1. a $\qquad \dot{\rho} = D\rho'' + C\rho$, with BC's

$$\dot{\rho}(-\tfrac{L}{2}, 0) = \dot{\rho}(+\tfrac{L}{2}, 0) = 0$$

reflecting

$\rho = T(t)\, X(x)$ means

$$\frac{T'}{DT} = -\kappa = \frac{X''}{X} + \frac{C}{D}$$

sep. constant

$$T_\kappa(t) = e^{-\kappa D t}$$

$$X''_\kappa = -\left(\kappa + \frac{C}{D}\right) X_\kappa \quad \text{with } \dot{X}(L/2, 0) = \dot{X}(-L/2, 0) = 0.$$

$X_\kappa = \cos\left(\frac{n\pi x}{L}\right)$ to satisfy BC's,

with $n = 0 \cdots \infty$

Then $\kappa + \frac{c}{D} = \frac{n^2\pi^2}{L^2}$.

If $n=0$, $\kappa = -\frac{c}{D} < 0$. Then

$T_{n=0}(t) = e^{+ct}$, which diverges

indep. of L.

(b) Our explicit method has

$$\rho_j^{n+1} = \rho_j^n + c\Delta t\, \rho_j^n + \frac{D\Delta t}{\Delta x^2}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n\right)$$

Our implicit method will have

$$\rho_j^{n+1} = \rho_j^n + \frac{c\Delta t}{2}\left(\rho_j^n + \rho_j^{n+1}\right)$$
$$+ \frac{D\Delta t}{2\Delta x^2}\left(\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n + \rho_{j+1}^{n+1} - 2\rho_j^{n+1} + \rho_{j-1}^{n+1}\right)$$

or

$$\left(\underline{\mathbb{1}} - \underline{M}/2\right)\vec{\rho}^{\,n+1} = \left(\underline{\mathbb{1}} + \underline{M}/2\right)\vec{\rho}^{\,n}$$

with

$$\underline{M} = \begin{cases} -\dfrac{2D\Delta t}{\Delta x^2} + C\Delta t & i = j \\[2ex] \dfrac{D\Delta t}{\Delta x^2} & |i-j| = 1 \\[2ex] 0 & \text{else.} \end{cases}$$

for $i = 0 \cdots N+1$.

To impose the B.C.'s, we also include two virtual grid points, -1 and N, and we set $\rho_{-1}^{n+1} = \rho_1^{n+1}$, $\rho_N^{n+1} = \rho_{N-2}^{n+1}$ after the update. This was easily done for FTCS, but is a bit more tedious for

CN: we need to account for the boundary term in the matrix (else we do not properly mix present & future).

$$\rho_0^{n+1} = \left(\rho_1^n - 2\rho_0^n + \rho_1^n\right) \cdot \frac{D\Delta t}{\Delta x^2}$$

$M_{01} = \underline{2 D\Delta t/\Delta x^2}$ ← twice the interaction, due to the lattice site

$M_{12} = D\Delta t/\Delta x^2$

$M_{23} = D\Delta t/\Delta x^2$

$\vdots$

and

$M_{N-2\,N-1} = 2 D\Delta t/\Delta x^2$

$M_{N-3\,N-2} = D\Delta t/\Delta x^2$

$\vdots$

2(b) Relaxation method has

$$\Phi_{ij}^{n+1} = \frac{1}{4}\left(\Phi_{i+1j}^{n} + \Phi_{i-1j}^{n} + \Phi_{ij+1}^{n} + \Phi_{ij-1}^{n}\right)$$

