The Property FW for the wreath products

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1 Introduction

The property FW was introduced by de Cornullier. It is a fixed point property for the action on wall spaces (for a detailed treatment of this property see [2]). For discrete groups, this property is implied by the Kazhdan property (T). The behavior of the Kazhdan Property (T) with the wreath product is well known:

Theorem 1.1 ([1, 3])**.** *Let G, H be two discrete groups and X a set on which H acts. The wreath product $G \wr_X H$ has the property (T) if and only if G and H have the property (T) and if X is finite.*

The same kind of result is true for the property FW

Theorem 1.2. *Let G, H be two discrete groups and X a set on which H acts transitively. The wreath product $G \wr_X H$ does not have the property FW if at least one of the following conditions is satisfied:*

1. *The group G does not have the property FW.*
2. *The group H does not have the property FW.*
3. *The set X is infinite.*

An action of a group on a CAT(0) cube complex is *essential* if all the orbits of vertices are unbounded and the action is transitive on the set of hyperplanes.

Corollary 1.3. *Let G, H be two discrete groups and X a set on which H acts transitively. If there exists an essential action of G or H on a CAT(0) cube complex or if X is infinite, then there exists an essential action of $G \wr_X H$ on a CAT(0) cube complex.*

2 Definitions

2.1 The Property FW

For the definition of the property FW, we will follow the survey of Y. de Cornullier [2].

Definition 2.1. Let G be a discrete group and X a discrete set on which G acts. A subset $M \subset X$ is *commensurated* by the G-action if

$$|gM \Delta M| < \infty$$

for all g in G.

An invariant G-subset is automatically commensurated. Moreover, for a subset M such that there exists an invariant G-subset N with $|M \Delta N| < \infty$ then M is commensurated. Such a set is called *transfixed*.

Definition 2.2. A group G has the property FW if all commensurable G-set are transfixed.

There are lot of equivalent characterizations of this property. We will give us without all the details and the precise definitions.

Proposition 2.3. *The following are equivalent:*

1. *G has the property FW;*
2. *every cardinal definite function on G is bounded;*
3. *every cellular action on a CAT(0) cube complex has bounded orbits for the ℓ^1-metric (the complexes can be infinite dimensional);*
4. *every cellular action on a CAT(0) cube complex has a fixed point;*
5. *every action on a connected median graph has bounded orbits;*
6. *every action on a nonempty connected median graph has a fixes point;*
7. *(if G is finitely generated) every Schreier graph of G has at most 1 end;*
8. *For every set Y endowed with a walling structure and compatible action on Y and on the index of the walling, the action on Y has bounded orbits for the wall distance;*
9. *every isometric action on an "integral Hilbert space" $\ell^2(X, \mathbf{Z})$ (X any discrete set), or equivalently on $\ell^2(X, \mathbf{R})$ preserving the integral points, has bounded orbits;*
10. *for every G-set X we have $H^1(G, \mathbf{Z}X) = 0$.*

Note that the name FW comes from the property of "fixed point" for the actions on the walling spaces. We will see in the following that a semi-splittable group does not have the property FW (see corollary **??**).

The property FW has links with other well known properties. For example, the property FH implies the characterisation 9. For discrete groups (and even for countable groups) the property FH is equivalent to the Kazhdan's property (T) by Delorme-Guichardet's Theorem. As trees are CAT(0) cube complexes, the property FW implies Serre's property FA.

2.2 Graphe de Schreier

Expliquer lien action de groupe, graphe de Schreier

2.3 Bouts

3 Proof of the Theorem

Proof. Let us begin by fixing the notation. We denote by S (and respectively by S') a finite generating set of G (and respectively of H). We choose an arbitrary point of x_0 of X. The group $\Gamma = G \wr_X H$ is generated by the set

$$\mathcal{S} = \{(\delta_{x_0}^s, e_h) : s \in S\} \cup \{(0, s') : s' \in S'\}$$

where

$$\delta_{x_0}^s = \begin{cases} e_g & x \neq x_0 \\ s & x = x_0 \end{cases} \quad \text{and} \quad 0(x) = e_g \quad \forall x \in X.$$

The idea of the proof is to construct, for each of the three cases, a Schreier graph of Γ with more than one end. To do this we will consider actions of Γ and associated graph's action. We will treat the 3 cases separately.

Suppose that X is an infinite set. We define $Y = G \times X$ and an action of Γ on Y as

$$(\varphi, h) \cdot (g, x) = (\varphi(hx)g, hx)$$

for (φ, h) in Γ and (g, x) in Y. This action is transitive. Indeed, let (g_1, x_1) and (g_2, x_2) be two elements of Y. By transitivity of the action of H on Y, there exists h in H such that $hx_1 = x_2$. We can always find φ in $\bigoplus_X G$ such that $\varphi(hx_1) = g_2 g_1^{-1}$. Then,

