferences with a continuous utility function. This will require the preferences to be complete, transitive, and most importantly ***continuous.*** Second, preferences over compound gambles need to be the same as the preferences over their induced simple gambles. This relies on the preferences to have a structure that is ***linear*** in probabilities.

9.4 Expanded Proof- Continuity

Let $\succsim$ be the preference relation on $\mathcal{G}$:

Axiom 1. ***Complete:*** *$\succsim$ is complete.*

Axiom 2. ***Transitive:*** *$\succsim$ is transitive.*

Assume $a_1 \succsim a_2 ... \succsim a_n$.

Axiom 3. ***Monotonic:*** For all $(\alpha\circ a_1,(1-\alpha)\circ a_n) \succsim (\beta\circ a_1,(1-\beta)\circ a_n)$ iff $\alpha \geq \beta$, Axiom 4. ***Continuous/Archemedian:*** For all g $\exists p \in [0,1]$ such that $g \sim (p \circ a_1, (1-p) \circ a_n)$

Under assumptions $1-4$, we can represent with a continuous utility function. Let $u(g)$ be defined this way. Find the simple gamble over the best and worst outcome that g is indifferent to. Let p be the probability of the best outcome in that simple gamble. Then $u(g) = p$. For instance, if $g \sim \left(\frac{1}{4} \circ a_1, \frac{3}{4} \circ a_n\right)$ then $u(g) = \frac{1}{4}$. Implicit, we can define $u(g)$ as:

$$u(g) : g \sim (u(g) \circ a_1, (1-u(g)) \circ a_n)$$

Let's see that:

$$u(g) \geq u(g') \Leftrightarrow g \succsim g'$$

Start with:

$$g \succsim g'$$

By continuity, these are indifferent to gambles over the best and worst outcome:

$$(p \circ a_1, (1-p) \circ a_n) \sim g \succsim g' \sim (p' \circ a_1, (1-p') \circ a_n)$$

By transitivity:

$$g \succsim g' \Leftrightarrow (p \circ a_1, (1-p) \circ a_n) \succsim (p' \circ a_1, (1-p') \circ a_n)$$

By monotonicity the right side if true if and only if $p \geq p'$. We now have:

$$g \succsim g' \Leftrightarrow (p \circ a_1, (1-p) \circ a_n) \succsim (p' \circ a_1, (1-p') \circ a_n) \Leftrightarrow p \geq p'$$

By construction, $u(g) = p$. Thus:

$$g \succsim g' \Leftrightarrow (p \circ a_1, (1-p) \circ a_n) \succsim (p' \circ a_1, (1-p') \circ a_n) \Leftrightarrow p \geq p' \Leftrightarrow u(g) \geq u(g')$$

9.5 Expanded Proof- Linearity

In the above utility construction we **measure utility through the probabilities of the indifferent gambles** over the best and worst outcome. But, under the above definition, the simple gamble $(p \circ a_1, (1-p) \circ a_n)$ that is indifferent to g *can be anything.* However, for a utility function with the expected utility property, $u(g) = u(g_s)$ since all that matters are the utility of the outcomes and the probabilities of those outcome. However, if we have enough structure to ensure that simple lottery over the best and worst outcomes that is indifferent to any gamble g is the same simple lottery over the best and worst outcomes that is indifferent to the simple lottery induced by g, then we can extend the above construction and show that it is linear.

We need two additional assumptions:

Axiom 5. ***Substitution:*** If $g = (p_1 \circ g_1, ..., p_k \circ g_k)$ and $h = (p_1 \circ h_1, ..., p_k \circ h_k)$ and if $g_i \sim h_i$ for all $i \in \{1, ..., k\}$ then $g \sim h$.

Axiom 6. ***Reduction:*** For any gamble g and the simple gamble it induces g_s, $g \sim g_s$.

Now, construct utility this way: let $u(a_i)$ be defined as above:

$$u(a_i) : a_i \sim (u(a_i) \circ a_1, (1 - u(a_i)) \circ a_n)$$

Note, instead of finding the simple gamble over the best and worst outcome **for every g.** Here, it is enough to know only those indifferent lotteries *for outcomes.* Then we extend the utility function to all gambles through expectation. Let p_i^g be the probability of outcome a_i in the simple gamble induced by g.

$$u(g) = \sum_{i=1}^{n} p_i^g u(a_i)$$

We have the following result, **under axioms 1,2,3,4,5,6:**

$$u(g) \geq u(g) \Leftrightarrow g \succsim g$$

Lets Prove it:

*By **Reduction**:*

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n,) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n, \right)$$

*By **Continuity** we know every outcome is indifferent to some simple gamble over the best and worst outcome:*

$$a_i \sim (\alpha \circ a_1, (1 - \alpha) \circ a_n)$$

*By Substitution we can glue these in to the above expression. The gamble on the right of the expression above is a **simple gamble over multiple outcomes**. By replacing every outcome by it's simple gamble over the best and worst outcome, we turn this into a **compound gamble over only the best and worst outcome**.*

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n \right)$$
$$\Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_i^{g_s} \circ (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_i^{g_s^{'}} \circ (\alpha_i a_1, (1 - \alpha_i) a_n), ..., p_n^{g_s^{'}} \circ a_n \right)$$

We can now make this a simple lottery again by applying reduction a second time:

$$g \succsim g' \Leftrightarrow g_s \succsim g_s^{'} \Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_n^{g_s^{'}} \circ a_n \right)$$
$$\Leftrightarrow (p_1^{g_s} \circ a_1, ..., p_i^{g_s} \circ (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), ..., p_n^{g_s} \circ a_n) \succsim \left(p_1^{g_s^{'}} \circ a_1, ..., p_i^{g_s^{'}} \circ (\alpha_i a_1, (1 - \alpha_i) a_n), ..., p_n^{g_s^{'}} \circ a_n \right)$$

$$\Leftrightarrow \left(\sum_{i=1}^{n}\alpha_i p_i^{g_s}\circ a_1, \sum_{i=1}^{n}(1-\alpha_i)\, p_i^{g_s}\circ a_n\right) \succsim \left(\sum_{i=1}^{n}\alpha_i p_i^{g_s'}\circ a_1, \sum_{i=1}^{n}(1-\alpha_i)\, p_i^{g_s'}\circ a_n\right)$$

We now have a simple gamble over the best and worst outcome again. By Monotonicity:

$$g \succsim g' \Leftrightarrow g_s \succsim g_s' \Leftrightarrow \left(p_1^{g_s}\circ a_1, ..., p_n^{g_s}\circ a_n\right) \succsim \left(p_1^{g_s'}\circ a_1, ..., p_n^{g_s'}\circ a_n\right)$$

$$\Leftrightarrow \left(p_1^{g_s}\circ a_1, ..., p_i^{g_s}\circ(\alpha_i\circ a_1, (1-\alpha_i)\circ a_n), ..., p_n^{g_s}\circ a_n\right) \succsim \left(p_1^{g_s'}\circ a_1, ..., p_i^{g_s'}\circ(\alpha_i a_1, (1-\alpha_i)\, a_n), ..., p_n^{g_s'}\circ a_n\right)$$

$$\Leftrightarrow \left(\sum_{i=1}^{n}\alpha_i p_i^{g_s}\circ a_1, \sum_{i=1}^{n}(1-\alpha_i)\, p_i^{g_s}\circ a_n\right) \succsim \left(\sum_{i=1}^{n}\alpha_i p_i^{g_s'}\circ a_1, \sum_{i=1}^{n}(1-\alpha_i)\, p_i^{g_s'}\circ a_n\right)$$

$$\Leftrightarrow \sum_{i=1}^{n}\alpha_i p_i^{g_s} \geq \sum_{i=1}^{n}\alpha_i p_i^{g_s'}$$

