ECONOMICS 8100

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Part 1. Budget

1. Consumption Set X

Assumptions: (Universe of Choice Objects): X
Bundles: Elements of X. $x \in X$

Assumptions about X.
1. $\emptyset \neq X \subseteq \mathbb{R}^n_+$.
2. X is closed.
3. X is convex.
4. $0 \in X$.

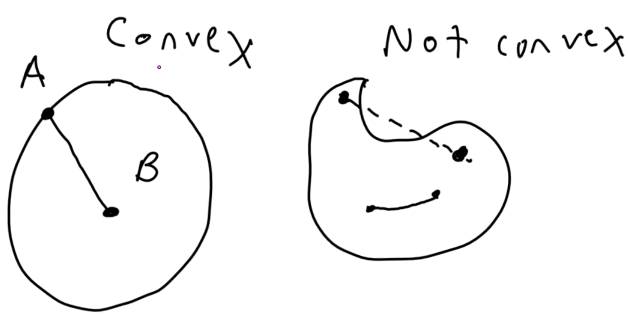

Figure 1.1. Examples of a Convex/Non-Convex Set.

2. Budget Set B

Budget Set: $B \subseteq X$

X defines the scope of the model. B is what an *individual consumer* chooses among.

Example. Budget Set with Prices and Income
$B = \{x | x \in X \,\&\, x_1 p_1 + x_2 p_2 \leq m\}$

Example. Ice Cream Bowls
Every ice cream bowl x has some non-negative number of scoops of Vanilla, Chocolate, Strawberry.

$$X = \mathbb{R}^3_+$$

Budget B is the set of bowls with *no more than one scoop of ice cream.*

$$B = \left\{ x | x \in R^3_+ \& \sum_{i=1}^{3} x_i \leq 1 \right\}$$

This is the unit-simplex in $\mathbb{R}_3$.
$(1, 0, 0) \in B$. (On the boundary.)
$(0.5, 0.5, 0) \in B$. (On the boundary.)
$(0.25, 0.25, 0.25) \in B$. (In the interior.)
$(2, 0, 0) \notin B$

Part 2. Preference

3. The Preference Relation

Preference Relation is a **Binary Relation**.
Formally, a binary relation on set X is a subset of the Cartesian product X with itself.

$$\succeq \subseteq X \times X$$

Another way to denote an ordered pair is "in" the relation:
If $(x, y) \in \succeq$ we can also write $x \succeq y$.
Informally we say "x" is at least as good as "y", or "x" preferred "y".

Axioms of $\succeq$.
Axiom 0 *(reflexive):* $\forall x \in X$, $x \succeq x$. This is implied by *axiom 1.*

Axiom 1 *(complete):* $\forall x, x' \in X$, either $x \succeq x'$ or $x' \succeq x$ (or both).
The consumer has "some" preference over every pair of objects.

Axiom 2 *(transitive):* $\forall x, x', x'' \in X$ *if* $x \succeq x'$ *and* $x' \succeq x'' \Rightarrow x \succeq x''$.

$\succeq$ is a "weak order" if it is complete and transitive.

4. Relations and Sets Related to $\succeq$

Subrelations:
$\sim$ is the indifference relation. $x \succeq y$ and $y \succeq x \Leftrightarrow x \sim y$.
$\succ$ is the strict relation. $x \succeq y$ and not $y \succeq x \Leftrightarrow x \succ y$.
Related Sets:
$\succeq (x)$ "upper contour set", "no worse than set"
$\precsim (x)$ "lower contour set", "no better than set"

5. From Preferences to Choice

Choice Correspondence.

We will assume that from a budget set B a consumer "chooses" *choice set* C according to their preference $\succeq$. $C = \{x | x \in B \,\&\, \forall x' \in B, x \succeq x'\}$.

Informally, C is the set of objects that are at least as good as anything else in the set.

Example With Transitive Preferences

$X = \{a, b, c\}$. $a \succeq b, c \succeq a, c \succeq b$.

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = c$$

$$C(\{a, b, c\}) = c$$

6. Cycles Lead to Empty Choice Sets

6.1. **The Problem with Intransitive Preferences.** $X = \{a, b, c\}$. $a \succeq b, c \succeq a, b \succeq c$. *This is intransitive!*

Choice correspondence:

$$C : P(X) / \emptyset \to X$$

$$C(\{a\}) = a, C(\{b\}) = b, C(\{c\}) = c$$

$$C(\{a, b\}) = a, C(\{a, c\}) = c, C(\{b, c\}) = b$$

$$C(\{a, b, c\}) = \emptyset$$

This consumer cannot make a choice from the set $\{a, b, c\}$.

