1. False. To accurately model the dynamics of a system, a small Δt should be used.

However, (1) we may instead care solely abt limiting points ($x(t=\infty)$ instead of $x(t)$ for finite t.)

also (2), if we choose Δt too small, rounding errors may occur, and computational time will be very large. Usually, We want a balance btwn accuracy (small enough Δt) and speed (small enough $T/\Delta t$).

2. Non adaptive: Δt is fixed for all iterations. This is not specific to any particular algorithm, verlet & rk4 are both fixed Δt.

Adaptive: Δt is changed at each timestep. There are many ways this can be accomplished, Garcia's alg. is not mandatory. - Generally, a larger Δt is used if a 1-step process gives the same result (within a tolerance) to a 2-step process with smaller Δt.

Uses: Adaptive is valuable for stiff eq's, where there is a mix of fast & slow dynamics. A nonadaptive alg req's a very small Δt to capture fast dynamics, and a very long sim to capture slow dynamics.

Disadv.: Adaptive methods require additional functional evaluations, which may be prohibitive.

3. See code. For (b), the power spectrum decreases for increasing f, meaning low freq. can pass through.

(4) (a) If $\underline{B} = \underline{N} - \underline{A}$, we are computing $\sum_k \underline{B}^k / k$.

We showed in the limit as $k \to \infty$ that $\underline{B}^k$ will be

$\underline{B}^k = \underline{V}\,\underline{\Lambda}^k\,\underline{V}^{-1} \sim \lambda^k_{max}$. This will diverge if B has an eigenvector $|\lambda_{max}| \geq 1$.

If A has evals μ_i, then $\underline{1} - A$ has evals $1-\mu_i$.

For $|\lambda_{max}| < 1$, that means $-1 < 1-\mu_i \leq 1$, or

$0 \leq \mu_i < 2$.

(b) The answer depends on the method used. If we perform a finite sum with large k, the calc. is multiple matrix mults, scaling as $n^3 \cdot k_{max}$

Or we can diagonalize the matrix, scaling as $N^3 + N K_{max}$.

computing $\log \lambda_i$

Or we can use a variable tolerance in the error, where

K_{max} depends on N (possibly).

Then $N^3 \cdot K_{max}(N)$

some unknown scaling.

(c-d) see code

5. (a) $m \frac{\partial^2 R_i}{\partial t^2} = G m^2 \frac{\partial}{\partial R_i} \left(\frac{1}{|R_1 - R_2|} + \frac{1}{|R_1 - R_3|} + \frac{1}{|R_2 - R_3|} \right)$

If we set $t' = t\sqrt{GM/R^3}$ and

$r_i = R_i/R$, for arb. R,

$$\frac{\partial^2 r_i}{\partial (t')^2} = \frac{\partial}{\partial r_i}\left(\frac{1}{|r_1 - r_2|} + \frac{1}{|r_2 - r_3|} + \frac{1}{|r_1 - r_3|}\right)$$

$G = 6.67 \times 10^{-11}\ m^3/kg\,s^2$, $M = 2 \times 10^{30}$ kg

If $R = 1$ AU (a natural choice for a star), $R = 1.5 \times 10^{8}$ km

To nondim 1 km/s, write

$$v = 1\,km/s \cdot \frac{1}{R\sqrt{GM/R^3}}$$

$$= \left(\frac{1.5 \times 10^{8}}{6.67 \times 10^{-11} \cdot 2 \times 10^{30}}\right)^{1/2}$$

$$= 1 \times 10^{-6} \quad \text{(unitless)}$$

(b) The fact that $r_1 = 0$ is fixed is easily seen if

$$\dot{v}_1 = -\frac{r_1 - r_2}{|r_1 - r_2|^3} - \frac{r_1 - r_3}{|r_1 - r_3|^3} \qquad \begin{pmatrix} r_1 = 0, \\ r_2 = r \\ r_3 = -r \end{pmatrix}$$

$$= -\frac{0 - r}{|r|^3} - \frac{0 - (-r)}{|r|^3} = 0$$

and

$\dot{r}_1 = v_1 = 0$. The particle won't move if $r_3 = -r_2 = r$.

