

# Economics 8100

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October 14, 2021

## Part I

## Budget

### 1 Consumption Set $X$

**Assumptions:** (Universe of Choice Objects):  $X$

**Bundles:** Elements of  $X$ .  $x \in X$

**Assumptions about  $X$ .**

1.  $\emptyset \neq X \subseteq \mathbb{R}_+^n$ .
2.  $X$  is closed.
3.  $X$  is convex.
4.  $0 \in X$ .

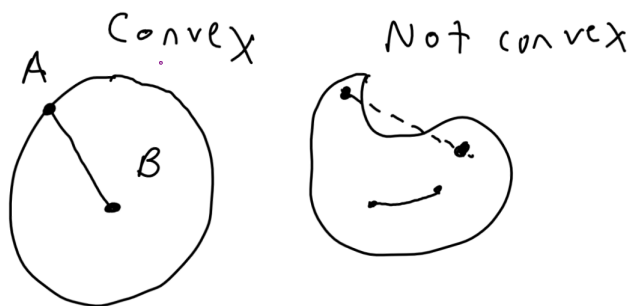


Figure 1.1: Examples of a Convex/Non-Convex Set.

### 2 Budget Set $B$

**Budget Set:**  $B \subseteq X$

$X$  defines the scope of the model.  $B$  is what an *individual consumer* chooses among.

**Example.** Budget Set with Prices and Income

$$B = \{x | x \in X \text{ \& } x_1 p_1 + x_2 p_2 \leq m\}$$

**Example.** Ice Cream Bowls

Every ice cream bowl  $x$  has some non-negative number of scoops of Vanilla, Chocolate, Strawberry.

$$X = \mathbb{R}_+^3$$

Budget  $B$  is the set of bowls with *no more than one scoop of ice cream*.

$$B = \left\{ x | x \in \mathbb{R}_+^3 \text{ \& } \sum_{i=1}^3 x_i \leq 1 \right\}$$

This is the unit-simplex in  $\mathbb{R}_3$ .

$(1, 0, 0) \in B$ . (On the boundary.)

$(0.5, 0.5, 0) \in B$ . (On the boundary.)

$(0.25, 0.25, 0.25) \in B$ . (In the interior.)

$(2, 0, 0) \notin B$

## Part II

# Preference

### 3 The Preference Relation

Preference Relation is a **Binary Relation**.

Formally, a binary relation on set  $X$  is a subset of the Cartesian product  $X$  with itself.

$$\succeq \subseteq X \times X$$

Another way to denote an ordered pair is “in” the relation:

If  $(x, y) \in \succeq$  we can also write  $x \succeq y$ .

Informally we say “ $x$ ” is at least as good as “ $y$ ”, or “ $x$ ” preferred “ $y$ ”.

**Axioms of  $\succeq$ .**

**Axiom 0** (*reflexive*):  $\forall x \in X, x \succeq x$ . This is implied by *axiom 1*.

**Axiom 1** (*complete*):  $\forall x, x' \in X$ , either  $x \succeq x'$  or  $x' \succeq x$  (or both).

The consumer has “some” preference over every pair of objects.

**Axiom 2** (transitive):  $\forall x, x', x'' \in X$  if  $x \succeq x'$  and  $x' \succeq x'' \Rightarrow x \succeq x''$ .

$\succeq$  is a “weak order” if it is complete and transitive.

## 4 Relations and Sets Related to $\succeq$

**Subrelations:**

$\sim$  is the indifference relation.  $x \succeq y$  and  $y \succeq x \Leftrightarrow x \sim y$ .

$\succ$  is the strict relation.  $x \succeq y$  and not  $y \succeq x \Leftrightarrow x \succ y$ .

**Related Sets:**

$\succeq(x)$  “upper contour set”, “no worse than set”

$\preceq(x)$  “lower contour set”, “no better than set”

## 5 From Preferences to Choice

**Choice Correspondence.**

We will assume that from a budget set  $B$  a consumer “chooses” *choice set*  $C$  according to their preference  $\succeq$ .  $C = \{x | x \in B \text{ \& } \forall x' \in B, x \succeq x'\}$ .

Informally,  $C$  is the set of objects that are at least as good as anything else in the set.

**Example With Transitive Preferences**

$X = \{a, b, c\}$ .  $a \succeq b, c \succeq a, c \succeq b$ .

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = c$$

$$C(\{a, b, c\}) = c$$

## 6 Cycles Lead to Empty Choice Sets

### 6.1 The Problem with Intransitive Preferences

$X = \{a, b, c\}$ .  $a \succeq b, c \succeq a, b \succeq c$ . *This is intransitive!*

Choice correspondence:

$$C : P(X) / \emptyset \rightarrow X$$

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = b$$

$$C(\{a, b, c\}) = \emptyset$$

This consumer cannot make a choice from the set  $\{a, b, c\}$ .

## 6.2 Cycles and Empty Choices

Notice in the previous example,  $a \succ b, a \succ c, c \succ a$ . We have proved (essentially) that if there is a cycle, there is an empty choice set.

In fact, suppose, there is an empty choice set **and**  $X$  is finite. There must be a cycle.

$$\forall x \in B, \#(\succ(x)) < \#(B)$$

By completeness,  $\forall x \exists x' \in X : x' \succ x$ . Choose an  $x_1$ , let  $x_2$  be any element of  $\succ(x_1)$ . We have  $x_2 \succ x_1$ . If there is an  $x_3 \in \succ(x_2)$  such that  $x_1 \succ x_3$  we have identified a cycle. Otherwise, we continue with an inductive step. Suppose we have  $x_n \succ \dots \succ x_1$ .  $\succ(x_n)$  is non-empty. Either it contains an element  $x_{n+1}$  such that there is an  $x_i \succ x_{n+1}$  in which case we have identified a cycle or it does not and we continue with another inductive step. Either we find a cycle or reach the  $N_{th}$  step with  $x_N \succ x_{n-1} \succ \dots \succ x_1$ .  $\succ(x_N)$  is non-empty.

So, the cycle condition is equivalence to a non-empty choice set. Transitivity of  $\succsim$  implies transitivity of  $\succ$  which implies no cycles (try this last step at home). But do no-cycles imply transitivity of  $\succsim$ ? No. Here is a counter-example:

$$x \succ y, y \sim z, z \succ x$$

## 7 Intransitivity: Empty Choices, Incoherent Choices: Pick One.

So if no-cycles of the strict preference is equivalent to non-empty choice (in finite sets), and transitivity of  $\succsim$  is not equivalent to no-cycles, why do we assume it?

**Finite non-emptiness:** For any  $B$  with  $\#(B) \in \mathbb{I}$ ,  $C(B) \neq \emptyset$

**Coherence:** For every  $x, y$  and  $B, B'$  such that  $x, y \in B \cap B'$ ,  $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$ .

Suppose there is an intransitive  $\succsim$ . There exists either a  $B$  where  $C(B) = \emptyset$  or there exists a  $x, y, B, B'$  where the choice correspondence is incoherent.

By intransitivity:

$$1) x \succ y, y \succ z, z \succ x$$

$$C(\{x, y, z\}) = \emptyset$$

$$2) x \sim y, y \sim z, z \succ x$$

$$3) x \sim y, y \succ z, z \succ x$$

$$x \notin C(\{x, y, z\})$$

$$y \in C(\{x, y, z\})$$

$$x \in C(\{x, y\})$$

$$4) x \succ y, y \sim z, z \succ x$$

Can you find the incoherent choice?

## 8 Indifference Sets

### 8.1 Indifference Maps

To understand preferences, we often draw sets of the form  $\sim(x)$ . Many times these are one dimension smaller than the space of bundles, in which case we often call them *indifference curves*, but they need not have any special structure, unless we make further assumptions about preferences. There is only one thing we really know about these sets.

