The Property FW for the wreath products

Paul-Henry Leemann, Grégoire Schneeberger

Thursday 19th November, 2020 17:25

1 Introduction

The property FW was introduced by Barnhill and Chatterji. It is a fixed point property for the action on wall spaces (for a detailed treatment of this property see [4]). For discrete groups, this property is implied by the Kazhdan property (T). The behavior of the Kazhdan Property (T) with the wreath product is well known:

Give ref (cf article de Cornullier)

Theorem 1.1 ([3, 6])**.** *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property (T) if and only if G and H have the property (T) and X is finite.*

The same kind of result is true for the property FW

Theorem 1.2. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property FW if and only if G and H have the property FW and X is finite.*

An action of a group on a CAT(0) cube complex is *essential* if all the orbits of vertices are unbounded and the action is transitive on the set of hyperplanes.

Corollary 1.3. *Let G, H be two discrete groups and X a set on which H acts transitively. If there exists an essential action of G or H on a CAT(0) cube complex or if X is infinite, then there exists an essential action of $G \wr_X H$ on a CAT(0) cube complex.*

2 Definitions

2.1 The Property FW

For the definition of the property FW, we will follow the survey of Y. de Cornullier [4].

Definition 2.1. Let G be a discrete group and X a discrete set on which G acts. A subset $M \subset X$ is *commensurated* by the G-action if

$$|gM \Delta M| < \infty$$

for all g in G.

An invariant G-subset is automatically commensurated. Moreover, for a subset M such that there exists an invariant G-subset N with $|M \Delta N| < \infty$ then M is commensurated. Such a set is called *transfixed*.

Definition 2.2. A group G has the property FW if all commensurable G-set are transfixed.

There are lot of equivalent characterizations of this property. We will give us without all the details and the precise definitions.

Proposition 2.3. *The following are equivalent:*

1. *G has the property FW;*
2. *every cardinal definite function on G is bounded;*
3. *every cellular action on a CAT(0) cube complex has bounded orbits for the ℓ^1-metric (the complexes can be infinite dimensional);*
4. *every cellular action on a CAT(0) cube complex has a fixed point;*
5. *every action on a connected median graph has bounded orbits;*
6. *every action on a nonempty connected median graph has a finite orbit;*
7. *(if G is finitely generated) every Schreier graph of G has at most 1 end;*
8. *For every set Y endowed with a walling structure and compatible action on Y and on the index of the walling, the action on Y has bounded orbits for the wall distance;*
9. *every isometric action on an "integral Hilbert space" $\ell^2(X, \mathbf{Z})$ (X any discrete set), or equivalently on $\ell^2(X, \mathbf{R})$ preserving the integral points, has bounded orbits;*
10. *for every G-set X we have $H^1(G, \mathbf{Z}X) = 0$.*

Note that the name FW comes from the property of "fixed point" for the actions on the walling spaces. We will see in the following that a semi-splittable group does not have the property FW (see corollary **??**).

The property FW has links with other well known properties. For example, the property FH implies the characterisation 9. For discrete groups (and even for countable groups) the property FH is equivalent to the Kazhdan's property (T) by Delorme-Guichardet's Theorem. As trees are CAT(0) cube complexes, the property FW implies Serre's property FA.

3 Median graphs

In this section, we will investigate the property FW via the action of groups on median graphs.

For u and v two vertices of a connected graph $\mathcal{G}$, we define the total interval $[u, v]$ as the set of vertices that lies on some shortest path between u and v. A connected graph $\mathcal{G}$ is *median* if for any three vertices u, v, w, the intersection

$[u,v]\cap[v,w]\cap[u,w]$ consists of a unique vertex, denoted $m(u,v,w)$. A graph is median if each of its connected components is median.

Recall that a group G has property FW if and only if every G action on a connected median graph has bounded orbits.

We now prove a series of results that will be generalized to a broader context in Section 4. The following easy result is folklore, and we provide a proof only for the sake of completeness.

Lemma 3.1. *Let G be a group and H be a finite index subgroup. If H has FW, then so does G.*

Proof. Suppose that G does not have FW and let X be a connected median graph on which G acts with unbounded orbits. Then H acts on X and every G-orbit is a union of at most $[G:H]$ orbits. This directly implies that H acts on X with unbounded orbit and that H does not have FW. $\square$

We also have the following lemma on semi-direct products:

Lemma 3.2. *Let $G=N\rtimes H$ be a semidirect product. Then*

1. *If G has FW, then so does H.*
2. *If both N and H have FW, then G also has FW.*

Proof. Suppose that G has FW and let X be a non-empty connected median graph on which H acts. Then G acts on X by $g.x:=h.x$ where $g=nh$ with $n\in N$ and $h\in H$. By assumption, the action of G on X has bounded orbits and so does the action of H.

