Wreath products of groups acting with bounded orbits

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Abstract

If S is a structure over metric spaces, we say that a group G has property BS if any action on a S-space has bounded orbits. Examples of such structures include metric spaces, Hilbert spaces, connected median graphs, trees or ultra-metric spaces. They correspond respectively to Bergman's property, property FH (which, for countable groups, is equivalent to the celebrated Kazdhan's property (T)), property FW, property FA and cof $\neq \omega$.

Our main result is that for a large class of structures S, the wreath product $G \wr_X H$ has property BS if and only if both G and H have property BS and X is finite. On one hand, this encompass in a general setting previously known results for properties FH and FW. On the other hand, this also apply to the Bergman property. Finally, we also obtain that $G \wr_X H$ has cof $\neq \omega$ if and only if both G and H have cof $\neq \omega$ and H acts on X with finitely many orbits.

Question 1 : que peut-on dire des groupes topologiques ? Les mêmes preuves doivent marcher mutatis mutandis. Ça serait bien de regarder cela avant publications (éventuellement avant de mettre sur arXiv).

Question 2 : que peut-on dire de la propriété F$\mathbf{R}$ (n.b, F$\mathbf{R}$ est strictement plus forte que FA) ? Et des autres dérivés du type FΛ ? À priori ni plus ni moins que FA. Mais on peut au moins en parler vite fait quelque part.

Question 3 : que peut-on dire des extensions $1 \to N \to G \to H$? Cf sous-section 3.1.

1 Introduction

When working with group properties, it is natural to ask if they are stable under "natural" group operations. For example, one may wonder when a property is stable by subgroups, quotients, direct products or semi-direct products. A slightly less common operation, but still of great use in geometric group theory, is the wreath product which stands in-between the direct and the semi-direct product, see Section 2 for all the relevant definitions.

In this context, the following result on the celebrated Kazdhan's property (T) was obtained in the mid's 2000:

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Theorem 1.1 ([**?**, **?**])**.** *Let G and H be two groups with G non-trivial and let X be a set on which H acts. The wreath product $G \wr_X H$ has property (T) if and only if G and H have property (T) and X is finite.*

For countable groups (and more generally for σ-compact locally-compact topological groups), property (T) is equivalent, by the Delorme-Guichardet's Theorem, to property FH, see [**?**, Thm. 2.12.4]. Hence, Theorem 1.1 can also be viewed, for countable groups, as a result on property FH.

The corresponding result for property FA is a little more convoluted and was obtained a few years later by Cornulier and Kar.

Theorem 1.2 ([**?**])**.** *Let G and H be two groups with G non-trivial and X a set on which H acts. Suppose that H acts on X with finitely many orbits and without fixed points. Then $G \wr_X H$ has property FA if and only if H has property FA, $G/[G:G]$ is finite and G has cof $\neq \omega$.*

Finally, in a recent note, the authors proved an analogous of Theorem 1.1 for property FW:

Theorem 1.3 ([**?**])**.** *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Suppose that all three of G, H and $G \wr_X H$ are finitely generated. Then the wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

Since the publication of Theorem 1.3, Y. Stalder let us know (private communication) that the arguments of [**?**] can be adapted to spaces with walls in order to replace the finite generation hypothesis of Theorem 1.3 by the fact that all three of G, H and $G \wr_X H$ are at most countable. On the other hand, A. Genevois that $G \wr_H H$ has property FW if and only if G has property FW and H is finite.

finir la phrase et mettre une référence

The above results on property FH, FW and FA were obtained with distinct methods even if the final results share a common flavor. On the other hand, all three of properties FH, FW and FA can be characterized by the fact that any action on a suitable metric space (respectively Hilbert space, connected median graph and tree) has bounded orbits. But more group properties can be characterized in terms of actions with bounded orbits. This is for example the case of the Bergman's property (actions on metric spaces) or of cof $\neq \omega$ (actions on ultrametric spaces).

By adopting the point of view of actions with bounded orbits, we obtain an unified proof of the following.

Theorem 1.4. *Let BS be any one of the following properties: Bergman's property, FH or FW. Let G and H be two groups with G non-trivial and let X a set on which H acts. Then the wreath product $G \wr_X H$ has property BS if and only if G and H have property BS and X is finite.*

With a little twist, we also obtain a similar result for groups with cof $\neq \omega$:

Theorem 1.5. *Let G and H be two groups with G non-trivial and let X a set on which H acts. Then the wreath product $G \wr_X H$ has cof $\neq \omega$ if and only if G and H have cof $\neq \omega$ and H acts on X with finitely many orbits.*

A crucial ingredient of our proofs, is that the spaces under consideration admit a natural notion of Cartesian products. In particular, some of our results do not work for trees and property FA. Nevertheless, we are still able to show that if $G \wr_X H$ has property FA, then H acts on X with finitely many orbits. Combining this with Theorem 1.2 we obtain

Theorem 1.6. *Let G and H be two groups with G non-trivial and X a set on which H acts. Suppose that H acts on X without fixed points. Then $G \wr_X H$ has property FA if and only if H has property FA, H acts on X with finitely many orbits, $G/[G:G]$ is finite and G has cof $\neq \omega$.*

Organization of the paper The next section contains all the definitions as well as some examples. Section 3 is devoted to the proof of Theorems 1.4 and 1.5 as well as to some related results.