6.2. **Cycles and Empty Choices.** Notice in the previous example, $a \succ b, a \succ c, c \succ a$. We have proved (essentially) that if there is a cycle, there is an empty choice set.

In fact, suppose, there is an empty choice set **and** X is finite. There must be a cycle.

$$\forall x \in B, \#\left(\succsim(x)\right) < \#(B)$$

By completeness, $\forall x \exists x' \in X : x' \succ x$. Choose an x_1, let x_2 be any element of $\succ(x_1)$. We have $x_2 \succ x_1$. If there is an $x_3 \in \succ(x_2)$ such that $x_1 \succ x_3$ we have identified a cycle. Otherwise, we continue with an inductive step. Suppose we have $x_n \succ ... \succ x_1$. $\succ(x_n)$ is non-empty. Either it contains an element x_{n+1} such that there is an $x_i \succ x_{n+1}$ in which case we have identified a cycle or it does not and we continue with another inductive step. Either we find a cycle or reach the N_{th} step with $x_N \succ x_{n-1} \succ ... \succ x_1$. $\succ (x_N)$ is non-empty.

So, the cycle condition is equivalence to a non-empty choice set. Transitivity of $\succsim$ implies transitivity of $\succ$ which implies no cycles (try this last step at home). But do no-cycles imply transitivity of $\succsim$? No. Here is a counter-example:

$$x \succ y, y \sim z, z \succ x$$

7. Intransitivity: Empty Choices, Incoherent Choices: Pick One.

So if no-cycles of the strict preference is equivalent to non-empty choice (in finite sets), and transitvity of $\succsim$ is not equivelent to no-cycles, why do we assume it?

Finite non-emptyness: *For any B with $\#(B) \in \mathbb{I}$, $C(B) \neq \emptyset$*

Coherence: *For every x, y and B, B' such that $x, y \in B \cap B'$, $x \in C(B) \wedge y \notin C(B) \Rightarrow y \notin C(B')$.*

Suppose there is an intransitive $\succsim$. There exists either a B where $C(B) = \emptyset$ or there exists a x, y, B, B' where the choice correspondence is incoherent.

By intransitivity:

$$1)\, x \succ y, y \succ z, z \succ x$$

$$C(\{x, y, z\}) = \emptyset$$

$$2)\, x \sim y, y \sim z, z \succ x$$

$$3)\, x \sim y, y \succ z, z \succ x$$

$$x \notin C(\{x, y, z\})$$

$$y \in C(\{x, y, z\})$$

$$x \in C(\{x, y\})$$

$$4)\, x \succ y, y \sim z, z \succ x$$

Can you find the incoherent choice?

8. Indifference Sets

8.1. **Indifference Maps.** To understand preferences, we often draw sets of the form $\sim(x)$. Many times these are one dimension smaller than the space of bundles, in which case we often call them *indifference curves*, but they need not have any special structure, unless we make further assumptions about preferences. There is only one things we really know about these sets.

8.2. **Complete, Transitive Preferences have Indifference Sets that Do Not Intersect. Result.** *Indifference curves do not cross.* For two bundles $x \succ y$, $\sim(x) \cap \sim(y) = \emptyset$.

Proof is given visually below:

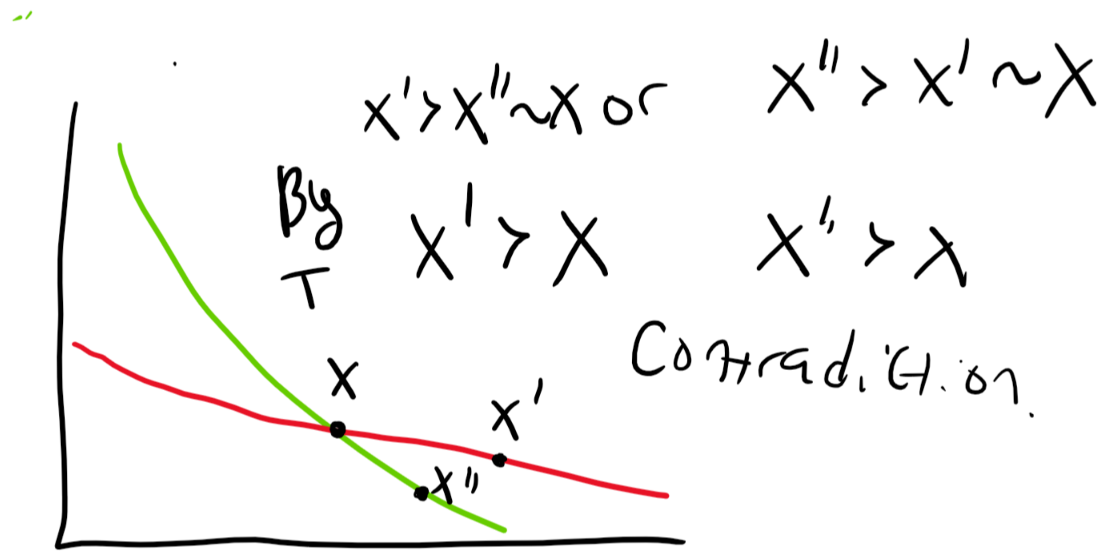

Figure 8.1. Distinct Indifference Sets do not Intersect

Part 3. From Preference to Utility

9. Utility Represents Preferences

Suppose there is some $U : X \to \mathbb{R}$ such that $U(x) \geq U(x') \Leftrightarrow x \succsim x'$ then we say $u()$ represents preference relation $\succsim$. When does such a representation exist?

9.1. Finite X.

Proposition 1. *A $U()$ exists that represents $\succsim \Leftrightarrow \succsim$ is complete and transitive.*

Proof. Let's start with $\Rightarrow$.

Because $\geq$ is complete on the real numbers, for every $x, y \in X$ either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ thus because $u()$ represents $\succsim$, it is complete.

By similar argument, $\succsim$ is transitive. For every three $x, y, z \in X$. If $u(x) \geq u(y)$ and $u(y) \geq u(z)$ then $u(x) \geq u(z)$ because $\geq$ is transitive on the real numbers.

Now we prove $\Leftarrow$:

Define $U(x) \equiv \#(\precsim(x))$

Example: $a \succ b, b \succ c$. $\precsim(a) = \{a, b, c\}.U(a) = 3$.

Lemma: For $x \succsim y$, $\precsim(y) \subseteq \precsim(x)$ (proved in PS1).

By this lemma, for $x \succsim y$, $\precsim(y) \subseteq \precsim(x)$ and thus $\# \precsim(y) \leq \# \precsim(x)$ and $u(x) \geq u(y)$. □

9.2. **Countably infinite X.** Pick any arbitrary order on the bundles: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim(x)} w(y)$$

Example: "*π shows up unexpectedly when eating ice cream.*"

An even number of scoops of ice cream are better than an odd number of scoops and otherwise more is better than less.

$$u(2) = \sum_{i=1}^{\infty} \left(\frac{1}{(2i-1)^2} \right) = \frac{\pi^2}{8}$$

$$u(4) = \frac{1}{4} + \frac{\pi^2}{8}$$

9.3. **Uncountable X.** The *Lexicographic* preferences have no utility representation:

$$X = \mathbb{R}_+^2$$

$(x_1, x_2) \succ (y_1, y_2)$ if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$.

$\succsim$ is complete, and transitive. [*Prove this for practice*].

Pick two real numbers $v_2 > v_1$ and construct four bundles $(v_1, 1), (v_2, 1), (v_1, 2), (v_2, 2)$.

$$(v_2, 2) \succ (v_2, 1) \succ (v_1, 2) \succ (v_1, 1)$$

Suppose there is a utility function representing these preferences, then we have two disjoint intervals:

$$[u(v_2, 1), u(v_2, 2)]$$

$$[u(v_1, 1), u(v_1, 2)]$$

For every real number, we can construct an interval like this. Because the rationals are dense in the reals, there is a rational number in each of these intervals. Thus, for every real, we can find a unique rational number. That is, we have a mapping from the reals into the rationals which implies that the cardinality of the rationals is at least as large as that of the reals. $\#\mathbb{Q} \geq \#\mathbb{R}$. This contradicts that the cardinality of the rationals is strictly smaller than the reals.

9.4. **An example of preference relation with a utility representation.** Cars have horse power in $[0, 999]$ and cup holders in $\mathbb{Z}_+$ (integers).

Suppose preferences are lexicographic and more cup holders are more important than more horsepower.