### 8.2 Complete, Transitive Preferences have Indifference Sets that Do Not Intersect

**Result.** *Indifference curves do not cross.* For two bundles  $x \succ y$ ,  $\sim(x) \cap \sim(y) = \emptyset$ .

Proof is given visually below:

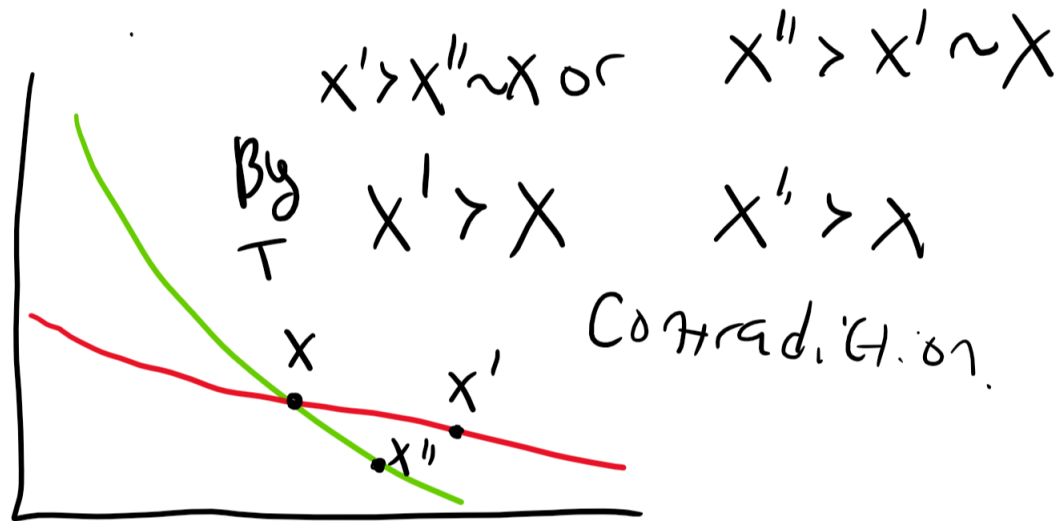


Figure 8.1: Distinct Indifference Sets do not Intersect

### Part III

## From Preference to Utility

### 9 Utility Represents Preferences

Suppose there is some  $U : X \rightarrow \mathbb{R}$  such that  $U(x) \geq U(x') \Leftrightarrow x \succsim x'$  then we say  $u()$  represents preference relation  $\succsim$ . When does such a representation exist?

#### 9.1 Finite $X$

**Proposition 1.** *A  $U()$  exists that represents  $\succsim \Leftrightarrow \succsim$  is complete and transitive.*

*Proof.* Let's start with  $\Rightarrow$ .

Because  $\geq$  is complete on the real numbers, for every  $x, y \in X$  either  $u(x) \geq u(y)$  or  $u(y) \geq u(x)$  thus because  $u()$  represents  $\succsim$ , it is complete.

*By similar argument,  $\succsim$  is transitive. For every three  $x, y, z \in X$ . If  $u(x) \geq u(y)$  and  $u(y) \geq u(z)$  then  $u(x) \geq u(z)$  because  $\geq$  is transitive on the real numbers.*

Now we prove  $\Leftarrow$ :

Define  $U(x) \equiv \#(\succsim(x))$

Example:  $a \succ b, b \succ c$ .  $\preceq(a) = \{a, b, c\}$ .  $U(a) = 3$ .

Lemma: For  $x \succsim y$ ,  $\preceq(y) \subseteq \preceq(x)$  (proved in PS1).

By this lemma, for  $x \succsim y$ ,  $\preceq(y) \subseteq \preceq(x)$  and thus  $\# \preceq(y) \leq \# \preceq(x)$  and  $u(x) \geq u(y)$ .  $\square$

## 9.2 Countably infinite $X$

Pick any arbitrary order on the bundles:  $(x_1, x_2, \dots)$ . And assign weights to those bundles  $w(x_i) = \frac{1}{i^2}$ . The following utility function represents preferences:

$$u(x) = \sum_{y \in \preceq(x)} w(y)$$

Example: “ $\pi$  shows up unexpectedly when eating ice cream.”

An even number of scoops of ice cream are better than an odd number of scoops and otherwise more is better than less.

$$u(2) = \sum_{i=1}^{\infty} \left( \frac{1}{(2i-1)^2} \right) = \frac{\pi^2}{8}$$

$$u(4) = \frac{1}{4} + \frac{\pi^2}{8}$$

## 9.3 Uncountable $X$

The *Lexicographic* preferences have no utility representation:

$$X = \mathbb{R}_+^2$$

$(x_1, x_2) \succ (y_1, y_2)$  if  $x_1 > y_1$  or  $x_1 = y_1$  and  $x_2 > y_2$ .

$\succsim$  is complete, and transitive. [Prove this for practice].

Pick two real numbers  $v_2 > v_1$  and construct four bundles  $(v_1, 1), (v_2, 1), (v_1, 2), (v_2, 2)$ .

$$(v_2, 2) \succ (v_2, 1) \succ (v_1, 2) \succ (v_1, 1)$$

Suppose there is a utility function representing these preferences, then we have two disjoint intervals:

$$[u(v_2, 1), u(v_2, 2)]$$

$$[u(v_1, 1), u(v_1, 2)]$$

For every real number, we can construct an interval like this. Because the rationals are dense in the reals, there is a rational number in each of these intervals. Thus, for every real, we can find a unique rational number. That is, we have a mapping from the reals into the rationals which implies that the cardinality of the rationals is at least as large as that of the reals.  $\#\mathbb{Q} \geq \#\mathbb{R}$ . This contradicts that the cardinality of the rationals is strictly smaller than the reals.

#### 9.4 An example of preference relation with a utility representation.

Cars have horse power in  $[0, 999]$  and cup holders in  $\mathbb{Z}_+$  (integers).

Suppose preferences are lexicographic and more cup holders are more important than more horse-power.

$u(c_i, h_i) = c_i + \frac{h_i}{1000}$  represents these preferences.

*See problem set 2 for example where we do not bound the horse power.*

#### 9.5 What ensures a utility representation in an uncountable universe?

A preference relation is representable by a utility function  $U(x)$  iff  $\forall x, y \in X$  s.t.  $x \succ y$ ,  $\exists x^* \in X^* \subset X$  s.t.  $x \succsim x^* \succ y$  and the set  $X^*$  is countable.

To construct the utility function,  $U(x)$ , Pick any arbitrary order on the bundles in  $X^*$ :  $(x_1, x_2, \dots)$ . And assign weights to those bundles  $w(x_i) = \frac{1}{i^2}$ . The following utility function represents preferences:

$$u(x) = \sum_{y \in \succsim(x) \cap X^*} w(y)$$

#### 9.6 Continuous $\succsim$ .

Preference relation  $\succsim$  is continuous if  $\forall x \in X$ ,  $\succsim(x)$  and  $\precsim(x)$  are closed in  $X$ .



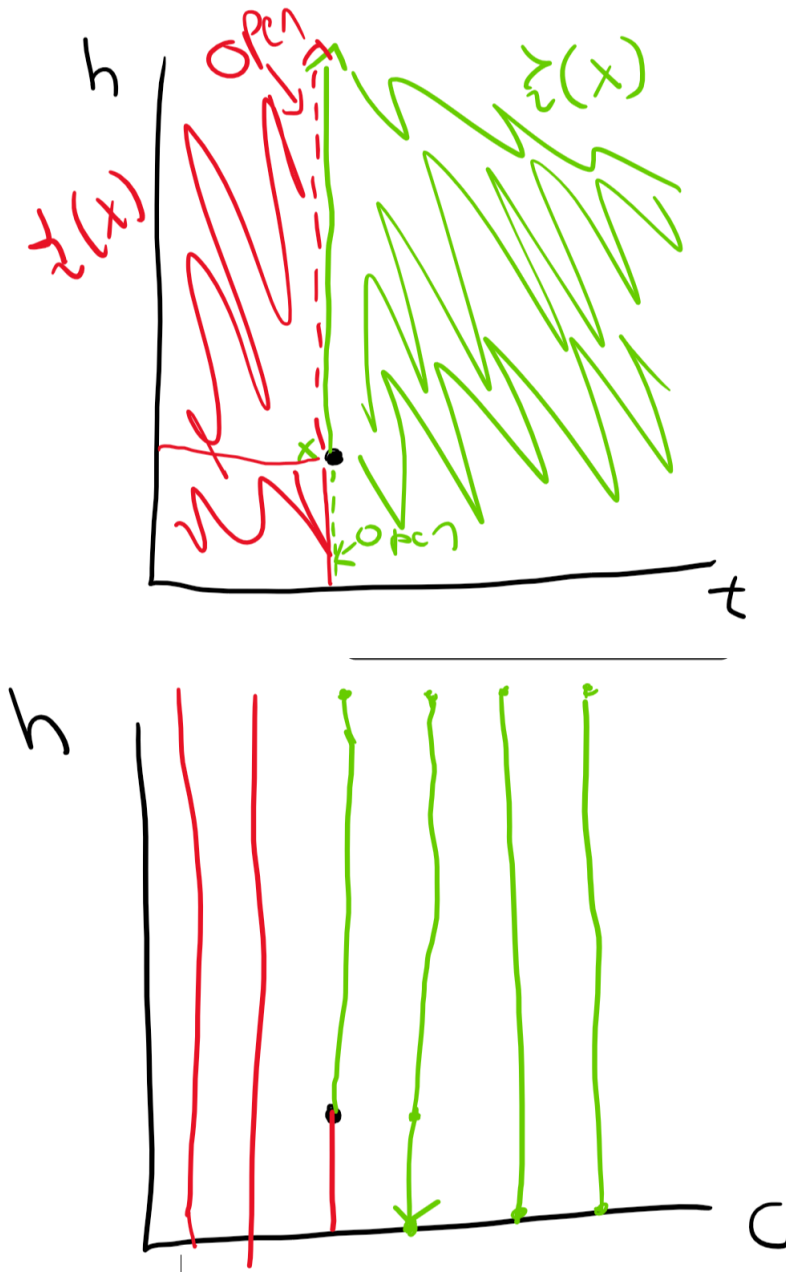


Figure 9.1: Not Continuous/Continuous Lexicographic Preferences.