By *Construction of* $u(g)$:

$$g \succsim g' \Leftrightarrow g_s \succsim g_s' \Leftrightarrow \left(p_1^{g_s}\circ a_1, ..., p_n^{g_s}\circ a_n\right) \succsim \left(p_1^{g_s'}\circ a_1, ..., p_n^{g_s'}\circ a_n\right)$$

$$\Leftrightarrow \left(p_1^{g_s}\circ a_1, ..., p_i^{g_s}\circ(\alpha_i\circ a_1, (1-\alpha_i)\circ a_n), ..., p_n^{g_s}\circ a_n\right) \succsim \left(p_1^{g_s'}\circ a_1, ..., p_i^{g_s'}\circ(\alpha_i a_1, (1-\alpha_i)\, a_n), ..., p_n^{g_s'}\circ a_n\right)$$

$$\Leftrightarrow \left(\sum_{i=1}^{n}\alpha_i p_i^{g_s}\circ a_1, \sum_{i=1}^{n}(1-\alpha_i)\, p_i^{g_s}\circ a_n\right) \succsim \left(\sum_{i=1}^{n}\alpha_i p_i^{g_s'}\circ a_1, \sum_{i=1}^{n}(1-\alpha_i)\, p_i^{g_s'}\circ a_n\right)$$

$$\Leftrightarrow \sum_{i=1}^{n}\alpha_i p_i^{g_s} \geq \sum_{i=1}^{n}\alpha_i p_i^{g_s'} \Leftrightarrow \sum_{i=1}^{n}u(a_i)\, p_i^{g_s} \geq \sum_{i=1}^{n}u(a_i)\, p_i^{g_s'} \Leftrightarrow u(g) \geq u(g')$$

9.6 Another Representation

There are other combinations of axioms that give the same result, many make use of the following axiom:

Axiom. ***Independence:*** For any gambles $g, g', g''$and any $\alpha \in (0,1]$, $\alpha g + (1-\alpha)\, g'' \succsim \alpha g + (1-\alpha)\, g' + (1-\alpha)\, g'' \Leftrightarrow g \succsim g'$.

A common form of the representation theorem is the following:

Complete and transitive preference admit an EU representation IFF they are Independent and Monotone.

The same proof we used above can be applied after showing that all of the assumption 1-6 are implied by *Completeness, Transitivity, Independence, and Monotonicity.*

9.7 Risk Preferences

We look at gambles over wealth. If a consumer is an expected utility maximizer, we can represent their preferences with the following utility function:

$$u(g) = E_g(v(w)) = \sum p(w_i)\, v(w_i)$$

Suppose $v(w) = \log(w)$ and consider the gamble $g = \left(\frac{1}{2} \circ 10, \frac{1}{2} \circ 20\right)$

$$u(g) = \frac{1}{2}(v(10)) + \frac{1}{2}(v(20))$$

$$u(g) = \frac{1}{2}(\log(10)) + \frac{1}{2}(\log(20)) = 2.64916$$

Let's compare this to the **utility of the expected wealth from the gamble**:

$$E_g(w) = \frac{1}{2}10 + \frac{1}{2}20 = 15$$

$$v(E_g(w)) = \log(15.0) \quad = \quad 2.70805$$

The expected wealth of the gamble is preferred to the gamble itself. We say this person is **risk averse.** Formally:

Risk Averse:

$$E_g(v(w)) < v(E_g(w))$$

Risk Loving:

$$E_g(v(w)) > v(E_g(w))$$

Risk Neutral:

$$E_g(v(w)) = v(E_g(w))$$

By **Jensen's inequality**: $E_g(v(w)) \leq (\geq)\, v(E_g(w))$ is true for all "random variables" (in this gambles g) if and only if v is concave (convex). If v is differentiable: $v''(w) \leq (\geq)\, 0$.

Note that this also gives: $E_g(v(w)) = v(E_g(w))$ if and only if v is both convex and concave. The only such functions are linear.

9.8 Certainty Equivalent / Risk Premium

The **certainty equivalent** is an amount of money such that having that money for sure is indifferent to gamble g. Formally:

$$CE(g) = v^{-1}(u(g))$$

Here, v^{-1} is the inverse of the utility function for wealth. If v is assumed strictly increasing, this is well defined. We can define risk averse in terms of the certainty equivalent:
A person is risk averse iff for all g:

$$CE(g) < E_g(w)$$

A person is risk loving iff for all g:

$$CE(g) > E_g(w)$$

From our previous example:

$$\frac{1}{2}(\log(10)) + \frac{1}{2}(\log(20.0)) \quad = \quad 2.64916$$

What is the certain wealth that gives this utility?

$$\log(CE) = 2.64916$$

$$CE = 14.1422$$

The expected wealth is 15 but the consumer would accept less than 15 in place of the gamble. Again this shows risk aversion.
The **risk premium** is the difference between the expected wealth and the certainty equivelent: $RP_g = E_g(w) - CE(g)$.

$$RP = 15 - 14.1422 = 0.8578$$

9.9 Two Measures of Risk Preferences and Differential Equations

Absolute Risk Aversion: $-\frac{v''(w)}{v'(w)}$
Relative Risk Aversion:$-\frac{wv''(w)}{v'(w)}$
Both of these are reasonable measures of the curvature of a utility function and have some nice properties. I bring them up to demonstrate a useful technique for finding functions that have properties given in terms of derivatives:

Suppose we want a constant relative risk aversion for some model. What utility functions can we use? We want to find all utility functions such that:

$$-\frac{wv''(w)}{v'(w)} = c$$

This is a differential equation. We can solve these easily with mathematica even if we are rusty on the various forms:

$$DSolve[-\frac{w * v''[w]}{v'[w]} == c, v[w], w]$$

$$\left\{\left\{v(w) \to \frac{c_1 w^{1-c}}{1-c} + c_2\right\}\right\}$$

This tells us every utility function for wealth with constant relative risk aversion is an affine transformation of:

$$v(w) = \frac{w^{1-c}}{1-c}$$

Part II

The Firm

10 The Firm's Problem

10.1 Technology

The most general way to represent the technology available to a firm is through a production possibilities set.

$$Y \subset \mathbb{R}^m$$

For instance the vector $(-1,-1,1)$ says:l take one input of goods 1 and 2 and create one output of good 3.

Example. Two apples and one crust make a pie.

$$Y = (-2,-1,1),(-4,-2,2),(-6,-3,3)...$$

If only one of the goods is ever an output and the rest are always inputs, we can represent this with a function. For instance, the production function for pies might be written:

$$f(\boldsymbol{x}) = min\left\{\frac{1}{2}x_1, x_2\right\}$$

We will work with production functions rather than possibilities set through the rest of the class, but it is nice to be aware more flexible language exists.

10.2 Assumptions on $f(\boldsymbol{x})$

1. Continuous

2. Strictly Increasing

Our pie example is strictly but not strongly increasing.

3. Strictly Quasi-Concave

4. $f(0) = 0$

Besides $f(0) = 0$**, this is essentially what we assumed about a utility function.**

10.3 Cardinal/Ordinal

The production function is a cardinal function. We can take the values seriously.

