Referee report for:

Property FW and wreath products of groups:
a simple approach using Schreier graphs

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The paper under review gives an elementary proof using Schreier graphs of the stability of property FW under wreath products. This stability result was originally proven by Cornulier. Property FW can be thought of a geometric weakening of property (T) (or property FH) where one doesn't require any action on a Hilbert space to have a fix point, but only particular ones, those coming from discrete wall spaces.

The paper is pleasant to read for someone (like the referee) who has never really thought of Schreier graphs, so it seems to fit the scope of Expositiones and I hence recommend publication, with the minor exposition changes suggested below:

~~P1 line 7 of the Intro which -> that~~
~~Definition 2.1: define orbit graphs and then put Cayley and Schreier as examples~~
~~P3 figure 3 H=?~~
~~P3 just before definition 2.2 "and all edges adjacent to them" -> "and all edges containing them"~~
~~P4 line 4 of section 2.2 "componentwise" unclear, isn't that pointwise?~~
~~P4 comments in the text below~~
~~P5 Proposition 2.6 is a pointless reformulation~~ and the top paragraph of the page could be reworked a bit.
~~P5 line 2 "is standard" -> "is a direct computation"~~

~~Remark: The words "easy" and "standard" should be avoided, especially in an expository paper, and replaced by more meaningful words like straightforward, direct computation, etc.~~

~~P5 Proof of Lemma 3.1: give a direct proof.~~
~~P6 comments in the text below~~
~~P7 Corollary 3.3 follows from quotients and from a direct argument without using Lemma 3.2~~
~~P7 proof of Lemma 3.5 that X is finite is already a direct argument~~
~~P8 "Suppose now…" should come before, and again make a direct argument.~~

Remark: Direct proofs (as opposed to proofs by contradiction) are usually more constructive and more pleasant to read, so I think that for such a paper it is worth putting in the extra effort.

purpose. A locally finite graph (i.e. such that every vertex has finite degree) is finite if and only if it has 0 end.

An important fact about the number of ends of a graph, is that it is an invariant of quasi-isometry, see [2]. In particular, if G is a finitely generated group it is possible to speak about the number of ends of the Schreier graph $\mathrm{Sch}(G, H; S)$ without specifying a particular finite generating set S. By a celebrated result of Hopf [9], the number of ends of a Cayley graph of a finitely generated group can only be 0, 1, 2 or infinite (in which case it is uncountable), see Figures 1 and 2 for some examples. On the other hand, Schreier graphs may have any number of ends in $\mathbf{N} \cup \{\infty\}$, see Figure 3 for an example of a graph with 4 ends. In fact, every regular graph of even degree is isomorphic to a Schreier graph, [8, 11].

We are now finally able to introduce property FW. Instead of giving the original definition in terms of actions on wall spaces, we will use an equivalent one for finitely generated groups, which essentially follows from [13], see [4] for a direct proof.

Definition 2.3. A finitely generated group G has *property FW* if all its Schreier graphs have at most one end.

It directly follows from the definition that all finite groups have property FW, but that $\mathbf{Z}$ does not have it. In fact, if G is a finitely generated group with an homomorphism onto $\mathbf{Z}$, then it does not have FW. Indeed, in this case $G \cong H \rtimes \mathbf{Z}$ for some H and the Schreier graph $\mathrm{Sch}(G, H; S)$ is isomorphic to a Cayley graph of $\mathbf{Z} \cong G/H$ and hence has 2 ends.

Property FW admits many distinct characterizations that allow to define it for groups that are non-necessarily finitely generated and even for topological groups. We refer the reader to [4] for a survey of these characterizations.

2.2 Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi\colon X \to G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

Pointwise ?

This is naturally a group, where multiplication is taken componentwise.

If H is a group acting on X, then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition.

Definition 2.4. Let G and H be groups and X be a set on which H acts. The *(retricted) wreath product* $G \wr_X H$ is the group $(\bigoplus_X G) \rtimes H$.

A prominent source of examples of wreath products is the ones of the form $G \wr_H H$, where H acts on itself by left multiplication. In particular, the group
is $(\mathbf{Z}/2\mathbf{Z}) \wr_{\mathbf{Z}} \mathbf{Z}$ has ~~become~~ well-known under the name of the *lamplighter group*. The
~~Other (trivial) examples of wreath products are~~ direct products $G \times H$ ~~which~~ corresponds to wreath products over a singleton $G \wr_{\{*\}} H$.

