Economics 8100

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Part I
Budget

1 Consumption Set X

Assumptions: (Universe of Choice Objects): X
Bundles: Elements of X. $x \in X$

Assumptions about X**.** 1. $\emptyset \neq X \subseteq \mathbb{R}^n_+$. 2. X is closed. 3. X is convex. 4. $0 \in X$.

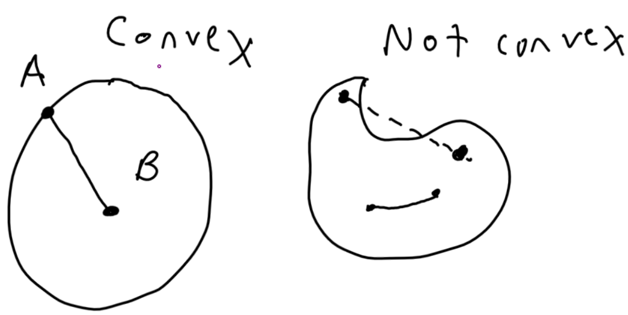

Figure 1.1: Examples of a Convex/Non-Convex Set.

2 Budget Set B

Budget Set: $B \subseteq X$

X defines the scope of the model. B is what an *individual consumer* chooses among.

Example. Budget Set with Prices and Income
$B = \{x | x \in X \,\&\, x_1 p_1 + x_2 p_2 \leq m\}$

Example. Ice Cream Bowls
Every ice cream bowl x has some non-negative number of scoops of Vanilla, Chocolate, Strawberry.

$$X = \mathbb{R}^3_+$$

Budget B is the set of bowls with *no more than one scoop of ice cream.*

$$B = \left\{ x | x \in R^3_+ \& \sum_{i=1}^{3} x_i \leq 1 \right\}$$

This is the unit-simplex in $\mathbb{R}_3$.
$(1, 0, 0) \in B$. (On the boundary.)
$(0.5, 0.5, 0) \in B$. (On the boundary.)
$(0.25, 0.25, 0.25) \in B$. (In the interior.)
$(2, 0, 0) \notin B$

Part II
Preference

3 The Preference Relation

Preference Relation is a **Binary Relation**.
Formally, a binary relation on set X is a subset of the Cartesian product X with itself.

$$\succeq \subseteq X \times X$$

Another way to denote an ordered pair is "in" the relation:
If $(x, y) \in \succeq$ we can also write $x \succeq y$.
Informally we say "x" is at least as good as "y", or "x" preferred "y".

Axioms of $\succeq$.
Axiom 0 *(reflexive):* $\forall x \in X$, $x \succeq x$. This is implied by *axiom 1.*

Axiom 1 *(complete):* $\forall x, x' \in X$, either $x \succeq x'$ or $x' \succeq x$ (or both).

The consumer has "some" preference over every pair of objects.

Axiom 2 *(transitive):* $\forall x, x', x'' \in X$ *if* $x \succeq x'$ *and* $x' \succeq x'' \Rightarrow x \succeq x''$.

$\succeq$ is a "weak order" if it is complete and transitive.

4 Relations and Sets Related to $\succeq$

Subrelations:

$\sim$ is the indifference relation. $x \succeq y$ and $y \succeq x \Leftrightarrow x \sim y$.

$\succ$ is the strict relation. $x \succeq y$ and not $y \succeq x \Leftrightarrow x \succ y$.

Related Sets:

$\succeq (x)$ "upper contour set", "no worse than set"

$\precsim (x)$ "lower contour set", "no better than set"

5 From Preferences to Choice

Choice Correspondence.

We will assume that from a budget set B a consumer "chooses" *choice set* C according to their preference $\succeq$. $C = \{x | x \in B \,\&\, \forall x' \in B, x \succeq x'\}$.

Informally, C is the set of objects that are at least as good as anything else in the set.

Example With Transitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, c \succeq b$.

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = c$$

$$C(\{a, b, c\}) = c$$

6 Cycles Lead to Empty Choice Sets

6.1 The Problem with Intransitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, b \succeq c$. *This is intransitive!*

Choice correspondence:

$$C : P(X)/\emptyset \to X$$

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a,b\}) = a, C(\{a,c\}) = c, C(\{b,c\}) = b$$

$$C(\{a,b,c\}) = \emptyset$$

This consumer cannot make a choice from the set $\{a, b, c\}$.

6.2 Cycles and Empty Choices

Notice in the previous example, $a \succ b, a \succ c, c \succ a$. We have proved (essentially) that if there is a cycle, there is an empty choice set.

In fact, suppose, there is an empty choice set **and** X is finite. There must be a cycle.

$$\forall x \in B, \#(\succsim(x)) < \#(B)$$

By completeness, $\forall x \exists x' \in X : x' \succ x$. Choose an x_1, let x_2 be any element of $\succ(x_1)$. We have $x_2 \succ x_1$. If there is an $x_3 \in \succ(x_2)$ such that $x_1 \succ x_3$ we have identified a cycle. Otherwise, we continue with an inductive step. Suppose we have $x_n \succ ... \succ x_1$. $\succ(x_n)$ is non-empty. Either it contains an element x_{n+1} such that there is an $x_i \succ x_{n+1}$ in which case we have identified a cycle or it does not and we continue with another inductive step. Either we find a cycle or reach the N_{th} step with $x_N \succ x_{n-1} \succ ... \succ x_1$. $\succ(x_N)$ is non-empty.