10.4 Cost Minimization / The Cost Function

We want to minimize cost of producing at least y output.

$$Min_{x:f(x)\geq y}\left(\sum x_i w_i\right)$$

Condition factor demands (analogous to the hicksian demands):

$$x_i^*(y, \boldsymbol{w})$$

Cost function (analogous to the consumer expenditure function):

$$c(y, \boldsymbol{w})$$

10.5 Properties of the Cost Function

0. $c(0, w) = 0$
1. Continuous.
2. For $w \gg 0$, strictly increasing and unbounded above in y.
3. Increasing in w.
4. Homogeneous of degree 1 in w.
5. Concave in w.
6. Shephard's lemma.

$$\frac{\partial c(w, y)}{\partial p_i} = x_i(w, y)$$

10.6 An Example

$$x_1^\alpha x_2^\alpha$$

The lagrangian:

$$x_1 w_1 + x_2 w_2 - \lambda \left(x_1^\alpha x_2^\alpha - y \right)$$

$$\frac{\partial \left(x_1 w_1 + x_2 w_2 - \lambda \left(x_1^\alpha x_2^\alpha - y \right) \right)}{\partial x_1} = w_1 - \alpha \lambda x_1^{\alpha-1} x_2^\alpha$$

$$\frac{\partial \left(x_1 w_1 + x_2 w_2 - \lambda \left(x_1^\alpha x_2^\alpha - y \right) \right)}{\partial x_2} = w_2 - \alpha \lambda x_1^\alpha x_2^{\alpha-1}$$

FOCs:

$$\frac{w_1}{\alpha x_1^{\alpha-1} x_2^\alpha} = \lambda$$

$$\frac{w_2}{\alpha x_1^\alpha x_2^{\alpha-1}} = \lambda$$

$$\frac{mp_1}{mp_2} = \frac{w_1}{w_2}$$

The $\frac{w_1}{\alpha x_1^{\alpha-1} x_2^\alpha}$ can be interpreted roughly as the cost of increasing output by one unit using factor demand 1.

The first order condition says: the cost output using any input is exactly the same.

λ is the "shadow cost" of production.

$$\frac{w_1}{\alpha x_1^{\alpha-1} x_2^\alpha} = \frac{w_2}{\alpha x_1^\alpha x_2^{\alpha-1}}$$

$$\frac{w_1 x_1}{\alpha x_1^\alpha x_2^\alpha} = \frac{w_2 x_2}{\alpha x_1^\alpha x_2^\alpha}$$

$$w_1 x_1 = w_2 x_2$$

Two conditions:

$$x_1 = \frac{w_2}{w_1} x_2$$

$$x_1^\alpha x_2^\alpha = y$$

Solve these for the condition factor demands:

$$x_1^{\alpha}\left(\frac{w_1}{w_2}x_1\right)^{\alpha}=y$$

Conditional Factor Demands:

$$x_1=\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}y^{\frac{1}{2\alpha}}$$

$$x_2=\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}y^{\frac{1}{2\alpha}}$$

These are:

$$=x_i\left(1,w\right)y^{\frac{1}{2\alpha}}$$

Cost Function:

$$w_1\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}y^{\frac{1}{2\alpha}}+w_2\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}y^{\frac{1}{2\alpha}}$$

$$\left(w_1w_2\right)^{\frac{1}{2}}y^{\frac{1}{2\alpha}}+\left(w_1w_2\right)^{\frac{1}{2}}y^{\frac{1}{2\alpha}}$$

$$c\left(y,w\right)=2\left(w_1w_2\right)^{\frac{1}{2}}y^{\frac{1}{2\alpha}}$$

Note that $2\left(w_1w_2\right)^{\frac{1}{2}}$ is $c\left(1,w\right)$ (the cost of producing one unit). Thus,

$$c\left(y,w\right)=c\left(1,w\right)y^{\frac{1}{2\alpha}}$$

10.7 Homogeneous/Homothetic Production

For any homogeneous production function homogeneous of degree β:

$$x=y^{\frac{1}{\beta}}x\left(1,w\right)$$

$$c=y^{\frac{1}{\beta}}c\left(1,w\right)$$

For any homothetic production function homogeneous of degree β:

$$x=g\left(y\right)x\left(1,w\right)$$

$$c=g\left(y\right)c\left(1,w\right)$$

10.8 A Homothetic Production Problem

$$ln\left(x_1\right)+ln\left(x_2\right)$$

The marginal products:

$$\frac{\partial\left(\log\left(x_1\right)+\log\left(x_2\right)\right)}{\partial x_1} = \frac{1}{x_1}$$

$$\frac{\partial\left(\log\left(x_1\right)+\log\left(x_2\right)\right)}{\partial x_2} = \frac{1}{x_2}$$

FOCs:

$$\frac{w_1}{\frac{1}{x_1}} = w_1 x_1 = \lambda$$

$$\frac{w_2}{\frac{1}{x_2}} = w_2 x_2 = \lambda$$

$$w_1 x_1 = w_2 x_2$$

$$x_1 = \frac{w_2}{w_1} x_2$$

$$y = ln\left(\frac{w_2}{w_1} x_2\right) + ln\left(x_2\right)$$

$$e^y = e^{ln\left(\frac{w_2}{w_1} x_2\right)+ln\left(x_2\right)}$$

$$e^y = e^{ln\left(\frac{w_2}{w_1} x_2\right)} e^{ln\left(x_2\right)}$$

$$\sqrt{\frac{w_1}{w_2}}\sqrt{e^y} = x_2\left(y, w\right)$$

$$\sqrt{\frac{w_2}{w_1}}\sqrt{e^y} = x_1\left(y, w\right)$$

$$x_1\left(1, w\right) = \sqrt{\frac{w_2}{w_1}}\sqrt{e}$$

$$x_2\left(1, w\right) = \sqrt{\frac{w_1}{w_2}}\sqrt{e}$$

$$x_1\left(y,w\right)=x_1\left(1,w\right)e^{\frac{1}{2}\left(y-1\right)}$$

$$x_2\left(y,w\right)=x_2\left(1,w\right)e^{\frac{1}{2}\left(y-1\right)}$$

$$c\left(1,w\right)=w_1\sqrt{\frac{w_2}{w_1}}\sqrt{e}+w_2\sqrt{\frac{w_1}{w_2}}\sqrt{e}=2\sqrt{w_1w_2}\sqrt{e}$$

$$c\left(y,w\right)=e^{\frac{1}{2}\left(y-1\right)}2\sqrt{w_1w_2}\sqrt{e}=2\sqrt{e^y}\sqrt{w_1w_2}$$

10.9 Separability

Suppose the inputs of a production function can be partitioned and the production function written:

$$f\left(g_1\left(x_1,x_2\right),g_2\left(x_3,x_4\right)\right)$$

When this is possible, x_1, x_2 the relative amounts of x_1 and x_2 can be optimized without reference to x_3, x_4 and vise versa. That is, an optimal mix of the inputs within each group can be chosen independently. We will see an example of this below.

When a production function can be partitioned this way into groups the production function is **weakly separable** on those groups. In this case, it is weakly separable on x_1, x_2 and x_3, x_4.

Note that:

$$\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}=\frac{\frac{\partial g}{\partial x_1}}{\frac{\partial g}{\partial x_2}},\frac{\frac{\partial f}{\partial x_3}}{\frac{\partial f}{\partial x_4}}=\frac{\frac{\partial g}{\partial x_3}}{\frac{\partial g}{\partial x_4}}$$

Another way to check for weak separability is to check that the ratio of partials of any two inputs in one group does not depend on the inputs in any other group.

A production function is **strongly separable** if it is weakly separable into any groups. Alternatively, if the ratio of partials of any two inputs only depends on those inputs.

$$x_1^a x_2^b x_3^c x_4^d$$

$$\frac{\frac{\partial\left(x_1^a x_2^b x_3^c x_4^d\right)}{\partial x_1}}{\frac{\partial\left(x_1^a x_2^b x_3^c x_4^d\right)}{\partial x_2}} = \frac{ax_2}{bx_1}$$

$$f\left(x_1,x_2,x_3,x_4\right)=\left(x_5ln\left(x_1\right)+x_5^2ln\left(x_2\right)\right)+x_5\left(x_3^{\frac{1}{2}}x_4^{\frac{1}{2}}\right)$$

$$\frac{\frac{\partial\left((x_5 \log(x_1)+x_5 \log(x_2))+x_5\left(x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)\right)}{\partial x_1}}{\frac{\partial\left((x_5 \log(x_1)+x_5 \log(x_2))+x_5\left(x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)\right)}{\partial x_2}} = \frac{x_2}{x_1}$$

$$\frac{\frac{\partial\left((x_5 \log(x_1)+x_5 \log(x_2))+x_5\left(x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)\right)}{\partial x_3}}{\frac{\partial\left((x_5 \log(x_1)+x_5 \log(x_2))+x_5\left(x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)\right)}{\partial x_4}} = \frac{x_4}{x_3}$$

10.10 A Separable Production Problem

$$f(x_1, x_2, x_3, x_4) = ln(x_1) + ln(x_2) + \left(x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}\right)$$

In this case, x_1,x_2 are separable from x_3, x_4. We can think of this problem as taking x_1, x_2 and producing intermediate input y_1 then taking x_3, x_4 and producing intermediate input y_2.