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2 Definitions and examples

This section contains all the definitions, as well as some useful preliminaries facts and some examples.

2.1 Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi\colon X \to G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken componentwise.

If H is a group acting on X, then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition

Definition 2.1. Let G and H be groups and X be a set on which H acts. The *(retricted) wreath product* $G \wr_X H$ is the group $(\bigoplus_X G) \rtimes H$.

A prominent source of examples of wreath products are the ones of the form $G \wr_H H$, where H acts on itself by left multiplication. They are sometimes called *standard wreath products* or simply *wreaths products*, while general $G \wr_X H$ are sometimes called *permutational wreath products*. The probably most well-kwon examples of wreath product is the so called *lamplighter group* $(\mathbf{Z}/2\mathbf{Z}) \wr_{\mathbf{Z}} \mathbf{Z}$. Another (trivial) examples of wreath products are direct products $G \times H$ which correspond to wreath products over a singleton $G \wr_{\{*\}} H$.

2.2 Actions with bounded orbits

Remind that a metric space (X, d) is *ultrametric* if and only if for any x, y and z in X we have $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

For u and v two vertices of a connected[1] graph $\mathcal{G}$, we define the total interval $[u, v]$ as the set of vertices that lie on some shortest path between u and v. A connected graph $\mathcal{G}$ is *median* if for any three vertices u, v, w, the intersection $[u, v] \cap [v, w] \cap [u, w]$ consists of a unique vertex, denoted $m(u, v, w)$. A graph is *median* if each of its connected components is median. For surveys on median graphs see [**?**, **?**].

Maybe add refs with respect to FW.

Add refs

Definition 2.2. Let G be a group. It is said to have

- *property SB* if any action on a metric space has bounded orbits,
- *property FH* if any action on a real Hilbert space has bounded orbits,
- *property FW* if any action on a connected median graph as bounded orbits,
- *property FA* if any action on a tree has bounded orbits,
- *cof* $\neq \omega$ if any action on an ultrametric space has bounded orbits.

In the above, *actions* are supposed to preserve the structure. In particular, actions on (ultra)metric spaces are by isometries, actions on graphs (including trees) are by graph isomorphisms and actions on Hilbert spaces are by linear isometries.

For countable groups (and more generally for σ-compact locally compact groups), property FH is equivalent to the celebrated Kazdhan's property (T) by the Delorme-Guichardet theorem, see for example [**?**], but this is not true in general [**?**].

Maybe expand a little more.

The names FH, FW and FA come from the fact that these properties admit a description in terms of existence of a Fixed point for actions on Hilbert spaces, on spaces with Walls and on trees (*Arbres* in french). On the other hand, SB stands for Strongly Bounded and is also called the *Bergman's property*. Finally, a group has cof $\neq \omega$ (that is *not countable cofinality*) if and only if it cannot be written has a countable increasing union of proper subgroups, see Lemma 2.4.

Give refs

We have the following strict implications between the properties of Definition 2.2 [**?**, **?**, **?**, **?**]

$$\text{SB} \implies \text{FH} \implies \text{FW} \implies \text{FA} \implies \text{cof} \neq \omega. \tag{\dagger}$$

Trees are the simplest examples of median graphs and a simple verification shows that if X and Y are both (connected) median graphs, then their cartesian product is also a (connected) median graph. On the other hand, the following example will be fundamental for us.

Example 2.3. Let X be a set and let $\mathcal{P}(X) = 2^X$ be the set of all subsets of X. Define a graph structure on $\mathcal{P}(X)$ by putting an edge between E and F if and only if $\#(E\Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $E\Delta F$ and the connected component of

[1]We will always assume that our connected graphs are non-empty. This is coherent with the definition that a connected graph is a graph with exactly one connected component.

E is the set of all subsets F with $E\Delta F$ finite. For E and F in the same connected component, $[E,F]$ consist of all subsets of X that both contain $E\cap F$ and are contained in $E\cup F$. In particular, $\mathcal{P}(X)$ is a median graph, with $m(D,E,F)$ being the set of all elements belonging to at least two of D, E and F. In other words, $m(D,E,F)=(D\cap E)\cup(D\cap F)\cup(E\cap F)$.

