1. a with $\vec{u} = \left(\frac{\alpha}{2} Re (1-y^2), 0, 0 \right)$

$p = p_0 - \alpha x$,

$\frac{\partial u}{\partial t} = 0$ $(\vec{u} \cdot \vec{\nabla}) \vec{u} = u_x \frac{\partial u_x}{\partial x} = 0$

$\vec{\nabla} p = -\alpha \hat{x}$ $\nabla^2 \vec{u} = -\alpha Re \, \hat{x}$

so $\frac{\partial u}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\nabla p + \frac{1}{Re} \nabla^2 u$.

also, $\vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} = 0$, as expected.

b. N.S. is $\frac{\partial u}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = -\nabla p + \frac{1}{Re} \nabla^2 \vec{u}$.

Curl of that is

$\frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{u} \cdot \vec{\nabla}) \vec{u} = \cancelto{0}{-\vec{\nabla} \times \vec{\nabla} p} + \frac{1}{Re} \nabla^2 \omega$

$$\vec{\nabla}\times(\vec{u}\cdot\vec{\nabla})\vec{u}, \quad \text{with} \quad \vec{u} = (u_x, u_y, 0).$$

$$\vec{\omega} = \left(0, 0, \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right)$$

$$(\vec{u}\cdot\vec{\nabla})\vec{u}_x = u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial}{\partial y}u_x$$

$$(\vec{u}\cdot\vec{\nabla})u_y = u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}$$

$$\vec{\nabla}\times(a, b, 0) = \left(0, 0, \frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\right), \quad \text{so}$$

$$\vec{\nabla}\times(\vec{u}\cdot\vec{\nabla})\vec{u} = \left(0, 0, \frac{\partial}{\partial y}\left[u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right] - \frac{\partial}{\partial x}\left[u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}\right]\right)$$

$$= \left(0, 0, \frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial x} + u_x\frac{\partial^2 u_x}{\partial x\partial y} + \frac{\partial u_y}{\partial y}\frac{\partial u_x}{\partial y} + u_y\frac{\partial^2 u_x}{\partial^2 y} - \frac{\partial u_x}{\partial x}\frac{\partial u_y}{\partial x} - u_x\frac{\partial^2 u_y}{\partial x^2} - \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} - u_y\frac{\partial^2 u_y}{\partial x\partial y}\right)$$

$= [0, 0, (\vec{u} \cdot \vec{\nabla})\omega + \omega(\vec{\nabla} \cdot \vec{u}))$

$\uparrow \vec{\nabla} \cdot \vec{u} = 0$

then $$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{\omega} = \frac{1}{Re} \nabla^2 \vec{\omega}$$

$\vec{u} = 0$ in all directions on the walls, but $\vec{\omega}$ is not! We know along a wall that $\vec{u}[x, \pm h] = 0$, meaning

$$\left.\frac{\partial \vec{u}}{\partial x}\right|_{y=\pm h} = 0.$$

then

$$\left.\vec{\omega}\right|_{y=\pm h} = \left.\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right|_{x=\pm h} = \left.\frac{\partial u_x}{\partial y}\right|_{x=\pm h}$$

There are many ways of

rewriting this (e.g. in terms of pressure) but any discussion will be fine if it is relevant.

(2) No, we can't really use the methods from before. There are many ways to discuss this.

We could write $\vec{u} = \vec{\nabla} \times \vec{A}$ with $\vec{A} = (0, 0, \psi)$, so $\vec{\omega} = \vec{\nabla} \times \vec{\nabla} \times \vec{u} =$

$= -\nabla^2 \psi \, \hat{z}$ then

$$\frac{\partial}{\partial t}(\nabla^2 \psi) + \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \nabla^2 \psi = \frac{1}{Re} \nabla^4 \psi .$$

This mixes the space & time derivs, so we would have to devise a new FTCS method.

However, we can work out 4 B.C.'s on ψ, and solve directly.

The answer "no" is also acceptable if the eqn is left in terms of vorticity & velocity. As written,

$\dot{\omega} + (\vec{u} \cdot \vec{\nabla}) \omega = \frac{1}{Re} \nabla^2 \omega$ depends on the unknown flow field u.

we could guess $\vec{u}$, then solve

$\dot{\omega} + (\vec{u}_{guess} \cdot \vec{\nabla}) \omega = \frac{1}{Re} \nabla^2 \omega$,

but we need to be able to update our guess $\vec{u}_{guess}$.

we can do that by recognizing

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} \times \vec{\nabla} \times \vec{u} = \nabla(\vec{\nabla} \cdot \vec{u}) - \nabla^2 \vec{u}$$

$$= -\nabla^2 u.$$

we can thus solve poisson's eqn,

$$\nabla^2 \vec{u} = -\vec{\nabla} \times \omega$$

to update $\vec{u}$.

Any reasonable discussion along these lines will recieve full credit, as long as the PDEs are complete.

2(a)

$$A = \int dr' dr'' \frac{n(r')n(r'')}{|r'-r''|}$$

$$\frac{\delta A}{\delta n} = \lim_{\epsilon \to 0} \left[\int dr' dr'' \frac{[n(r') + \epsilon\delta(r-r')][n(r'') + \epsilon\delta(r-r'')]}{|r'-r''|} - \frac{n(r')n(r'')}{|r'-r''|} \right] / \epsilon$$

$$= \lim_{\epsilon \to 0} \int dr' dr'' \left(n(r') \frac{\delta(r-r'')}{|r'-r''|} + \frac{n(r'')\delta(r-r')}{|r'-r''|} \right) + O(\epsilon)$$

$$= \int dr' \frac{n(r')}{|r'-r|} + \int dr'' \frac{n(r'')}{|r-r''|}$$

$$= 2 \int dr' \frac{n(r')}{|r-r'|}$$

$$A = \int dr' \, |\nabla n(r')|^2$$

$$\frac{\delta A}{\delta n} = \lim_{\epsilon \to 0} \left[\int dr' \left| \vec{\nabla}\left(n(r') + \epsilon\, \delta(r-r')\right)\right|^2 - \int dr' \, |\vec{\nabla} n(r')|^2 \right] \Big/ \epsilon$$

$$= 2 \int dr' \, \vec{\nabla} n(r') \cdot \vec{\nabla} \delta(r-r') \qquad + O(\epsilon^2)$$

$$= 2\, \delta(r-r') \nabla n(r') \Big|_{r' \in \text{bdry}} - 2 \int dr' \, \nabla^2 n(r') \, \delta(r-r')$$

$$= -2 \nabla^2 n(r)$$ (since $\delta(r-r') = 0$ for r in the volume).