Suppose you are product y_1 from $\log(x_1) + \log(x_2)$. What is the cost of producing y_1? We already solved this.

$$c(y_1, w) = 2\sqrt{e^{y_1}}\sqrt{w_1 w_2}$$

Suppose you are producing y_2 from $x_3^{\frac{1}{2}} x_4^{\frac{1}{2}}$ what is the cost of y_2?

$$c(y_2, w) = 2y_2\sqrt{w_3 w_4}$$

What is the cheapest way to produce final output y from y_1, y_2?

$$f(y_1, y_2) = y_1 + y_2$$

Cost using intermediate inputs y_1 y_2 is:

$$2\sqrt{e^{y_1}}\sqrt{w_1 w_2} + 2y_2\sqrt{w_3 w_4}$$

What is the cheapest way to produce y? Lets assume all input prices are $w_i = 1$:

$$Min_{y_1, y_2} 2\sqrt{e^{y_1}} + 2y_2 - \lambda(y_1 + y_2 - y)$$

$$\frac{\partial 2\sqrt{e^{y_1}}}{\partial y_1} = \sqrt{e^{y_1}}$$

$$\frac{\partial\left(2\sqrt{e^{y_1}}+2y_2-\lambda\left(y_1+y_2-y\right)\right)}{\partial y_1} = \sqrt{e^{y_1}}-\lambda$$

$$\frac{\partial\left(2\sqrt{e^{y_1}}+2y_2-\lambda\left(y_1+y_2-y\right)\right)}{\partial y_2} = 2-\lambda$$

FOC:

$$\sqrt{e^{y_1}}=2$$

To be at an interior solution, enough y_1 has to be used to bring it's marginal cost of output down to 2 so that it makes sense to put some money into y_2. Otherwise, the firm will **only** use y_1.

An interior solution has:

$$y_1 = 1.38629$$

$$y_2 = y - 1.38629$$

The cost function for y:

$$c\left(y\right) = 6.77258 + 2y$$

This only holds if $y > 1.38629$. Otherwise:

$$y_1 = y$$

$$y_2 = 0$$

$$c\left(y\right) =$$

10.11 A CObb DOuglass PrOblem

$$x_1^{\alpha}x_2^{\beta}x_3^{\gamma}x_4^{\delta}$$

Let's assume the the prices are all the same equal to 1. Let's split this into $y_1 = f_1\left(x_1, x_2\right)$ and $y_2 = f_2\left(x_3, x_4\right)$

$$\frac{\partial\left(x_1^{\alpha}x_2^{\beta}x_3^{\gamma}x_4^{\delta}\right)}{\partial x_1} = \frac{\alpha x_1^{\alpha}x_2^{\beta}x_4^{\delta}x_3^{\gamma}}{x_1}$$

$$\frac{\partial\left(x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta\right)}{\partial x_2} = \beta x_1^\alpha x_2^{\beta-1} x_4^\delta x_3^\gamma$$

$$\frac{\alpha x_1^\alpha x_2^\beta x_4^\delta x_3^\gamma}{x_1} = \frac{\beta x_1^\alpha x_2^\beta x_4^\delta x_3^\gamma}{x_2}$$

$$\frac{\alpha}{x_1} = \frac{\beta}{x_2}$$

$$x_1 = \frac{\alpha}{\beta} x_2$$

$$y_1 = \left(\frac{\alpha}{\beta}\right) x_2^{\alpha+\beta}$$

$$y_1 = \left(\frac{\alpha}{\beta}\right)^\alpha x_2^{\alpha+\beta}$$

$$x_1 = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} y_1^{\frac{1}{\alpha+\beta}}$$

$$x_2 = \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} y_1^{\frac{1}{\alpha+\beta}}$$

$$c\left(y_1\right) = \left(\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right) y_1^{\frac{1}{\alpha+\beta}}$$

Do the same for y_2.

$$y = f\left(y_1, y_2\right) = y_1 y_2$$

11 Profit

11.1 The Problem

Let's ignore (for now) the fact that price changes when output changes:

$$\pi = py - wx = pf\left(x\right) - wx$$

$$Max_x pf\left(x\right) - wx$$

$$\frac{\partial \pi}{\partial x_i} = p\frac{\partial f}{\partial x_i} - w_i$$

FOC:

$$p\frac{\partial f}{\partial x_i} = w_i$$

$$\frac{w_i}{MP_i} = p$$

This implies that cost is optimized for producing the optimal level of output. In fact, we know that the "shadow cost" λ of output at the profit maximizing point is exactly the price out output. Otherwise you could increase profit by changing output.

11.2 Example

$$f(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$$

From some example long ago:

$$c(y, w) = 2(w_1 w_2)^{\frac{1}{2}} y^2$$

We can now re-write the profit function in terms of y:

$$\pi(y, w_1, w_2) = py - 2(w_1 w_2)^{\frac{1}{2}} y^2$$

$$\frac{\partial \left(p * y - 2(w_1 w_2)^{\frac{1}{2}} y^2\right)}{\partial y} = p - 4\sqrt{w_1 w_2} y$$

$$y^* = \frac{p}{4\sqrt{w_1 w_2}}$$

$$\pi = p\frac{p}{4\sqrt{w_1 w_2}} - 2(w_1 w_2)^{\frac{1}{2}} \left(\frac{p}{4\sqrt{w_1 w_2}}\right)^2$$

$$= \frac{p^2}{4\sqrt{w_1 w_2}} - \frac{p^2}{8\sqrt{w_1 w_2}} = \frac{1}{8}\frac{p^2}{\sqrt{w_1 w_2}}$$

11.3 The Profit Function

$$\pi(p,w) = max_y py - c(y) = Max_x\, pf(x) - wx$$

$$py - (w_1x_1 + w_2x_2)$$

When well defined:

1. Increasing in p,
2. Decreasing in w,
3. Homogeneous of degree one in p, w
4. Convex in p, w (why)
5. Hotelling: $\frac{\partial\pi}{\partial p} = y(p,w), -\frac{\partial\pi}{\partial w_i} = x_i(p,w)$

Combining 4 and 5 we can prove that, output is increasing in price (weakly), and any input is weakly decreasing in it's own wage (this is the substitution effect for production).

$$\frac{\partial y}{\partial p} \geq 0, \frac{\partial x_i}{\partial w_i} \leq 0$$

Part III
Markets

12 Price Taking

For the next few chapters, we will use (when possible) a running example of firms with production function

$$f(x_1,x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$$

As we have seen, the firm's cost function is:

$$c(y,w) = 2(w_1w_2)^{\frac{1}{2}} y^2$$

We will also use a demand function

$$Y_d = \frac{100}{p}$$

12.1 A Simple Market With Price Taking

Suppose a firm has $f(x_1,x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$ and $w_1 = w_2 = 1$. Market demand is $Y_d = \frac{100}{p}$. (This might come from cobb douglass consumers for instance.)

Our equilibrium condition is that the total production at price p has to be the same as the total demand $Y_d = \frac{100}{p}$ at price p.