We denote by $\mathcal{P}_{\mathrm{f}}(X)$, respectively $\mathcal{P}_{\mathrm{cof}}(X)$ the set of all finite, respectively cofinite, subsets of X. They are connected components of $\mathcal{P}(X)$, which coincide if and only if X is finite. More generally, the connected components of $\mathcal{P}(X)$ are hypercubes and it turns out that every connected median graph is a retract of a connected component of some $\mathcal{P}(X)$, see [**?**].

These graphs will be fundamental for us due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}(X)$ by graph homomorphisms: $g.\{x_1,\dots,x_n\}=\{g.x_1,\dots,g.x_n\}$. Be careful that the action of G on $\mathcal{P}(X)$ may exchange the connected components. In fact, the connected component of $E\subset X$ is stabilized by G if and only if E is *commensurated* by G, that is if for every $g\in G$ the set $E\Delta gE$ is finite. For example, both $\mathcal{P}_{\mathrm{f}}(X)$ and $\mathcal{P}_{\mathrm{cof}}(X)$ are always preserved by the action of G.

The following characterization of groups with cof $\neq\omega$ is well-known and we include a proof only for the sake of complexity.

Lemma 2.4. *Let G be a group. Then the following are equivalent:*

1. *There exists a G-invariant (for the action by left multiplication) ultrametric d on G such that $G\curvearrowright G$ has an unbounded orbit,*
2. *There exists an ultrametric space X on which G acts with an unbounded orbit,*
3. *G does not have cof $\neq\omega$, that is it can be written as a countable increasing union of proper subgroups.*

Proof. It is clear that the first item implies the second.

Suppose that (X,d) is a ultrametric space on which G acts with an unbounded orbit and let x_0 be an element of X. For any $n\in\mathbf{N}$ let H_n be the subset of G defined by

$$H_n\coloneqq\{g\in G\mid d(x_0,h.x_0)\leq n\}.$$

It is clear that G is the increasing union of the (countably many) H_n. On the other hand, the H_n are subgroups of G. Indeed, H_n is trivially closed under taking the inverse, and it is also closed under taking products since $d(x_0,gh.x_0)\leq\max\{d(x_0,g.x_0),d(g.x_0,gh.x_0)\}=\max\{d(x_0,g.x_0),d(x_0,h.x_0)\}$.

Finally, suppose that $G=\bigcup_{n\in\mathbf{N}}H_n$ where the H_n form an increasing sequence of proper subgroups. It is always possible to suppose that $H_1=\{1\}$. Define d on G by $d(g,h)\coloneqq\min\{n\mid g^{-1}h\in H_n\}$. One easily verify that d is a G-invariant ultrametric. Moreover, the orbit of 1 contains all of G and is hence unbounded. □

A slight variation of the above lemma gives us

Example 2.5. Let $(G_i)_{i\geq1}$ be non-trivial groups and let $G\coloneqq\bigoplus_{i\geq1}G_i$ be their direct sum. Then $d_\infty(f,g)\coloneqq\max\{i\mid f(i)\neq g(i)\}$ is an ultra-distance on G which is G-invariant for the action of G on itself by left multiplication.

Finally, it is possible to define other properties in the spirit of Definition 2.2.

Definition 2.6. An *additional structure on metric spaces* is, a category **SMet** together with a faithful functor $F_S\colon \mathbf{SMet} \to \mathbf{Met}$, where **Met** is the category of metric spaces with short maps. The objects of **SMet** are called *S-spaces.* A *G-action on a S-space X* is simply an homomorphism $\alpha\colon G \to \mathrm{Aut}_{\mathbf{SMet}}(X)$. It has *bounded orbits* if $F_S \circ \alpha\colon G \to \mathrm{Aut}_{\mathbf{Met}}(X)$ has bounded orbits.

We can now, formally define the groupe property *BS* as: *Every G-action on a S-space has bounded orbits.*

In other words, an additional structure on metric spaces, is concrete category over **Met**. Since **Met** itself is concrete, that is we have a faithful functor $F\colon \mathbf{Met} \to \mathbf{Set}$, the category **SMet** is also concrete and its objects can be thoughts as sets with "extra structure".

Remark 2.7. The only use we will do of category theory is as a language that allows to formally define what is an *additional structure on metric spaces* and to prove things in this general setting. A reader unfamiliar with category theory and interested only in one specific structure (as for example Banach spaces), might forget all this general non-sense and only verify that the arguments of Section 3 apply for this specific structure.