$$(\varphi, h)(g_1, x_1) = (\varphi(hx_1)g, hx_1) = (g_2, x_2).$$

The graph of the action of Γ on Y is isomorphic to the Schreier graph $\mathrm{Sch}(\Gamma, \mathrm{Stab}(e_G, x_0), \mathcal{S})$. We decompose the graph into leaves of the form $Y_g = \{g\} \times G$. There are two types of edges on this graph which are coming from the two types of generators. The first one, of the form $(0, s')$, give us on each leaf a copy of the graph of the action of H on X. Indeed, we have

$$(0, s')(g, x) = (g, s'x).$$

Tthe second one, of the form $(\delta_{x_0}^s, 0)$, give us loops everywhere excepting on vertices with x_0 as second coordinate. By direct computation, we see that a leaf Y_g is connected to Y_{sg} by the vertices (g, x_0) and (sg, x_0):

$$(\delta_{x_0}^s, 0)(g, x) = \begin{cases} (g, x) & x \neq x_0 \\ (sg, x) & x = x_0 \end{cases}$$

Structure du Schreier et des feuilles

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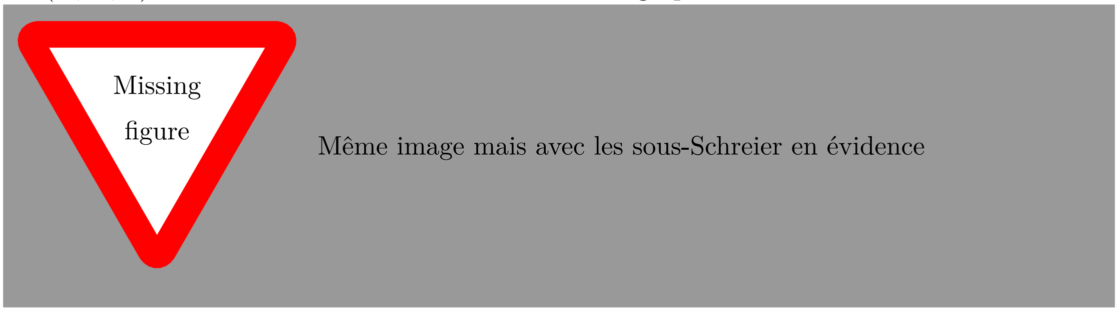
If we remove a vertex (g, x_0) we disconnect the leaf Y_g to the graph. As X is infinite, the number of ends is strictly greater than 1.

Suppose that G does not have the property FW. By the point 7 of the Proposition 2.3 there exists a subgroup K of G such that $\mathrm{Sch}(G, K, S)$ has more than one end. The group Γ acts on $G/K \times X$:

$$(\varphi, h)(gK, x) = (\varphi(hx)gK, hx).$$

As above, the action is transitive and the graph of this action is isomorphic to a Schreier graph. We decompose this graph into leaves in the same way. Now we look at the subgraph made up of vertices (g, x_0) and edges $(\delta^s_{x_0}, 0)$ and we remark that it is isomorphic the Schreier graph $\mathrm{Sch}(G, K, S)$ which has more than one end. Then our graph has also more than one end.



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Suppose that H does not have the property FW. There is a subgroup K of H such that $\mathrm{Sch}(H, K, S')$ has more than one end. We consider the action of Γ on H/K defined as

$$(\varphi, h)h'K = hh'K.$$

This is a transitive action. All the edges of type $(\delta^s_{x_0}, 0)$ are loops and the edges $(0, s')$ are the edges of $\mathrm{Sch}(H, K, S')$ which has more than one end. Then this Schreier graph has also more than one end. $\square$

4 Median graphs

In this section, we will investigate the FW property via the action of groups on median graphs.

For u and v two vertices of a connected graph $\mathcal{G}$, we define the total interval $[u, v]$ as the set of vertices that lies on some shortest path between u and v. A connected graph $\mathcal{G}$ is *median* if for triple of distinct vertices u, v, w, the intersection $[u, v] \cap [v, w] \cap [u, w]$ consist of a unique vertex, denoted $m(u, v, w)$. A graph is median if each of its connected components is median.

A simple verification shows that if $\mathcal{G}$ and $\mathcal{H}$ are both median graph, then their cartesian product is also median.

Example 4.1. Let X be a subset and $\mathcal{P}_f(X)$ the set of all its finite subsets. Define a graph structure on $\mathcal{P}_f(X)$ by putting an edge between E and F if and only if $\#(E\Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $E\Delta F$, while $[E, F]$ consist of all subsets of X that both contain $E \cap F$ and are contained in $E \cup F$. In particular $\mathcal{P}_f(X)$ is a connected median, with $m(D, E; F)$ being the set of all elements belonging to at least two of D, E and F. In other words, $m(D, E; F) = (D \cap E) \cup (D \cap F) \cup (E \cap F)$.

The same construction endows $\mathcal{P}(X)$ the set of all subsets of X with a structure of median graph, where the connected component of E consists of all F such that $E\Delta F$ is finite.

Any action of a group G on a set X naturally extend to an action of G on $\mathcal{P}_f(X)$ by graph homomorphisms: $g.\{x_1, \ldots, x_n\} = \{g.x_1, \ldots, g.x_n\}$.

Building on this example, we obtain that no infinite sum of groups has the property FW.