So, the cycle condition is equivalence to a non-empty choice set. Transitivity of $\succsim$ implies transitivity of $\succ$ which implies no cycles (try this last step at home). But do no-cycles imply transitivity of $\succsim$? No. Here is a counter-example:

$$x \succ y, y \sim z, z \succ x$$

7 Intransitivity: Empty Choices, Incoherent Choices: Pick One.

So if no-cycles of the strict preference is equivalent to non-empty choice (in finite sets), and transitvity of $\succsim$ is not equivelent to no-cycles, why do we assume it?

Finite non-emptyness: *For any B with $\#(B) \in \mathbb{I}$, $C(B) \neq \emptyset$*

Coherence: *For every x, y and B, B' such that $x, y \in B \cap B'$, $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$.*

Suppose there is an intransitive $\succsim$. There exists either a B where $C(B) = \emptyset$ or there exists a x, y, B, B' where the choice correspondence is incoherent.

By intransitivity:

$$1)\, x \succ y, y \succ z, z \succ x$$

$$C(\{x, y, z\}) = \emptyset$$

$$2)\, x \sim y, y \sim z, z \succ x$$

$$3)\, x \sim y, y \succ z, z \succ x$$

$$x \notin C(\{x, y, z\})$$

$$y \in C(\{x, y, z\})$$

$$x \in C(\{x, y\})$$

$$4)\, x \succ y, y \sim z, z \succ x$$

Can you find the incoherent choice?

8 Indifference Sets

8.1 Indifference Maps

To understand preferences, we often draw sets of the form $\sim (x)$. Many times these are one dimension smaller than the space of bundles, in which case we often call them *indifference curves*, but they need not have any special structure, unless we make further assumptions about preferences. There is only one things we really know about these sets.

8.2 Complete, Transitive Preferences have Indifference Sets that Do Not Intersect

Result. *Indifference curves do not cross.* For two bundles $x \succ y$, $\sim (x) \cap \sim (y) = \emptyset$.

Proof is given visually below:

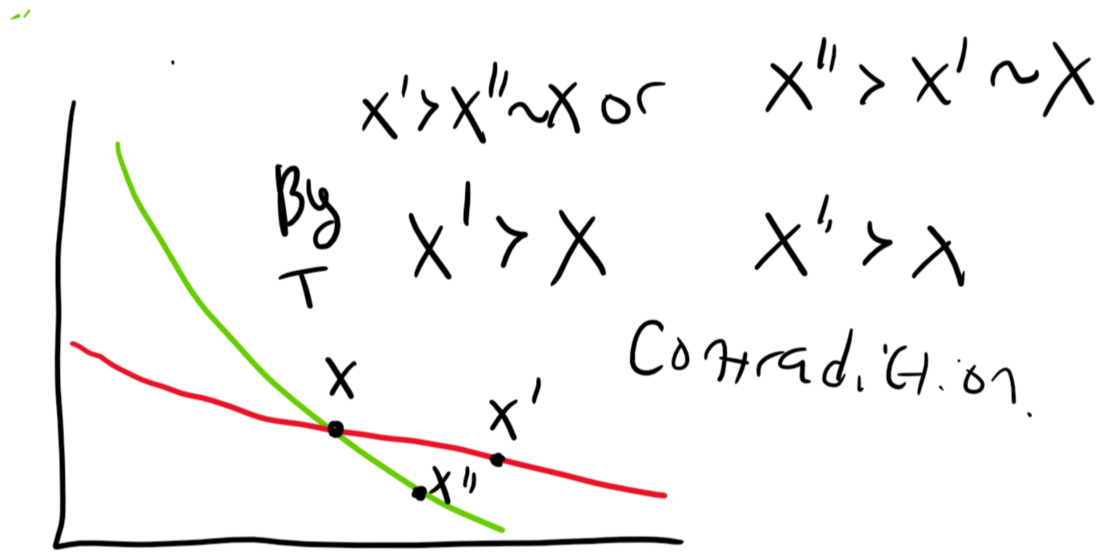

Figure 8.1: Distinct Indifference Sets do not Intersect

Part III
From Preference to Utility

9 Utility Represents Preferences

Suppose there is some $U : X \to \mathbb{R}$ such that $U(x) \geq U(x') \Leftrightarrow x \succsim x'$ then we say $u()$ represents preference relation $\succsim$. When does such a representation exist?

9.1 Finite X

Proposition 1. *A $U()$ exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.*

Proof. Let's start with $\Rightarrow$.

Because $\geq$ is complete on the real numbers, for every $x, y \in X$ either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ thus because $u()$ represents $\succsim$, it is complete.

By similar argument, $\succsim$ is transitive. For every three $x, y, z \in X$. If $u(x) \geq u(y)$ and $u(y) \geq u(z)$ then $u(x) \geq u(z)$ because $\geq$ is transitive on the real numbers.

Now we prove $\Leftarrow$:

Define $U(x) \equiv \#(\precsim(x))$

Example: $a \succ b, b \succ c$. $\precsim (a) = \{a, b, c\} . U(a) = 3$.
Lemma: For $x \succsim y$, $\precsim (y) \subseteq \precsim (x)$ (proved in PS1).
By this lemma, for $x \succsim y$, $\precsim (y) \subseteq \precsim (x)$ and thus $\# \precsim (y) \leq \# \precsim (x)$ and $u(x) \geq u(y)$. $\square$

9.2 Countably infinite X

Pick any arbitrary order on the bundles: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim (x)} w(y)$$

Example: "π shows up unexpectedly when eating ice cream."
An even number of scoops of ice cream are better than an odd number of scoops and otherwise more is better than less.