$u(c_i, h_i) = c_i + \frac{h_i}{1000}$ represents these preferences.

See problem set 2 for example where we do not bound the horse power.

9.5. **What ensures a utility representation in an uncountable universe?** A preference relation is representable by a utility function $U(x)$ iff $\forall x, y \in X \; s.t. \; x \succ y$, $\exists x^* \in X^* \subset X \; s.t. \; x \succsim x^* \succ y$ and the set X^* is countable.

To construct the utility function, $U(x)$, Pick any arbitrary order on the bundles in X^*: $(x_1, x_2, ...)$. And assign weights to those bundles $w(x_i) = \frac{1}{i^2}$. The following utility function represents preferences:

$$u(x) = \sum_{y \in \precsim(x) \cap X^*} w(y)$$

9.6. **Continuous $\succsim$.** Preference relation $\succsim$ is continuous if $\forall x \in X$, $\succsim(x)$ and $\precsim(x)$ are closed in X.

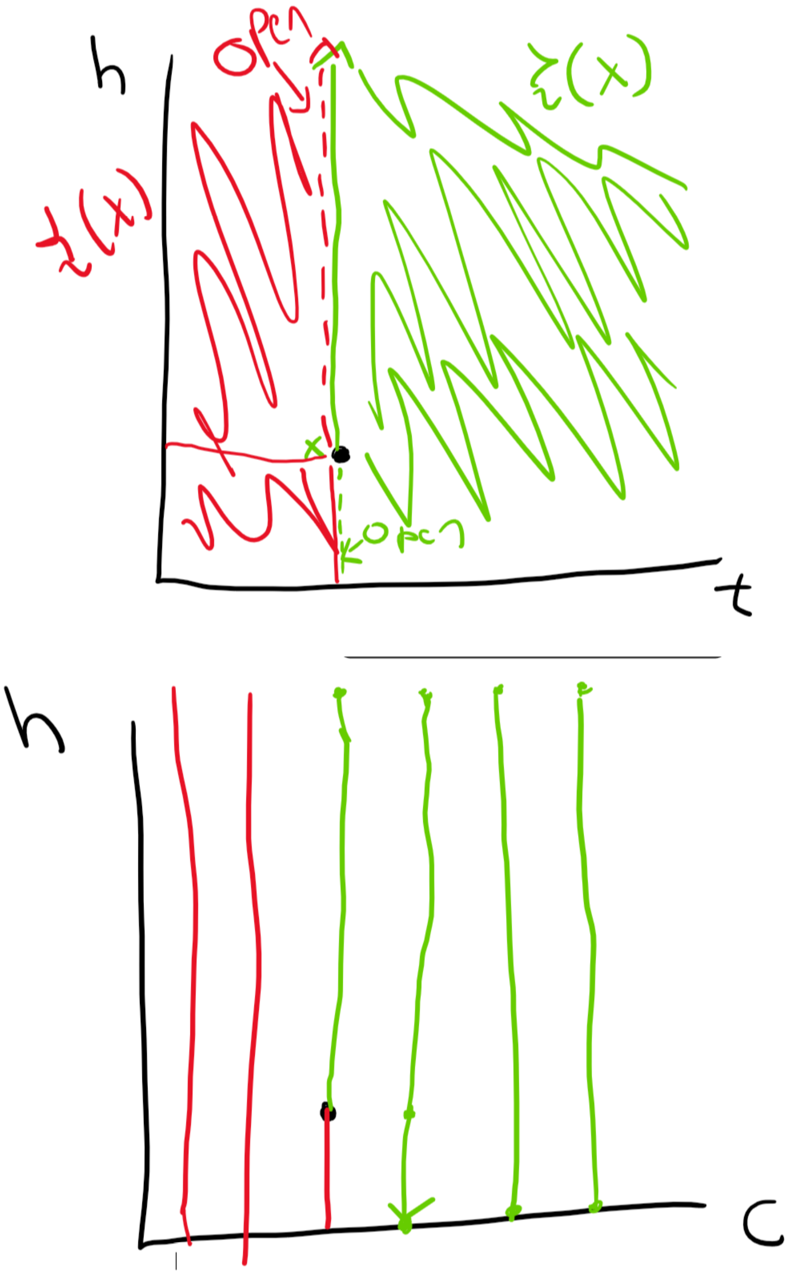

Figure 9.1. Not Continuous/Continuous Lexicographic Preferences.