## 9.7 What ensures a continuous utility representation?

A *complete*, *transitive*, and *continuous* preference relation  $\succsim$  can be represented by a continuous utility function  $U(x)$  and, a continuous utility function represented C,T,C preferences.

## 10 Other Properties of $\succsim$

### 10.1 Monotonicity

Ensure consumers consume on the boundary of the budget set.

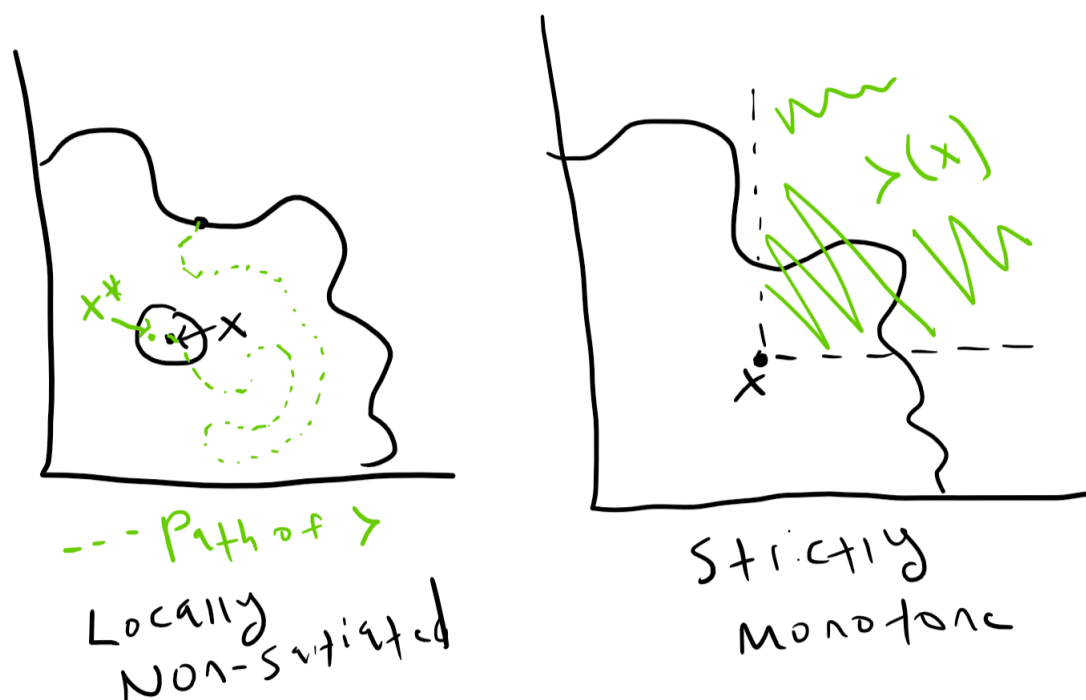


Figure 10.1: Locally Non-satiated vs. Strictly Monotone

#### Strict Monotonicity

*More stuff is better.*

First, some notation:

For  $X \subseteq \mathbb{R}^n$

$x \geq x'$  iff  $x_i \geq x'_i$  for all  $i \in \{1, 2, \dots, n\}$

$x \gg x'$  iff  $x_i > x'_i$  for all  $i \in \{1, 2, \dots, n\}$

For example:  $(2, 2) \gg (1, 1)$ ,  $(2, 1) \geq (1, 1)$ ,  $(1, 1) \geq (1, 1)$

**Definition. Strict Monotonicity.**  $x \geq x' \Rightarrow x \succsim x'$  and  $x \gg x' \Rightarrow x \succ x'$

### Local Nonsatiation

**Definition. Local Nonsatiation.**  $\forall x \in X$  and  $\forall \varepsilon > 0$ ,  $\exists x^* \in B_\varepsilon(x)$  such that  $x^* \succ x$ .

A consumer can always change the bundle a “little bit” no matter how small that little bit is, and find something strictly better.

## 10.2 Convex Sets, Convex/Concave Functions, Quasi-Convex/Concave Functions

### Convex Sets

In a subset of euclidean space  $X$ , the line between  $x \in X$  and  $x' \in X$  is another point in the set  $X$  given by  $tx + (1 - t)x'$  where  $t \in [0, 1]$ . We call points like this **Convex Combinations** of  $x$  and  $x'$ .

For example:  $x = (1, 0)$ ,  $x' = (0, 1)$ . If we take  $t = 0.5$ . The convex combination is  $0.5(1, 0) + 0.5(0, 1) = (0.5, 0.5)$ .

A **convex set**  $S \subseteq X$  is a set of points that contains all of its convex combinations.

Formally,  $\forall x, x' \in S$ ,  $\forall t \in [0, 1]$ ,  $tx + (1 - t)x' \in S$ .

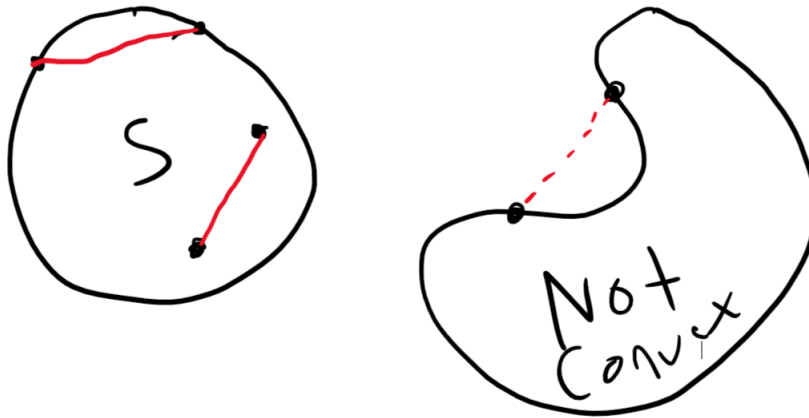


Figure 10.2: A Convex and Non-Convex Set

## Convex Functions

A line between two points “on the function” lies above the function itself.

**Convex Function:**

$$\forall x, x' \in X, t \in [0, 1], tf(x) + (1-t)f(x') \geq f(tx + (1-t)x')$$

**Strictly Convex Function:**

$$\forall x, x' \in X, t \in (0, 1), tf(x) + (1-t)f(x') > f(tx + (1-t)x')$$

**Contour Sets:**

A convex function has **convex lower contour sets**.

## Concave Functions

A line between two points “on the function” lies below the function itself.

**Concave Function:**

$$\forall x, x' \in X, t \in [0, 1], tf(x) + (1-t)f(x') \leq f(tx + (1-t)x')$$

**Strictly Concave Function:**

$$\forall x, x' \in X, t \in (0, 1), tf(x) + (1-t)f(x') < f(tx + (1-t)x')$$

A concave function has **convex upper contour sets**.

## Quasi-Concave Functions

A function  $f(x)$  is quasi-concave if **and only** it has convex upper contour sets.

A function  $f(x)$  is quasi-concave if and only if it is a monotonic transformation of a concave function.

A function  $f(x)$  is quasi-concave if and only if  $f(tx + (1-t)x') \geq \min\{f(x), f(x')\}$  for  $t \in [0, 1]$ .

A function  $f(x)$  is **strictly quasi-concave** if and only if  $f(tx + (1-t)x') > \min\{f(x), f(x')\}$  for  $t \in (0, 1)$ .

Notice that, for a strictly quasi-concave utility function, let  $x' \succ x$ , then the set  $tx + (1-t)x' \in \succ(x)$  for  $t \in (0, 1)$ . Thus, there is a small enough ball around that point  $B_\epsilon(tx + (1-t)x') \in \succ(x)$ . Thus, these points are in the interior of  $\succ(x)$  and  $\succ$  is **strictly convex**.

## Quasi-Convex Functions

A function  $f(x)$  is quasi-convex if **and only** it has convex lower contour sets.

A function  $f(x)$  is quasi-convex if and only if it is a monotonic transformation of a convex function.

A function  $f(x)$  is quasi-convex if and only if  $f(tx + (1-t)x') \leq \max\{f(x), f(x')\}$  for  $t \in [0, 1]$ .

A function  $f(x)$  is **strictly quasi-convex** if and only if  $f(tx + (1-t)x') < \max\{f(x), f(x')\}$  for  $t \in (0, 1)$ .

### 10.3 Convexity of $\succsim$ .

**Convex Preferences:**  $x \succsim x' \Rightarrow t(x) + (1-t)x' \succsim x', t \in [0, 1]$

$x \in \succsim(x') \Rightarrow t(x) + (1-t)x' \in \succsim(x')$

Thus,  $\succsim(x)$  are convex if  $\succsim$  is a convex preference relation.

