6 (a)

If we have a map $x_{n+1} = f(x_n)$, a stable fixed point satisfies

(1) $x^* = f(x^*)$ and

(2) $\epsilon_{n+1} \approx f'(x^*)\,\epsilon_n$ converges,

where (2) implies $|f'(x^*)| < 1$

For $x_{n+1} = \dfrac{r x_n}{1 + x_n^2}$, we have

(2) $f'(x^*) = \dfrac{r}{1+(x^*)^2} - \dfrac{2r(x^*)^2}{\left(1+(x^*)^2\right)^2}$

This can be simplified knowing that $\dfrac{1}{1+(x^*)^2} = \dfrac{1}{r}$ from (1), So $$f'(x^*) = 1 - \frac{2(x^*)^2}{r}.$$

Then $|f'(x^*)| \le 1$ when

$$-1 < 1 - \frac{2(x^*)^2}{r} < 1.$$

is satisfied for all x^*,

so the only constraint is

$$(x^*)^2 < r, \quad \text{or} \quad x^* < \sqrt{r}.$$

(1) implies $$x^* = \frac{r x^*}{1+(x^*)^2}, \quad \text{or}$$

$$x^* = \sqrt{r-1} < \sqrt{r}.$$