Obvious examples of additional structures on metric spaces include, metric spaces, Hilbert spaces and ultrametric spaces. For connected graphs (and hence for connected median graphs and for trees), one look at the category **Graph** where objects are connected simple graphs $G = (V, E)$ and where a morphism $f\colon (V, E) \to (V', E')$ is a function between the vertex sets such that if (x, y) is an edge then either $f(x) = f(y)$ or $(f(x), f(y))$ is an edge. The functor $F_S\colon \mathbf{Graph} \to \mathbf{Met}$ sends a connected graph to its vertex set together with the graph distance: $d(x, y)$ is the minimum number of edges on a path between x and y.

An example of an uninteresting property BS is given by taking **SMet** to be the metric spaces of bounded diameter (together with short maps). Indeed, in this case, any group has BS.

While we will be able to obtain some results for a general **SMet**, we will sometimes need to restrict ourself to structure with a suitable notion of cartesian product.

Definition 2.8. A structure S on metric spaces has *compatible cartesian powers* if for any S-space X and any integer n, there exists an S-object called the *n^{th} cartesian power of X* and written X^n such that:

1. X^n is compatible with the cartesian product of sets. That is $F \circ F_S(X^n)$ is the set cartesian power and there exists S-morphisms $\pi_i\colon X^n \to X$ such that $F \circ F_S(\pi_i)$ are the usual set projections.

2. X^n is compatible with the topology. That is, the underlying topology of X^n is the product topology on $F \circ F_S(X^n)$.

3. X^n is compatible with the bornology. That is, the underlying bornology of X^n is the product bornology.[2]

[2] In practice, we will only need that if $E \subset X$ is unbounded, then the diagonal $\mathrm{diag}(E) \subset X^n$ is unbounded.

4. $\mathrm{Aut}_{\mathbf{SMet}}(X)^n \rtimes \mathrm{Sym}(n)$ is a subgroup of $\mathrm{Aut}_{\mathbf{SMet}}(X^n)$.

For (ultra)metric spaces, the categorical product (corresponding to the metric $d_\infty = \max\{d_X, d_Y\}$) works fine, but any product metric of the form $d_p = (d_X^p + d_Y^p)^{\frac{1}{p}}$ for $p \in [1, \infty]$ works as well. For Hilbert spaces, we take the usual cartesian product (which is also the categorial product), which corresponds to the metric $d_2 = \sqrt{d_X^2 + d_Y^2}$. For connected median graph, the usual cartesian product (which is not the categorical product![3]) with $d_1 = d_X + d_Y$ does the job. On the other hand, trees do not have compatible cartesian powers.

We conclude this section by a remark on a variation of Definition 2.2. One might wonder what happens if in Definition 2.2 we replace the requirement of having bounded orbits by having uniformly bounded orbits. It turns out that this is rather uninteresting as a group G is trivial if and only if any G-action on a metric space (respectively on an Hilbert space, on a connected median graph, on a tree or on an ultrametric space) has uniformly bounded orbits. Indeed, if G is non-trivial, then for the action of G on the Hilbert space $\ell^2(G)$ the orbit of $n \cdot \delta_g$ as diameter $n\sqrt{2}$. For a tree (and hence also for a connetected median graph), one may look at the tree T obtained by taking a root r on which we glue an infinite ray for each elements of G. Then G naturally acts on T by permuting the rays. The orbits for this action are the $\mathcal{L}_n = \{v \mid \text{at distance } n \text{ of } r\}$ which have diameter $2n$. Finally, it is possible to put an ultradistance on the vertices of T by $d_\infty(x, y) \coloneqq \max\{d(x, r), d(y, r)\}$ if $x \neq y$. Then the orbits are still the $\mathcal{L}_n$, but this time with diameter n.

3 Proofs of the main results

We begin this section with the following trivial but useful result.

Lemma 3.1. *Let G be a group and H be a quotient. If G has property BS, then so does H.*

Proof. We have $H \cong G/N$. If H acts on some S-space X with an unbounded orbit, then the G action on X defined by $g.x \coloneqq gN.x$ has also an unbounded orbit. □

We also have the following lemma on semi-direct products:

Lemma 3.2. *Let $N \rtimes H$ be a semidirect product. Then*

1. *If $N \rtimes H$ has property BS, then so does H.*
2. *If both N and H have property BS, then $N \rtimes H$ also has property BS.*

Proof. The first part is Lemma 3.1.

On the other hand, suppose that N and H have BS and let X be a S-space on which G acts. Then both N and H acts on X with bounded orbits. Let x be an element of X, D_1 be the diameter of $H.x$ and D_2 be the diameter of $N.x$. Since G acts by isometries, for every h in H the set $N.(h.x) = Nh.x = hN.x = h.(N.x)$ has also diameter D_2. Therefore, every element of $G.x = NH.x$ is at distance at most $D_1 + D_2$ of x, which implies that the orbit $G.x$ is bounded. □

[3]The categorial product in **Graph** is the strong product.

