The Property FW for the wreath products

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1 Introduction

The property FW was introduced by Barnhill and Chatterji. It is a fixed point property for the action on wall spaces (for a detailed treatment of this property see [2]). For discrete groups, this property is implied by the Kazhdan property (T). The behavior of the Kazhdan Property (T) with the wreath product is well known:

Give ref (cf article de Cornullier)

Theorem 1.1 ([1, 3]). *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property (T) if and only if G and H have the property (T) and X is finite.*

The same kind of result is true for the property FW

Theorem 1.2. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property FW if and only if G and H have the property FW and X is finite.*

An action of a group on a CAT(0) cube complex is *essential* if all the orbits of vertices are unbounded and the action is transitive on the set of hyperplanes.

Corollary 1.3. *Let G, H be two discrete groups and X a set on which H acts transitively. If there exists an essential action of G or H on a CAT(0) cube complex or if X is infinite, then there exists an essential action of $G \wr_X H$ on a CAT(0) cube complex.*

2 Definitions

2.1 The Property FW

For the definition of the property FW, we will follow the survey of Y. de Cornullier [2].

Definition 2.1. Let G be a discrete group and X a discrete set on which G acts. A subset $M \subset X$ is *commensurated* by the G-action if

$$|gM\Delta M| < \infty$$

for all g in G.

An invariant G-subset is automatically commensurated. Moreover, for a subset M such that there exists an invariant G-subset N with $|M\Delta N| < \infty$ then M is commensurated. Such a set is called *transfixed*.

Definition 2.2. A group G has the property FW if all commensurable G-set are transfixed.

There are lot of equivalent characterizations of this property. We will give us without all the details and the precise definitions.

Proposition 2.3. *The following are equivalent:*

1. *G has the property FW;*
2. *every cardinal definite function on G is bounded;*
3. *every cellular action on a CAT(0) cube complex has bounded orbits for the ℓ^1-metric (the complexes can be infinite dimensional);*
4. *every cellular action on a CAT(0) cube complex has a fixed point;*
5. *every action on a connected median graph has bounded orbits;*
6. *every action on a nonempty connected median graph has a finite orbit;*
7. *(if G is finitely generated) every Schreier graph of G has at most 1 end;*
8. *For every set Y endowed with a walling structure and compatible action on Y and on the index of the walling, the action on Y has bounded orbits for the wall distance;*
9. *every isometric action on an "integral Hilbert space" $\ell^2(X, \mathbf{Z})$ (X any discrete set), or equivalently on $\ell^2(X, \mathbf{R})$ preserving the integral points, has bounded orbits;*
10. *for every G-set X we have $H^1(G, \mathbf{Z}X) = 0$.*

Note that the name FW comes from the property of "fixed point" for the actions on the walling spaces. We will see in the following that a semi-splittable group does not have the property FW (see corollary **??**).

The property FW has links with other well known properties. For example, the property FH implies the characterisation 9. For discrete groups (and even for countable groups) the property FH is equivalent to the Kazhdan's property (T) by Delorme-Guichardet's Theorem. As trees are CAT(0) cube complexes, the property FW implies Serre's property FA.

3 Ends of Schreier graph

Ajouter les petits lemmes sur les sous-groupes

For finitely generated groups, the point 7 of the Proposition 2.3 gives us a nice geometrical characteriation of property FW. We will present more explicit and constructive proofs of the Proposition **??** in this context.

We will begin by a short recall on Schreier graph. .

Utile de redéfinir les Schreier ? Oui !

Def de bouts, mettre au moins deux versions

Definition 3.1. Let G be a group, H a subgroup of G and S a symmetric[1] generating set. The *(left) Schreier graph* $\mathrm{Sch}(G,H;S)$ is the graph with vertices the left cosets $gH=\{gh\mid h\in H\}$ and where two vertices gH and $g'H$ are adjacent if there exists a generator s such that $g'H=sgH$.

Observe that with the above definition, a Schreier graph may have loops, but never has multiple edges. While, depending on the situation, some authors allow multiple edges, when studying end of graphs we only need to know if two vertices are adjacent and the exact number of edges between them does not matter.

