

ECONOMICS 8100

GREG LEO

Part 1. Budget

1. CONSUMPTION SET X

Assumptions: (Universe of Choice Objects): X

Bundles: Elements of X . $x \in X$

Assumptions about X .

1. $\emptyset \neq X \subseteq \mathbb{R}_+^n$.
2. X is closed.
3. X is convex.
4. $0 \in X$.

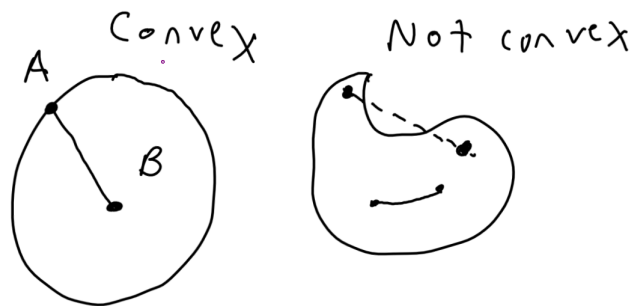


FIGURE 1.1. Examples of a Convex/Non-Convex Set.

2. BUDGET SET B

Budget Set: $B \subseteq X$

X defines the scope of the model. B is what an *individual consumer* chooses among.

Example. Budget Set with Prices and Income

$$B = \{x \mid x \in X \text{ \& } x_1 p_1 + x_2 p_2 \leq m\}$$

Example. Ice Cream Bowls

Every ice cream bowl x has some non-negative number of scoops of Vanilla, Chocolate, Strawberry.

$$X = \mathbb{R}_+^3$$

Budget B is the set of bowls with *no more than one scoop of ice cream*.

$$B = \left\{ x \mid x \in \mathbb{R}_+^3 \text{ \& } \sum_{i=1}^3 x_i \leq 1 \right\}$$

This is the unit-simplex in \mathbb{R}_3 .

$(1, 0, 0) \in B$. (On the boundary.)

$(0.5, 0.5, 0) \in B$. (On the boundary.)

$(0.25, 0.25, 0.25) \in B$. (In the interior.)

$(2, 0, 0) \notin B$

Part 2. Preference

3. THE PREFERENCE RELATION

Preference Relation is a **Binary Relation**.

Formally, a binary relation on set X is a subset of the Cartesian product X with itself.

$$\succeq \subseteq X \times X$$

Another way to denote an ordered pair is “in” the relation:

If $(x, y) \in \succeq$ we can also write $x \succeq y$.

Informally we say “ x ” is at least as good as “ y ”, or “ x ” preferred “ y ”.

Axioms of \succeq .

Axiom 0 (*reflexive*): $\forall x \in X, x \succeq x$. This is implied by *axiom 1*.

Axiom 1 (*complete*): $\forall x, x' \in X$, either $x \succeq x'$ or $x' \succeq x$ (or both).

The consumer has “some” preference over every pair of objects.

Axiom 2 (*transitive*): $\forall x, x', x'' \in X$ if $x \succeq x'$ and $x' \succeq x'' \Rightarrow x \succeq x''$.

\succeq is a “weak order” if it is complete and transitive.

4. RELATIONS AND SETS RELATED TO \succeq

Subrelations:

\sim is the indifference relation. $x \succeq y$ and $y \succeq x \Leftrightarrow x \sim y$.

\succ is the strict relation. $x \succeq y$ and not $y \succeq x \Leftrightarrow x \succ y$.

Related Sets:

$\succeq(x)$ “upper contour set”, “no worse than set”

$\preceq(x)$ “lower contour set”, “no better than set”

5. FROM PREFERENCES TO CHOICE

Choice Correspondence.

We will assume that from a budget set B a consumer “chooses” *choice set* C according to their preference \succeq . $C = \{x | x \in B \text{ \& } \forall x' \in B, x \succeq x'\}$.

Informally, C is the set of objects that are at least as good as anything else in the set.

Example With Transitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, c \succeq b$.

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = c$$

$$C(\{a, b, c\}) = c$$

6. CYCLES LEAD TO EMPTY CHOICE SETS

6.1. The Problem with Intransitive Preferences. $X = \{a, b, c\}$. $a \succeq b, c \succeq a, b \succeq c$. *This is intransitive!*

Choice correspondence:

$$C : P(X) / \emptyset \rightarrow X$$

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = b$$

$$C(\{a, b, c\}) = \emptyset$$

This consumer cannot make a choice from the set $\{a, b, c\}$.

6.2. Cycles and Empty Choices. Notice in the previous example, $a \succ b, a \succ c, c \succ a$. We have proved (essentially) that if there is a cycle, there is an empty choice set.

In fact, suppose, there is an empty choice set **and** X is finite. There must be a cycle.

$$\forall x \in B, \#(\succ(x)) < \#(B)$$

By completeness, $\forall x \exists x' \in X : x' \succ x$. Choose an x_1 , let x_2 be any element of $\succ(x_1)$. We have $x_2 \succ x_1$. If there is an $x_3 \in \succ(x_2)$ such that $x_1 \succ x_3$ we have identified a cycle. Otherwise, we continue with an inductive step. Suppose we have $x_n \succ \dots \succ x_1$. $\succ(x_n)$ is non-empty. Either it contains an element x_{n+1} such that there is an $x_i \succ x_{n+1}$ in which case we have identified a cycle or it does not and we continue with another inductive step. Either we find a cycle or reach the N_{th} step

with $x_N \succ x_{n-1} \succ \dots \succ x_1$. $\succ (x_N)$ is non-empty.