$$u(2) = \sum_{i=1}^{\infty} \left(\frac{1}{(2i-1)^2} \right) = \frac{\pi^2}{8}$$

$$u(4) = \frac{1}{4} + \frac{\pi^2}{8}$$

9.3 Uncountable X

The *Lexicographic* preferences have no utility representation:

$$X = \mathbb{R}_+^2$$

$(x_1, x_2) \succ (y_1, y_2)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$.
$\succsim$ is complete, and transitive. [*Prove this for practice*].
Pick two real numbers $v_2 > v_1$ and construct four bundles $(v_1, 1), (v_2, 1), (v_1, 2), (v_2, 2)$.

$$(v_2, 2) \succ (v_2, 1) \succ (v_1, 2) \succ (v_1, 1)$$

Suppose there is a utility function representing these preferences, then we have two disjoint intervals:

$$[u(v_2, 1), u(v_2, 2)]$$

$$[u(v_1, 1), u(v_1, 2)]$$

For every real number, we can construct an interval like this. Because the rationals are dense in the reals, there is a rational number in each of these intervals. Thus, for every real, we can find a unique rational number. That is, we have a mapping from the reals into the rationals which implies that the cardinality of the rationals is at least as large as that of the reals. $\#\mathbb{Q} \geq \#\mathbb{R}$. This contradicts that the cardinality of the rationals is strictly smaller than the reals.

9.4 An example of preference relation with a utility representation.

Cars have horse power in $[0, 999]$ and cup holders in $\mathbb{Z}_+$ (integers).

Suppose preferences are lexicographic and more cup holders are more important than more horsepower.

$u(c_i, h_i) = c_i + \frac{h_i}{1000}$ represents these preferences.

See problem set 2 for example where we do not bound the horse power.

9.5 What ensures a utility representation in an uncountable universe?

A preference relation is representable by a utility function $U(x)$ iff $\forall x, y \in X\, s.t.\, x \succ y$, $\exists x^* \in X^* \subset X\, s.t.\, x \succsim x^* \succ y$ and the set X^* is countable.

To construct the utility function, $U(x)$, Pick any arbitrary order on the bundles in X^*: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim(x) \cap X^*} w(y)$$

9.6 Continuous $\succsim$.

Preference relation $\succsim$ is continuous if $\forall x \in X$, $\succsim(x)$ and $\precsim(x)$ are closed in X.

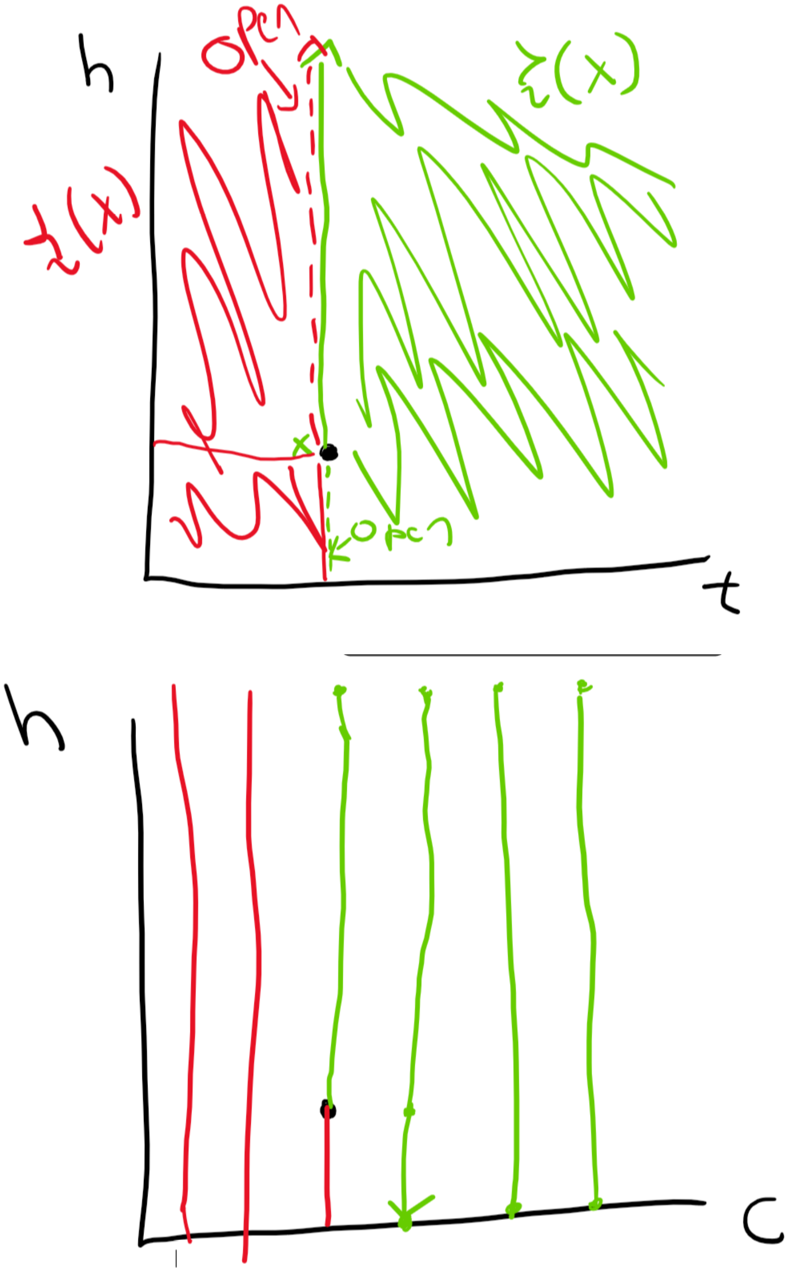

Figure 9.1: Not Continuous/Continuous Lexicographic Preferences.