If X is a set with a left action $G\curvearrowright X$ and S is a symmetric generating set for G, the corresponding (left) *orbital graph* $\mathrm{Sch}_{\mathcal{O}}(G,X;S)$ is the graph with vertex set X and with an edge between x and y if there exists s in S such that $s.x=y$. As the notation suggest, these two definitions are the two faces of the same coin. Indeed, we have $\mathrm{Sch}(G,H;S)=\mathrm{Sch}_{\mathcal{O}}(G,G/H;S)$, while for every $x\in X$, the graph $\mathrm{Sch}(G,\mathrm{Stab}_G(x);S)$ is equal to the connected component of x in $\mathrm{Sch}_{\mathcal{O}}(G,X;S)$.

Schreier graphs are generalizations of the well-known *Cayley graphs*, whith $\mathrm{Cayl}(G;S)=\mathrm{Sch}(G,\{1\};S)$.

Examples of Schreier graphs with number of ends.

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support :

$$\bigoplus_X G=\{\varphi\colon X\to G\mid \varphi(x)=1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken componentwise. If H is a group acting on X, then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x)=\varphi(h^{-1}.x)$. This leads to the following definition

Definition 3.2. Let G and H be groups and X be a set on which H acts. The *(retricted) wreath product* $G\wr_X H$ is the group $(\bigoplus_X G)\rtimes H$.

Let S be a generating set of G and T a generating set of H. Suppose that H acts transitively on X and let $x\in X$ be any point. Finally, let δ_x^s be the element of $\bigoplus_X G$ defined by $\delta_x^s(x)=s$ and $\delta_x^s(y)=1_G$ if $y\neq x$ and let $\mathbf{1}$ be the constant function with value 1. It is then standard that

$$\{(\delta_x^s,1_H)\mid s\in S\}\ \cup\{(\mathbf{1},t\mid t\in T\}$$

is a generating set for $G\wr_X H$.

On the other hand, it follows from the definition of the wreath product that we have

$$G\wr_X H=\bigoplus_{Y \text{ is an } H\text{-orbit}}\big(G\wr_Y H\big).$$

We hence have as a corollary

Lemma 3.3. *The group $G\wr_X H$ is finitely generated if and only if both G and H are finitely generated and H acts on X with finitely many orbits.*

[1]That is, $s\in S$ if and only if $s^{-1}\in S$.

We know begin our result on the number of ends of Schreier graphs.

Lemma 3.4. *Let N and H be two finitely generated groups and $N \rtimes H$ a semi-direct product. Then*

1. *If $N \rtimes H$ has FW, then so does H,*

2. *If both N and H have FW, then G also has FW.*

Proof. Let S, respectively T, denotes a finite generating set of N, respectively H. It is well known that $G = N \rtimes H$ is finitely generated by $U = (S \times \{1\}) \cup (\{1\} \times T)$.

Suppose that H does not have property FW. Then there exists a Schreier graph $X = \mathrm{Sch}(H, K; T)$ of H with more than one end. The group $G = N \rtimes H$ acts on X via $(n, h).x := h.x$. Since H acts transitively on X, so does G. In fact, the graph of the action $G \curvearrowright X$ is isomorphic to the graph X with some additional loops for generators in $S \times \{1\}$. As adding loops does not change the number of ends, this Schreier graph has more than one end and therefore G does not have property FW.

Suppose now that both N and H have property FW. We want to show that every Schreier graphs of G have at most one end. If they are all finite, then there is nothing to prove (and G is finite). So let X be an infinite Schreier graph of G with respect to the generating set U. The groups N and H acts on X by restriction of the action of G. That is, $n.x = (n, 1).x$ and $h.x = (1, h).x$. For each vertex x we define X_x^H (and respectively X_x^N) as the Schreier graph obtained from the action of H (respectively N) on the H-orbit (respectively N-orbit) of x. These are subgraphs of X. As N and H have property FW, the graphs X_x^H and X_x^N are either finite or one-ended. We want to prove that in this case X has exactly one end.

Let K be a finite set of vertices of X. If x is in K and X_x^H is finite, add all vertices of X_x^H to K. By doing so for every x in K, we obtain a new finite set $K \subset K'$ of vertices of X. We will show that $X \setminus K'$ has only one infinite connected component. By definition of K', if x is not in K', then either X_x^H has one end or X_x^H does not contains vertices in K'.

