(1a)
We know $r = \sqrt{x^2+y^2}$, so

$$\frac{\partial r}{\partial t} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial x}{\partial t} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial y}{\partial t}$$

$$= \frac{1}{r}\Big[x\big(ax + y - x(x^2+y^2)\big)$$
$$+ y\big(-x + ay^2 - y(x^2+y^2)\big)\Big]$$

$$= \frac{1}{r}\Big(ax^2 + ay^2 - (x^2+y^2)^2\Big]$$

$$= r(a - r^2).$$

This has fixed pts

$$r = 0, \quad r = \pm\sqrt{a}$$

If $a<0$, the only real root is at $r=0$. This a stable fixed pt, since

$$\dot{r} = r(a-r^2)$$
$$\approx ra = -|a|r$$

(expon. decaying for $a<0$).

If $a>0$, $\dot{r} \approx +|a|r$ near 0, so unstable. near $r=\sqrt{a}$,

$r = \sqrt{a}+\epsilon$, so

$$\dot{\epsilon} \approx (\epsilon+\sqrt{a})(a-(\epsilon+\sqrt{a})^2)$$
$$= (\epsilon+\sqrt{a})(-2\epsilon\sqrt{a}-\epsilon^2)$$
$$\approx -2a\epsilon + O(\epsilon^2)$$

which decays exp. for $a>0$.

(2a) Say $(\underset{\sim}{V})_{ij} = (\vec{v}_j)_i$ with

$\underline{M}\vec{v}_j = \lambda_j \vec{v}_j$.

If $\underset{\sim}{M} = \underline{V}\,\underline{L}\,\underset{\sim}{V}^{-1}$, then

$\underset{\sim}{M}\underline{V} = \underline{V}\,\underline{L}$.

$$(\underset{\sim}{M}\underset{\sim}{V})_{ij} = \sum_k M_{ik} V_{kj}$$

$$= \sum_k M_{ik} (v_j)_k$$

$$= (\underset{\sim}{M}\vec{v}_j)_i = \lambda_j (\vec{v}_j)_i$$

$$(\underline{V}\,\underline{L})_{ij} = \sum_k V_{ik} L_{kj}$$

$$= \sum_k V_{ik} \lambda_j \delta_{jk}$$

$= v_{ij}\lambda_j = \lambda_j (\vec{v}_j)_i$.

Thus equal.

(b) $\underline{M}^n = \underline{M}\cdot\underline{M}\cdot \cdots \cdot\underline{M}$ (n times)

$= \underline{V}\,\underline{L}\,\underline{V}^{-1}\,\underline{V}\,\underline{L}\,\underline{V}^{-1}\cdots$ (n times)

$= \underline{V}\,\underbrace{\underline{L}\,\underline{L}\,\underline{L}\cdots}_{n \text{ times}}\,\underline{V}^{-1}$

$= \underline{V}\,\underline{L}^n\,\underline{V}^{-1}$

this is same evecs as for $\underline{M}$.

elts here are λ_j^n on diagonal, so are evals of $\underline{M}^n$.

For any $\vec{b}$,

$$(M^n b)_i = \sum_j M_{ij} b_j$$

$$= \sum_{jkl} V_{ij} L^n_{jk} (V^{-1})_{kl} b_l$$

$$= \sum_{jkl} v_{ij} \lambda_j^n \delta_{jk} (v^{-1})_{kl} b_l$$

$$= \sum_{jk} v_{ij} \lambda_j^n (v^{-1})_{jk} b_k$$

$$= \sum_j (v_j)_i \lambda_j^n \sum_k (v^{-1})_{jk} b_k$$

Then

$$\underset{\sim}{M}^n \vec{b} = \lambda_1^n \vec{v}_1 \cdot \sum_k (v^{-1})_{1k} b_k$$

$$+\lambda_2^n \vec{v}_2 \cdot \sum_k (V^{-1})_{2k} b_k$$

$$+ \cdots$$

Then:

$$c_i = (V^{-1} \vec{b})_i$$

Note that $\vec{v}_i$ & c_i are indep of n, so

$$\underline{M}^n \vec{b} = \lambda_{max}^n \Big(\overset{c_{max}}{\vec{v}_{max}} + c_{second} \frac{\lambda_{second}^n}{\lambda_{max}^n} \vec{v}_{second}$$

$$+ c_{third} \frac{\lambda_{third}^n}{\lambda_{max}^n} \vec{v}_{third}$$

$$+ \cdots \Big)$$

So

$$\lim_{n \to \infty} M^n \vec{b} = c_{max} \lambda_{max}^n \vec{v}_{max}.$$

Note: a slight oversight in the problem is that we will converge on the evec largest in <u>abs. value</u>, not just largest. For our example, all evals are > 0.