As a direct corollary, we have

Corollary 3.3. *Let G and H be two groups. Then $G \times H$ has property BS if and only if both G and H have property BS.*

By iterating Lemma 3.2, we obtain

Corollary 3.4. *Let G and H be two groups and X a set on which H acts. Then,*

1. *If $G \wr_X H$ has property BS, then so does H,*
2. *If both G and H have property BS and X is finite, then $G \wr_X H$ has property BS.*

On the other hand, we have the following result on infinite direct sums. It is of course possible to prove it using the characterization of cof $\neq \omega$ in terms of subgroups. However, we find enlightening to prove it using the characterization in terms of actions on ultrametric spaces.

Lemma 3.5. *Suppose that BS implies cof $\neq \omega$. Then*

1. *An infinite direct sum of non-trivial groups does not have BS,*
2. *If $G \neq \{1\}$, then $\bigoplus_X G$ has BS if and only if G has BS and X is finite,*
3. *If $G \wr_X H$ has BS, then H acts on X with finitely many orbits.*

Proof. By Corollary 3.3, it is enough to prove the first assertion for countable direct sums of groups. So let $G \coloneqq \bigoplus_{i\geq 1} G_i$ and for each i, choose $g_i \neq 1$ in G_i. Let $d_\infty(f,g) \coloneqq \max\{i \mid f(i) \neq g(i)\}$ be the G-invariant ultrametric of Example 2.5. Then for every integer n, the orbit $G.1_G$ contains $\{g_1, \dots, g_n, 1, \dots\}$ which is at distance n of 1_G for d_∞. In particular, an infinite direct sum of non-trivial groups does not have cof $\neq \omega$, nor does it have BS.

The second assertion follows of the first assertion combined with Corollary 3.3.

The last assertion is a simple variation on the first. Indeed, we have

$$G \wr_X H \cong (\bigoplus_{Y \in X/H} L_Y) \rtimes H \qquad \text{with} \qquad L_Y \cong \bigoplus_{y\in Y} G_y,$$

where X/H is the set of H-orbits. The important fact for us is that H fixes the decomposition into L_Y factors: for all Y we have $H.L_Y = L_Y$. Up to regrouping some of the L_Y together we hence have $G \wr_X H \cong \big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$ with $H.L_i = L_i$ for all i. Now, we have an ultradistance d_∞ on $L \coloneqq \bigoplus_{i\geq 1} L_i$ as above and we can put the discrete distance d on H. Then $d'_\infty = \max\{d_\infty, d\}$ is an ultradistance on $\big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$, which is $\big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$-invariant (for the action by left multiplication). From a practical point of view, we have $d'_\infty\big((f,h),(g,h')\big) \coloneqq \max\{i \mid f(i) \neq g(i)\}$ if $f \neq g$ and $d'_\infty\big((f,h),(f,h') = 1$ if $h \neq h'$ Since the action of L on itself has an unbouded orbit for d_∞, the action of $\big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$ on itself has an unbounded orbit for d'_∞. ☐

While the statement (and the proof) of Lemma 3.5 is expressed in terms of cof $\neq \omega$, it is also possible to state it and prove it for a structure S without a priori knowing if BS is stronger than cof $\neq \omega$. The main idea is to find a "natural" S-space on which $G = \bigoplus_{i\geq 1} G_i$ acts. For example, for Hilbert spaces, one can take $\bigoplus_{i\geq 1} \ell^2(G_i)$. For connected median graphs, one take the connected component of $\{1_{G_1}, 1_{G_2}, \dots\}$ in $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$. For trees, it is possible to put a forest structure on $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$ in the following way. For $E \in \mathcal{P}(\bigsqcup_{i\geq 1} G_i)$, and for each i such that $E \cap G_j$ is empty for all $j \leq i$, add an edge from E to $E \cup \{g\}$ for each $g \in G_i$. The graph obtained this way is a G-invariant subforest of the median graph on $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$.

Lemma 3.6. *Suppose that BS implies FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. If $G \wr_X H$ has BS, then X is finite.*

Proof. It is enough to prove the assertion for property FW. Suppose that X is infinite. The group $\bigoplus_X G$ acts coordinatewise on $\bigsqcup_X G$: the group G_x acting by left multiplication on G_x and trivially on G_y for $y \neq x$. On the other hand, H acts on $\bigsqcup_X G$ by permutation of the factors. Altogether we have an action of $G \wr_X H$ on $\bigsqcup_X G$ and hence on the median graph $\mathcal{P}(\bigsqcup_X G)$. Let $\mathbf{1} := \bigcup_{x\in X} 1_G$ be the subset of $\mathcal{P}(\bigsqcup_X G)$ consisting of the identity elements of all the copies of G. Since every element of $\bigoplus_X G$ has only a finite number of non-trivial coordinates, the action of $G \wr_X H$ preserves the connected components of $\mathbf{1}$ (and in fact every connected component of $\mathcal{P}(\bigsqcup_X G)$).