Let x and y be two vertices, each of them lying in some infinite connected component of $X \setminus K'$. We will construct a path from x to y in $X \setminus K'$ as a concatenation of three smaller paths as follow, see Figure 1. First, a path in $X_x^H \setminus K'$ from x to some z, then a path in $X_z^N \setminus K'$ from z to some $z' \in (X_z^N \cap X_y^H) \setminus K'$, and finally a path in $X_y^H \setminus K'$ from z' to y. In order to finish the proof, it remains to exhibit elements z and z' and the three desired paths.

The action of G on X being transitive, there exists an element (n_0, h_0) of $N \rtimes H$ such that $(n_0, h_0).x = y$. Since K' is finite, the set $X_x^H \setminus K'$ is infinite. Moreover, there is infinitely many z in $X_x^H \setminus K'$ such that either X_z^N is one-ended or X_z^N does not intersect K'. For such a z there exists h such that $(1, h).x = z$. Now, the vertex $z' := (hh_0^{-1}n_0, h).x$ is both equal to $(hh_0^{-1}.n_0, 1)(1, h_0).x = (hh_0^{-1}.n_0, 1).z$ and to $(1, hh_0^{-1})(n_0, h_0).x = (1, hh_0^{-1}).y$. That is, z' is in $X_z^N \cap X_y^H$. A simple computation show us that the map $z \mapsto z'$ is injective: $z_1' = z_2'$ if and only if $z_1 = z_2$. Since K' is finite, there is only finitely many z' in K' and hence there is infinitely many $z \in X_x^H$ such that both z and z' are not in K' and either X_z^N is one-ended or X_z^N does not intersect K'.

In order to finish the proof, observe that the three graphs X_x^H, X_y^H and X_z^N are all either one-ended or do not intersect K'. Therefore, there is a path in

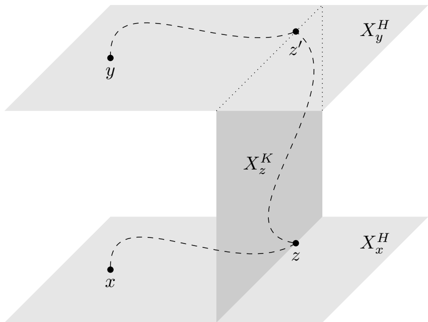

Figure 1: The path between x and y.

$X_x^H \setminus K'$ from x to z as desired, and so on for the paths from z to z' and z' to y. We just have proved that for any finite K the graph $X \setminus K$ has only one infinite connected component and therefore that X is one-ended. $\square$

By iterating the last lemma, we obtain

Corollary 3.5. *Let G and H be two finitely generated groups and X a H-set such that the number of orbits is finite. Then,*

1. *If $G \wr_X H$ has property FW, then so does H,*
2. *If both G and H have property FW and X is finite, then $G \wr_X H$ has property FW.*

Lemma 3.6. *Let G and H be two finitely generated groups such that $G \neq \{1\}$ and H acts on some set X with finitely many orbits. If $G \wr_X H$ has property FW, then*

1. *G has property FW,*
2. *X is finite.*

Proof. We will prove the contrapositives. The idea is to construct, for each of the two cases, a Schreier graph of $G \wr_X H$ with more than one end by exhibiting some well-chosen action of $G \wr_X H$. We fix some finite generating sets S and T of G and H and let

$$U := \{(\delta_x^s, 1_H) \mid s \in S\} \cup \{(\mathbf{1}, t \mid t \in T\}$$

be the standard generating set of $G \wr_X H$.

Suppose that X is an infinite set. Since H acts on X with finitely many orbits, there exists an infinite orbit X'. Let x_0 be an arbitrary vertex of X'. The group G acting on itself by left multiplications, we have the so-called *imprimitive action* of the wreath-product $G \wr_X H$ on $Y := G \times X'$:

$$(\varphi, h).(g, x) := (\varphi(h.x)g, h.x).$$

Since both $G \curvearrowright G$ and $H \curvearrowright X'$ are transitive, the action $G \wr_X H \curvearrowright Y$ is also transitive. Therefore, the orbital Schreier graph of $G \wr_X H \curvearrowright Y$ is isomorphic to the Schreier graph $\Gamma \coloneqq \mathrm{Sch}(G \wr_X H, \mathrm{Stab}(1_G, x_0), U)$. We decompose this graph into leaves of the form $Y_g = \{g\} \times X'$. There are two types of edges in Γ, which are coming from the two sets of generators, see Figure 2. The first ones, of the form $(\mathbf{1}, t)$, give us on each leaf a copy of the orbital Schreier graph of $H \curvearrowright X'$. Indeed,

