

# The Property FW for the wreath products

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## 1 Introduction

The property FW was introduced by de Cornuillier. It is a fixed point property for the action on wall spaces (for a detailed treatment of this property see [2]). For discrete groups, this property is implied by the Kazhdan property (T). The behavior of the Kazhdan Property (T) with the wreath product is well known:

**Theorem 1.1** ([1, 3]). *Let  $G, H$  be two discrete groups and  $X$  a set on which  $H$  acts. The wreath product  $G \wr_X H$  has the property (T) if and only if  $G$  and  $H$  have the property (T) and if  $X$  is finite.*

The same kind of result is true for the property FW

**Theorem 1.2.** *Let  $G, H$  be two discrete groups and  $X$  a set on which  $H$  acts transitively. The wreath product  $G \wr_X H$  does not have the property FW if at least one of the following conditions is satisfied:*

1. *The group  $G$  does not have the property FW.*
2. *The group  $H$  does not have the property FW.*
3. *The set  $X$  is infinite.*

An action of a group on a CAT(0) cube complex is *essential* if all the orbits of vertices are unbounded and the action is transitive on the set of hyperplanes.

**Corollary 1.3.** *Let  $G, H$  be two discrete groups and  $X$  a set on which  $H$  acts transitively. If there exists an essential action of  $G$  or  $H$  on a CAT(0) cube complex or if  $X$  is infinite, then there exists an essential action of  $G \wr_X H$  on a CAT(0) cube complex.*

## 2 Definitions

### 2.1 The Property FW

For the definition of the property FW, we will follow the survey of Y. de Cornuillier [2].

*Definition 2.1.* Let  $G$  be a discrete group and  $X$  a discrete set on which  $G$  acts. A subset  $M \subset X$  is *commensurated* by the  $G$ -action if

$$|gM \Delta M| < \infty$$

for all  $g$  in  $G$ .

An invariant  $G$ -subset is automatically commensurated. Moreover, for a subset  $M$  such that there exists an invariant  $G$ -subset  $N$  with  $|M\Delta N| < \infty$  then  $M$  is commensurated. Such a set is called *transfixed*.

**Definition 2.2.** A group  $G$  has the property FW if all commensurable  $G$ -set are transfixed.

There are lot of equivalent characterizations of this property. We will give us without all the details and the precise definitions.

**Proposition 2.3.** *The following are equivalent:*

1.  $G$  has the property FW;
2. every cardinal definite function on  $G$  is bounded;
3. every cellular action on a  $CAT(0)$  cube complex has bounded orbits for the  $\ell^1$ -metric (the complexes can be infinite dimensional);
4. every cellular action on a  $CAT(0)$  cube complex has a fixed point;
5. every action on a connected median graph has bounded orbits;
6. every action on a nonempty connected median graph has a fixes point;
7. (if  $G$  is finitely generated) every Schreier graph of  $G$  has at most 1 end;
8. For every set  $Y$  endowed with a walling structure and compatible action on  $Y$  and on the index of the walling, the action on  $Y$  has bounded orbits for the wall distance;
9. every isometric action on an "integral Hilbert space"  $\ell^2(X, \mathbf{Z})$  ( $X$  any discrete set), or equivalently on  $\ell^2(X, \mathbf{R})$  preserving the integral points, has bounded orbits;
10. for every  $G$ -set  $X$  we have  $H^1(G, \mathbf{Z}X) = 0$ .

Note that the name FW comes from the property of "fixed point" for the actions on the walling spaces. We will see in the following that a semi-splittable group does not have the property FW (see corollary ??).

The property FW has links with other well known properties. For example, the property FH implies the characterisation 9. For discrete groups (and even for countable groups) the property FH is equivalent to the Kazhdan's property (T) by Delorme-Guichardet's Theorem. As trees are  $CAT(0)$  cube complexes, the property FW implies Serre's property FA.

## 2.2 Graphe de Schreier

Expliquer lien action de groupe, graphe de Schreier

## 2.3 Bouts

## 3 Proof of the Theorem

*Proof.* Let us begin by fixing the notation. We denote by  $S$  (and respectively by  $S'$ ) a finite generating set of  $G$  (and respectively of  $H$ ). We choose an arbitrary point of  $x_0$  of  $X$ . The group  $\Gamma = G \wr_X H$  is generated by the set

$$\mathcal{S} = \{(\delta_{x_0}^s, e_h) : s \in S\} \cup \{(0, s') : s' \in S'\}$$

where

$$\delta_{x_0}^s = \begin{cases} e_g & x \neq x_0 \\ s & x = x_0 \end{cases} \quad \text{and} \quad 0(x) = e_g \quad \forall x \in X.$$

The idea of the proof is to construct, for each of the three cases, a Schreier graph of  $\Gamma$  with more than one end. To do this we will consider actions of  $\Gamma$  and associated graph's action. We will treat the 3 cases separately.