So, the cycle condition is equivalence to a non-empty choice set. Transitivity of \succsim implies transitivity of \succ which implies no cycles (try this last step at home). But do no-cycles imply transitivity of \succsim ? No. Here is a counter-example:

$$x \succ y, y \sim z, z \succ x$$

7. INTRANSITIVITY: EMPTY CHOICES, INCOHERENT CHOICES: PICK ONE.

So if no-cycles of the strict preference is equivalent to non-empty choice (in finite sets), and transitivity of \succsim is not equivalent to no-cycles, why do we assume it?

Finite non-emptiness: For any B with $\#(B) \in \mathbb{I}$, $C(B) \neq \emptyset$

Coherence: For every x, y and B, B' such that $x, y \in B \cap B'$, $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$.

Suppose there is an intransitive \succsim . There exists either a B where $C(B) = \emptyset$ or there exists a x, y, B, B' where the choice correspondence is incoherent.

By intransitivity:

$$1) x \succ y, y \succ z, z \succ x$$

$$C(\{x, y, z\}) = \emptyset$$

$$2) x \sim y, y \sim z, z \succ x$$

$$3) x \sim y, y \succ z, z \succ x$$

$$x \notin C(\{x, y, z\})$$

$$y \in C(\{x, y, z\})$$

$$x \in C(\{x, y\})$$

$$4) x \succ y, y \sim z, z \succ x$$

Can you find the incoherent choice?

Part 3. From Preference to Utility

8. UTILITY REPRESENTS PREFERENCES

Suppose there is some $U : X \rightarrow \mathbb{R}$ such that $U(x) \geq U(x') \Leftrightarrow x \succsim x'$ then we say $u(\cdot)$ represents preference relation \succsim . When does such a representation exist?

8.1. Finite X .

Proposition 1. *A $U(\cdot)$ exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.*

Proof. Let's start with \Rightarrow .

Because \geq is complete on the real numbers, for every $x, y \in X$ either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ thus because $u(\cdot)$ represents \succsim , it is complete.

By similar argument, \succsim is transitive. For every three $x, y, z \in X$. If $u(x) \geq u(y)$ and $u(y) \geq u(z)$ then $u(x) \geq u(z)$ because \geq is transitive on the real numbers.

Now we prove \Leftarrow :

Define $U(x) \equiv \#(\prec(x))$

Example: $a \succ b, b \succ c$. $\prec(a) = \{a, b, c\}$. $U(a) = 3$.

Lemma: For $x \succ y$, $\prec(y) \subseteq \prec(x)$ (proved in PS1).

By this lemma, for $x \succ y$, $\prec(y) \subseteq \prec(x)$ and thus $\# \prec(y) \leq \# \prec(x)$ and $u(x) \geq u(y)$. \square

8.2. Countably infinite X . Pick any arbitrary order on the bundles: (x_1, x_2, \dots) . And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \prec(x)} w(y)$$

Example: “ π shows up unexpectedly when eating ice cream.”

An even number of scoops of ice cream are better than an odd number of scoops and otherwise more is better than less.

$$u(2) = \sum_{i=1}^{\infty} \left(\frac{1}{(2i-1)^2} \right) = \frac{\pi^2}{8}$$

$$u(4) = \frac{1}{4} + \frac{\pi^2}{8}$$

8.3. Uncountable X . The *Lexicographic* preferences have no utility representation:

$$X = \mathbb{R}_+^2$$

$(x_1, x_2) \succ (y_1, y_2)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$.

\succsim is complete, and transitive. [Prove this for practice].

Pick two real numbers $v_2 > v_1$ and construct four bundles $(v_1, 1), (v_2, 1), (v_1, 2), (v_2, 2)$.

$$(v_2, 2) \succ (v_2, 1) \succ (v_1, 2) \succ (v_1, 1)$$

Suppose there is a utility function representing these preferences, then we have two disjoint intervals:

$$[u(v_2, 1), u(v_2, 2)]$$

$$[u(v_1, 1), u(v_1, 2)]$$

For every real number, we can construct an interval like this. Because the rationals are dense in the reals, there is a rational number in each of these intervals. Thus, for every real, we can find a unique rational number. That is, we have a mapping from the reals into the rationals which implies that the cardinality of the rationals is at least as large as that of the reals. $\#\mathbb{Q} \geq \#\mathbb{R}$. This contradicts that the cardinality of the rationals is strictly smaller than the reals.

8.4. An example of preference relation with a utility representation. Cars have horse power in $[0, 999]$ and cup holders in \mathbb{Z}_+ (integers).

Suppose preferences are lexicographic and more cup holders are more important than more horsepower.

$u(c_i, h_i) = c_i + \frac{h_i}{1000}$ represents these preferences.

See problem set 2 for example where we do not bound the horse power.

8.5. What ensures a utility representation in an uncountable universe?

Proposition 2. $U(x)$ exists that represents \succsim iff there exists a countable set $X^* \subset X$ s.t. $\forall x, y$ where $x \succ y \exists x^* \in X^*, x \succsim x^* \succ y$.

8.6. Continuous \succsim .

Definition. Continuity of \succsim : $\forall x \in R_+^n, \succsim(x)$ and $\precsim(x)$ are closed in R_+^n .

OTHER PROPERTIES OF \succsim