9.7 What ensures a continuous utility representation?

A *complete*, *transitive,* and *continuous* preference relation $\succsim$ can be represented by a continuous utility function $U(x)$ and, a continuous utility function represented C,T,C preferences.

10 Other Properties of $\succsim$

10.1 Monotonicity

Ensure consumers consume on the boundary of the budget set.

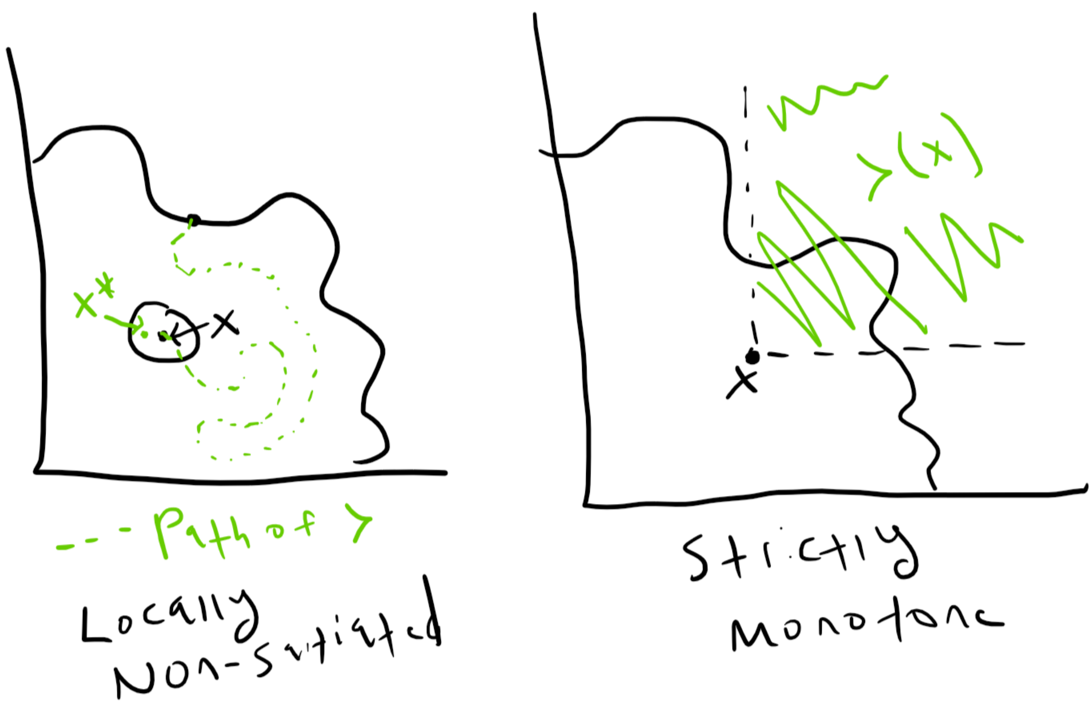

Figure 10.1: Locally Non-satiated vs. Strictly Monotone

Strict Monotonicity

More stuff is better.

First, some notation:

For $X \subseteq \mathbb{R}^n$

$x \geq x'$ iff $x_i \geq x_i'$ for all $i \in \{1, 2, ..., n\}$

$x \gg x'$ iff $x_i > x_i'$ for all $i \in \{1, 2, ..., n\}$

Fro example: $(2,2) >> (1,1), (2,1) \geq (1,1), (1,1) \geq (1,1)$

Definition. Strict Monotonicity. $x \geq x' \Rightarrow x \succsim x'$ and $x \gg x' \Rightarrow x \succ x'$

Local Nonsatiation

Definition. Local Nonsatiatin. $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x^* \in B_\varepsilon(x)$ such that $x^* \succ x$.

A consumer can always change the bundle a "little bit" no matter how small that little bit is, and find something strictly better.

10.2 Convex Sets, Convex/Concave Functions, Quasi-Convex/Concave Functions

Convex Sets

In a subset of euclidean space X, the line between $x \in X$ and $x' \in X$ is another point in the set X given by $tx + (1-t)x'$ where $t \in [0,1]$. We call points like this **Convex Combinations** of x and x'.

For example: $x = (1,0), x' = (0,1)$. If we take $t = 0.5$. The convex combination is $0.5(1,0) + 0.5(0,1) = (0.5, 0.5)$.

A **convex set** $S \subseteq X$ is a set of points that contains all of its convex combinations.

Formally, $\forall x, x' \in S$, $\forall t \in [0,1]$, $tx + (1-t)x' \in S$.

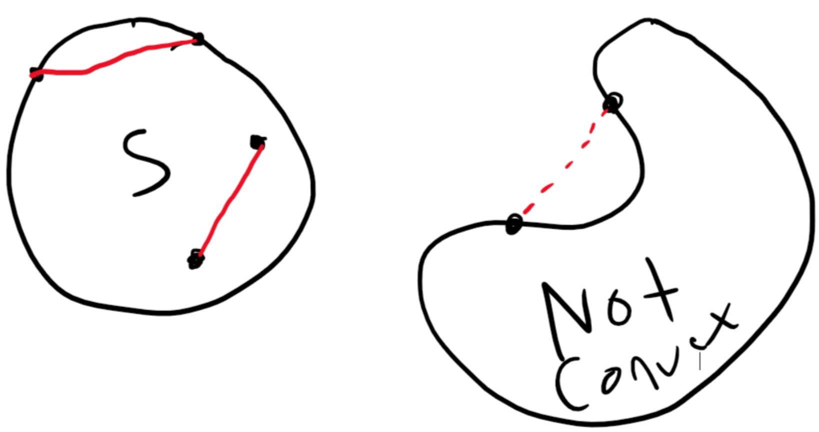

Figure 10.2: A Convex and Non-Convex Set

Convex Functions

A line between two points "on the function" lies above the function itself.
Convex Function*:*

$$\forall x, x' \in X, t \in [0,1], tf(x) + (1-t) f(x') \geq f(tx + (1-t) x')$$

Strictly Convex Function*:*

$$\forall x, x' \in X, t \in (0,1), tf(x) + (1-t) f(x') > f(tx + (1-t) x')$$

Contour Sets:
A convex function has **convex lower contour sets**.