**Strictly Convex Preferences:**  $x \succ x' \Rightarrow t(x) + (1-t)x' \succ x', t \in (0, 1)$

The upper contour sets  $\succsim(x)$  are *strictly* convex.

### 10.4 Utility and Preference Relationships

If  $U$  represents  $\succsim$ :

- 1)  $\succsim$  (strictly) **convex**  $\Leftrightarrow U$  is (strictly) **quasi-concave**.
- 2)  $\succsim$  are **strictly monotonic**  $\Leftrightarrow U$  is **strictly** increasing.
- 3)  $\succsim$  are **strictly monotonic**  $\Leftarrow U$  is **strongly** increasing.

## 11 The Consumer Problem

### 11.1 Choice

The set of all “best things” in the budget set. This is what we are looking for:

$$C(B) = \{x | x \in B \wedge x \succsim x', \forall x' \in B\}$$

**Competitive Budgets:**

$$B = \{x | x \in \mathbb{R}_+^n, p \cdot x \leq m\}$$

$p$  is the vector of prices.

$m$  is the “income”.

Constrained problem:

$$\text{Max}_{x \in X} U(x) \text{ s.t. } p \cdot x \leq m$$

## 11.2 The Lagrange Method- Some Intuition.

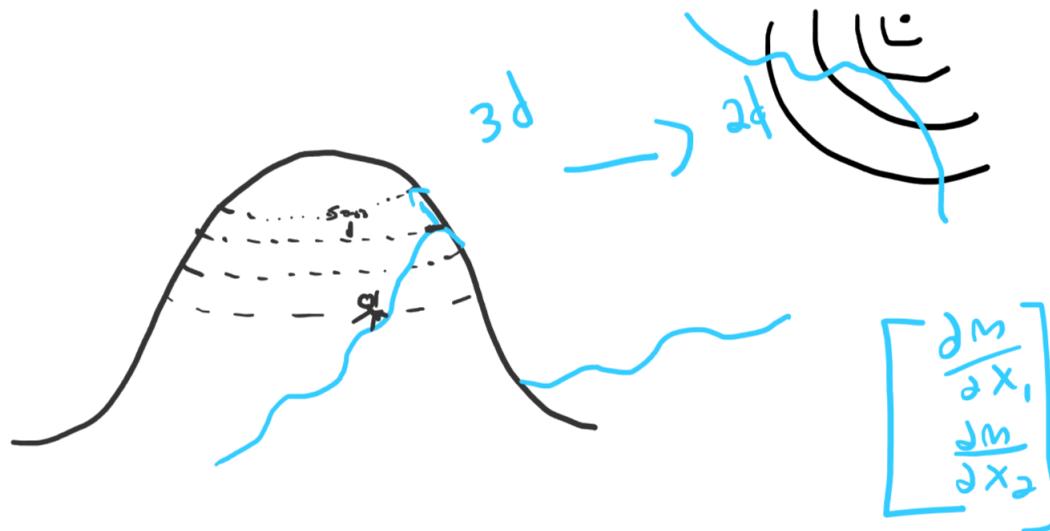


Figure 11.1: Finding the best spot for a selfie.

If both the objective and the constraint are smooth, at the optimal **the *direction* of the gradient of the objective has to be equal to the *direction* of the gradient of the constraint.** Otherwise, moving along the constraint boundary in *some* direction will yield a larger value of the objective! (Caveat: this assumes we *can* move in every direction along the constraint. That will only be true at non-boundary points.)

Thus, for smooth functions, the equality of the direction of the gradients of the objective and the constraint are **necessary** for an non-boundary optimum.

Since the direction of the gradient is just a scaling of the gradient, suppose  $U$  is our objective and  $G$  is the function for the boundary of the constraint. Then,

$$\nabla U(x) = \lambda \nabla G(x)$$

Can we write a function such that the first order condition will yield this gradient condition? *Sure:*

$$\mathcal{L} = U(x) - \lambda (G(x) - c)$$

Let's treat this as an unconstrained problem. The FOC. of this function is:

$$\nabla U(x) - \lambda \nabla G(x) = 0$$

$$\nabla U(x) = \lambda \nabla G(x)$$

This is precisely the necessary condition we need for the constrained problem.

**Thus, FOC for unconstrained optimization of the Lagrangian is the necessary constrained optimization condition.**

### 11.3 Example (Two Constraints)

$$Max_x (x_1 x_2)$$

$$(x_1^2 + x_2^2)^{\frac{1}{2}} \leq 10$$

$$2x_1 + x_2 \leq m$$

**One Binds.**  $m = 40$ .

After plotting the two constraints, we can see that the distance constraint is entirely contained on the interior of the budget constraint. The only constraint that could possibly bind is the distance constraint:

$$x_1 x_2 - \lambda \left( (x_1^2 + x_2^2)^{\frac{1}{2}} - 10 \right)$$

$$\frac{\partial \left( x_1 x_2 - \lambda \left( (x_1^2 + x_2^2)^{\frac{1}{2}} - 10 \right) \right)}{\partial x_1} = x_2 - \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial \left( x_1 x_2 - \lambda \left( (x_1^2 + x_2^2)^{\frac{1}{2}} - 10 \right) \right)}{\partial x_2} = x_1 - \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}}$$

$$x_2 \frac{\sqrt{x_1^2 + x_2^2}}{x_1} = \lambda$$

$$x_1 \frac{\sqrt{x_1^2 + x_2^2}}{x_2} = \lambda$$

$$x_2^2 = x_1^2$$

$$x_1 = x_2$$

$$x_1 = x_2 = \frac{10}{\sqrt{2}}$$

$m = 15$ .

Now neither constraint is contained in the other. Let's set up the Lagrangian with both constraints:

$$x_1 x_2 - \lambda \left( (x_1^2 + x_2^2)^{\frac{1}{2}} - 10 \right) - \mu (2x_1 + x_2 - 15)$$

The FOCs:

$$\frac{\partial \left( x_1 x_2 - \lambda \left( (x_1^2 + x_2^2)^{\frac{1}{2}} - 10 \right) - \mu (2x_1 + x_2 - m) \right)}{\partial x_1} = -\frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} - 2\mu + x_2$$

$$\frac{\partial \left( x_1 x_2 - \lambda \left( (x_1^2 + x_2^2)^{\frac{1}{2}} - 10 \right) - \mu (2x_1 + x_2 - m) \right)}{\partial x_2} = -\frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} - \mu + x_1$$

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

**Let suppose both constraints bind, then  $\lambda \geq 0$ ,  $\mu \geq 0$ .**

Since we are on the boundary of both constraints:

$$2x_1 + x_2 = 15$$

$$(x_1^2 + x_2^2)^{\frac{1}{2}} = 10$$

The only point on the boundary of both constraints is:

$$x_1 \approx 2.68338, x_2 \approx 9.63325$$

Can the FOCs hold at this point? Let's see what  $\lambda$  and  $\mu$  have to be. Both of these have to be true:

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}} + 2\mu$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}} + \mu$$

Plugging in  $x_1$  and  $x_2$  and solving we get:



$$\mu \approx 5.16181, \lambda \approx -2.57279$$

Notice the negative value of  $\lambda$ . This tells us, to get both to bind, we need to change the direction of the distance constraint from a  $\leq$  to a  $\geq$  constraint. But we can't do this. The reason both cannot bind is because that the point both bind, the slope of the boundary of both constraints is shallower than the slope of the indifference curve at that point. We cannot take a linear combination of the constraints can have the slope of that linear combination be equal to the slope of the indifference curve.

**Let's suppose only the distance constraint binds:**

$$x_2 = \frac{\lambda x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$x_1 = \frac{\lambda x_2}{\sqrt{x_1^2 + x_2^2}}$$

We've already seen for this FOC,  $x_1 = x_2$

$$x_1 = x_2 = \frac{10}{\sqrt{2}}$$

However, at this point, we are not on the interior of the budget constraint. This point costs  $15\sqrt{2}$  but we only have  $m = 15$ . Thus, when  $m = 15$ , the budget constraint is violated on any point that could possibly be optimal on the distance constraint.