9.7. **What ensures a continuous utility representation?** A *complete*, *transitive,* and *continuous* preference relation $\succsim$ can be represented by a continuous utility function $U(x)$ and, a continuous utility function represented C,T,C preferences.

10. Other Properties of $\succsim$

10.1. **Monotonicity.** Ensure consumers consume on the boundary of the budget set.

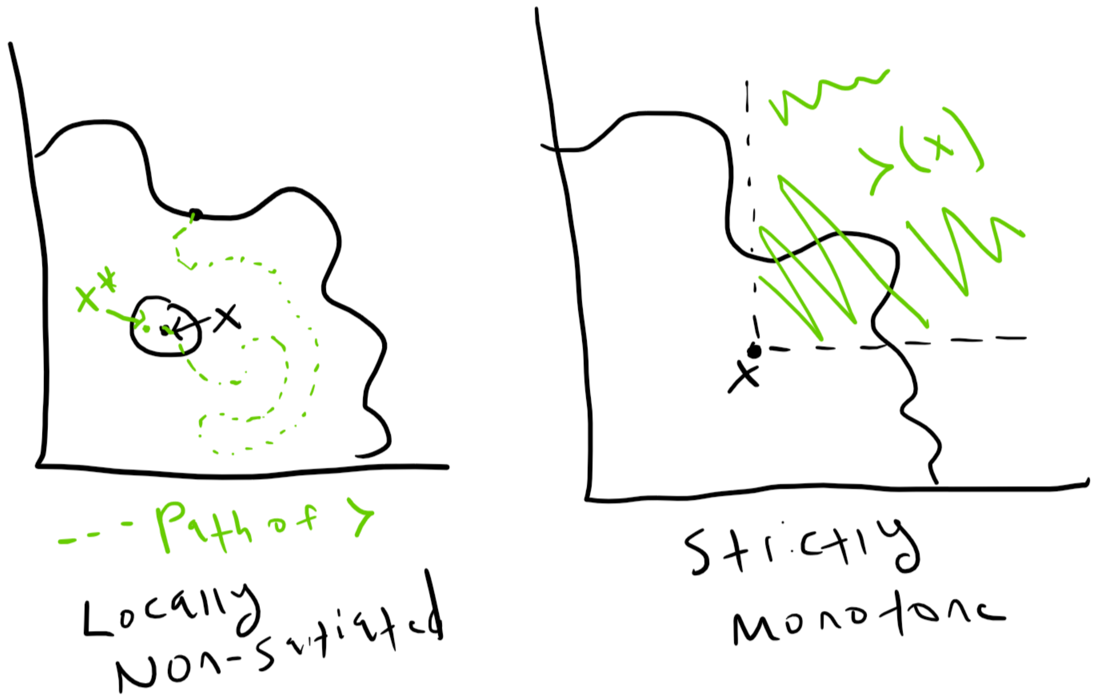

Figure 10.1. Locally Non-satiated vs. Strictly Monotone

10.1.1. *Strict Monotonicity.* ***More stuff is better.***

First, some notation:

For $X \subseteq \mathbb{R}^n$

$x \geq x'$ iff $x_i \geq x_i'$ for all $i \in \{1, 2, ..., n\}$

$x \gg x'$ iff $x_i > x_i'$ for all $i \in \{1, 2, ..., n\}$

Fro example: $(2,2) >> (1,1), (2,1) \geq (1,1), (1,1) \geq (1,1)$

Definition. Strict Monotonicity. $x \geq x' \Rightarrow x \succsim x'$ and $x \gg x' \Rightarrow x \succ x'$

10.1.2. *Local Nonsatiation.* **Definition. Local Nonsatiatin.** $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x^* \in B_\varepsilon(x)$ such that $x^* \succ x$.

A consumer can always change the bundle a "little bit" no matter how small that little bit is, and find something strictly better.

10.2. **Convex Sets, Convex/Concave Functions, Quasi-Convex/Concave Functions.**

10.2.1. *Convex Sets.* In a subset of euclidean space X, the line between $x \in X$ and $x' \in X$ is another point in the set X given by $tx + (1-t)\, x'$ where $t \in [0,1]$. We call points like this **Convex Combinations** of x and x'.

For example: $x = (1,0)\,, x' = (0,1)$. If we take $t = 0.5$. The convex combination is $0.5\,(1,0) + 0.5\,(0,1) = (0.5, 0.5)$.