If we begin with $r_2 = -r_3$ and $v_2 = -v_3$, we will have

$$\underbrace{\dot{r}_2 = +v_2 \qquad \dot{r}_3 = +v_3}$$

equiv to $\dot{r} = +v$

and

$$\dot{v}_2 = -\frac{r_2}{|r_2|^3} - \frac{r_2 - r_3}{|r_2 - r_3|^3}$$

$$= -\frac{r}{|r|^3} - \frac{2r}{8|r|^3} = -\frac{5r}{4|r|^3}$$

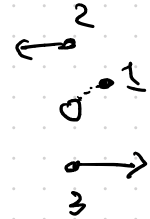
and likewise

$$\dot{v}_3 = +\frac{5r}{4|r|^3}.$$

A solution for $r_2 = -r_3$, $v_2 = -v_3$ thus exists, so the first star will be stationary at the origin.

For this to be stable, we need the star to return to the origin when $|r_1| = \vec{\epsilon} = \epsilon$

$$\dot{v}_1 = -\frac{\vec{\epsilon} - \vec{r}_2}{|\vec{\epsilon} - \vec{r}_2|^3} - \frac{\vec{\epsilon} - \vec{r}_3}{|\vec{\epsilon} - \vec{r}_3|^3}$$



$$= \vec{r}\left(\frac{1}{|\vec{\epsilon} - \vec{r}|^3} - \frac{1}{|\vec{\epsilon} + \vec{r}|^3}\right)$$

$$-\vec{\epsilon}\left(\frac{1}{|\epsilon + r|^3} + \frac{1}{|\epsilon - r|^3}\right)$$

$$\approx r\left(\frac{1}{|r|^3} + \frac{3\vec{\epsilon}\cdot\vec{r}}{|r|^5} + \dots\right.$$

$$\left. - \frac{1}{|r|^3} + \frac{3\vec{\epsilon}\cdot\vec{r}}{|r|^5} + \dots\right)$$

$$-\cancel{4}\,\vec{\epsilon}\left(\frac{1}{|r|^3} + \frac{1}{|r|^3} + \dots\right)$$

$$= \frac{2}{|r|^3}\left(3\hat{r}\cdot\vec{\epsilon}\ \hat{r} - \vec{\epsilon}\right).$$

This can be >0 or <0, so sometimes the particle will move towards the origin, and sometimes away. However:

$$\dot{v}_2 = -\frac{r_2 - \vec{\epsilon}}{|r_2 - \epsilon|^3} - \frac{r_2 - r_3}{|r_2 - r_3|^3}$$

$$\approx -\frac{5\vec{r}}{4|r|^3} + \frac{\vec{\epsilon}}{|r|^3} - \frac{3\hat{r}\cdot\vec{\epsilon}\,\hat{r}}{|r|^3}$$

$$\dot{v}_3 \approx +\frac{5\vec{r}}{4|r|^3} + \frac{\vec{\epsilon}}{|r|^3} + \frac{3\hat{r}\cdot\vec{\epsilon}\,\hat{r}}{|r|^3}$$

breaks the symmetry of r_3, r_2; so that it will no longer be true that $r_2 = -r_3$. Thus, $r_1 = 0$ is unstable!

RK4 code

$$\dot{X} = F(X)$$

$$\tilde{X} = \begin{pmatrix} x_2 \\ v_{x2} \\ y_2 \\ v_{y2} \\ \\ x_1 \\ v_{x1} \\ y_1 \\ v_{y1} \\ \\ x_3 \\ v_{x3} \\ y_3 \\ v_{y3} \end{pmatrix}$$

$$\dot{x}_i = v_{xi}$$

$$\dot{y}_i = v_{yi}$$

$$\log(\underline{A}) = V \log(\underline{\Lambda}) V^{-1}$$

$$= V \begin{pmatrix} \log \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \log \lambda_N \end{pmatrix} V^{-1}$$

$$\log(\lambda_i) = -\sum_{i=1}^{\infty} \frac{(1-\lambda_i)^k}{k}$$

$$\approx -\sum_{i=1}^{k_{max}} \frac{(1-\lambda_i)^k}{k} + \text{tol}$$