or $(f(x), f(y))$ is an edge. There are two natural ways to see **Graph** as a subcategory of **Met**. The first one consists to look at the vertex set V endowed with the shortest-path metric: $d(v, w)$ is the minimal number of edges on a path between v and w. The second one, consists at looking at the so called geometric realization of (V, E), where each edge is seen as an isometric copy of the segment [0, 1]. Similarly to what happens for cube complexes (see the discussion after Definition 2.4), the geometric realization of a graph gives an embedding **Graph** $\hookrightarrow$ **Met** which is not full. Nevertheless, for our purpose, the particular choice of one of the above two embeddings **Graph** $\hookrightarrow$ **Met** will make no difference.

We can now formally define the group property B**S**.

Definition 2.7. Let **S** be a subcategory of **PMet**. A group G has *property* B**S** if every G-action by **S**-automorphisms on an **S**-space has all its orbits bounded. A pair (G, H) of a group and a subgroup has *relative property* B**S** if for every G-action by **S**-automorphisms on an **S**-space, the H-orbits are bounded.

Observe that a group G has property B**S** if and only if any G-action on an **S**-space has at least one bounded orbit.

All the properties of Definition 2.4 are of the form B**S**. Another example of property of the form B**S** can be found in [14, Definition 6.22]: a group has property (FHyp$_{\mathbf{C}}$) if any action on a real or complex hyperbolic space of finite dimension has bounded orbits. This property is implied by property FH, but does not imply property FA [14, Corollary 6.23 and Example 6.24]. One can also want to look at the category of all Banach spaces (the corresponding property BB hence stands between the Bergman's property and property FB$_{\mathrm{r}}$), or the category of L^p-spaces for p fixed [4] (if $p \notin \{1, \infty\}$, then BL^p is implied by FB$_{\mathrm{r}}$).

Another interesting example of a property of the form B**S** is the fact to have no quotient isomorphic to **Z**; see Example 2.8. The main interest for us of this example is that property FA is the conjunction of three properties, two of them (uncountable cofinality and having no quotient isomorphic to **Z**) still being of the form B**S**.

Example 2.8. Let Z be the 2-regular tree, or in other words the Cayley graph of **Z** for the standard generating set. Then $\mathrm{Aut}_{\mathbf{Graph}}(Z) = \mathbf{Z} \rtimes (\mathbf{Z}/2\mathbf{Z})$ is the infinite dihedral group and its subgroup of orientation preserving isometries is isomorphic to **Z**. Let **S** be the category with one object Z and with morphisms the orientation preserving isometries. Hence, we obtain that a group G has no quotient isomorphic to **Z** if and only if every G-action on **S**-space has bounded orbits. Let us denote by B**Z** this property.

Since Z is a tree, property FA implies property B**Z**. This implication is strict as demonstrated by **Q**. In fact, the counterexample **Q** shows that B**Z** does not imply

The orbits for this action are the

$$\mathcal{L}_n = \{v \mid d(v,r) = n\},$$

which have diameter $2n$. Finally, it is possible to put an ultradistance on the vertices of T by

$$d_\infty(x,y) := \max\{d(x,r), d(y,r)\}$$

if $x \neq y$. Then the orbits are still the $\mathcal{L}_n$, but this time with diameter n.

Topological groups. One can wonder what happens for topological groups. While, the wreath product of topological groups is not in general a topological group, this is the case if G is discrete and X is a discrete set endowed with a continuous H-action. In this particular context, Theorem 3.1, as well as its proof, remains true. The details are left to the interested reader.

Categorical generalizations. In the above, we defined property B**S** for **S** a subcategory of **PMet**. It is possible to generalize this definition to more general categories. We are not aware of any example of the existence of a group property arising in this general context that is not equivalent to a property B**S** in the sense of Definition 2.7, but still mention it fur the curious reader.