Concave Functions

A line between two points "on the function" lies above the function itself.
Concave Function*:*

$$\forall x, x' \in X, t \in [0,1], tf(x) + (1-t) f(x') \leq f(tx + (1-t) x')$$

Strictly Concave Function*:*

$$\forall x, x' \in X, t \in (0,1), tf(x) + (1-t) f(x') < f(tx + (1-t) x')$$

A concave function has **convex upper contour sets**.

Quasi-Concave Functions

A function $f(x)$ is quasi-concave if **and only** it has convex upper contour sets.
A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a concave function.
A function $f(x)$ is quasi-concave if and only if $f(tx + (1-t) x') \geq min\{f(x), f(x')\}$ for $t \in [0,1]$.
A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t) x') > min\{f(x), f(x')\}$ for $t \in (0,1)$.

Notice that, for a strictly quasi-concave utility function, let $x' \succ x$, then the set $tx+(1-t) x' \in \succ (x)$ for $t \in (0,1)$. Thus, there is a small enough ball around that point $B_\epsilon (tx + (1-t) x') \in \succsim (x)$. Thus, these points are in the interior of $\succsim (x)$ and $\succsim$ is **strictly convex.**

Quasi-Convex Functions

A function $f(x)$ is quasi-convex if **and only** it has convex lower contour sets.
A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a convex function.
A function $f(x)$ is quasi-convex if and only if $f(tx + (1-t) x') \leq max\{f(x), f(x')\}$ for $t \in [0,1]$.
A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t) x') < max\{f(x), f(x')\}$ for $t \in (0,1)$.

10.3 Convexity of $\succsim$.

Convex Preferences: $x \succsim x' \Rightarrow t(x) + (1-t)x' \succsim x', t \in [0,1]$
$x \in \succsim (x') \Rightarrow t(x) + (1-t)x' \in \succsim (x')$
Thus, $\succsim (x)$ are convex if $\succsim$ is a convex preference relation.
Strictly Convex Preferences: $x \succsim x' \Rightarrow t(x) + (1-t)x' \succ x', t \in (0,1)$
The upper contour sets $\succsim (x)$ are *strictly* convex.

10.4 Utility and Preference Relationships

If U represents $\succsim$:
1) $\succsim$ (strictly) **convex** $\Leftrightarrow U$ is (strictly) **quasi-concave**.
2) $\succsim$ are **strictly monotonic** $\Leftrightarrow$ U is **strictly** increasing.
3) $\succsim$ are **strictly monotonic** $\Leftarrow$ U is **strongly** increasing.

11 The Consumer Problem

11.1 Choice

The set of all "best things" in the budget set. This is what we are looking for:
$C(B) = \{x | x \in B \wedge x \succsim x', \forall x' \in B\}$

Competitive Budgets:

$$B = \left\{x | x \in \mathbb{R}^n_+, p \cdot x \leq m\right\}$$

p is the vector of prices.
m is the "income".
Constrained problem:

$$Max_{x \in X} U(x) \, s.t. \, p \cdot x \leq m$$

11.2 The Lagrange Method- Some Intuition.

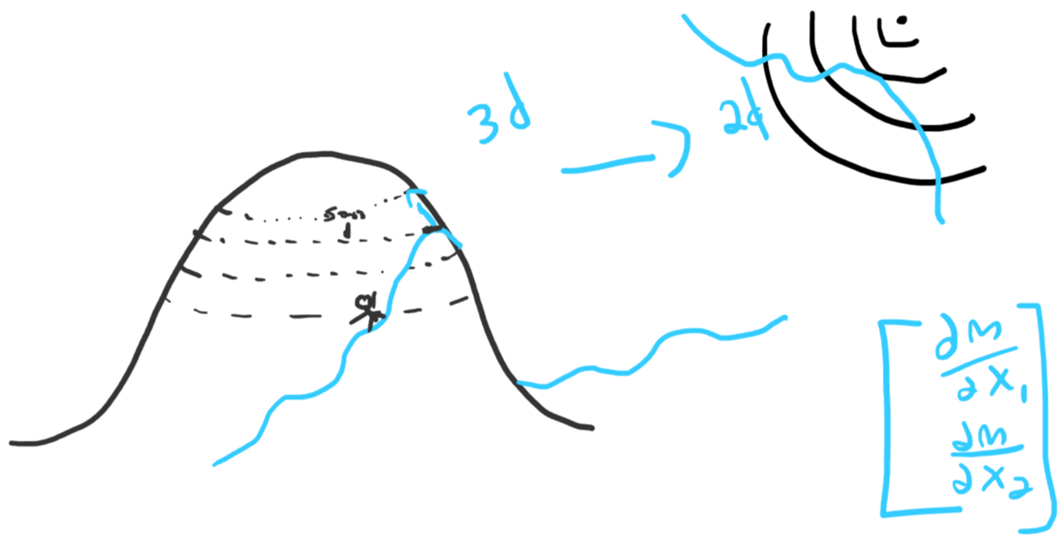

Figure 11.1: Finding the best spot for a selfie.

