1. $$p(x) = \frac{1/x}{\int_1^T dx\, \frac{1}{x}} = \frac{1}{x \log(T)}.$$

$$P_c = \int_1^x dy\ p(y) = \frac{\log(x)}{\log(T)}.$$

$P(x) = r$ means that

$$r = T^x.$$

uniform btwn 0 - 1

distrib. of $\frac{1}{r \log T}$

btwn 1 & T.

(a) $U = \frac{1}{2} \sum k_1 r_i^2 + \sum_{i<j} V_{int}(|r_i - r_2|)$.

$$m\ddot{r}_i = -\xi \dot{r}_i - k_1 r_i - \sum_{j \neq i} U'(|r_{ij}|)\, \hat{r}_{ij} + \vec{F}_i .$$

Then

$$\frac{m}{N} \sum \ddot{r}_i = m\ddot{R} = -\xi \sum \dot{r}_i - k_1 \sum r_i + \sum F_i$$

$$- \sum_i \sum_{j \neq i} U'(|r_{ij}|)\, \hat{r}_{ij}$$

$$-\xi \dot{R} - k_1 R + F_c - \Big[U(r_{12})(\hat{r}_{12} + \hat{r}_{21}) + U(r_{23})(\hat{r}_{12} + \hat{r}_{21}) + \cdots \Big]$$

$r_{ij} = -r_{ji}$, so

$$m\ddot{R} = -\xi \dot{R} - k_1 R + F_c$$

$P(\vec{F}_c)$ = prob that $\sum_i \vec{F}_i = \vec{F}_c$

$$= \int d^3F_1 \, d^3F_2 \cdots d^3F_p \quad P(F_1) P(F_2) \cdots P(F_p)$$

$$\times \; \delta\left(\sum_i \vec{F}_i - \vec{F}_c\right)$$

$$\propto \int \frac{d^dk}{2\pi} \, e^{ikF_c} \int \prod_i d^dF_i \quad \mathrm{Exp}\left[-\frac{\Delta t}{4\xi k_BT} \sum_{ij} F_{ij}^2 \right.$$

$$\left. - i \sum_{ij} k_j F_{ij} \right]$$

$$\propto \int \frac{d^dk}{2\pi} \, e^{ikF_c} \int \prod_i d^dF_i \quad \mathrm{Exp}\left[-\frac{\Delta t}{4\xi k_BT} \left(F_{ij} - i \frac{2k\xi k_BT}{\Delta t}\right)^2 - \frac{N k^2 \xi k_BT}{\Delta t}\right]$$

integrating over F_{ij} gives a const, indep. of k or F_c

$$\propto \int \frac{d^2k}{2\pi} \, e^{i\vec{k}\cdot F_c - k^2 \xi k_BT/\Delta t}$$

$$= \int \frac{d^dk}{2\pi} \, e^{-N\xi k_BT/\Delta t \left(k - i\frac{\Delta t F_c}{2N\xi k_BT}\right)^2 - \frac{\Delta t F_c^2}{2N\xi k_BT}}$$

$\propto e^{-\delta t\, F_c^2 / 2N\xi k_B T}$.

The effective ran. force on R & r are both G. distr. with 0 mean and variance $\langle F_c^2 \rangle = 2\xi k_B T / N\delta t$.

In the cont limit, this gives the expected $\langle F_c \rangle = 0$ & $\langle F_c(t) F_c(t') \rangle =$

$$\frac{2\xi k_B T}{N} \delta(t-t').$$

(b) Laplace transform $t \to \lambda$

$$\mathcal{L}(\dot{x}) = \lambda \tilde{x} - x_0$$

$$\mathcal{L}(\ddot{x}) = \lambda^2 \tilde{x} - \lambda \dot{x}_0 + x_0$$

so

$$\mathcal{L}(m\ddot{x} + \xi \dot{x} + k_1 x) = (m\lambda^2 + \xi\lambda + k_1)\tilde{x}$$

$$= \tilde{F}_c.$$

since $x_0 = \dot{x}_0 = 0$

Then $$\tilde{x} = \frac{\tilde{F}_c}{m\lambda^2 + \xi\lambda + k},$$

and via the conv. theorem,

$$x(t) = \int_0^t dt' \, F_c(t') \, \mathcal{L}^{-1}_{t-t'}\left[\frac{1}{m\lambda^2 + \xi\lambda + k}\right]$$

$$\int_0^t dt' \, F_c(t') \left[\frac{2m}{\tau_2} e^{-(t-t')/2\tau_1} \sinh\left(\frac{t-t'}{2\tau_2}\right)\right]$$

with $\tau_1 = m/\xi$ and $\tau_2 = \frac{m}{\sqrt{\xi^2 - 4mk}}$

Note that τ_2 is imaginary if

$\xi < 2\sqrt{mk}$, so $\frac{1}{\tau_2}\sinh\left(\frac{t-t'}{2\tau_2}\right) \to$

$\frac{1}{|\tau_2|}\sin\left(\frac{t-t'}{2|\tau_2|}\right)$ in that limit.

$$\langle R^2\rangle = 3\langle R_x^2\rangle = 3\int_0^t dt'\,dt''\; K(t-t')\,K(t-t'')$$

$$\cdot\left\langle F_c(t')\,F_c(t'')\right\rangle$$

$$= \frac{12\,\xi\,k_B T\,m^2}{N\tau_2^2}\int_0^t dt'\; K^2(t-t')$$

with $K(t-t') = e^{-t/2\tau_1}\sinh\left(\frac{t}{2\tau_2}\right)$

The integral is straightforward,

since $K(t) = \frac{1}{2}\left(e^{-t\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)} - e^{-t\left(1/\tau_1 - 1/\tau_2\right)}\right)$.

Integration & some algebra gives

$$\langle R^2\rangle = \frac{3\tau_1\tau_2^2 k_B T\,\xi}{m^2}\left(2e^{-t/\tau_1} + \frac{\tau_2}{\tau_1-\tau_2}e^{-t/\tau_1+t/\tau_2} - \frac{\tau_2}{\tau_1+\tau_2}e^{-t/\tau_1-t/\tau_2} - \frac{2\tau_1^2}{\tau_1^2-\tau_2^2}\right)$$

Note that if $\tau_2 \to 0$, this does not diverge, instead $\frac{1}{\tau_2} \sinh\left(\frac{(t-t')}{2\tau_2}\right)$

$$\approx \frac{(t-t')}{2}.$$

This does agree with the sims.