**Suppose only the budget constraint binds:**

$$\frac{x_2}{2} = \mu$$

$$x_1 = \mu$$

$$x_1 = \frac{x_2}{2}$$

$$4x_1 = 15$$

$$x_1 = \frac{15}{4}$$

$$x_2 = \frac{30}{4}$$

## 11.4 Example: Non-Negativity Constraints and Complementary Slackness

$$u = \log(x_1) + \sqrt{x_2} + x_3$$

$$x_1 + x_2 + x_3 \leq m$$

Let's set up the Lagrangian function while putting the non-negativity constraints in explicitly:

$$\log(x_1) + \sqrt{x_2} + x_3 - \lambda(x_1 + x_2 + x_3 - m) - \mu_1(-x_1) - \mu_2(-x_2) - \mu_3(-x_3)$$

The first order conditions are:

$$\mu_1 + \frac{1}{x_1} = \lambda$$

$$\mu_2 + \frac{1}{2\sqrt{x_2}} = \lambda$$

$$\mu_3 + 1 = \lambda$$

Suppose none of our non-negativity constraints bind. By complementary slackness:  $\mu_1, \mu_2, \mu_3 = 0$ . The first order conditions become:

$$\frac{1}{x_1} = \lambda$$

$$\frac{1}{2\sqrt{x_2}} = \lambda$$

$$1 = \lambda$$

Solving these for  $x_1, x_2, x_3$ :

$$1 = x_1$$

$$\frac{1}{4} = x_2$$

$$x_3 = m - \frac{5}{4}$$

If  $m \geq \frac{5}{4}$ , this is a valid solution to the problem and since the utility function is concave, it is sufficient for the optimum. However, note that if  $m < \frac{5}{4}$ , this is not a feasible solution since it violates the non-negativity constraint for  $x_3$ . What is the optimal solution in that case?

Lets suppose  $x_1 \geq 0$  is binding. Thus,  $x_1 = 0$ . The first order condition on  $x_1$  requires  $\frac{1}{x_1} = \lambda - \mu_1$ . However, at  $x_1 = 0$  this equation cannot hold. Thus,  $x_1 > 0$  in any solution. Similarly, we can show that  $x_2 \geq 0$  cannot bind since it's first order condition requires  $\frac{1}{2\sqrt{x_2}} = \lambda - \mu_2$  which is not true at  $x_2 = 0$ . The only alternative is that  $x_1 > 0$ ,  $x_2 > 0$  and  $x_3 = 0$ . Since the non-negativity constraints on  $x_1$  and  $x_2$  do not bind,  $\mu_1 = \mu_2 = 0$ . The first order conditions are:

$$\begin{aligned}\frac{1}{x_1} &= \lambda \\ \frac{1}{2\sqrt{x_2}} &= \lambda \\ x_1 &= 2\sqrt{x_2}\end{aligned}$$

Solving these:

$$\begin{aligned}x_1 &= 2(\sqrt{m+1} - 1), x_2 = m - 2\sqrt{m+1} + 2 \\ x_3 &= 0\end{aligned}$$

Let's look at the multipliers:

$$\begin{aligned}\lambda &= \frac{1}{2(\sqrt{m+1} - 1)} \\ \mu_3 &= \frac{1}{2(\sqrt{m+1} - 1)} - 1\end{aligned}$$

Let's suppose  $m = 1$  which meets the condition:

$$\lambda \approx 1.20711$$

$$\mu_3 = 0.20711$$

The rate at which utility increases if you relax the budget constraint is about 1.20711 (it is 1 if  $m > \frac{5}{4}$ ). Why is it that the utility increases at a rate one less than this if we relax the non-negativity constraint on  $x_3$ . If it is relaxed we can decrease  $x_3$  which in-turn relaxes the budget constraint at a rate of one (the price of  $x_3$ ). This allows us to increase utility at a rate of 1.20711 but decreasing  $x_3$  decreases utility at a rate of 1 so the net effect is 0.20711.

## 11.5 Example: Two Budget Constraints

$$u = x_1 x_2$$

Suppose there are two budgets:

$$\frac{1}{2}x_1 + 2x_2 \leq 100$$

$$3x_1 + 2x_2 \leq 250$$

Would you rather add \$1 to budget 1 or \$1 to budget 2?

$$x_1 x_2 - \lambda \left( \frac{1}{2}x_1 + 2x_2 - 100 \right) - \mu (3x_1 + 2x_2 - 250)$$

$$\frac{\partial (x_1 x_2 - \lambda (\frac{1}{2}x_1 + 2x_2 - 100) - \mu (3x_1 + 2x_2 - 250))}{\partial x_1} = -\frac{\lambda}{2} - 3\mu + x_2$$

$$\frac{\partial (x_1 x_2 - \lambda (\frac{1}{2}x_1 + 2x_2 - 100) - \mu (3x_1 + 2x_2 - 250))}{\partial x_2} = -2\lambda - 2\mu + x_1$$

$$x_2 = \frac{\lambda}{2} + 3\mu$$

$$x_1 = 2\lambda + 2\mu$$

$$x_1 = 60, x_2 = 35$$

$$\lambda = 22, \mu = 8$$

If budget one is relaxed, consumer can add utility at a rate of 22 while the rate is only 8 for budget 2. Budget one is more constraining. Think about why this is. Try to do the same problem with an OR constraint. That is, only one of the two constraints need to hold.

## 11.6 Example CD Utility

$$u(x_1, x_2) = x_1^\alpha x_2^\beta$$

$$p_1 x_1 + p_2 x_2 \leq m$$

$$x_1^\alpha x_2^\beta - \lambda (p_1 x_1 + p_2 x_2 - m)$$

$$\frac{\partial \left( x_1^\alpha x_2^\beta - \lambda (p_1 x_1 + p_2 x_2 - m) \right)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1$$

$$\alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 = 0$$

$$\frac{\partial \left( x_1^\alpha x_2^\beta - \lambda (p_1 x_1 + p_2 x_2 - m) \right)}{\partial x_2} = \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2$$

$$\beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 = 0$$

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \lambda$$

$$\frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2} = \lambda$$

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

$$\frac{\alpha}{\beta} x_2 p_2 = x_1 p_1$$

Marshallian Demands:

$$x_1^* = \frac{\frac{\alpha}{\alpha+\beta} m}{p_1}$$

$$x_2^* = \frac{\frac{\beta}{\alpha+\beta} m}{p_2}$$

## 11.7 Indirect Utility

$$\text{Max}_x u(x) \text{ s.t. } px \leq y$$

$$V(p, y) = u(x^*(p, y))$$

For  $x_1^\alpha x_2^\beta$  this is:

$$\left( \frac{\frac{\alpha}{\alpha+\beta} y}{p_1} \right)^\alpha \left( \frac{\frac{\beta}{\alpha+\beta} y}{p_2} \right)^\beta$$

If  $\alpha = \beta = 1$

$$\left( \frac{\frac{1}{2} y}{p_1} \right) \left( \frac{\frac{1}{2} y}{p_2} \right) = \frac{\frac{1}{4} y^2}{p_1 p_2}$$

**Homogeneous Functions:**

$$f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$$

$f$  is homogeneous of degree  $\alpha$ .

**Properties.**

1. Continuous.

*Berge's Maximum Theorem*

2. Homogeneous of degree zero in prices and income.

$$V(t\mathbf{p}, ty) = t^0 V(\mathbf{p}, y) = V(\mathbf{p}, y)$$

3. Strictly increasing in income.

Due to local non-satiation.

$$V = u(x_1^*, x_2^*) - \lambda(p_1 x_1^* + p_2 x_2^* - y)$$

$$V = u(x_1^*, x_2^*) - \lambda^*(p_1 x_1^* + p_2 x_2^* - y)$$

$$\frac{\partial V}{\partial y} = \lambda^*$$

$$\frac{MU_i}{p_i} = \lambda$$

4. Decreasing (weakly) in prices.

$$V = u(x_1^*, x_2^*) - \lambda^*(p_1 x_1^* + p_2 x_2^* - y)$$

$$\frac{\partial (u(x_1^*, x_2^*) - \lambda^*(p_1 x_1^* + p_2 x_2^* - y))}{\partial p_1} = (-x_1^*) \lambda^*$$

5. Quasi-convex in  $(p, y)$ .

$$(p, y), (p', y') \quad (tp + (1-t)p', ty + (1-t)y')$$

$$(tp + (1 - t)p', ty + (1 - t)y')$$

Because (by homework) anything achievable in budget  $(tp + (1 - t)p', ty + (1 - t)y')$  is achievable in one of the two other budgets, then either  $\max\{V((p, y)), V((p', y'))\} \geq V((tp + (1 - t)p', ty + (1 - t)y'))$

6. Roy's Identity. (An envelope condition)

The ratio of the way utility changes with price  $i$  to the way it changes with income is proportional to the amount of  $i$  consumed. This is because as price  $i$  changes, it changes effective income by  $(\Delta p_i)x_i$  and locally, there is no need to worry about changes in consumption level.

$$-\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial y}} = -\frac{-x_i^* \lambda}{\lambda} = x_i^*$$

## 12 Cost Minimization

*This is the dual to utility maximization.*

### 12.1 The Dual Consumer Problem

$$\min (px) \text{ s.t. } u(x) \geq \bar{u}$$

### 12.2 Example CD Utility

$$\min. p_1 x_1 + p_2 x_2$$

$$\text{s.t. } u(x_1, x_2) = x_1^\alpha x_2^\beta \geq \bar{u}$$

$$\text{Min.}_{x \in X} (p_1 x_1 + p_2 x_2) - \lambda (x_1^\alpha x_2^\beta - \bar{u})$$

$$\frac{\partial \left( (p_1 x_1 + p_2 x_2) - \lambda (x_1^\alpha x_2^\beta - \bar{u}) \right)}{\partial (x_1)} = p_1 - \lambda \alpha x_1^{\alpha-1} x_2^\beta$$

$$\frac{\partial \left( (p_1 x_1 + p_2 x_2) - \lambda (x_1^\alpha x_2^\beta - \bar{u}) \right)}{\partial (x_2)} = p_2 - \beta \lambda x_1^\alpha x_2^{\beta-1}$$