On the other hand, suppose that N and H have FW and let X be a non-empty connected median graph on which G acts. Then both N and H acts on X, with orbits bounded respectively by d_N and d_H. Now, for every $x\in X$ and $g\in G$, there is $n\in N$ and $h\in H$ such that $g=nh$ ans thus $g.x=n.(h.x)$ is at distance at most d_N+d_H from x. $\square$

Multiple applications of the above Lemma give us

Corollary 3.3. *Let $G\wr_X H$ be the wreath product of G and $H\curvearrowright X$. Then*

1. *If $G\wr_X H$ has FW, then so does H.*
2. *If G and H have FW and H acts on a finite set X, then $G\wr_X H$ has FW.*

We now turn back our attention on results that will rely more on the fact that we are using median graphs.

Trees are the simplest examples of median graphs and a simple verification shows that if $\mathcal{G}$ and $\mathcal{H}$ are both median graphs, then their cartesian product is also median. On the other hand, the following example will be fundamental for us.

Example 3.4. Let X be a set and let $\mathcal{P}_f(X)$ be the set of all its finite subsets. Define a graph structure on $\mathcal{P}_f(X)$ by putting an edge between E and F if and only if $\#(E\Delta F)=1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $E\Delta F$, while $[E,F]$ consist of all subsets of X that both contain $E\cap F$ and are contained in $E\cup F$. In particular $\mathcal{P}_f(X)$ is a connected median graph, with $m(D,E;F)$ being the set of all elements belonging to at least two of D, E and F. In other words, $m(D,E;F) = (D\cap E)\cup(D\cap F)\cup(E\cap F)$.

The same construction endows $\mathcal{P}(X)$ the set of all subsets of X with a structure of median graph, where the connected component of E consists of all F such that $E\Delta F$ is finite.

The graphs $\mathcal{P}_f(X)$ are exactly the hypercube and it turns out that every connected median graph is a retract of some $\mathcal{P}_f(X)$, see [1].

These graphs will be fundamental for us due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}_f(X)$ by graph homomorphisms: $g.\{x_1,\dots,x_n\} = \{g.x_1,\dots,g.x_n\}$.

Building on Example 3.4, we obtain that no infinite sum of groups has the property FW.

Lemma 3.5. *An infinite sum of non-trivial groups $G = \bigoplus_{i=1}^{\infty} G_i$ does not have FW.*

Proof. Let $X = \bigsqcup_{i=1}^{\infty} G_i$. There is a natural action of G on X: G_i acts by left multiplication on G_i and trivially on G_j for $j\neq i$. Therefore, we have an action of G on the connected median graph $\mathcal{P}_f(X)$. For every i, choose a non-trivial $g_i\in G_i$. Then the orbit of the vertex $\{1_{G_1},\dots,1_{G_n}\}$ contains the point $\{g_1,\dots,g_n\}$ which is at distance $2n$ of $\{1_{G_1},\dots,1_{G_n}\}$. That is the action of G on $\mathcal{P}_f(X)$ has unbounded orbits. □

As a direct corollary of Lemmas 3.5 and 3.2, we obtain

Corollary 3.6. *The group $\otimes_X G$ has FW if and only if X is finite and G has FW.*

We also have a converse of Lemma 3.1.

Lemma 3.7. *Let G be a group and H be a finite index subgroup. If G has FW, then so does H.*

Proof. Suppose that H does not have FW and let X be a connected median space on which H acts with unbounded orbits. This induces an action of G on $X^{G/H}$ by □ Finish the proof

We finally characterize which wreath products do have FW and hence provide a proof of Theorem 1.2.

Proposition 3.8. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G\wr_X H$ has the property FW if and only if G and H have the property FW and X is finite.*

Proof. In view of Corollary 3.3 it remains to show that if $G\wr_X H$ has FW, then G has FW and X is finite.

First, suppose that G does not have FW and let Y be a connected median graph on which G acts with unbounded orbits. Then $G\wr_X H$ acts on the connected median graph $Y\times\mathcal{P}_f(X)$ by

$$(\varphi,h).(y,E) = \begin{cases}(\varphi(h.t).y, h.\{t\}) & \text{if } E=\{t\}\\ (y, h.E) & \text{if } E \text{ is not a singleton}\end{cases}$$

But then the orbit of $(y, \{x\})$ is unbounded for every $x \in X$, which implies that $G \wr_X H$ does not have FW. Indeed, it contains

$$\{(\delta_x^g, 1).(y, \{x\}) = (g.y, \{x\}) \,|\, g \in G\},$$

where $\delta_x^g(x) = g$ and $\delta_x^g(z) = 1$ if $z \neq x$. And since all the $\{g.y, \{x\}\}_{g\in G}$ are in the same slice, the distance between them is the same as the distance between the $\{g.y\}_{g\in G}$.