12.2 We have already done a lot of this problem above:

$$\pi(y, w_1, w_2) = py - 2y^2$$

$$\frac{\partial\left(p * y - 2(w_1 w_2)^{\frac{1}{2}} y^2\right)}{\partial y} = p - 4y$$

$$y^* = \frac{p}{4}$$

Suppose there are J firms. Output Y_s is:

$$Y_s = Jy^* = J\frac{p}{4}$$

In equilibrium, $Y_S = Y_d$

$$J\frac{p}{4} = \frac{100}{p}$$

$$p^* = \sqrt{\frac{400}{J}} = \frac{20}{\sqrt{J}}$$

$$Y^* = \frac{100}{\frac{20}{\sqrt{J}}} = 5\sqrt{J}$$

$$y^* = \frac{5\sqrt{J}}{J} = \frac{5}{\sqrt{J}}$$

$$\pi(y, w_1, w_2) = \frac{20}{\sqrt{J}}\left(\frac{5}{\sqrt{J}}\right) - 2\left(\frac{5}{\sqrt{J}}\right)^2 = \frac{100}{J} - \frac{50}{J} = \frac{50}{J}$$

12.3 Entry With Price Taking

Suppose it costs \$1 to enter this market. How many firms will enter?

To enter, a firm will have make at least a dollar:

$$\frac{50}{J} \geq 1$$

$$50 \geq J$$

A **Nash** Equilibrium of this game is a set of strategies where no firm has incentive to change strategies. The possible strategies are {enter,don't enter}. There are two types of equilibrium here:

If 49 firms enter, then all 49 make strictly postive profit. There is no incentive for any to leave. There is no incentive for a firm to enter since they will be the 50th firm and earn 0 profit anyway.

If 50 firms enter, then all earn 0 profit. There is no incentive for any firm to enter or leave.

In summary: we expect 49 or 50 firms to enter this market in Nash equilibrium.

13 Markets without Price Taking

13.1 How Reasonable is Price Taking?

In the previous problem the firms assume profit is:

$$\pi = py - 2y^2$$

Price is actually a function of total output Y. This is given by the inverse demand function:

$$Y_d = \frac{100}{p}$$

Solve this for Y to get the inverse demand function:

$$p(y) = \frac{100}{Y}$$

The inverse demand function is the relationship between price and total output. If we plug this into the profit function for each firm, we have the precise profit function without the price-taking assumption.

Denote $\sum_{j \neq i} y_j = Y_{-i}$. For example if 10 firms all produce 10 units, $y_i = 10$ and $Y_{-i} = 90$ for all the firms.

$$\pi_i(y_i, Y_{-i}) = p(y_i + Y_{-i})\, y_i - c(y_i)$$

Note: $\frac{\partial p(Y)}{\partial y_i} = \frac{\partial p(Y)}{\partial Y}\frac{\partial Y}{\partial y_i} = \frac{\partial p(Y)}{\partial Y}$

The first order condition for each firm is.

$$\frac{\partial p}{\partial Y} y_i + p = \frac{\partial c}{\partial y_i}$$

Thus, the first-order condition without the assumption of price taking differs from the price taking first-order condition by the term $\frac{\partial p}{\partial Y} y_i$. How much will this distort the firms choice depends on the size of $\frac{\partial p}{\partial Y}$. Let's explore this in the context of our working example.

13.2 Back to the Example

Let's write the profit function without the price-taking assumption:

$$\pi_i(y_i, Y_{-i}) = \frac{100}{y_i + Y_{-i}} y_i - 2y_i^2$$

The first derivative is:

$$\frac{\partial\left(\frac{100}{y_i + Y_{-i}} y_i - 2y_i^2\right)}{\partial y_i} = -\frac{100 y_i}{(y_i + Y_{-i})^2} + \frac{100}{y_i + Y_{-i}} - 4y_i$$

The first-order condition is:

$$\frac{100}{y_i + Y_{-i}} - \frac{100y_i}{(y_i + Y_{-i})^2} = 4y_i$$

$$\frac{100}{Y} - \frac{100}{Y^2}y_i = 4y_i$$

Notice that the term $\frac{\partial p}{\partial Y} = \frac{100}{Y^2}$. Thus, the distortion of price taking is relative to the size of Y. To push this further, we really need to be able to figure out what firms actually produce without the price taking assumption. But to solve this, we need to know how to deal with a situation where firm i's output decision y_i depends on the output decisions of all the other firms Y_{-i}. This is the realm of game theory. We won't be too formal about the game theory we do in this chapter, but we will get a feel for some of the fundamentals of game theory through these exercises.

13.3 A Simple Cournot Market

This market is a game. There are players (the firms), strategies (quantity choices), outcomes (price), and preferences (profit function). These are the key elements of a game.

We will solve this game using Nash Equilibrium. A nash equilibrium requires that all players are best responding to each-others strategies. In this case, it will be that all firms output decisions are optimal given what the other firms are doing.

We have already seen the first order condition for the firms is:

$$\frac{100}{y_i + Y_{-i}} - \frac{100y_i}{(y_i + Y_{-i})^2} = 4y_i$$

If we solve this for y_i, this defines a "best response function", that is condition on the other's actions, what is the optimal decision for me?

13.4 Groping Around for a Nash Equilibrium:

Assume there are two firms. The first order condition for firm i is:

$$\frac{100}{y_i + y_j} - \frac{100y_i}{(y_i + y_j)^2} = 4y_i$$

If we plug in a number for y_j and solve for y_i, we will have i's best response to j producing y_j

Suppose, for no reason at all:

$$y_j = 10$$

Using a *Mathematica* to solve the first order condition for y_i, the best response is:

$$y_i = 1.79652$$

Is $y_j = 10$,$y_i = 1.79652$ a Nash equilibrium? Let's see what j's best response is to $y_i = 1.79652$. Using mathematica again, it is $y_j = 2.46875$. Thus, *this is not an equilibrium.* We could keep doing this until we get a number that is a best response to itself. If this process converges, it is nice numerical way to find a *symmetric equilibrium* (all players choose the same strategy), but here we can actually solve for it analytically pretty easily.

13.5 A Symmetric Equilibrium of the Cournot game.

Once we have the first order condition,we can impose that we are looking for a symmetric equilibrium where $y_i = y$ for all i.

$$\frac{100}{y + (J-1)\,y} - \frac{100y}{(y + (J-1)\,y)^{\,2}} = 4y$$

Solving this for y:

$$y = \frac{5\sqrt{J-1}}{J}$$

For $J = 2$ for instance:

$$\frac{5\sqrt{2-1}}{2.0} \quad = \quad 2.5$$

Our first Nash Equilibrium!

13.6 Comparison to Equilibrium under Price Taking

With price taking we found:

$$y_{price-taking} = \frac{5}{\sqrt{J}}$$

In the symmetric cournot equilibrium:

$$y_{cournot} = \frac{5\sqrt{J-1}}{J}$$

Let's compare for various J. The Mathematica code I'll use is:

$$TableForm[Round[Table[\{J, \frac{5.0\sqrt{J-1}}{J}, \frac{5.0\sqrt{J}}{J}\}, \{J, \{2, 10, 50, 1000\}\}], 0.001]]$$

2.	2.5	3.536
10.	1.5	1.581
50.	0.7	0.707
1000.	0.158	0.158

Notice how for 2 firms, we are about 30% off. But for 50 we are not far off at all. Recall that 50 was the expected number of firms in this market (in equilibrium of the entry game).

13.7 Entry With Cournot

Suppose it costs \$1 to enter this market and all firms compete in Cournot competition. How many firms will enter when costs is $2y^2$?

$$\pi = \left(\frac{100}{y_i + Y_{-j}}\right) y_i - 2y_i^2$$

In Cournot equilibrium:

$$y = \frac{5\sqrt{J-1}}{J}$$

The profit of each firm under Cournot equilibrium as a function of J is given by:

$$\pi(J) = \left(\frac{100}{J\frac{5\sqrt{J-1}}{J}}\right)\left(\frac{5\sqrt{J-1}}{J}\right) - 2\left(\frac{5\sqrt{J-1}}{J}\right)^2$$

$$= \frac{100}{J} - \frac{50(J-1)}{J^2}$$

When can a firm justify entering? When this profit is greater than \$1.

$$\frac{100}{J} - \frac{50(J-1)}{J^2} \geq 1$$

$$J \leq 50.9808$$

This this case the only equilibrium is 50 firms entering. If that's the case, the are all earning positive profit and none have incentive to leave. If another firms enters, then it will become the 51st and earn negative profit, so no firm has incentive to enter.