If both the objective and the constraint are smooth, at the optimal **the *direction* of the gradient of the objective has to be equal to the *direction* of the gradient of the constraint.** Otherwise, moving along the constraint boundary in *some* direction will yield a larger value of the objective! (Caveat: this assumes we *can* move in every direction along the constraint. That will only be true at non-boundary points.)

Thus, for smooth functions, the equality of the direction of the gradients of the objective and the constraint are **necessary** for an non-boundary optimum.

Since the direction of the gradient is just a scaling of the gradient, suppose U is our objective and G is the function for the boundary of the constraint. Then,

$$\nabla U(x) = \lambda \nabla G(x)$$

Can we write a function such that the first order condition will yield this gradient condition? *Sure*:

$$\mathcal{L} = U(x) - \lambda (G(x) - c)$$

Let's treat this as an unconstrained problem. The FOC. of this function is:

$$\nabla U(x) - \lambda \nabla G(x) = 0$$

$$\nabla U(x) = \lambda \nabla G(x)$$

This is precisely the necessary condition we need for the constrained problem.

Thus, FOC for unconstrained optimization of the Lagrangian is the necessary constrained optimization condition.

11.3 Example (Two Constraints)

$$Max_x\,(x_1x_2)$$

$$\left(x_1^2+x_2^2\right)^{\frac{1}{2}}\leq 10$$

$$2x_1+x_2\leq m$$

One Binds. $m=40$**.**

After plotting the two constraints, we can see that the distance constraint is entirely contained on the interior of the budget constraint. The only constraint that could possibly bind is the distance constraint:

$$x_1x_2-\lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)$$

$$\frac{\partial\left(x_1x_2-\lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)\right)}{\partial x_1} = x_2-\frac{\lambda x_1}{\sqrt{x_1^2+x_2^2}}$$

$$\frac{\partial\left(x_1x_2-\lambda\left(\left(x_1^2+x_2^2\right)^{\frac{1}{2}}-10\right)\right)}{\partial x_2} = x_1-\frac{\lambda x_2}{\sqrt{x_1^2+x_2^2}}$$

$$x_2\frac{\sqrt{x_1^2+x_2^2}}{x_1}=\lambda$$

$$x_1\frac{\sqrt{x_1^2+x_2^2}}{x_2}=\lambda$$

$$x_2^2=x_1^2$$

$$x_1=x_2$$

$$x_1=x_2=\frac{10}{\sqrt{2}}$$

$m = 15$**.**

Now neither constraint is contained in the other. Let's set up the Lagrangian with both constraints:

$$x_1x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right) - \mu\left(2x_1 + x_2 - 15\right)$$

The FOCs:

$$\frac{\partial\left(x_1x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right) - \mu\left(2x_1 + x_2 - m\right)\right)}{\partial x_1} = -\frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} - 2\mu + x_2$$

$$\frac{\partial\left(x_1x_2 - \lambda\left(\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} - 10\right) - \mu\left(2x_1 + x_2 - m\right)\right)}{\partial x_2} = -\frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} - \mu + x_1$$

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

Let suppose both constraints bind, then $\lambda \geq 0$**,** $\mu \geq 0$**.**
Since we are on the boundary of both constraints:

$$2x_1 + x_2 = 15$$

$$\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} = 10$$

The only point on the boundary of both constraints is:

$$x_1 \approx 2.68338, x_2 \approx 9.63325$$

Can the FOCs hold at this point? Let's see what λ and μ have to be. Both of these have to be true:

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

Plugging in x_1 and x_2 and solving we get:

$$\mu \approx 5.16181, \lambda \approx -2.57279$$

Notice the negative value of λ. This tells us, to get both to bind, we need to change the direction of the distance constraint from a $\leq$ to a $\geq$ constraint. But we can't do this. The reason both cannot bind is because that the point both bind, the slope of the boundary of both constraints is shallower than the slope of the indifference curve at that point. We cannot take a linear combination of the constraints can have the slope of that linear combination be equal to the slope of the indifference curve.

Let's suppose only the distance constraint binds:

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}}$$

We've already seen for this FOC, $x_1 = x_2$

$$x_1 = x_2 = \frac{10}{\sqrt{2}}$$

However, at this point, we are not on the interior of the budget constraint. This point costs $15\sqrt{2}$ but we only have $m = 15$. Thus, when $m = 15$, the budget constraint is violated on any point that could possible be optimal on the distance constraint.

Suppose only the budget constaint binds:

$$\frac{x_2}{2} = \mu$$

$$x_1 = \mu$$

$$x_1 = \frac{x_2}{2}$$

$$4x_1 = 15$$

$$x_1 = \frac{15}{4}$$

$$x_2 = \frac{30}{4}$$

11.4 Example CD Utility

$$u\left(x_1, x_2\right) = x_1^{\alpha} x_2^{\beta}$$

$$p_1 x_1 + p_2 x_2 \leq m$$

$$x_1^{\alpha} x_2^{\beta} - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)$$

$$\frac{\partial\left(x_1^{\alpha} x_2^{\beta} - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1$$

$$\alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0$$

$$\frac{\partial\left(x_1^{\alpha} x_2^{\beta} - \lambda\left(p_1 x_1 + p_2 x_2 - m\right)\right)}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2$$

$$\beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0$$

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \lambda$$

$$\frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2} = \lambda$$

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

$$\frac{\alpha}{\beta} x_2 p_2 = x_1 p_1$$

Marshallian Demands:

$$x_1^* = \frac{\frac{\alpha}{\alpha+\beta} m}{p_1}$$

$$x_2^* = \frac{\frac{\beta}{\alpha+\beta} m}{p_2}$$

11.5 Indirect Utility

$$Max_x u(x) \; s.t. \, px \leq y$$

$$V(p,y) = u(x^*(p,y))$$

For $x_1^\alpha x_2^\beta$ this is:

$$\left(\frac{\frac{\alpha}{\alpha+\beta}y}{p_1}\right)^\alpha \left(\frac{\frac{\beta}{\alpha+\beta}y}{p_2}\right)^\beta$$

If $\alpha = \beta = 1$

$$\left(\frac{\frac{1}{2}y}{p_1}\right)\left(\frac{\frac{1}{2}y}{p_2}\right) = \frac{\frac{1}{4}y^2}{p_1 p_2}$$

Homogeneous Functions:

$$f(t\boldsymbol{x}) = t^\alpha f(x)$$

f is homogeneous of degree α.