Let S be a generating set of G and T a generating set of H. Let $\{x_i\}_{i \in I}$ be a choice of a representative in each H-orbit. Finally, let δ_x^s be the element of

Follows from

Proof. The first part ~~is~~ Lemma 3.1.

For the second part, let S, respectively T, denote~~s~~ a finite generating set of N, respectively H. Then the group $G := N \rtimes H$ is finitely generated by $U = (S \times \{1\}) \cup (\{1\} \times T)$.

Suppose that both N and H have property FW. We want to show that every Schreier graph~~s~~ of G has at most one end. If they are all finite, then there is nothing to prove (and G is finite). So let Γ be an infinite Schreier graph of G with respect to the generating set U. The groups N and H act on the vertices of Γ by restriction of the action of G. That is, $n.x = (n, 1).x$ and $h.x = (1, h).x$. For x in … For each vertex x we define Γ_x^H (and respectively Γ_x^N) as the Schreier graph obtained from the action of H (respectively N) on the H-orbit (respectively N-orbit) of x. These are subgraphs of Γ. As N and H have property FW, the graphs Γ_x^H and Γ_x^N are either finite or one-ended. We want to prove that in this case Γ has exactly one end. Refers to what?

Let K be a finite set of vertices of Γ. If x is in K and Γ_x^H is finite, add all vertices of Γ_x^H to K. By doing so for every x in K, we obtain a new finite set $K \subset K'$ of vertices of Γ. K'⊃K We will show that $\Gamma \setminus K'$ has only one infinite connected component. By definition of K', if x is not in K', then either Γ_x^H has one end or Γ_x^H does not contain vertices of K'.

Let x and y be two vertices, each of them lying in some infinite connected component of $\Gamma \setminus K'$. We will construct a path from x to y in $\Gamma \setminus K'$ as a concatenation of three smaller paths, see Figure 4, as follows. First, a path in $\Gamma_x^H \setminus K'$ from x to some z, then a path in $\Gamma_z^N \setminus K'$ from z to some $z' \in (\Gamma_z^N \cap \Gamma_y^H) \setminus K'$, and finally a path in $\Gamma_y^H \setminus K'$ from z' to y. In order to finish the proof, it remains to exhibit elements z and z' and the three desired paths.

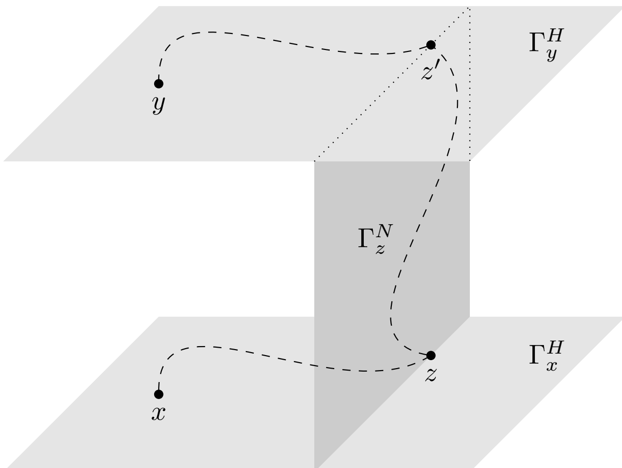

Figure 4: The path between x and y.

The action of G on Γ being transitive, there exists an element (n_0, h_0) of $N \rtimes H$ such that $(n_0, h_0).x = y$. Since K' is finite, the set $\Gamma_x^H \setminus K'$ is infinite. Moreover, there is infinitely many z in $\Gamma_x^H \setminus K'$ such that either Γ_z^N is one-ended or Γ_z^N does not intersect K'. For such a z there exists h such that $(1, h).x = z$. Now, the vertex $z' := (hh_0^{-1}.n_0, h).x$ is both equal to $(hh_0^{-1}.n_0, 1)(1, h_0).x = (hh_0^{-1}.n_0, 1).z$ and to $(1, hh_0^{-1})(n_0, h_0).x = (1, hh_0^{-1}).y$. That is, z' is in $\Gamma_z^N \cap \Gamma_y^H$. A simple computation shows us that the map $z \mapsto z'$ is injective: $z_1' = z_2'$ if and