Lemma 4.2. *An infinite sum of non-trivial groups $G = \bigoplus_{i=1}^{\infty} G_i$ does not have FW.*

Proof. Let $X = \bigsqcup_{i=1}^{\infty} G_i$. There is a natural action of G on X: G_i acts by left multiplication on G_i and trivially on G_j for $j \neq i$. Therefore, we have an action of G on the connected median graph $\mathcal{P}_f(X)$. For every i, choose a non-trivial $g_i \in G_i$. Then the orbit of the vertex $\{1_{G_1}, \dots, 1_{G_n}\}$ contains the point $\{g_1, \dots, g_n\}$ which is at distance $2n$ of $\{1_{G_1}, \dots, 1_{G_n}\}$. That is the action of G on $\mathcal{P}_f(X)$ has unbounded orbits. $\square$

We also have the following easy result on semi-direct products:

Lemma 4.3. *Let $G = N \rtimes H$ be a semidirect product. Then*

1. *If G has FW, then so does H.*
2. *If both N and H have FW, then G also has FW.*

Proof. Suppose that G has FW and let X be a non-empty connected median graph on which H acts. Then G acts on X by $g.x \coloneqq h.x$ where $g = nh$ with $n \in N$ and $h \in H$. By assumption, the action of G on X has bounded orbits and so does the action of H.

On the other hand, suppose that G does not have FW and let X be a non-empty connected median graph on which G acts. Then both N and H acts on X, with orbits bounded respectively by d_N and d_H. Now, for every $x \in X$ and $g \in G$, there is $n \in N$ and $h \in H$ such that $g = nh$ ans thus $g.x = n.(h.x)$ is at distance at most $d_N + d_H$ from x. $\square$

Multiple applications of the above Lemma give us

Corollary 4.4. *Let $G \wr_X H$ be the wreath product of G and $H \curvearrowright X$. Then*

1. *If $G \wr_X H$ has FW, then so does H.*
2. *If G and H have FW and H acts on a finite set X, then $G \wr_X H$ has FW.*

It also follow from Lemmas 4.2 and 4.3 that

Corollary 4.5. *The group $\otimes_X G$ has FW if and only if X is finite and G has FW.*

We finally characterize which wreath products do have FW.

Proposition 4.6. *Let G be a non-trivial group. Then the group $G \wr_X H$ has FW if and only if both G and H have FW and X is finite.*

Proof. In view of Corollary 4.4 it remains to show that if $G \wr_X H$ has FW, then G has FW and X is finite.

First, suppose that G does not have FW and let Y be a connected median graph on which G acts with unbounded orbits. Then $G \wr_X H$ acts on the connected median graph $Y \times \mathcal{P}_f(X)$ by

$$(\varphi, h).(y, E) = \begin{cases} (\varphi(h.t).y, h.\{t\}) & \text{if } E = \{t\} \\ (y, h.E) & \text{if } E \text{ is not a singleton} \end{cases}$$

But then the orbit of $(y, \{x\})$ is unbounded for every $x \in X$, which implies that $G \wr_X H$ does not have FW. Indeed, it contains

$$\{(\delta_x^g, 1).(y, \{x\}) = (g.y, \{x\}) \mid g \in G\},$$

where $\delta_x^g(x) = g$ and $\delta_x^g(z) = 1$ if $z \neq x$. And since all the $\{g.y, \{x\}\}_{g \in G}$ are in the same slice, the distance between them in the same as the distance between the $\{g.y\}_{g \in G}$.

Finally, suppose that X is infinite and let g be a non-trivial element of G. Then G acts naturally on the connected median graph $P_f(\bigsqcup_X G)$ and therefore $G \wr_X H$ acts on $\mathcal{P}_f(\bigsqcup_X G) \times \mathcal{P}_f(X)$. Let v be the the vertex $(\{1_1, \dots, 1_n\}, \emptyset)$, where $\{1_1, \dots, 1_n\}$ consists of n times the element $1_G \in G$ but living in distinct copies of G. The orbit of v contains the point $(\{g_1, \dots, g_n\}, \emptyset)$ which is at distance $2n$ of v. That is the action of $G \wr_X H$ on $\mathcal{P}_f(\bigsqcup_X G) \times \mathcal{P}_f(X)$ has unbounded orbits and $G \wr_X H$ has not FW. □

We conclude this section by the following important remark that some of the above result applies in a more generality.

Remark 4.7. Let Q be a property of metric spaces (for example *be a connected median graph*) and P be the group property: *Every G-action on a space with Q has bounded orbits.* Then both Lemma 4.3 and Corollary 4.4 are true for groups with P.

Example of such properties P are: property FW, property FA (every action on a tree has bounded orbits) or property FH (every isometric action on a real Hilbert space has bounded orbits) which for countable group is equivalent to Kazdhan's property (T).

Dire qqch (même juste références) sur l'équivalent de la proposition 4.6 pour FA et FH

Regarder le cas PW, Haagerup,... i.e. action propre. Rappel: une action isométrique de G est propre si pour tout x (de manière équivalente il existe x), pour tout $r \in \mathbf{R}$ l'ensemble $\{g \in G \mid d(x, g.x)\}$ est fini.

References

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