Let $I = \{i_1, i_2, \dots\}$ be a countable subset of X and for every $i \in I$, choose a non-trivial $g_i \in G_i$. Then the orbit of the vertex $\mathbf{1}$ contains the point $\{g_{i_1}, \dots, g_{i_n}\} \cup \left(\bigcup_{j>n} 1_{G_{j_i}}\right) \cup \left(\bigcup_{x\notin I} 1_{G_x}\right)$ which is at distance $2n$ of $\mathbf{1}$. That is the action of $G \wr_X H$ on the connected component of $\mathbf{1}$ has an unbounded orbit. □

Once again, given a suitable S, it is sometimes possible to give a direct proof of Lemma 3.6. For example, for Hilbert spaces one can take $\bigoplus_X \ell^2(G)$ with $\bigoplus_X G$ acting coordinatewise and H by permutations. On the other hand, both the forest structure on $\mathcal{P}(\bigsqcup_X G)$ and the ultrametric structure on $\bigoplus_X G$ are in general not invariant under the natural action of H by permutations.

In fact, it follows from Theorems 1.2 and 1.5 than in the assumptions of Lemma 3.6 it is not possible to replace property FW by property FA or by cof $\neq \omega$.

Remark 3.7. A reader familiar with wreath products might have recognized that we used the primitive action of the wreath product in the proof of Lemma 3.6.

Indeed, G acts on itself by left multiplication. It hence acts on the set $G' := G \sqcup \{\varepsilon\}$ by fixing ε, and we have the primitive action of $G \wr_X H$ on G'^X. Now, the set $\bigsqcup_X G$ naturally embeds as the subset of G'^X consisting of all functions $\varphi \colon X \to G'$ such that $\varphi(x) = \varepsilon$ for all but one $x \in X$. This subset is $G \wr_X H$ invariant, which gives us the desired action of $G \wr_X H$ on $\bigsqcup_X G$.

We now turn our attention to properties that behave well under products in the sense of Definition 2.8. We first describe the comportement of property BS under finite index subgroups.

Lemma 3.8. *Let G be a group and let H be a finite index subgroup.*

1. *If H has property BS, then so does G,*

2. *If S has compatible cartesian powers and G has property BS, then H has property BS.*

Proof. Suppose that G does not have BS and let X be a S-space on which G acts with an unbounded orbit $\mathcal{O}$. Then H acts on X and $\mathcal{O}$ is a union of at most $[G:H]$ orbits. This directly implies that H has an unbounded orbit and therefore does not have BS.

On the other hand, suppose that $H \leq G$ is a finite index subgroup of G without property BS. Let $\alpha\colon H \curvearrowright X$ be an action of H on a S-space (X, d_X) such that there is an unbounded orbit $\mathcal{O}$. Similarly to the classical theory of representation of finite groups, we have the induced action $\operatorname{Ind}_H^G(\alpha)\colon G \curvearrowright X^{G/H}$ on the set $X^{G/H}$. Since H has finite index, $X^{G/H}$ is a S-space and the action is by S-automorphisms. On the other hand, the subgroup $H \leq G$ acts diagonally on $X^{G/H}$, which implies that $\operatorname{diag}(\mathcal{O})$ is contained in a G-orbit. Since $\operatorname{diag}(\mathcal{O})$ is unbounded, G does not have property BS.

For readers that are not familiar with representations of finite groups, here is the above argument in more details. Let $(f_i)_{i=1}^n$ be a transversal for G/H. The natural action of G on G/H gives rise to an action of G on $\{1, \dots, n\}$. Hence, for any g in G and i in $\{1, \dots, n\}$ there exists a unique $h_{g,i}$ in H such that $gf_i = f_{g.i}h_{g,i}$. That is, $h_{g,i} = gf_if_{g.i}^{-1}$. We then define $g.(x_1, \dots, x_n) \coloneqq (h_{g,g^{-1}.1}.x_{g^{-1}.1}, \dots, h_{g,g^{-1}.n}.x_{g^{-1}.n})$. This is indeed an action, which, by Condition 4 of Definition 2.8, is by S-automorphisms, of $X^{G/H}$. Moreover, every element $h \in H$ acts diagonally by $h.(x_1, \dots, x_n) = (h.x_1, \dots, h.x_n)$. In particular, this G action has an unbounded orbit. □

We now prove one last lemma that will be necessary fo the proof of Theorem 1.4.