On one hand, we can replace **PMet** with a more general category. For example, one can look at the category **M** of sets X endowed with a map $d\colon X \times X \to \mathbf{R}_{\geq 0}$ satisfying the triangle inequality. That is, d is a pseudo-distance, except that it is not necessary symmetric and $d(x,x)$ may be greater than 0. All the statements and the proofs remain true for **S** a subcategory of **M**.

On the other hand, we can define property B**S** for any category **S** over **PMet**, that is for any category **S** endowed with a faithful functor $F\colon \mathbf{S} \to \mathbf{PMet}$. Such a couple $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{PMet})$ is sometimes called a *structure over* **PMet**, and F is said to be *forgetful*. In this context, we need to be careful to define Cartesian powers (Definitions 2.13 and 2.14) using F, but apart for that all the statements and all the proofs remain unchanged. An example of such an **S** that cannot be expressed as a subcategory of **PMet** is the category of edge-labeled graphs, where the morphisms are graph morphisms that induce a permutation on the set of labels. However, in this case the property B**S** is equivalent to the Bergman's property.

One can also combine the above two examples and look at couples $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{M})$, with F faithful.

Finally, in view of Definitions 2.7, 2.13 and 2.14, the reader might ask why we are working in **PMet** or **M** instead of **Born**, the category of bornological spaces together with bounded maps. The reason behind this is the forthcoming Lemma 3.3 and its corollaries, which fail for general bornological spaces. In fact, all the statements and

[2] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's property (T)*. New Math. Monogr. 11, Cambridge University Press, Cambridge, 2008. Zbl 1146.22009 MR 2415834

[3] G. M. Bergman, Generating infinite symmetric groups. *Bull. London Math. Soc.* **38** (2006), no. 3, 429–440. Zbl 1103.20003 MR 2239037

[4] M. Bourdon, Un théorème de point fixe sur les espaces L^p. *Publ. Mat.* **56** (2012), no. 2, 375–392. Zbl 1334.31006 MR 2978328

[5] N. Brown and E. Guentner, Uniform embeddings of bounded geometry spaces into reflexive Banach space. *Proc. Amer. Math. Soc.* **133** (2005), no. 7, 2045–2050. Zbl 1069.46003 MR 2137870

[6] I. Chatterji, C. Druţu and F. Haglund, Kazhdan and Haagerup properties from the median viewpoint. *Adv. Math.* **225** (2010), no. 2, 882–921. Zbl 1271.20053 MR 2671183

[7] I. Chatterji and G. Niblo, From wall spaces to $CAT(0)$ cube complexes. *Internat. J. Algebra Comput.* **15** (2005), no. 5-6, 875–885. Zbl 1107.20027 MR 2197811

[8] V. Chepoi, Graphs of some CAT(0) complexes. *Adv. in Appl. Math.* **24** (2000), no. 2, 125–179. Zbl 1019.57001 MR 1748966

[9] P.-A. Cherix, F. Martin and A. Valette, Spaces with measured walls, the Haagerup property and property (T). *Ergodic Theory Dynam. Systems* **24** (2004), no. 6, 1895–1908. Zbl 1068.43007 MR 2106770

2016

3) [10] Y. Cornulier, Group actions with commensurated subsets, wallings and cubings. ~~2013~~, arXiv:1302.5982~~v1~~. v2

2) [11] Y. Cornulier, Irreducible lattices, invariant means, and commensurating actions. *Math. Z.* **279** (2015), no. 1-2, 1–26. Zbl 1311.43001 MR 3299841

4) [12] Y. Cornulier and A. Kar, On property (FA) for wreath products. *J. Group Theory* **14** (2011), no. 1, 165–174. Zbl 1228.20027 MR 2764930

1) [13] Y. de Cornulier, Strongly bounded groups and infinite powers of finite groups. *Comm. Algebra* **34** (2006), no. 7, 2337–2345. Zbl 1125.20023 MR 2240370

relative order

[14] P. de la Harpe and A. Valette, La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger). Astérisque 175, Société Mathématique de France, Paris, 1989. Zbl 0759.22001 MR 1023471