A **convex set** $S \subseteq X$ is a set of points that contains all of its convex combinations.

Formally, $\forall x, x' \in S$, $\forall t \in [0,1]$, $tx + (1-t)\, x' \in S$.

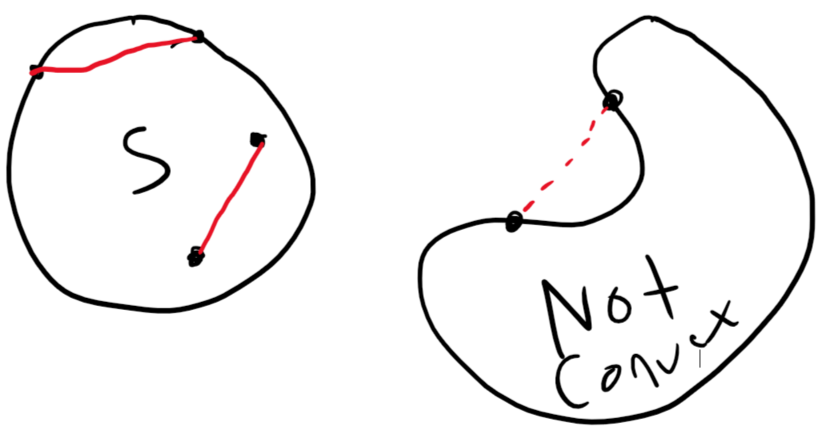

FIGURE 10.2. A Convex and Non-Convex Set

10.2.2. *Convex Functions. A line between two points "on the function" lies above the function itself.*

Convex Function:

$$\forall x, x' \in X, t \in [0,1]\,, tf\,(x) + (1-t)\, f\,(x') \geq f\,(tx + (1-t)\, x')$$

Strictly Convex Function:

$$\forall x, x' \in X, t \in (0,1)\,, tf\,(x) + (1-t)\, f\,(x') > f\,(tx + (1-t)\, x')$$

Contour Sets:

A convex function has **convex lower contour sets**.

10.2.3. *Concave Functions.* A line between two points "on the function" lies above the function itself.

Concave Function:

$$\forall x, x' \in X, t \in [0,1], tf(x) + (1-t) f(x') \leq f(tx + (1-t) x')$$

Strictly Concave Function:

$$\forall x, x' \in X, t \in (0,1), tf(x) + (1-t) f(x') < f(tx + (1-t) x')$$

A concave function has **convex upper contour sets**.

10.2.4. *Quasi-Concave Functions.* A function $f(x)$ is quasi-concave if **and only** it has convex upper contour sets.

A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a concave function.

A function $f(x)$ is quasi-concave if and only if $f(tx + (1-t) x') \geq min\{f(x), f(x')\}$ for $t \in [0,1]$.

A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t) x') > min\{f(x), f(x')\}$ for $t \in (0,1)$.

Notice that, for a strictly quasi-concave utility function, let $x' \succ x$, then the set $tx + (1-t) x' \in \succ (x)$ for $t \in (0,1)$. Thus, there is a small enough ball around that point $B_\epsilon (tx + (1-t) x') \in \succsim (x)$. Thus, these points are in the interior of $\succsim (x)$ and $\succsim$ is **strictly convex.**

10.2.5. *Quasi-Convex Functions.* A function $f(x)$ is quasi-convex if **and only** it has convex lower contour sets.

A function $f(x)$ is quasi-concave if and only if is a monotonic transformation of a convex function.

A function $f(x)$ is quasi-convex if and only if $f(tx + (1-t) x') \leq max\{f(x), f(x')\}$ for $t \in [0,1]$.

A function $f(x)$ is **strictly quasi-concave** if and only if $f(tx + (1-t) x') < max\{f(x), f(x')\}$ for $t \in (0,1)$.

10.3. **Convexity of $\succsim$. Convex Preferences:** $x \succsim x' \Rightarrow t(x) + (1-t) x' \succsim x', t \in [0,1]$

$x \in \succsim (x') \Rightarrow t(x) + (1-t) x' \in \succsim (x')$

Thus, $\succsim (x)$ are convex if $\succsim$ is a convex preference relation.

Strictly Convex Preferences: $x \succsim x' \Rightarrow t(x) + (1-t) x' \succ x', t \in (0,1)$

The upper contour sets $\succsim (x)$ are *strictly* convex.