Lemma 3.9. *Let S be a structure that has compatible cartesian powers. If X is finite and $G \wr_X H$ has property BS, then G has property BS.*

Proof. Suppose that G does not have BS and let (Y, d_Y) be a S-space on which G acts with an unbounded orbit $G.y$. Then (Y^X, d) is a S-space and we have the *primitive action* of the wreath product $G \wr_X H$ on Y^X:

$$\big((\varphi, h).\psi\big)(x) = \varphi(h^{-1}.x).\psi(h^{-1}.x).$$

By Condition 4 of Definition 2.8, this action is by S-automorphisms. The orbit $G.y$ embeds diagonally and hence $\operatorname{diag}(G.y)$ is an unbounded subset of some $G \wr_X H$-orbit, which implies that $G \wr_X H$ does not have property BS. □

By combining Corollary 3.4 and Lemmas 3.6 and 3.9 we obtain the following result which implies Theorem 1.4.

Theorem 3.10. *Let S be a structure that has compatible cartesian powers and such that BS implies FW. Let G and H be two groups with G non-trivial and X a set on which H acts. Then the wreath product $G \wr_X H$ has property BS if and only if G and H have property BS and X is finite.*

We now proceed to prove Theorem 1.5. As for Lemma 3.5, it is also possible to prove it using the characterization of $\operatorname{cof} \neq \omega$ in terms of subgroups, but we will only give a proof using the characterization in terms of actions on ultrametric spaces.

Theorem 3.11. *Let G and H be two groups with G non-trivial and let X a set on which H acts. Then the wreath product $G \wr_X H$ has cof $\neq \omega$ if and only if G and H have cof $\neq \omega$ and H acts on X with finitely many orbits.*

Proof. By Corollary 3.4 and Lemma 3.5 we already know that if $G \wr_X H$ has cof $\neq \omega$, then H has cof $\neq \omega$ and it acts on X with finitely many orbits. We will now prove that if $G \wr_X H$ has cof $\neq \omega$ so does G. Let us suppose that G has countable cofinality. Then there exists an ultrametric d on G such that the action of G on itself by left multiplication has an unbounded orbit. But then we have the primitive action of the wreath product $G \wr_X H$ on $G^X \cong \prod_X G$, which preserves $\bigoplus_X G$. It is easy to check that the map $d_\infty \colon \bigoplus_X G \times \bigoplus_X G \to \mathbf{R}$ defined by $d_\infty(\psi_1, \psi_2) \coloneqq \max\{d(\psi_1(x), \psi_2(x)) \mid x \in X\}$ is a $G \wr_X H$-invariant ultrametric. Finally, let $h \in G$ be an element of unbounded G-orbit for d and let x_0 be any element of X. Then for any g in G we have $(\delta^g_{x_0}, 1).\delta^h_{x_0} = \delta^{gh}_{x_0}$ and hence $d_\infty(\delta^h_{x_0}, \delta^{gh}_{x_0}) = d(h, gh)$ is unbounded.

Suppose now that both G and H have cof $\neq \omega$ and that H acts on X with finitely many orbits. We want to prove that $G \wr_X H$ has cof $\neq \omega$.

Let (Y, d) be an ultrametric space on which $G \wr_X H$ acts. Then H and all the G_x act on Y with bounded orbits. Let $\mathcal{O}_1, \dots, \mathcal{O}_n$ be the H-orbits on X and for each $1 \leq i \leq n$ choose an element x_i in $\mathcal{O}_i$. Let y be any element of Y. Then $H.y$ has finite diameter D_0 while $G_{x_i}.y$ has finite diameter D_i. For any $x \in X$, there exists $1 \leq i \leq n$ and $h \in H$ such that $x = h.x_i$. We have

$$\begin{aligned} d\big((\delta^g_{x_i}, h^{-1}).y, y\big) &\leq \max\{d\big((\delta^g_{x_i}, h^{-1}).y, (\delta^g_{x_i}, 1).y\big), d\big((\delta^g_{x_i}, 1).y, y\big)\} \\ &= \max\{d\big((1, h^{-1}).y, y\big), d\big((\delta^g_{x_i}, 1).y, y\big)\} \\ &\leq \max\{D_0, D_i\}, \end{aligned}$$

which implies that the diameter of $G_{x_i} h^{-1}.y$ is bounded by $\max\{D_0, D_i\}$. But $G_{x_i} h^{-1}.y$ has the same diameter as $hG_{x_i} h^{-1}.y = G_{h.x_i}.y = G_x.y$.