13.8 Monopoly

To find the monopoly solution (J=1) we can reuse the above analysis. If we set $J = 1$ we are essentially just solving the profit maximizing level of output for a sole firm in a market.

Notice when we set $J = 1$ in the above Cournot equibrlium we get that output is **zero?!?** What's going on?

The issue is that demand is always unit-elastic.

At any point where demand is either elastic or unit-elastic, a monopolist will have incentive to cut output by 1%, this will let them increase price by at least 1% which will keep revenue the same (or increase it) and lower cost. **This will always increase profit.**

We need to be a bit more careful with market demand functions with monopoly (or even colluding firms who act similarly to a monopoly).

13.9 Colluding Firms

Let's change our demand for a moment so it is not always unit-elastic. Suppose demand is given by:

$$y = 100 - p$$

Inverse demand is:

$$p = 100 - y$$

Let's assume cost is the same as our working example $2y_i^2$. Profit is given by:

$$\pi = (100 - Y)\, y_i - 2y_i^2$$

The first order condition is:

$$\frac{100 - Y_{-i}}{6} = y_i$$

Impose symmetry and solve to get the Cournot equilibrium:

$$\frac{100 - (J-1)\, y}{6} = y$$

$$\frac{100}{5 + J} = y_{cournot}$$

What if instead of imposing symmetry on the first-order condition, we imposed symmetry before taking the first order condition?

$$\pi = (100 - Jy)\, y - 2y^2$$

Notice the profit of all firms is:

$$J\pi = J\,(100 - Jy)\, y - J2y^2$$

Since this is just a scaling of the profit of a single firm, maximizing the individual firms profit by setting y is also maximizing joint profits of *all the firms*. That is, by imposing symmetry before taking the first-order condition, we are essentially asking the firms to collude. Let's solve this for the collusive output:

$$\frac{\partial \left(100y - (J+2)\, y^2\right)}{\partial y} = 100 - 2(J+2)y$$

$$\frac{100}{2J + 4} = y_{collusive}$$

If all of the firms are producing at $y_{collusive}$, is $y_{collusive}$ a best response? The best response comes from the first order condition we found above:

$$\frac{100 - Y_{-i}}{6} = y_i$$

Plugging in the collusive output for all $J-1$ other firms:

$$\frac{100 - (J-1)\left(\frac{100}{2J+4}\right)}{6} = y_i$$

$$\frac{1}{6}\left(100 - \frac{100(J-1)}{2J+4}\right) = y_{best-response}$$

Let's look at $J = 2$

$$y_{cournot} \approx 14.2857$$

$$y_{collusive} = 12.5$$

$$y_{best-response} \approx 14.5833$$

We can now calculate the profits of the firms:

$$\pi_{cournot} = 612.25$$

$$\pi_{collusive} = 625$$

If a firm deviates from the collusive agreement by best-reponding:

$$\pi_{deviating} \approx 638.02$$

$$\pi_{non-deviating} \approx 598.96$$

Preview- Repeated Games
Suppose these firms play this game indefinitely. A firm can earn an extra (approximately) 13.02 by deviating from the collusive agreement in any period. However, if this leads to a loss of the collusive agreement, the will loose about 12.75 every period after cheating on the collusive agreement. These are almost the same value. How impatient would a firm have to be to cheat on this collusive agreement? They would have to discount each period by about 0.50 to be willing to cheat. This type of analysis in common in repeated game. However, we will leave it for next semester.

13.10 Asymmetric Cournot

Let's return to our working example with inverse demand $p(Y) = \frac{100}{Y}$. Suppose there are two firms with different cost functions:

$$c_1(y_1) = 2y_1^2$$

$$c_2(y_2) = 2y_2^{\frac{3}{2}}$$

The profit functions for the firms is:

$$\pi_1 = \left(\frac{100}{y_1 + y_2}\right) y_1 - 2y_1^2$$

$$\pi_2 = \left(\frac{100}{y_1 + y_2}\right) y_2 - 2y_2^{\frac{3}{2}}$$

Let's find the firms first-order conditions:

$$0 = -\frac{100y_1}{(y_1 + y_2)^2} - 4y_1 + \frac{100}{y_1 + y_2}$$

$$0 = -\frac{100y_2}{(y_1 + y_2)^2} - 3\sqrt{y_2} + \frac{100}{y_1 + y_2}$$

Solving these simultaneously:

$$y_1 = 2.42755, y_2 = 3.95231$$

Let's compare to symmetric costs with $2y^2$ there we had:

$$y_1 = y_2 = 2.5$$

13.11 Stackleberg: A Challenge

Let's return to our working example with demand $\frac{100}{Y}$ and both firms again have $c(y) = 2y^2$. Suppose there are two firm. Both have profit function:

$$\pi_i = \left(\frac{100}{y_i + y_j}\right) y_i - 2y_i^2$$

Unlike before, suppose firm 2 learns firm one's choice y_1 and then maximizes profit. It's first order condition is:

$$0 = -\frac{100y_2}{(y_1 + y_2)^2} - 4y_2 + \frac{100}{y_1 + y_2}$$

Solving this for y_2, we get firm 2's best response to the choice y_1:

$$y_2 = \frac{1}{3}\left(\frac{\sqrt[3]{2}y_1^2}{\sqrt[3]{2y_1^3 + 675y_1 + 15\sqrt{3}\sqrt{4y_1^4 + 675y_1^2}}} - 2y_1 + \frac{\sqrt[3]{2y_1^3 + 675y_1 + 15\sqrt{3}\sqrt{4y_1^4 + 675y_1^2}}}{\sqrt[3]{2}}\right)$$

Let's call this $y_2^*(y_1)$. What does y_1 do? They know that firm 2 will use this best response function and so firm one can incorporate this into their profit:

$$\pi_1 = \left(\frac{100}{y_1 + y_2^*(y_1)}\right) y_1 - 2y_1^2$$

Substitution $y_2^*(y_1)$:

$$\pi_1 = \left(\frac{100}{y_1 + \frac{1}{3}\left(\frac{\sqrt[3]{2}y_1^2}{\sqrt[3]{2y_1^3+675y_1+15\sqrt{3}\sqrt{4y_1^4+675y_1^2}}} - 2y_1 + \frac{\sqrt[3]{2y_1^3+675y_1+15\sqrt{3}\sqrt{4y_1^4+675y_1^2}}}{\sqrt[3]{2}}\right)}\right) y_1 - 2y_1^2$$

The derivative of this thing is ridiculous. I have not yet managed to get Mathematica to give me a solution to this the first-order condition. Want to try? There's a Pentel P205 in it for the first student to provide an approximate solution to the first order condition and prove that it is approximately a Nash equilibrium of the Cournot game.

13.12 Stackleberg Redux

Let's move away from the running example and linearize the demand function. Suppose cost is $2y^2$ with demand $100 - Y$.

$$\pi_i = (100 - (y_i + y_j))\, y_i - 2y_i^2$$

Let's find firm 2's first order condition since firm 2 maximizes subject to firm one's chocie:

$$-y_1 - 6y_2 + 100 = 0$$

The best response of firm 2 is:

$$y_2 = \frac{100 - y_1}{6}$$

Firm one knows firm two will use this best response so we can write firm one's profit using this in place of y_2.

$$\pi_1(y_1) = \left(100 - \left(y_1 + \frac{100 - y_1}{6}\right)\right) y_1 - 2y_1^2$$

The first order condition is:

$$0 = \frac{1}{6}(y_1 - 100) - \frac{35y_1}{6} + 100$$

$$y_1 = 14.7059$$

Plug this back into firm 2's best response to get y_2:

$$y_2 = 14.2157$$

Notice, the first mover has an advantage here. Despite being identical, firm one earns more profit.

$$\pi_1 = 612.745$$

$$\pi_2 = 606.257$$

14 Surplus

Surplus is the difference between what an agent pay/gets for a unit of some good and what they were willing to pay/get.