Finally, suppose that X is infinite and let g be a non-trivial element of G. Then G acts naturally on the connected median graph $P_f(\bigsqcup_X G)$ and therefore $G \wr_X H$ acts on $\mathcal{P}_f(\bigsqcup_X G) \times \mathcal{P}_f(X)$. Let v be the the vertex $(\{1_1, \dots, 1_n\}, \emptyset)$, where $\{1_1, \dots, 1_n\}$ consists of n times the element $1_G \in G$ but living in distinct copies of G. The orbit of v contains the point $(\{g_1, \dots, g_n\}, \emptyset)$ which is at distance $2n$ of v. That is the action of $G \wr_X H$ on $\mathcal{P}_f(\bigsqcup_X G) \times \mathcal{P}_f(X)$ has unbounded orbits and $G \wr_X H$ has not FW. □

4 Related concepts and generalizations

This short section is devoted to generalize some of the results of Section 3 as well as to presente some properties related to the FW property.

Property FW does not stand alone and there are other similar properties that are of great interest.

Definition 4.1. Let G be a group. It is said to have *property SB* if any action on a metric space has bounded orbits. It has *property FH* if any action on a real Hilbert space has bounded orbits and *property FA* if any action on a tree has bounded orbits. Finally, G is said to have (cofinality$\neq \omega$) if any action on a ultrametric space has bounded orbits.

For countable groups (and more generally for σ-compact locally compact groups), property FH is equivalent to the celebrated Kazdhan's property (T) by the Delorme-Guichardet theorem, see for example [**?**], but this is not true in general [**?**].

Maybe expand a little more, or at least give some references

The names FH, FW and FA come from the fact that this property admit a description in terms of existence of a Fixed point for action on Hilbert spaces, on spaces with Walls and on trees (*Arbres* in french). On the other hand, SB stands for Strongly Bounded and is sometimes called the Bergman property, while a group has cofinality$\neq \omega$ if and only if it cannot be written has an increasing union of proper subgroups.

We have the following strict implications [**?**, **?**, **?**, **?**]

$$\text{SB} \implies \text{FH} \implies \text{FW} \implies \text{FA} \implies (\text{cofinality} \neq \omega).$$

It is possible to define other properties in the spirit of Definition 4.1. Let P be a property of metric spaces (for example *be a connected median graph*) and BP be the group property: *Every G-action on a space with P has bounded orbits*, where a G-action is supposed to "preserve the P-structure". Lemmas 3.1 and 3.2 as well as Corollary 3.3 as well as their proofs remain true for groups with property BP.

On the other hand, the other results of Section **??** require a specific construction and do not generalize straightforward to groups with property BP. Nevertheless, it is possible to extract the main ingredients of the proof and to adapt them in some specific cases. We will now give a raw outline of this process, but let the details to the interested reader.

Réécrire correctement ce qui suit. Veux-t-on parler de foncteurs ?

In order to generalize Example 3.4, we will need to construct from a G action on a set X a G action on a P-space Y. More precisely, we say that it is *possible to extend G actions to P-spaces* if there is a G-equivariant map i that associate to every G-set X a P-space Y endowed with a G action and a G-equivariant map $\iota\colon X \hookrightarrow Y$. For connected median graphs we took $Y = \mathcal{P}_f(X)$ in Example 3.4, while for real Hilbert spaces it is possible to take Y to be the real Hilbert space generated by X. That is,

$$Y = \ell^2(X) = \left\{ f\colon X \to \mathbf{R} \,\middle|\, \sum_{x \in X} f(x)^2 < \infty \right\}.$$

For property SB, we can take Y to be $\mathcal{P}_f(X)$ or $\ell^2(X)$, as well as many other possibilities. In the context of trees, is is possible to take $Y = X \sqcup \{*\}$ with for every $x \in X$ an edge between x and $*$, while fo ultrametric spaces it is always possible to take $Y = X$ with the discrete metric.