Properties.

1. Continuous.
Berge's Maximum Theorem
2. Homogeneous of degree zero in prices and income.
$V(t\boldsymbol{p}, ty) = t^0 V(\boldsymbol{p}, y) = V(p,y)$
3. Strictly increasing in income.
Due to local non-satiation.

$$V = u(x_1^*, x_2^*) - \lambda(p_1 x_1^* + p_2 x_2^* - y)$$

$$V = u(x_1^*, x_2^*) - \lambda^*(p_1 x_1^* + p_2 x_2^* - y)$$

$$\frac{\partial V}{\partial y} = \lambda^*$$

$$\frac{MU_i}{p_i} = \lambda$$

4. Decreasing (weakly) in prices.

$$V = u(x_1^*, x_2^*) - \lambda^*(p_1 x_1^* + p_2 x_2^* - y)$$

$$\frac{\partial\left(u\left(x_1^*,x_2^*\right)-\lambda^*\left(p_1x_1^*+p_2x_2^*-y\right)\right)}{\partial p_1}=\left(-x_1^*\right)\lambda^*$$

5. Quasi-convex in (p, y).
$(p,y),(p',y')\ (tp+(1-t)p',ty+(1-t)y')$

$$(tp+(1-t)p',ty+(1-t)y')$$

Because (by homework) anything achieveable in budget $(tp+(1-t)p',ty+(1-t)y')$ is achievable in one of the two other budgets, then either $max\left\{V((p,y)),V((p',y'))\right\}\geq V((tp+(1-t)p',ty+(1-t)y'))$

6. Roy's Identity. (An envelope condition)

The ratio of the way utility changes with price i to the way it changes with income is proportional to the amount of i consumed. This is because as price i changes, it changes effective income by $(\Delta p_i)\,x_i$ and locally, there is no need to worry about changes in consumption level.

$$-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial y}}=-\frac{-x_i^*\lambda}{\lambda}=x_i^*$$

12 Cost Minimization

This is the dual to utility maximization.

12.1 The Dual Consumer Problem

$$min\,(px)\ s.t.\,u(x)\geq\bar{u}$$

12.2 Example CD Utility

$$min.\,p_1x_1+p_2x_2$$

$$s.t.\,u(x_1,x_2)=x_1^{\alpha}x_2^{\beta}\geq\bar{u}$$

$$Min._{\boldsymbol{x}\in\boldsymbol{X}}\,(p_1x_1+p_2x_2)-\lambda\left(x_1^{\alpha}x_2^{\beta}-\bar{u}\right)$$

$$\frac{\partial\left((p_1x_1+p_2x_2)-\lambda\left(x_1^{\alpha}x_2^{\beta}-u\right)\right)}{\partial(x_1)}=p_1-\lambda\alpha x_1^{\alpha-1}x_2^{\beta}$$

$$\frac{\partial\left((p_1x_1+p_2x_2)-\lambda\left(x_1^{\alpha}x_2^{\beta}-u\right)\right)}{\partial(x_2)}=p_2-\beta\lambda x_1^{\alpha}x_2^{\beta-1}$$

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{1}{\lambda}$$

$$\frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2} = \frac{1}{\lambda}$$

λ is the relative cost of increasing utility.

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

Tangency Condition (same as for utility maximization):

$$\frac{\alpha}{\beta} x_2 p_2 = x_1 p_1$$

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

Utility Condition:

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

$$\left(x_2 \frac{\alpha p_2}{\beta p_1}\right)^{\alpha} x_2^{\beta} = u$$

$$x_2 = \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{\alpha}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

$$x_1 = \left(\frac{\alpha p_2}{\beta p_1}\right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

12.3 Marshallian / Hicksian Demand.

Marshallian Demand (Demand):
Amount of good i that is optimal given prices and income.

$$x_i^* \left(\boldsymbol{p}, y\right)$$

Hicksian Demand:
The amount of good i you choose to achieve utiliy u in the cheapest way:

$$x_i^h \left(\boldsymbol{p}, u\right)$$

12.4 Expenditure Function

$$e(\boldsymbol{p}, u) = \sum_{i=1}^{n} p_i x_i^h(\boldsymbol{p}, u)$$

This is the "optimized" value of the cost minimization objective subject to the utility constraint. It is analogous to the indirect utility function for utility maximization.

Properties.

1. Continuous.

Berge's Maximum Principle.

2. For $p \gg 0$, strictly increasing and unbounded above in u.

3. Increasing in p.

4. Homogeneous of degree 1 in p.

5. Concave in p.

The meaning of this in terms of economics:

If x^* is optimal at p, u and prices change. x^* still achieves the utility u. The cost of x^* thus represents an upper bound on the expenditure I need to achieve u at the new prices.

Let's talk about this one.