Consumer Surplus is defined as the area under the demand curve but above price.

Producer Surplus is defined as the area above marginal cost curve and below price.

14.1 Example

$$p(q) = 50 - 2q$$

$$c(q) = 10 + 2q$$

15 General Equilibrium- A Minimal Introduction

15.1 2x2 Economies, Pure Exchange Economy, Edgeworth Box

Two consumers. Endowments ω_i. Preferences u_i. Prices p.

The budget equation for each consumer is:

$$p_1 x_{11} + p_2 x_{12} = p_1 \omega_{11} + p_2 \omega_{12}$$

$$p_1 x_{21} + p_2 x_{22} = p_1 \omega_{21} + p_2 \omega_{22}$$

An equilibrium is a set of prices and allocations such that.
1) Maximization. For $i \in \{1, 2\}$:

$$x_i = Arg.Max\, u(x_i)\ s.t.\, x_i p \leq \omega_i p$$

2) Market Clearing (**feasibility**). For $j \in \{1, 2\}$

$$x_{1j} + x_{2j} \leq \omega_{1j} + \omega_{2j}$$

15.2 Welfare Theorems (First)

Pareto Efficient Allocation:

A feasible (x_1, x_2) is pareto efficient is there does not exist an alternative fesible $(\tilde{x}_1, \tilde{x}_2)$ such that $\tilde{x}_i \succsim x_i$ for $i \in \{1, 2\}$ and the relation is strict for at least one.

First Welfare Theorem:

Under local non-statiation, any equilibrium is Pareto efficient.

Proof. By maximization and local nonsatiation the budget equation binds for both consumers. Thus, if $\tilde{x}_i \succ x_i$ then $p_1 x_{i1} + p_2 x_{i2} > p_1\omega_{i1} + p_2\omega_{i2}$. Otherwise, there exists a strictly preferred and affordable bundle, contradicting maximization. If $\tilde{x}_j \succsim x_j$ then $p_1 x_{j1} + p_2 x_{j2} \geq p_1\omega_{j1} + p_2\omega_{j2}$. Otherwise (by local nonsationation) there is a strictly preferred and affordable bundle, contradicting maximization. Combining these, if there is a pareto improving allocation:

$$p_1 x_{j1} + p_2 x_{j2} \geq p_1\omega_{j1} + p_2\omega_{j2}$$

$$p_1 x_{i1} + p_2 x_{i2} > p_1\omega_{i1} + p_2\omega_{i2}$$

Summing:

$$p_1 (x_{i1} + x_{j1}) + p_w (x_{i2} + x_{j2}) \geq p_1 (w_{i1} + w_{j1}) + p_2 (w_{i2} + w_{j2})$$

By feasibility $w_{ik} + w_{jk} \geq x_{ik} + x_{jk}$

$$p_1 (w_{i1} + w_{j1}) + p_2 (w_{i2} + w_{j2}) > p_1 (x_{i1} + x_{j1}) + p_w (x_{i2} + x_{j2}) > p_1 (w_{i1} + w_{j1}) + p_2 (w_{i2} + w_{j2})$$

$$1 > 1$$

Thus, every equilibrium is Pareto efficient. But s ever Pareto efficient allocation an equilibrium.

15.3 Welfare Theorems (Second)

First Welfare Theorem:

Under local non-statiation, and if u_i is strictly-quasiconvex for both consumers, any pareto efficient allocation an equilibrium from some starting endowments.

Here is a sketch of the proof:

Since u_i is strictly-quasiconvex for both consumers, the set $\succsim (x_i)$ is a convex set for both i containing the point (x_1, x_2). By the separating hyperplane theorem, there is a hyperplane (price vector) that separates the interior of $\succ (x_i)$ from $\succsim (x_j)$ and vise-versa. x_i, x_j is an equilibrium allocation from any endowment on this hyperplane.

15.4 Example

$U_1 = x_{11}^1 x_{12}^2$, $U_2 = x_{21}^2 x_{22}^1$, $\omega_{11} = 10$,$\omega_{22} = 10$.

$$x_{11}^* = \frac{\frac{1}{3}(10p_1)}{p_1}, x_{12}^* = \frac{\frac{2}{3}(10p_1)}{p_2}, x_{21}^* = \frac{\frac{2}{3}(10p_2)}{p_1}, x_{22}^* = \frac{\frac{1}{3}(10p_2)}{p_2}$$

$$x_{11}^* + x_{21}^* = \frac{\frac{1}{3}(10p_1)}{p_1} + \frac{\frac{2}{3}(10p_2)}{p_1} = \frac{\frac{1}{3}(10p_1) + \frac{2}{3}(10p_2)}{p_1} = 10$$

$$Solve[\{\frac{\frac{1}{3}(10p_1) + \frac{2}{3}(10p_2)}{p_1} == 10, \frac{\frac{2}{3}(10p_1) + \frac{1}{3}(10p_2)}{p_2} == 10\}, \{p_1, p_2\}]$$

$$\frac{\frac{1}{3}(10) + \frac{2}{3}(10p_2)}{1} = 10$$

$$\frac{2}{3}(10p_2) = \frac{2}{3}10$$

$$p_2 = 1$$

Part IV

Price Discrimination and Optimal Mechanisms

16 Mechanism Design

16.1 First Degree Price Discrimination

Motivation.

16.2 Pricing a Single Indivisible Good

Buyer

Type θ. (Willingness to pay).

$u = \theta - t$ (if they get the good)

$u = 0$ (otherwise)

Seller

Maximize revenue.

Believes $\theta \in [\underline{\theta}, \bar{\theta}]$

$\theta \sim F(\theta)$ (has a belief about the distribution of θ)

16.3 Mechanisms

Mechanism $\Gamma(\Sigma, t, q)$**:**

Σ is the set of strategies available to the consumer with elements σ.

$t(\sigma) : \Sigma \to \mathbb{R}_+$ (transfers/payments)

$q(\sigma) : \Sigma \to [0, 1]$ (probability of getting the good).

Posted price:

$\Sigma = \{y, n\}$ *"yes" "no"*

$t(y) = p$. $t(n) = 0$ (pay p if "yes" pay 0 if "no")

q(y) = 1. $q(n) = 0$ (gets the good for sure if "yes)

Optimal strategies:

$\theta \geq p$ "yes"

$\theta \leq p$ "no"

Direct Mechanism $\Gamma(\Theta, t, q)$**:**

A mechanism where the strategy space is the set of types.

$\Sigma = \Theta$

$t(\theta) = p$ if $\theta \geq p$, $t(\theta) = 0$ if $\theta < p$.

$q(\theta) = 1$ if $\theta \geq p$ and 0 otherwise.

This is incentive compatible because θ is a optimal for a player with type θ. Telling the truth is optimal. There's no reason not to tell the truth. Lying about your type cannot make you strictly better off.

Incentive Compatibility

$\theta q(\theta) - t(\theta) \geq \theta q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right)$ for all $\tilde{\theta} \in \Theta$.

16.4 Revelation Principle and Properties of Direct Mechanisms

Revelation Principle: *For every mechanism Γ, there is a an incentive compatible direct mechanism $\tilde{\Gamma}$ where a and $t(\sigma(\theta)) = \tilde{t}(\theta)$ where $\sigma(\theta)$ is the optimal strategy for a player with type θ in the mechanism Γ.*

The intuition for this is we can always ask a type and promise to play the optimal strategy for the player. The resulting thing is an incentive compatible direct mechanism.