Let P be a property such that G actions extend to P-spaces. Observe that the map $\iota\colon X \hookrightarrow Y$ naturally extends to $\iota_*\colon \mathcal{P}_f(X) \hookrightarrow \mathcal{P}_f(Y)$, where $\iota_*(\{x_1, \dots, x_n\}) = \{\iota(x_1), \dots, \iota(x_n)\}$. Define

$$d_n \coloneqq \inf\{d(\iota_*(\{x_1, \dots, x_n\}), \iota_*(\{y_1, \dots, y_n\}) \mid \{x_1, \dots, x_n\} \cap \{y_1, \dots, y_n\} = \emptyset\}$$

Regarder comment écrire correctement la condition pour que le lemme 3.5 fonctionne.

We will say that P is a property such that G actions extend *unboundedly* to P-spaces if

Vérifier ce qui suit. Écrire le thm pour les SB spaces et regarder pour des références (Cornulier)

On the other hand, Lemma 3.7 only need that a product of two P-spaces is still a P-space, which holds for real Hilbert spaces, but once again fails for trees.

Finally, Proposition 3.8 needs both that a product of two P-spaces is still a P-space and that is is possible to extend G actions to P-spaces. In particular, we have

Proposition 4.2. *Let G, H be discret groups and X a H-set. Then the wreath product $G \wr_X H$ has the property (FH) if and only if G and H have the property (FH) and if X is finite*

Regarder si on peut enlever groupe discret et ne pas passer par le résultat pour (T)

The proof is a direct application of the equivalence of the properties (FH) and (T) for discrete groups (see [2]) and of the Theorem 1.1 .

Theorem 4.3 ([5]). *Let G, H be two groups and X be a non-empty H set which has only a finite number of orbits and no fixed point. Then the wreath product $G \wr_X H$ has the property (FA) if and only if H has the property (FA) and if G is a group with a finite abelianization, which cannot be expressed as an union of proper increasing sequence of subgroups.*

regarder si c'est plus fort que H possède la propriété (FA) et le lien avec le cardinal de X

Regarder le cas PW, Haagerup,... i.e. action propre. Rappel: une action isométrique de G est propre si pour tout x (de manière équivalente il existe x), pour tout $r \in \mathbf{R}$ l'ensemble $\{g \in G \,|\, d(x, g.x)\}$ est fini.

5 Ends of Schreier graph

Ajouter les petits lemmes sur les sous-groupes

For finitely generated groups, the point 7 of the Proposition 2.3 gives us a nice geometrical characteriation of the property FW. We will present more explicit and constructive proofs of the Proposition 3.8 in this contex.

We will begin by a short recall on Schreier graph. .

Utile de redéfinir les Schreier ?

Definition 5.1. Let G be a finitely generated group, H a subgroup of G and S a finite generating set. The *Cayley graph* of G with respect to H and S is the graph where the vertex are the cosets gH and two vertices gH and $g'H$ are adjacent if there exists a generator s such that $g'H = sgH$.

If a group G acts on a set X, we can define the graph of the action of G on the orbit of an element x as the graph where the vertex are the element of the orbit of x and two vertices are adjacent if they are linked by the action of a generator. The Schreier graphs are intimitely linked with group actions by the classical following lemma.

Lemma 5.2. *Let G be a finitely generated groups, S be a finite generating set and X be a G-set. Then for each element x of X, the graph of the action of G on the orbit of x is isomorphic to the Schreier graph $\mathrm{Sch}(G, \mathrm{Stab}(x), S)$.*

In fact all the Schreier graph can be viewed as graphs of actions, by considering the action of G on this Schreier graph.

As we want to work with finitely generated groupm the folowing lemma is useful.

Lemma 5.3. *Let G and H be groups and X be a H-set. The wreath product $G \wr_X H$ is finitely generated if and only if G and H are finitely generated and if the number of orbits of the action $H \curvearrowright X$ is finite.*

Proof. Suppose that G and H are finitely generated and that the number of orbits is finite. Let S be a finite generating set of G, S' a finite generating set of H and $\{x_0, \dots, x_n\}$ a representative system of the orbits. For s in S and x in X, we define δ_x^s as

$$\delta_x^s(y) = \begin{cases} e_G & y \neq x \\ s & y = x \end{cases}.$$

It is straightforward to prove that the set

$$\{(\delta_x^s, e_H) : s \in S, x \in \{x_0, \dots, x_n\}\} \ \cup \{(0, s') : s' \in S'\}$$

is a finite generating set of $G \wr_X H$, where $0(x) = e_g$ for all x in X.