10.4. **Utility and Preference Relationships.** If U represents $\succsim$:

1) $\succsim$ (strictly) **convex** $\Leftrightarrow$ U is (strictly) **quasi-concave**.

2) $\succsim$ are **strictly monotonic** $\Leftrightarrow$ U is **strictly** increasing.

3) $\succsim$ are **strictly monotonic** $\Leftarrow$ U is **strongly** increasing.

11. The Consumer Problem

11.1. **Choice.** The set of all "best things" in the budget set. This is what we are looking for:
$C(B) = \{x | x \in B \wedge x \succsim x', \forall x' \in B\}$

Competitive Budgets:

$$B = \left\{x | x \in \mathbb{R}_+^n, p \cdot x \leq m\right\}$$

p is the vector of prices.
m is the "income".
Constrained problem:

$$Max_{x \in X} U(x) \, s.t. \, p \cdot x \leq m$$

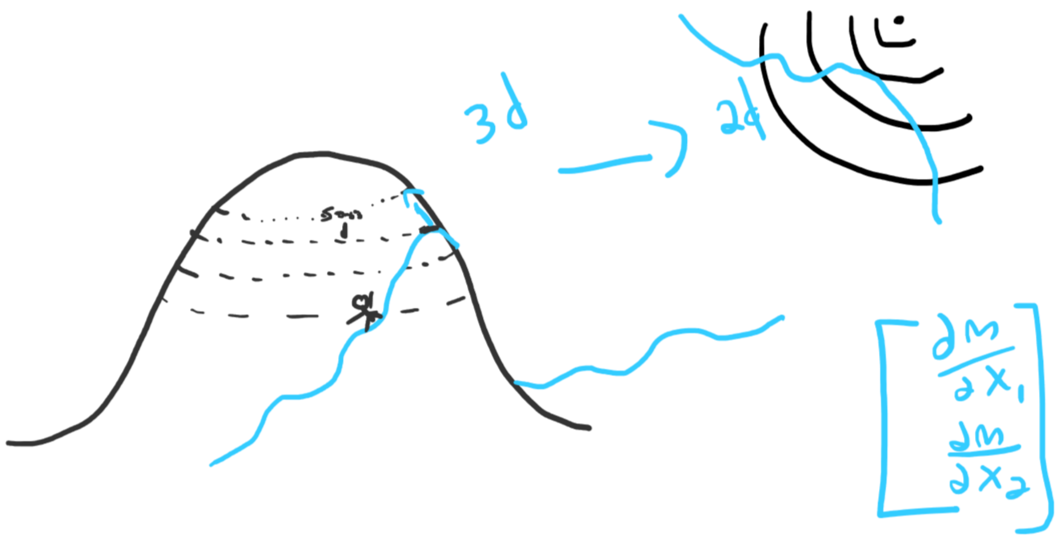

Figure 11.1. Finding the best spot for a selfie.

11.2. **The Lagrange Method.** If both the objective and the constraint are smooth, at the optimal **the *direction* of the gradient of the objective has to be equal to the *direction* of the gradient of the constraint.** Otherwise, moving along the constraint boundary in *some* direction will yield a larger value of the objective! (Caveat: this assumes we *can* move in every direction along the constraint. That will only be true at non-boundary points.)

Thus, for smooth functions, the equality of the direction of the gradients of the objective and the constraint are **necessary** for an non-boundary optimum.

Since the direction of the gradient is just a scaling of the gradient, suppose U is our objective and G is the function for the boundary of the constraint. Then,

$$\nabla U\left(x\right)=\lambda\nabla G\left(x\right)$$

Can we write a function such that the first order condition will yield this gradient condition? *Sure*:

$$\mathcal{L}=U\left(x\right)-\lambda\left(G\left(x\right)-c\right)$$

Let's treat this as an unconstrained problem. The FOC. of this function is:

$$\nabla U\left(x\right)-\lambda\nabla G\left(x\right)=0$$

$$\nabla U\left(x\right)=\lambda\nabla G\left(x\right)$$

This is precisely the necessary condition we need for the constrained problem. **Thus, FOC for unconstrained optimization of the Lagrangian is the necessary constrained optimization condition.**

11.3. **Indirect Utility.**

11.3.1. *Properties.* 1. Continuous.

2. Homogeneous of degree zero in (p, y).
3. Strictly increasing in y.
4. Decreasing (weakly) in p.
5. Quasi-convex in (p, y).
6. Roy's Identity.