On the other hand, the diameter of $\bigoplus_X G.y$ is bounded by the supremum of the diameters of the $G_x.y$, and hence bounded by $\max\{D_0, D_1, \dots, D_n\}$. Finally, for (φ, h) in $G \wr_Y H$ we have

$$\begin{aligned} d\big(y, (\varphi, h).y\big) &\leq \max\{d\big(y, (\varphi, 1).y\big), d\big((\varphi, 1).y, (\varphi, h).y\big)\} \\ &= \max\{d\big(y, (\varphi, 1).y\big), d\big(y, (1, h).y\big)\} \\ &\leq \max\{\max\{D_0, D_1, \dots, D_n\}, D_0\}. \end{aligned}$$

That is, the diameter of $G \wr_Y H.z$ is itself bounded by $\max\{D_0, D_1, \dots, D_n\}$, which finishes the proof. □

While the fact that being a tree is not compatible with products is an obstacle to our methods, we still have the following weak version of Theorem 3.10 for property FA.

Proposition 3.12. *Let G and H be two groups with G non-trivial and X a set on which H acts. Then*

1. *If $G \wr_X H$ has property FA, then H has property FA, H acts on X with finitely many orbits, $G/[G : G]$ is finite and G has cof $\neq \omega$,*

2. *If both G and H have property FA and X is finite, then $G \wr_X H$ has property FA.*

Proof. The only things that is not a consequence of Corollary 3.4, Lemma 3.5 and Theorem 1.5 is the fact that if $G \wr_X H$ has property FA, then the abelianization $G^{\mathrm{ab}} = G/[G,G]$ is finite.

If $G \wr_X H$ has property FA, so does its abelianization $(G \wr_X H)^{\mathrm{ab}} = (G^{\mathrm{ab}})^{X/H} \times H^{\mathrm{ab}}$. Since we already know that H acts on X with finitely many orbits, we conclude that G^{ab} is an abelian group with property FA. But an infinite abelian group either has a quotient which is isomorphic to $\mathbf{Z}$, or a quotient which is an infinite direct sum of non-trivial groups. But neither $\mathbf{Z}$, nor an infinite direct sum of non-trivial groups has FA, and hence G^{ab} is finite. □

Moreover, by using Lemma 3.5 we can get ride of the "finitely many orbits" hypothesis in Theorem 1.2 in order to obtain Theorem 1.6. By Proposition 3.12 (more precisely Lemma 3.5), it is possible to get ride of the "finitely many orbits" hypothesis in Theorem 1.2 to obtain

Finally, observe that, by the characterization of Serre [?], a group G has property FA if and only if it has no morphism onto $\mathbf{Z}$, it has cof $\neq \omega$ and it is not a non-trivial amalgam. In particular, a finitely generated group G has property FA if and only if $G/[G:G]$ is finite, G has cof $\neq \omega$ and G is not a non-trivial amalgam. We hence obtain

Proposition 3.13. *Let G and H be two groups with G non-trivial and finitely generated and X a set on which H acts. If $G \wr_X H$ has property FA, then H has property FA, H acts on X with finitely many orbits, $G/[G:G]$ is finite and G has cof $\neq \omega$.*

Proof. We already know that H acts with finitely many orbits, which implies that the group $G \wr_Y H$ has property FA for every H-orbit Y. Since the result holds if H has no fixed points, we can hence suppose that there is one orbit reduced to a point x_0. But in this case $G \wr_{\{x_0\}} H \cong G \times H$ and G has property FA. Since G is finitely generated, this implies the desired result. □

Observe that, due to Corollary 3.3, the converse of Proposition 3.13 is false in general.

3.1 On group extensions

Ce qui suit est une note interne. Si on arrive à en faire quelque chose, on le poussera dans le corps du texte.

Definition 3.14. Let G be a group and H be a subgroup. The pair (G,H) has the *relative property BS* if for any G-action on a S-space, the H-orbits are bounded.

Observe that G has property BS if and only if (G,G) has relative property BS. On the other hand, if G has property BS, then for any $H \leq G \leq L$ both (L,G) and (G,H) have relative property BS.

Lemma 3.15. *Let $1 \to N \to G \to H \to 1$ be a group extension. If G has property BS, then H has property BS and (G,N) has relative property BS.*

Proof. One part is Lemma 3.1, while the other part is trivial. □

Question 3.16. *For which properties BS, the converse of Lemma 3.15 holds?*

By [**?**], this is the case for property FW. A crude idea would be the following: let X be a S-space on which G acts. Then the action of N on X has bounded orbits. Moreover, $H = G/N$ acts on X/N

C'est là qu'il y a un problème. En effet, rien ne garantit que X/N soit un P-espace. Il faudrait déjà que ce soit un espace métrique (en général un quotient d'un espace métrique est seulement pseudo-métrique). Et même si on regarde le quotient métrique (on identifie les points à distance 0) de l'espace pseudo-métrique X/N ce n'est à priori pas un P-espace. Par example, on peut regarder $G := \mathbf{Z}$ agissant sur lui-même, vu comme un arbre (et donc un graphe médian), par translation. Si on quotiente par $3\mathbf{Z}$, on obtient un 3-cycle qui n'est donc ni un arbre, ni un graphe médian :-/