Suppose that  $X$  is an infinite set. We define  $Y = G \times X$  and an action of  $\Gamma$  on  $Y$  as

$$(\varphi, h) \cdot (g, x) = (\varphi(hx)g, hx)$$

for  $(\varphi, h)$  in  $\Gamma$  and  $(g, x)$  in  $Y$ . This action is transitive. Indeed, let  $(g_1, x_1)$  and  $(g_2, x_2)$  be two elements of  $Y$ . By transitivity of the action of  $H$  on  $Y$ , there exists  $h$  in  $H$  such that  $hx_1 = x_2$ . We can always find  $\varphi$  in  $\bigoplus_X G$  such that  $\varphi(hx_1) = g_2g_1^{-1}$ . Then,

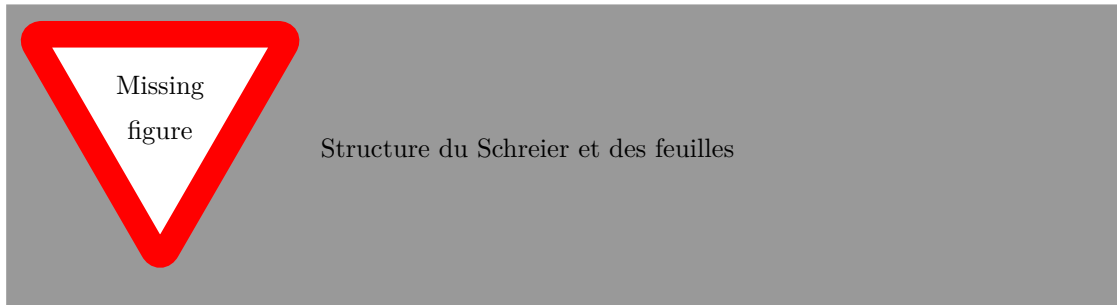
$$(\varphi, h)(g_1, x_1) = (\varphi(hx_1)g_1, hx_1) = (g_2, x_2).$$

The graph of the action of  $\Gamma$  on  $Y$  is isomorphic to the Schreier graph  $\text{Sch}(\Gamma, \text{Stab}(e_G, x_0), \mathcal{S})$ . We decompose the graph into leaves of the form  $Y_g = \{g\} \times G$ . There are two types of edges on this graph which are coming from the two types of generators. The first one, of the form  $(0, s')$ , give us on each leaf a copy of the graph of the action of  $H$  on  $X$ . Indeed, we have

$$(0, s')(g, x) = (g, s'x).$$

The second one, of the form  $(\delta_{x_0}^s, 0)$ , give us loops everywhere excepting on vertices with  $x_0$  as second coordinate. By direct computation, we see that a leaf  $Y_g$  is connected to  $Y_{sg}$  by the vertices  $(g, x_0)$  and  $(sg, x_0)$ :

$$(\delta_{x_0}^s, 0)(g, x) = \begin{cases} (g, x) & x \neq x_0 \\ (sg, x) & x = x_0 \end{cases}$$

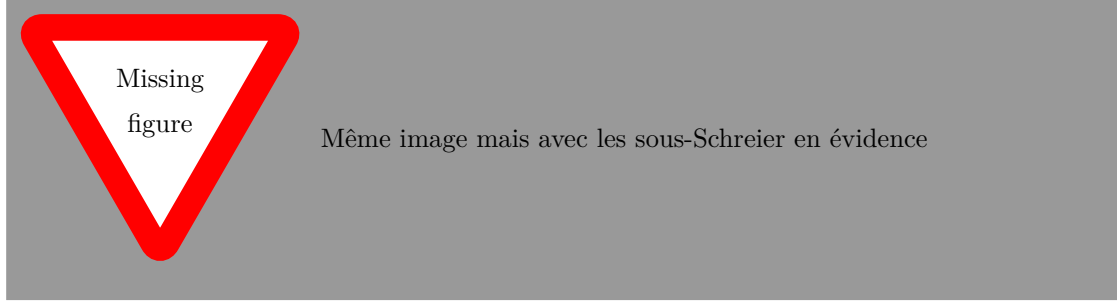


If we remove a vertex  $(g, x_0)$  we disconnect the leaf  $Y_g$  to the graph. As  $X$  is infinite, the number of ends is strictly greater than 1.

Suppose that  $G$  does not have the property FW. By the point 7 of the Proposition 2.3 there exists a subgroup  $K$  of  $G$  such that  $\text{Sch}(G, K, S)$  has more than one end. The group  $\Gamma$  acts on  $G/K \times X$ :

$$(\varphi, h)(gK, x) = (\varphi(hx)gK, hx).$$

As above, the action is transitive and the graph of this action is isomorphic to a Schreier graph. We decompose this graph into leaves in the same way. Now we look at the subgraph made up of vertices  $(g, x_0)$  and edges  $(\delta_{x_0}^s, 0)$  and we remark that it is isomorphic the Schreier graph  $\text{Sch}(G, K, S)$  which has more than one end. Then our graph has also more than one end.



Suppose that  $H$  does not have the property FW. There is a subgroup  $K$  of  $H$  such that  $\text{Sch}(H, K, S')$  has more than one end. We consider the action of  $\Gamma$  on  $H/K$  defined as

$$(\varphi, h)h'K = hh'K.$$

This is a transitive action. All the edges of type  $(\delta_{x_0}^s, 0)$  are loops and the edges  $(0, s')$  are the edges of  $\text{Sch}(H, K, S')$  which has more than one end. Then this Schreier graph has also more than one end.  $\square$

## References

- [1] P. A. Cherix, F. Martin, and A. Valette. Spaces with measured walls, the Haagerup property and property (T). *Ergod. Theory Dyn. Syst.*, 24(6):1895–1908, dec 2004.
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- [3] M. Neuhauser. Relative property (T) and related properties of wreath products. *Math. Zeitschrift*, 251(1):167–177, sep 2005.