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \frac{1}{\lambda}$$

$$\frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2} = \frac{1}{\lambda}$$

$\lambda$  is the relative cost of increasing utility.

$$\frac{\alpha x_1^{\alpha-1} x_2^\beta}{p_1} = \frac{\beta x_1^\alpha x_2^{\beta-1}}{p_2}$$

$$\frac{\alpha}{\beta} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

Tangency Condition (same as for utility maximization):

$$\frac{\alpha}{\beta} x_2 p_2 = x_1 p_1$$

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

Utility Condition:

$$x_2 \frac{\alpha p_2}{\beta p_1} = x_1$$

$$\left( x_2 \frac{\alpha p_2}{\beta p_1} \right)^\alpha x_2^\beta = u$$

$$x_2 = \left( \frac{\beta p_1}{\alpha p_2} \right)^{\frac{\alpha}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

$$x_1 = \left( \frac{\alpha p_2}{\beta p_1} \right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

### 12.3 Marshallian / Hicksian Demand.

Marshallian Demand (Demand):

Amount of good  $i$  that is optimal given prices and income.

$$x_i^*(\mathbf{p}, y)$$

Hicksian Demand:

The amount of good  $i$  you choose to achieve utility  $u$  in the cheapest way:

$$x_i^h(\mathbf{p}, u)$$



## 12.4 Expenditure Function

$$e(\mathbf{p}, u) = \sum_{i=1}^n p_i x_i^h(\mathbf{p}, u)$$

This is the “optimized” value of the cost minimization objective subject to the utility constraint. It is analogous to the indirect utility function for utility maximization.

### Properties.

1. Continuous.

*Berge’s Maximum Principle.*

2. For  $p \gg 0$ , strictly increasing and unbounded above in  $u$ .
3. Increasing in  $p$ .
4. Homogeneous of degree 1 in  $p$ .
5. Concave in  $p$ .

The meaning of this in terms of economics:

If  $x^*$  is optimal at  $p, u$  and prices change.  $x^*$  still achieves the utility  $u$ . The cost of  $x^*$  thus represents an upper bound on the expenditure I need to achieve  $u$  at the new prices.

*Let’s talk about this one.*

6. Shephard’s lemma.

$$p_1 x_1^h + p_2 x_2^h - \lambda (u(x_1^h, x_2^h) - \bar{u})$$

$$\frac{\partial (p_1 x_1^h + p_2 x_2^h - \lambda (u(x_1^h, x_2^h) - \bar{u}))}{\partial p_i} = x_i^h$$

$$\frac{\partial e(u, \mathbf{p})}{\partial p_i} = x_i^h$$

Another envelope condition that has no name:

$$\frac{\partial e(\bar{u}, \mathbf{p})}{\partial \bar{u}} = \lambda$$

## 12.5 Duality of Indirect Utility/Expenditure

In general:

$$e(p, v(p, y)) \leq y$$

$$v(p, e(p, \bar{u})) \leq \bar{u}$$

But with continuous, **strictly monotonic** utility:

$$v(p, e(p, \bar{u})) = \bar{u}$$

$$e(p, v(p, y)) = y$$

$$x_i(p, y) = x_i^h(p, v(p, y))$$

## 12.6 Example. Cobb-Douglas

Suppose  $u = x_1x_2$

**Marshallian demands:**

$$L = x_1x_2 - \lambda(x_1 + x_2 - m)$$

$$\frac{\partial (x_1x_2 - \lambda(p_1x_1 + p_2x_2 - m))}{\partial x_1} = x_2 - \lambda p_1$$

$$\frac{\partial (x_1x_2 - \lambda(p_1x_1 + p_2x_2 - m))}{\partial x_2} = x_1 - \lambda p_2$$

$$\frac{x_2}{p_1} = \lambda$$

$$\frac{x_1}{p_2} = \lambda$$

*Notice, the Lagrange Multiplier is exactly equal to the amount utility increases when the consumer spends marginally more on either good. Setting these equal,*

$$x_1 = x_2$$

Plugging this into the budget equation:  $x_1 + x_2 = m$

$$x_1 = x_2 = \frac{1}{2} \frac{m}{p_i}$$

**Indirect Utility:**

Plug the Marshallian demands into the utility function:

$$u = \frac{1}{2} \frac{m}{p_1} \frac{1}{2} \frac{m}{p_2}$$

$$v(p_1, p_2, m) = \frac{1}{4} \frac{m^2}{p_1 p_2}$$

**Expenditure:**

Invert the indirect utility function, solve for  $m$ :

$$e = \sqrt{4up_1p_2}$$

**Hicksian Demands:**

Leverage the envelope condition, take derivative of the expenditure function to get Hicksian demands:

$$\frac{\partial (\sqrt{4up_1p_2})}{\partial p_1} = x_1^h$$

$$x_1^h = \sqrt{u} \sqrt{\frac{p_2}{p_1}}$$

$$x_2^h = \sqrt{u} \sqrt{\frac{p_1}{p_2}}$$

## 13 Decomposition

$$T.E. = S.E. + I.E.$$

### 13.1 Two Types.

#### Hicksian

Hick's formalized the Substitution effect in the following way:

The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old utility level**:

$$SE : x_i(p, y) - x_i(p', e(p', v(p, y)))$$

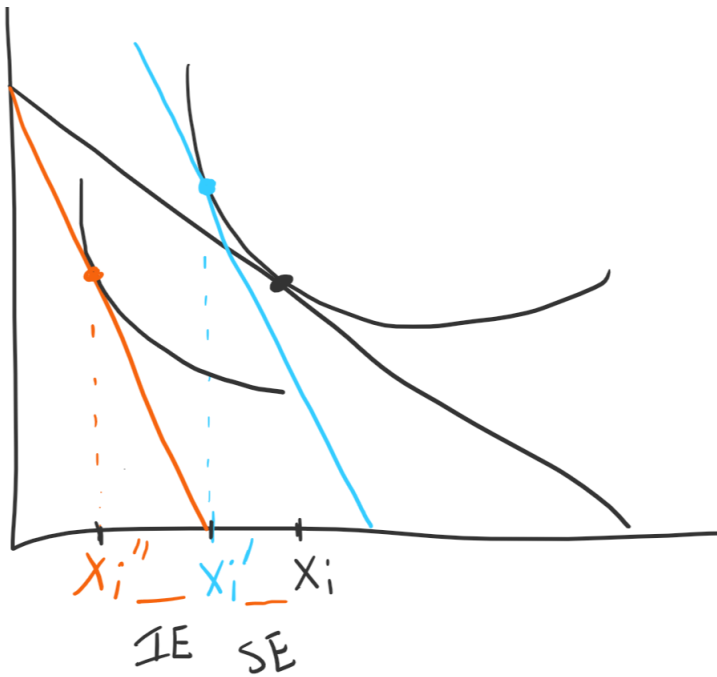


Figure 13.1: An example of the Hicksian decomposition.

The substitution effect has to be negative because **any demand on the compensated budget that lies below the old budget equation was previously strictly affordable**. That is, *if the price of a good increases, a consumer will demand less of it due to the substitution effect*.

### Slutsky

The substitution effect is the difference between original demand and the demand a consumer would choose at the new prices but with **enough income to afford the old bundle**:

$$SE : x_i(p, y) - x_i(p', p \cdot x^*(p, y))$$

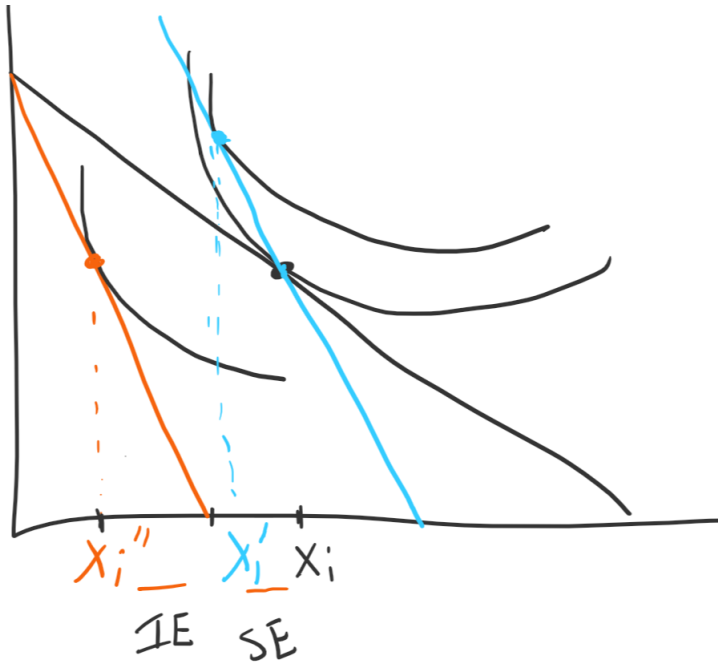


Figure 13.2: Example of the Slutsky decomposition.