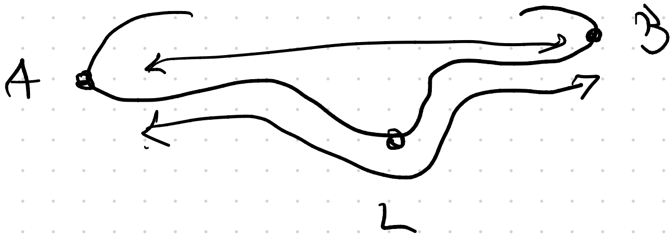
$$+ \Delta t\, \rho/\epsilon_0$$

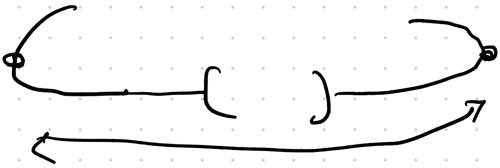
$$\rho = \begin{cases} \dfrac{Q}{\Delta x^2} & \text{at } x = x_0 \\ 0 & \text{else} \end{cases}$$

$$\Phi_{j}^{n+1} = \Phi_{j}^{n} + \Delta t\, D\left[\frac{\Phi_{ij+1}^{n} - \Phi_{ij}^{n} + \Phi_{ij-1}^{n} + \cdots}{\Delta x^2} + \rho/\epsilon_0\right]$$

$$\Phi^{n+1}_{ij} = \Phi^{n+1}_{ij}\Big|_{Q=0} + \begin{cases} Q/\varepsilon_0 & x = x_0 \\ 0 & \text{else} \end{cases}$$

$x_{charge} \simeq 0$ $\qquad y_{charge} = \frac{1}{2}$

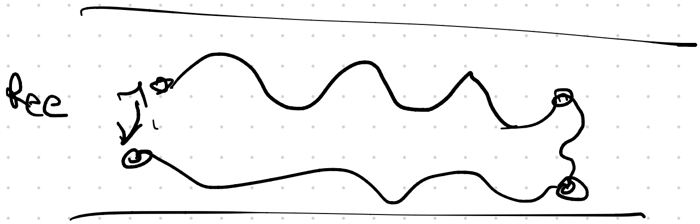


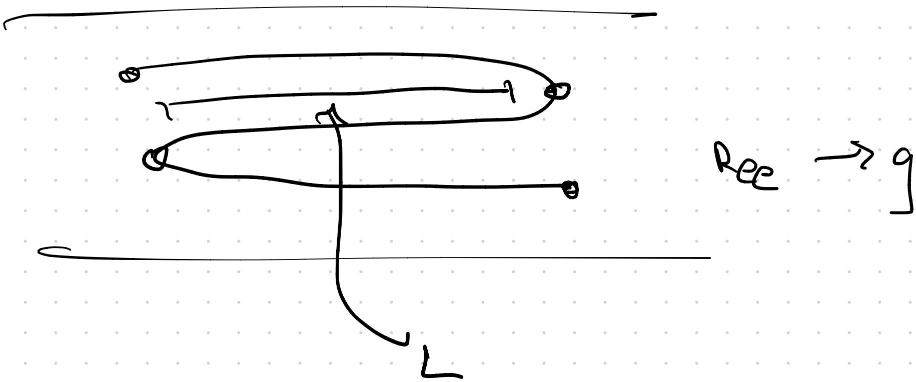


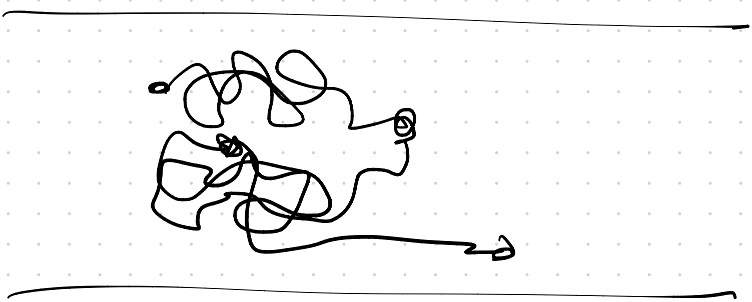
$$D_{AB}^{\text{with Tr.}} = L$$

$$D_{AB}^{\text{without Tr}} = 2 - L_z < L$$

$$\langle R_{ee}^2 \rangle \approx (\dots)\, L\, \ell_p \left(1 - \frac{\ell_p}{L}\left(e^{-L/\ell_p} - 1\right)\right)$$


Ree


Ree → g
L



$$\langle R_x^2 \rangle = \frac{1}{2}$$