Profit Max:
Let's ignore the fact that price changes when output changes:

$$\pi = py - wx = pf\left(x\right) - wx$$

$$Max_x pf\left(x\right) - wx$$

The first order conditions:

$$pf_1\left(x\right) = w_1$$

$$pf_2\left(x\right) = w_2$$

$$\frac{w_1}{f_1\left(x\right)} = p$$

$$\frac{w_2}{f_2\left(x\right)} = p$$

$$\frac{w_1}{f_1\left(x\right)} = \frac{w_2}{f_2\left(x\right)}$$

FOCs for cost min:

$$\frac{w_1}{f_1\left(x\right)} = \lambda$$

$$\frac{w_2}{f_2\left(x\right)} = \lambda$$

$$\frac{w_1}{f_1\left(x\right)} = \frac{w_2}{f_2\left(x\right)}$$

Thus, cost minimization is implied by profit maximization. This implies we can write the profit function where c is the cost minimized cost function.

$$\pi^*\left(p, w_1, w_2\right) = Max_y\left[py - c\left(y, w_1, w_2\right)\right]$$

Profit Max Example

$$x_1^\alpha x_2^\alpha$$

Assume firm is a price taker:

$$\pi\left(x_1, x_2\right) = p\left(x_1^\alpha x_2^\alpha\right) - \left(w_1 x_1 + w_2 x_2\right)$$

This is a concave function under the condition that $\alpha \leq \frac{1}{2}$.

$$\frac{\partial\left(p\left(x_1^\alpha x_2^\alpha\right) - \left(w_1 x_1 + w_2 x_2\right)\right)}{\partial x_1} = \alpha p x_1^{\alpha-1} x_2^\alpha - w_1$$

$$\frac{\partial \left(p\left(x_1^\alpha x_2^\alpha\right) - \left(w_1 x_1 + w_2 x_2\right)\right)}{\partial x_2} = \alpha p x_1^\alpha x_2^{\alpha-1} - w_2$$

First order conditions on profit:

$$\frac{p\alpha x_1^{\alpha-1} x_2^\alpha}{w_1} = 1$$

$$\frac{p\alpha x_1^\alpha x_2^{\alpha-1}}{w_2} = 1$$

Solving these:

$$\frac{p\alpha x_1^{\alpha-1} x_2^\alpha}{w_1} = \frac{p\alpha x_1^\alpha x_2^{\alpha-1}}{w_2}$$

$$\frac{\alpha x_1^{\alpha-1} x_2^\alpha}{w_1} = \frac{\alpha x_1^\alpha x_2^{\alpha-1}}{w_2}$$

Interpreting the above fractions. They are the increase in production for increasing expenditure on each input by a small amount.

$$-\frac{\alpha x_1^{\alpha-1} x_2^\alpha}{\alpha x_1^\alpha x_2^{\alpha-1}} = -\frac{w_1}{w_2}$$

This version above is the marginal rate of technical substitution equal to the slope of an isocost. This is precisely the cost minimization first order condition.

$$\frac{x_1^\alpha x_2^\alpha}{x_1 w_1} = \frac{x_1^\alpha x_2^\alpha}{x_2 w_2}$$

$$x_2 = x_1 \frac{w_1}{w_2}$$

To calculate the optimal level of x_1, plug this into a first order condition:

$$\frac{p\alpha x_1^{\alpha-1} \left(x_1 \frac{w_1}{w_2}\right)^\alpha}{w_1} = 1$$

$$x_1^{2\alpha-1} = \frac{1}{p\alpha} \left(\frac{w_2}{w_1}\right)^\alpha w_1$$

We now have the optimal input levels:

$$x_1^* = \left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_2^{\frac{\alpha}{2\alpha-1}} w_1^{\frac{1-\alpha}{2\alpha-1}}$$

$$x_2^* = \left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_1^{\frac{\alpha}{2\alpha-1}} w_2^{\frac{1-\alpha}{2\alpha-1}}$$