6. Shephard's lemma.

$$p_1 x_1^h + p_2 x_2^h - \lambda\left(u\left(x_1^h, x_2^h\right) - \bar{u}\right)$$

$$\frac{\partial\left(p_1 x_1^h + p_2 x_2^h - \lambda\left(u\left(x_1^h, x_2^h\right) - \bar{u}\right)\right)}{p_i} = x_i^h$$

$$\frac{\partial e(u, \boldsymbol{p})}{\partial p_i} = x_i^h$$

Another envelope condition that has no name:

$$\frac{\partial e(\bar{u}, \boldsymbol{p})}{\partial \bar{u}} = \lambda$$

12.5 Duality of Indirect Utility/Expenditure

In general:

$e(p, v(p, y)) \leq y$

$v(p, e(p, \bar{u})) \leq \bar{u}$

But with continuous, **strictly monotonic** utility:

$v(p, e(p, \bar{u})) = \bar{u}$

$e(p, v(p, y)) = y$

$x_i(p, y) = x_i^h(p, v(p, y))$

13 Decomposition

$$T.E. = S.E. + I.E.$$

13.1 Two Types.

Hicksian.

Hick's formalized the Substitution effect in the following way:
The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old utility level:**

$$\boldsymbol{SE:} \, x_i\left(p, y\right) - x_i\left(p', e\left(p', v\left(p, y\right)\right)\right)$$

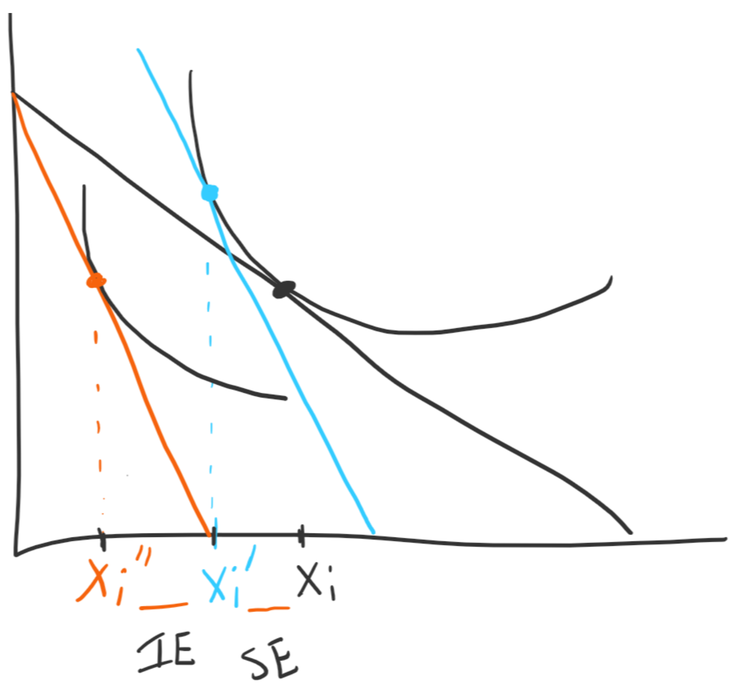

Figure 13.1: An example of the Hicksian decomposition.

The substitution effect has to be negative. That is, *if the price of a good increases, a consumer will demand less of it due to the substitution effect.*

Slutsky:

The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old bundle:**

$$\boldsymbol{SE:}\, x_i\left(p,y\right)-x_i\left(p',p\cdot x^*\left(p,y\right)\right)$$

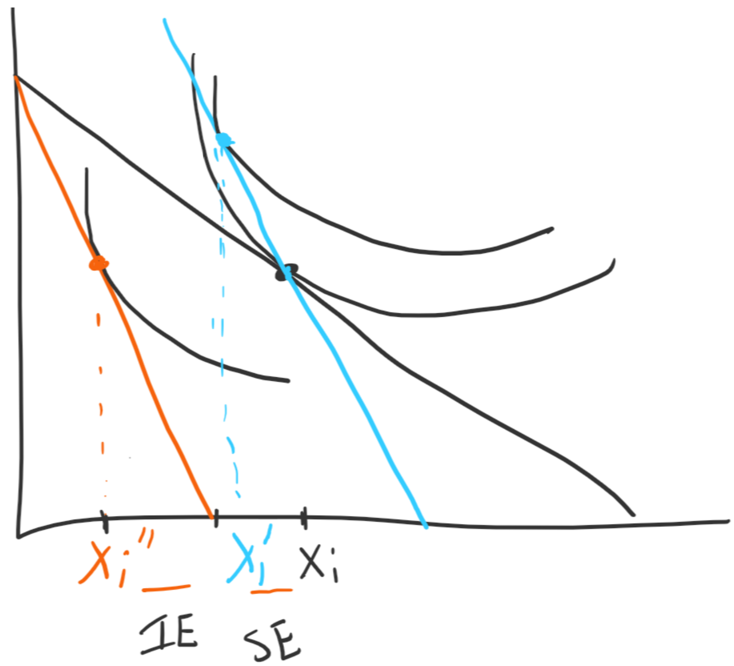

Figure 13.2: Example of the Slutsky decomposition.

The substitution effect has to be (non-positive) because any bundle with more x_i (when p_i is the price that is increasing) was previously strictly affordable.

13.2 Slutsky Equation

Start here:

$$x_i^h\left(p,\bar{u}\right)=x_i\left(p,e\left(p,\bar{u}\right)\right)$$

13.3 Neagtive Own-Substitution Effect

13.4 Elasticities

$$\eta_i=\frac{\frac{\partial x_i}{x_i}}{\frac{\partial y}{y}}$$

$$\epsilon_{ij}=\frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}}$$

13.5 Elasticity Relations

$$\sum s_i \eta_i = 1$$

$$\sum s_i \epsilon_{i,j} = -s_j$$