[15] A. Genevois, Lamplighter groups, median spaces and Hilbertian geometry. *Proc. Edinb. Math. Soc. (2)* **65** (2022), no. 2, 500–529. Zbl 1514.20150 MR 4449680

[16] V. N. Gerasimov, Semi-splittings of groups and actions on cubings. In *Algebra, geometry, analysis and mathematical physics (Novosibirsk, 1996)*, pp. 91–109, Institut Matematiki im. S. L. Soboleva, Novosibirsk, 1997. Zbl 0906.20025 MR 1624115

[17] V. N. Gerasimov, Fixed-point-free actions on cubings. *Siberian Adv. Math.* **8** (1998), no. 3, 36–58. Zbl 0912.20028 MR 1663779

[18] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I*. Second edn., Grundlehren Math. Wiss. 115, Springer, Berlin-New York, 1979. Zbl 0416.43001 MR 0551496

[19] S. Klavžar and H. M. Mulder, Median graphs: Characterizations, location theory and related structures. *J. Combin. Math. Combin. Comput.* **30** (1999), 103–127. Zbl 0931.05072 MR 1705337

[20] P.-H. Leemann and G. Schneeberger, Property FW and wreath products of groups: A simple approach using Schreier graphs. *Expo. Math.* **40** (2022), no. 4, 1261–1270. Zbl 1525.20023 MR 4518652

[21] A. Minasyan, New examples of groups acting on real trees. *J. Topol.* **9** (2016), no. 1, 192–214. Zbl 1365.20021 MR 3465847

[22] L. Nebeský, Median graphs. *Comment. Math. Univ. Carolinae* **12** (1971), 317–325. Zbl 0215.34001 MR 0286705

[23] M. Neuhauser, Relative property (T) and related properties of wreath products. *Math. Z.* **251** (2005), no. 1, 167–177. Zbl 1139.22002 MR 2176470

[24] L. Nguyen Van Thé and V. G. Pestov, Fixed point-free isometric actions of topological groups on Banach spaces. *Bull. Belg. Math. Soc. Simon Stevin* **17** (2010), no. 1, 29–51. Zbl 1207.22002 MR 2656670

[25] B. Nica, Cubulating spaces with walls. *Algebr. Geom. Topol.* **4** (2004), 297–309. Zbl 1131.20030 MR 2059193

[26] P. W. Nowak, Group actions on Banach spaces. In *Handbook of group actions. Vol. II*, pp. 121–149, Adv. Lect. Math. (ALM) 32, International Press, Somerville, MA, 2015. MR 3382026

[27] M. Roller, Poc sets, median algebras and group actions. PhD thesis, Universität Regensburg, 1998.

[28] J.-P. Serre, *Arbres, amalgames,* SL_2. Astérisque 46, Société Mathématique de France, Paris, 1977. Zbl 0369.20013 MR 0476875

[29] R. Tessera and A. Valette, Locally compact groups with every isometric action bounded or proper. *J. Topol. Anal.* **12** (2020), no. 2, 267–292. Zbl 1444.22006 MR 4119107

[30] E. R. Verheul, *Multimedians in metric and normed spaces*. CWI Tract 91, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1993. Zbl 0790.46008 MR 1244813

(Reçu le 25 juin 2021)

11, rue

Paul-Henry Leemann, Institut de Mathématiques, Université de Neuchâtel, ~~Rue~~ Emile-Argand, ~~11,~~ 2000 Neuchâtel, Switzerland; ~~*e-mail:* paul-henry.leemann@unine.ch;~~ remove the comma?
e-mail: paulhenry.leemann@xjtlu.edu.cn

Grégoire Schneeberger, Section de mathématiques, Université de Genève, 7-9, rue du Conseil Général, 1205 Genève, Switzerland; *e-mail:* gregoire.schneeberger@unige.ch