This is an important result. It tells us that if there is not an incentive compatible direct mechanism that implements some $q()$ and $t()$, then there is no mechanism at all with those allocation and transfer functions. We can restrict our attention to direct mechanisms!

Incentive Compatibility [I.C.]

$\theta q(\theta) - t(\theta) \geq \theta q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right)$ for all $\tilde{\theta} \in \Theta$.

Utility Under Incentive Compatible Direct Mechanism

$u(\theta) = \theta q(\theta) - t(\theta)$

Individually Rational [I.R.]

$u(\theta) \geq 0$ for all $\theta \in \Theta$ (participating is at least as good as not participating).

16.5 Some Examples

Posted Price

Pick some p:

$$q(\theta) = \begin{cases} 1 & \theta \geq p \\ 0 & \theta < p \end{cases}$$

$$t(\theta) = \begin{cases} p & \theta \geq p \\ 0 & \theta < p \end{cases}$$

Lottery Selling

$$q(\theta) = \begin{cases} 1 & \theta \geq \frac{1}{2} \\ \frac{1}{2} & \theta \in [\frac{1}{4}, \frac{1}{2}) \\ 0 & \theta < \frac{1}{4} \end{cases}$$

$$t(\theta) = \begin{cases} \frac{1}{2} & \theta \geq \frac{1}{2} \\ \frac{1}{4} & \theta \in [\frac{1}{4}, \frac{1}{2}) \\ 0 & \theta < \frac{1}{4} \end{cases}$$

Is this incentive compatible? $\theta = \frac{1}{4}$ $u\left(\frac{1}{4}|\frac{1}{4}\right) = \frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{4} = -\frac{1}{8}$

$u\left(0|\frac{1}{4}\right) = 0$

Not incentive compatible. Because this is not incentive compatible, there is no mechanism implementing this $q()$ and $t()$.

16.6 Characterizing IC IR Mechanisms

Our goal is to characterize t and q for suitable mechanisms. In this section we will show that incentive compatible and individually rational mechanisms much have a monotonic $q()$. Once q is chosen, $t()$ is determined (up to a constant) by a result known as revenue equivalence. We will prove that in the next section.

I.C. implies Monotonic $q()$:
Proof.
Assume $\theta > \tilde{\theta}$ without loss of generality.
$\theta q(\theta) - t(\theta) \geq \theta q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right)$ (θ doesn't want to pretend to be $\tilde{\theta}$)
$\tilde{\theta} q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right) \geq \tilde{\theta} q(\theta) - t(\theta)$ (the reverse)
Re-arranging these:
$\theta q(\theta) - t(\theta) - \theta q\left(\tilde{\theta}\right) + t\left(\tilde{\theta}\right) \geq 0$
$\tilde{\theta} q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right) - \tilde{\theta} q(\theta) + t(\theta) \geq 0$
Adding the inequalities:
$q(\theta)\left(\theta - \tilde{\theta}\right) + q\left(\tilde{\theta}\right)\left(\tilde{\theta} - \theta\right) \geq 0$
Since $\theta - \tilde{\theta} > 0$:
$q(\theta) \geq q\left(\tilde{\theta}\right)$
Thus, q in monotonic in θ.

16.7 Revenue Equivalence

$$Max_{\tilde{\theta}\in\Theta} u\left(\tilde{\theta}|\theta\right)$$

$$Max_{\tilde{\theta}\in\Theta}\left(\theta q\left(\tilde{\theta}\right) - t\left(\tilde{\theta}\right)\right)$$

Because this is a family of functions indexed by $\tilde{\theta}$ that is both increasing and convex, the value function has to be increasing and convex.
$u(\theta)$ is both increasing and convex in θ.
$q(\theta) = 1$ and $t(\theta) = \frac{1}{2}$ if $\theta \geq \frac{1}{2}$ and both are 0 otherwise.
$u(\theta) = 0$ if $\theta < \frac{1}{2}$. $u(\theta) = \theta - \frac{1}{2}$ if $\theta \geq \frac{1}{2}$.
Through the envelope condition:

$$u'(\theta) = q(\theta)$$

A convex function is differentiable almost everywhere.

$$u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u'(x)\, dx$$

By the envelope condition we get **payoff equivalence.**

$$u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)\, dx$$

Confirm this for posted price:

$$u(\underline{\theta}) = 0$$

For $\theta < \frac{1}{2}$

$$u(\theta) = 0 + \int_{\underline{\theta}}^{\theta} 0 dx = 0$$

For $\theta \geq \frac{1}{2}$

$$u(\theta) = 0 + \int_{\underline{\theta}}^{\frac{1}{2}} (0)\, dx + \int_{\frac{1}{2}}^{\theta} (1)\, dx$$

$$u(\theta) = \int_{\frac{1}{2}}^{\theta} (1)\, dx = \theta - \frac{1}{2}$$

Revenue Equivalence:
Start with our result above:

$$u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)\, dx$$

$$\theta q(\theta) - t(\theta) = \underline{\theta} q(\underline{\theta}) - t(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)\, dx$$

$$\theta q(\theta) - \left(\underline{\theta} q(\underline{\theta}) - t(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)\, dx \right) = t(\theta)$$

$$\theta q(\theta) - \underline{\theta} q(\underline{\theta}) + t(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} q(x)\, dx = t(\theta)$$

$$t(\theta) = t(\underline{\theta}) + [\theta q(\theta) - \underline{\theta} q(\underline{\theta})] - \int_{\underline{\theta}}^{\theta} q(x)\, dx$$

16.8 Individual Rationality

Because $u(\theta)$ is increasing in θ, an incentive compatible mechanism is individually rational if and only if $u(\underline{\theta}) \geq 0$.

Let's assume $u(\underline{\theta}) \geq 0$

$$u(0) = -t(0)$$

Unless we pay the lowest type to participate, $t(0) = 0$. If this is the case, it simplifies our rev. eq.

$$t(\theta) = \theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(x)\, dx$$

The second term comes up often in mechanism design. It is called the "information rent".

Let's suppose the mechanism deisgner wants to try to implement first degree price discrimination. It sells to everyone. What is the most is can charge? Set $q() = 1$

$$t(\theta) = \theta - \int_0^{\theta} 1dy = 0$$

16.9 Examples.

$\Theta = [0,1], q = \theta.\ \theta \sim u[0,1]$

$$t(\theta) = \theta^2 - \int_0^{\theta} xdx$$

$$t(\theta) = \frac{\theta^2}{2}$$

$$= \frac{1}{6}$$

$\Theta = [0,1],\ q(\theta) = 1$ for $\theta \geq \frac{1}{2}$. and $q(\theta) = 0$ otherwise.

For types above $\frac{1}{2}$, transfers are:

$$t(\theta) = \theta - \int_{\frac{1}{2}}^{\theta} (1)\, dz = \frac{1}{2}$$

16.10 Optimal Mechanisms

The goal of the mechanism designer is to maximize revenue $\int_{\Theta} f(\theta) \left(\theta q(\theta) - \int_0^{\theta} q(\theta)\right)$ by choosing an I.C. I.R. mechanism. Thus, the designer needs to pick a $q()$ to maximize this from the set of monotonic q.

Lemma. *The set of monotonic q is a convex set.*

Theorem. *Bauer Maximum Principle.*

A continuous convex function attains its maximum on a compact convex set at an extreme point.

Lemma. *Extreme Points of the Set of I.C. and I.R. q are Steps.*

The extreme points of q are all q such that $q(\theta) = 0$ for $\theta < \theta^*$ and $q(\theta) = 1$ for $\theta > \theta^*$.

Revenue is maximized by using some step function for q. The step functions are posted price mechanisms.

For uniform $[0,1]$. The best posted price mechanism charges a price of $\frac{1}{2}$.

Proposition. *A Revenue Maximizing Mechanism is a Posted Price Mechanism with $p^* = Arg.Max_{p \in [\underline{\theta}, \overline{\theta}]} p(1 - F(p))$.*