Plugging these into production:

$$y^* = \left(\left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_2^{\frac{\alpha}{2\alpha-1}} w_1^{\frac{1-\alpha}{2\alpha-1}}\right)^{\alpha} \left(\left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_1^{\frac{\alpha}{2\alpha-1}} w_2^{\frac{1-\alpha}{2\alpha-1}}\right)^{\alpha}$$

And finally the profit function (yikes):

$$\pi = p\left(\left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_2^{\frac{\alpha}{2\alpha-1}} w_1^{\frac{1-\alpha}{2\alpha-1}}\right)^{\alpha} \left(\left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_1^{\frac{\alpha}{2\alpha-1}} w_2^{\frac{1-\alpha}{2\alpha-1}}\right)^{\alpha}$$
$$- w_1\left(\left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_2^{\frac{\alpha}{2\alpha-1}} w_1^{\frac{1-\alpha}{2\alpha-1}}\right) - w_2\left(\left(\frac{1}{p\alpha}\right)^{\frac{1}{2\alpha-1}} w_1^{\frac{\alpha}{2\alpha-1}} w_2^{\frac{1-\alpha}{2\alpha-1}}\right)$$

The Profit Function

$$\pi(p,w) = max_y py - c(y) = Max\, pf(x) - wx$$

When well defined:

1. Increasing in p,
2. Decreasing in w,
3. Homogeneous of degree one in p, w
4. Convex in p, w (why),
5. Hotelling:

$$\frac{\partial \pi}{\partial p} = y(p,w), -\frac{\partial \pi}{\partial w_i} = x_i(p,w)$$

Profit maximization through cost minimization.

Let's do the same profit maximization in two steps. First, find the cost function (cheapest way of producing y). We know that the first first order conditions for cost minimization will imply:

$$\frac{w_1}{w_2} x_1 = x_2$$

What is the cost of producing one unit of output? Plug this condition into:

$$x_1^{\alpha} x_2^{\alpha} = 1$$

$$x_1^{\alpha}\left(\frac{w_1}{w_2} x_1\right)^{\alpha} = 1$$

This gives us conditional factor demands for producing one unit:

$$x_1(1, w_1, w_2) = \left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}$$

$$x_1\left(1,w_1,w_2\right)=\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}$$

Thus the cost of producing one unit of output is:

$$c\left(1,w_1,w_2\right)=w_1\left(\frac{w_2}{w_1}\right)^{\frac{1}{2}}+w_2\left(\frac{w_1}{w_2}\right)^{\frac{1}{2}}=2\sqrt{w_2}\sqrt{w_1}$$

Leveraging the properties of homogeneous production functions we learned last week:

$$c\left(y,w_1,w_2\right)=2\sqrt{w_2}\sqrt{w_1}y^{\frac{1}{2\alpha}}$$

Now write the profit function as a function of y:

$$\pi\left(y\right)=py-2\sqrt{w_2}\sqrt{w_1}y^{\frac{1}{2\alpha}}$$

This is concave under the assumption that $\alpha\leq\frac{1}{2}$ (that should look familiar):

$$\frac{\partial\left(p*y-2\sqrt{w_2}\sqrt{w_1}y^{\frac{1}{2\alpha}}\right)}{\partial y}\quad=\quad p-\frac{\sqrt{w_1}\sqrt{w_2}y^{\frac{1}{2\alpha}-1}}{\alpha}$$

The optimal output level is:

$$y^*=\left(p\left(w_1w_2\right)^{\frac{1}{2}}\alpha\right)^{\frac{2\alpha}{1-2\alpha}}$$

Plug this into $\pi\left(y\right)=py-2\sqrt{w_2}\sqrt{w_1}y^{\frac{1}{2\alpha}}$:

$$\pi=p\left(p\left(w_1w_2\right)^{\frac{1}{2}}\alpha\right)^{\frac{2\alpha}{1-2\alpha}}-2\sqrt{w_2}\sqrt{w_1}\left(\left(p\left(w_1w_2\right)^{\frac{1}{2}}\alpha\right)^{\frac{1}{1-2\alpha}}\right)$$