The substitution effect has to be (non-positive) because any bundle with more  $x_i$  (when  $p_i$  is the price that is increasing) was previously strictly affordable.

### 13.2 Slutsky Equation

At the margin, the amount of money I need to maintain the same utility level is the same as the money I need to maintain the same bundle. Why? (Think first-order condition). Thus, at the margin, both methods are the same.

Let's look at the Hicksian decomposition at the margin.

$$S.E. = T.E. - I.E.$$

The substitution effect is:

$$\frac{\partial x_i^h(p, \bar{u})}{\partial p_j}$$

By duality:

$$x_i(p, e(p, \bar{u})) = x_i^h(p, \bar{u})$$

Thus, T.E.-I.E. must be:

$$\frac{\partial (x_i(p, e(p, \bar{u})))}{\partial p_j}$$

Set them equal and expand this:

$$\frac{\partial (x_i(p, e(p, \bar{u})))}{\partial p_j} + \frac{\partial (x_i(p, e(p, \bar{u})))}{\partial y} \frac{\partial e(p, \bar{u})}{\partial p_j} = \frac{\partial (x_i^h(p, \bar{u}))}{\partial p_j}$$

Let's label the various effects to keep them straight.

$$(T.E.) \frac{\partial (x_i(p, e(p, \bar{u})))}{\partial p_j} + (-I.E.) \frac{\partial (x_i(p, e(p, \bar{u})))}{\partial y} \frac{\partial e(p, \bar{u})}{\partial p_j} = (S.E.) \frac{\partial (x_i^h(p, \bar{u}))}{\partial p_j}$$

Use envelope condition to replace  $\frac{\partial e(p, \bar{u})}{\partial p_j}$  with  $x_j^h$ .

$$(T.E.) \frac{\partial (x_i(p, e(p, \bar{u})))}{\partial p_j} + (-I.E.) \left( \frac{\partial (x_i(p, e(p, \bar{u})))}{\partial y} x_j^h \right) = (S.E.) \frac{\partial (x_i^h(p, \bar{u}))}{\partial p_j}$$

Let  $e(p, \bar{u}) = y$ .

$$(T.E.) \frac{\partial (x_i(p, y))}{\partial p_j} + (-I.E.) \frac{\partial (x_i(p, y))}{\partial y} x_j^h = (S.E.) \frac{\partial (x_i^h(p, \bar{u}))}{\partial p_j}$$

**Rearranging this, we get the Slutsky Equation:**

$$\frac{\partial (x_i(p, y))}{\partial p_j} = \frac{\partial (x_i^h(p, \bar{u}))}{\partial p_j} - \frac{\partial (x_i(p, y))}{\partial y} x_j^h$$

$$(T.E.) \frac{\partial (x_i(p, y))}{\partial p_j}$$

$$(S.E.) \frac{\partial (x_i^h(p, \bar{u}))}{\partial p_j}$$

$$(I.E.) \left( -\frac{\partial (x_i(p, y))}{\partial y} x_j^h \right)$$

### 13.3 Example Cobb Douglass

Let's decompose demand:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial x_i^h}{\partial p_i} - \frac{\partial x_i}{\partial m} \frac{\partial e}{\partial p_i}$$

Cobb Douglass utility:  $x_1 x_2$

$$x_1^* = \frac{\frac{1}{2}m}{p_1}$$

$$x_2^* = \frac{\frac{1}{2}m}{p_2}$$

Indirect Utility:

$$V(p, m) = \frac{\frac{1}{2}m}{p_1} \frac{\frac{1}{2}m}{p_2} = \frac{\frac{1}{4}m^2}{p_1 p_2}$$

Leverage Duality:

$$\bar{u} = \frac{\frac{1}{4}e(p, \bar{u})^2}{p_1 p_2}$$

$$\sqrt{4\bar{u}p_1 p_2} = e(p, \bar{u})$$

Use Shephards Lemma:

$$x_1^h = \frac{\partial e(p, \bar{u})}{\partial p_j} = \frac{\partial (\sqrt{4\bar{u}p_1 p_2})}{\partial p_1}$$

$$x_1^h = \frac{\partial (\sqrt{4\bar{u}p_1 p_2})}{\partial p_1} = \frac{p_2 \bar{u}}{\sqrt{p_1 p_2 \bar{u}}}$$

$$x_2^h = \frac{\partial (\sqrt{4\bar{u}p_1 p_2})}{\partial p_2} = \frac{p_1 \bar{u}}{\sqrt{p_1 p_2 \bar{u}}}$$

Substitution Effect for good 1 when  $p_1$  changes:

$$\frac{\partial \left( \frac{p_2 \bar{u}}{\sqrt{p_1 p_2 \bar{u}}} \right)}{\partial p_1} = -\frac{p_2^2 \bar{u}^2}{2(p_1 p_2 \bar{u})^{3/2}}$$

Income Effect:

$$-\frac{\partial \left( \frac{\frac{1}{2}m}{p_1} \right)}{\partial m} \frac{\partial (\sqrt{4\bar{u}p_1 p_2})}{\partial p_1} = -\frac{p_2 \bar{u}}{2p_1 \sqrt{p_1 p_2 \bar{u}}}$$

$$-\frac{p_2 \bar{u}}{2p_1 \sqrt{p_1 p_2 \bar{u}}} = -\frac{p_2^2 \bar{u}^2}{2(p_1 p_2 \bar{u})^{3/2}}$$

$$-\frac{p_2 \bar{u}}{2p_1 \sqrt{p_1 p_2 \bar{u}}} = -\frac{\sqrt{p_2 \bar{u}}}{2p_1 (p_1)^{1/2}}$$

Income and Substitution effect are the same for the consumer at all levels of income and prices.

### 13.4 Negative Own-Substitution Effect

$$\frac{\partial x_i^h}{\partial p_i} \leq 0$$

$$\frac{\partial^2 (e)}{(\partial p_i)^2} = \frac{\partial x_i^h}{\partial p_i} \leq 0$$

Because  $e$  is concave, it has to be concave in any direction. Thus, the second derivative must be non-positive.

### 13.5 Elasticities

Suppose the price of a good changes from 1 to 2. Consumer 1's demand changes from 100 to 50 and consumer 2's changes from 10 to 5. Their behavior in terms of absolute changes in demand  $\frac{\Delta x_i}{\Delta p_i}$  is wildly different, but their behavior in terms of percentage terms  $\frac{\frac{\Delta x_i}{x_i}}{\frac{\Delta p_i}{p_i}}$  is identical. Elasticity is simply a way of quantifying comparative statics in unit-free percentage terms.

$$\eta_i = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial y}{y}} = \frac{\partial x_i}{\partial y} \frac{y}{x_i}$$

*Example:*  $x_1^\alpha x_2^\alpha$

$$\frac{\partial \left( \frac{A*y}{p_i} \right)}{\partial y} \frac{y}{\left( \frac{A*y}{p_i} \right)} = 1$$

Price and Cross-Price Elasticity.

$$\epsilon_{ij} = \frac{\frac{\partial x_i}{x_i}}{\frac{\partial p_j}{p_j}} = \frac{\partial x_i}{\partial p_j} \frac{p_j}{x_i}$$

*Example:*  $x_1^\alpha x_2^\alpha$

$$\frac{\partial \left( \frac{A*y}{p_i} \right)}{\partial p_i} \frac{p_i}{\left( \frac{A*y}{p_i} \right)} = -1$$

$$\frac{\partial \left( \frac{A*y}{p_i^2} \right)}{\partial p_i} \frac{p_i}{\left( \frac{A*y}{p_i^2} \right)} = -2$$