$$(\mathbf{1}, t).(g, x) = (g, t.x).$$

The second ones, of the form $(\delta^s_{x_0}, 1)$, give us loops everywhere except on vertices of the form (g, x_0). By direct computation, we see that the vertices (g, x_0) and (sg, x_0) connect the leaves Y_g and Y_{sg},

$$(\delta^s_{x_0}, 1)(g, x) = \begin{cases} (g, x) & \text{if } x \neq x_0, \\ (sg, x) & \text{if } x = x_0. \end{cases}$$

If we remove a vertex (g, x_0) we disconnect the leaf Y_g from the rest of Γ. As

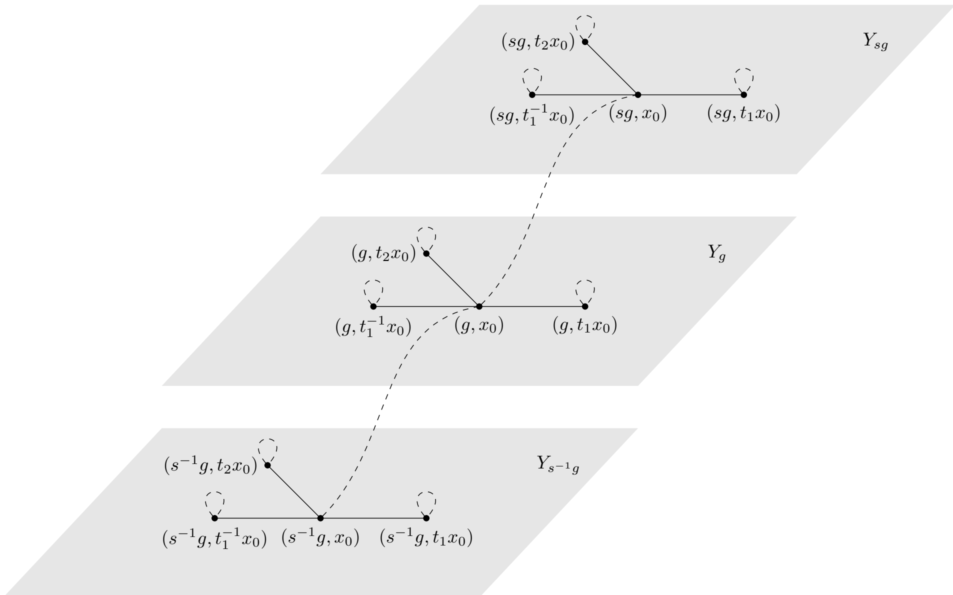

Figure 2: The leaf structure of the orbital Schreier graph of $G \wr_X H \curvearrowright Y$. Plain edges correspond to generators of the form $(\mathbf{1}, t)$ while dotted edges correspond to generators of the form $(\delta^s_{x_0}, 1)$.

X' is infinite each leave is infinite and the number of ends of Γ is then at least $|G| \geq 2$. We just proved that if X is infinite the group $G \wr_X H$ does not have property FW.

Suppose now that G does not have property FW. There exists a subgroup K of G such that $\mathrm{Sch}(G, K, S)$ has more than one end. Let x_0 be any point of X and X' be its orbit under the action of H. We have the imprimitive action of $G \wr_X H$ on $G/K \times X$, which we could restrict to an action on $G/K \times X'$:

$$(\varphi, h)(gK, x) = (\varphi(h.x)gK, h.x).$$

As above, the action is transitive and the orbital Schreier graph of this action is isomorphic to a Schreier graph Γ. We decompose this graph into leaves in the same way. Now we look at the subgraph made up of vertices (g, x_0) and edges $(\delta^s_{x_0}, 1)$ and we remark that it is isomorphic to the Schreier graph $\mathrm{Sch}(G, K, S)$ which has more than one end. Then Γ has also more than one end, which finishes the proof. $\square$

The following proposition is a direct application of Corollary 3.5 and Lemma 3.6.

Proposition 3.7. *Let G be a non trivial finitely generated group, H be a finitely generated group and X a set on which H acts with a finite number of orbit. The wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

References

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