## 13.6 Elasticity Relations

$$y = \sum_{j \in I} p_j x_j(p, y)$$

Derivative w.r.t. Price:

$$\frac{\partial y}{\partial p_i} = \frac{\partial \sum_{j \in I} p_j x_j(p, y)}{\partial p_i}$$

$$0 = \frac{\partial \sum_{j \in I} p_j x_j(p, y)}{\partial p_i}$$

$$0 = \sum_{j \neq i} p_j \frac{\partial x_j}{\partial p_i} + p_i \frac{\partial x_i}{\partial p_i} + x_i$$

$$0 = \sum_{j=1}^n p_j \frac{\partial x_j}{\partial p_i} + x_i$$

$$-x_i = \sum_{i=1}^n p_j \frac{\partial x_j}{\partial p_i} \left( \frac{p_i}{x_j} \frac{x_j}{p_i} \right)$$

$$-x_i = \sum_{j=1}^n p_j \left( \frac{\partial x_j}{\partial p_i} \frac{p_i}{x_j} \right) \left( \frac{x_j}{p_i} \right)$$

$$-x_i = \sum_{j=1}^n p_j \left( \frac{x_j}{p_i} \right) \varepsilon_{j,i}$$

$$-x_i = \sum_{j=1}^n \left( \frac{p_j x_j}{p_i} \right) \varepsilon_{j,i}$$

$$-x_i = \frac{1}{p_i} \sum_{j=1}^n \left( \frac{p_j x_j}{1} \right) \varepsilon_{j,i}$$

$$-\left( \frac{1}{y} \right) \frac{x_i p_i}{1} = \left( \frac{1}{y} \right) \sum_{j=1}^n \left( \frac{p_j x_j}{1} \right) \varepsilon_{j,i}$$

$$-s_i = \sum_{j=1}^n \left( \frac{p_j x_j}{y} \right) \varepsilon_{j,i}$$

$$-s_i = \sum_{j=1}^n s_j \varepsilon_{j,i}$$

Derivative w.r.t. Income:

$$y = \sum_{j \in I} p_j x_j(p, y)$$

$$\frac{\partial y}{\partial y} = \frac{\partial \sum_{j \in I} p_j x_j(p, y)}{\partial y}$$

$$1 = \sum_{j=1}^n p_j \frac{\partial x_j}{\partial y}$$

Multiply by a “one”

$$1 = \sum_{j=1}^n p_j \frac{\partial x_j}{\partial y} \left( \frac{y}{x_j} \frac{x_j}{y} \right)$$

$$1 = \sum_{j=1}^n \frac{p_j x_j}{y} \eta_j$$

$$1 = \sum_{j \in I} s_j \eta_j$$

## 14 Decisions Under Uncertainty

Outcomes:

$$A \equiv \{a_1, \dots, a_n\}$$

Simple gambles:

Probability distribution over outcomes.

$$\mathcal{G}_s \equiv \left\{ (p_1 \circ a_1, p_2 \circ a_2, \dots, p_n \circ a_n) \mid \sum p_i = 1, p_i \geq 0 \right\}$$

Let  $g_1, \dots, g_n \in \mathcal{G}_s$

*Example:*  $A = \{\$10, \$5, \$0\}$

$$g_1 = \left( \frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0 \right)$$

$$g_2 = (1 \circ \$5) \text{ (degenerate gamble)}$$

*First-order Compound Gambles:*

for all  $i$ ,  $g_i \in \mathcal{G}_s$

$$\mathcal{G}_{c_1} \equiv \left\{ (p_1 \circ g_1, p_2 \circ g_2, \dots, p_n \circ g_n) \mid \sum p_i = 1 \right\}$$

*Example:*

$$\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ (1 \circ \$5)\right)$$

*Nth-order Compound Gambles:*

Let  $g_i \in \mathcal{G}_{c_j}$  for  $j \in \mathbb{Z}_+$ .

*Compound gambles:*

$$\mathcal{G} = \left\{ (p_1 \circ g_1, p_2 \circ g_2, \dots, p_m \circ g_m) \mid \sum p_i = 1 \right\}$$

*Example:*

$$\left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2} \circ \$7\right)\right)$$

## 14.1 Expected Utility

$$\left(\left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \left(\frac{1}{2} \circ \$10, \frac{1}{2} \circ \$0\right), \frac{1}{2} \circ \$5\right)\right), \left(\frac{1}{2} \circ \$7\right)\right)$$

The probability of each outcome:

$$\frac{1}{8} \circ \$10, \frac{1}{8} \circ \$0, \frac{1}{4} \circ \$5, \frac{1}{2} \circ \$7$$

We'd like to represent their preferences over that compound gamble by:

$$\frac{1}{8}u(\$10) + \frac{1}{8}u(\$0) + \frac{1}{4}u(\$5) + \frac{1}{2}u(\$7)$$

This is the **simple gamble induced** by compound gamble  $g$ .

When we can do this, we say preferences have the **expected utility property**.

Suppose we have  $u(a_i)$ :

$$U(g) = E_g(u(a_i))$$

Suppose gamble  $g$  induced simple gamble  $g_s$ . Let  $p_i^{g_s}$  be the probability of outcome  $i$  in  $g_s$ .

$$U(g) = \sum_{i=1}^n p_i^{g_s} u(a_i)$$

This utility function is of the expected value form.

**When can we represent preferences over  $\mathcal{G}$  with an expected utility function?**

## 14.2 Expected Utility Theorem

To get this, we really need two key things. *continuity* and *linearity*.

### 14.3 Expanded Proof

Let  $\succsim$  be the preference relation on  $\mathcal{G}$ :

Axiom 1. **Complete:**  $\succsim$  is complete.

Axiom 2. **Transitive:**  $\succsim$  is transitive.

Axiom 3. **Monotonic:** For all  $(\alpha \circ a_1, (1 - \alpha) \circ a_n) \succsim (\beta \circ a_1, (1 - \beta) \circ a_n)$  iff  $\alpha \geq \beta$ ,

Axiom 4. **Continuous:** For all  $g \exists p \in [0, 1]$  such that  $g \sim (p \circ a_1, (1 - p) \circ a_n)$

Axiom 5. **Substitution:** If  $g = (p_1 \circ g_1, \dots, p_k \circ g_k)$  and  $h = (p_1 \circ h_1, \dots, p_k \circ h_k)$  and if  $g_i \sim h_i$  for all  $i \in \{1, \dots, k\}$  then  $g \sim h$ .

Axiom 6. **Reduction:** For any gamble  $g$  and the simple gamble it induces  $g_s$ ,  $g \sim g_s$ .

Under assumptions 1 – 4, we have a utility representation:

$$\begin{aligned} u(g) : g &\sim (u(g) \circ a_1, (1 - u(g)) \circ a_n) \\ g &\succsim g' \Leftrightarrow u(g) \geq u(g') \end{aligned}$$

**With 3,4,5,6 there is additional structure:**

Let  $p_i^g$  be the probability of outcome  $a_i$  in the simple gamble induced by  $g$ .  $\succsim$  can be represented by:

$$u(g) = \sum_{i=1}^n p_i^g u(a_i)$$

**Prove it:**

Let  $u(a_i) = \alpha : a_i \sim (\alpha \circ a_1, (1 - \alpha) \circ a_n)$ .

$$g \succsim g'$$

By *Reduction*:

$$g \succsim g' \Leftrightarrow g_s \succsim g'_s \Leftrightarrow (p_1^{g_s} \circ a_1, \dots, p_n^{g_s} \circ a_n) \succsim (p_1^{g'_s} \circ a_1, \dots, p_n^{g'_s} \circ a_n)$$

By *Continuity*:

$$a_i \sim (\alpha \circ a_1, (1 - \alpha) \circ a_n)$$

By *Substitution*:

$$\begin{aligned} g \succsim g' &\Leftrightarrow g_s \succsim g'_s \Leftrightarrow (p_1^{g_s} \circ a_1, \dots, p_n^{g_s} \circ a_n) \succsim (p_1^{g'_s} \circ a_1, \dots, p_n^{g'_s} \circ a_n) \\ &\Leftrightarrow (p_1^{g_s} \circ a_1, \dots, p_i^{g_s} \circ (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), \dots, p_n^{g_s} \circ a_n) \succsim (p_1^{g'_s} \circ a_1, \dots, p_i^{g'_s} \circ (\alpha_i \circ a_1, (1 - \alpha_i) \circ a_n), \dots, p_n^{g'_s} \circ a_n) \end{aligned}$$

By *Reduction*:

$$\Leftrightarrow \left( \sum_{i=1}^n \alpha_i p_i^{g_s} \circ a_1, \sum_{i=1}^n (1 - \alpha_i) p_i^{g_s} \circ a_n \right) \succsim \left( \sum_{i=1}^n \alpha_i p_i^{g'_s} \circ a_1, \sum_{i=1}^n (1 - \alpha_i) p_i^{g'_s} \circ a_n \right)$$

By *Monotonicity*:

$$\Leftrightarrow \sum_{i=1}^n \alpha_i p_i^{g_s} \geq \sum_{i=1}^n \alpha_i p_i^{g'_s}$$

By *Construction of  $u(g)$* :

$$\Leftrightarrow \sum_{i=1}^n p_i^g u(a_i) \geq \sum_{i=1}^n p_i^{g'} u(a_i)$$

These assumptions are actually a little strong. Notice we did not need to assume 1 and 2 to get this. 1 and 2 were implied by the others. What is the least we could get away with assuming if we wanted to assume 1 and 2 from the start?

Axiom. ***Independence***: For any gambles  $g, g', g''$  and any  $\alpha \in (0, 1]$ ,  $\alpha g + (1 - \alpha) g'' \succsim \alpha g + (1 - \alpha) g' + (1 - \alpha) g'' \Leftrightarrow g \succsim g'$ .