If G or H are not finitely generated, it is clear that $G \wr_X H$ is not either. Suppose that there are infinitely many orbits. If φ is a function of $\bigoplus_X G$ whose support is include in $X_1 \sqcup \dots X_n$, then for every h in H, the support of $h.\varphi$ is also contain in $X_1 \sqcup \dots X_n$, where $X = X_1 \sqcup X_2 \sqcup \dots$ is a decomposition on disjoint orbits. As all the elements of $\bigoplus_X G$ have finite support, it is necessary to have infintely many such element to generate the whole group $G \wr_X H$. $\square$

Lemma 5.4. *Let $G = N \rtimes H$ be a finitely generated semidirect product. Then*

1. *If G has FW, then so does H.*
2. *If both N and H have FW, then G also has FW.*

Proof. It is well known that if N is generated by a set S and H by a set S', then G is generated by $S \cup S'$.

Suppose that H does not have the property FW. There exists a Schreier graph X of H with more than one end. The group G acts on X via

$$(n, h).x = hx$$

for a vertex x of X. This action $G \curvearrowright \Gamma$ is transitive because the action $H \curvearrowright \Gamma$ is. The graph of the action $G \curvearrowright X$ is exactly the graph X with some additional loops for generators of S'. As adding loops does not change the number of ends, this Schreier graph has more than one ends and the G does not have the property FW.

Suppose now that N and H have the property FW. We want to show that every Schreier graphs of G have at most one end. Let X be an infinite Schreier graph of G with respect to the generating set $S \cup S'$. The groups N and H acts on X via

$$n.x = (n, 0)x$$
$$h.x = (0, h)x$$

for a vertex x of X. For each such vertex x we define X_x^H (and respectively X_x^N) the Schreier graph obtained from the action of N (resp. H) on the orbit of x. This is subgraphs of X. As N and H have the property FW, the graphs X_x^H and X_x^N are finite or one-ended. We want to prove that in this case, X has at most one end.

Let K be a finite set of vertices of X. We will construct a path between every pair of vertices which are in infinite connected components. First, for every x in K, if X_x^H is finite, we add all the vertices of this subgraph in K. Adding vertices in K can not decrease the number of ends. Moreover we add a finite number of vertices in K. With this new set K, we have that for every x in $X \setminus K$, the subgraph $X_x^H \setminus K$ is equal to X_x^H or has one end.

Let x and y be two vertices of an infinite component of $X \setminus K$. The action of G on X is transitive, then there exists an element (n, h) of $N \rtimes H$ such that $(n, h).x = y$. We choose a vertex z in X_x^H such that X_z^N does not contain an element of K or is one-ended and such that $z' = (h'h^{-1}.n, 0)z$ is not in K, where h' is an element of H and $h'.x = z$. Such an element exists as X_x^H and X_y^H are infinite and K is finite. The vertex z' is in X_y^H because

$$(0, hh'^{-1})z' = (0, hh'^{-1})(h'h^{-1}.n, 0)(0, h')x = (n, h)x = y.$$

We will construct a path on $X \setminus K$ between x and y as follows. The subgraph X_x^H is one-ended, then there exists a path between x and z. In the same way, X_z^N has one end or has no vertex in K, then there is a path which join z and z'. There exists a path between z' and y in X_y^H which is one-ended. Then x and y are path-connected and then X has one end. We proved that all the Schreier graph of G

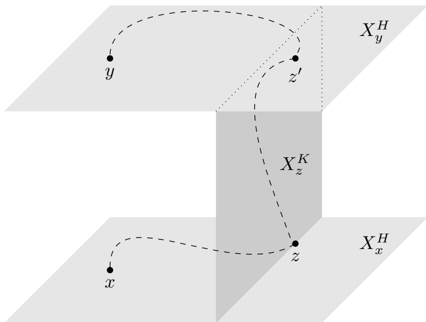

Figure 1: The path between x and y.

$\square$

Corollary 5.5. *Let G and H be two finitely generated groups and X a H-set such that the number of orbits is finite. Then,*

1. *If $G \wr_X H$ has the property FW, then so does H.*
2. *If G and H have the property FW and X is finite, then $G \wr_X H$ has FW.*

The proof is an iterated application of the previous lemma.

Lemma 5.6. *Let $\Gamma = G \wr_X H$ be a finitely generated wreath product and G be a non-trivial group. If Γ has the property FW, then*

1. *G has the property FW.*
2. *X is finite.*

Proof. We will prove the contrapositives. The idea is to construct, for each of the three cases, a Schreier graph of Γ with more than one end by using actions of Γ. We will suppose, as in the Lemma 5.2 that $G = \langle S \rangle$, $H = \langle S' \rangle$ and if there is no more condition x_0 is an arbitrary point of X.

Suppose that X is an infinite set. There exsits an infinite orbit. Ideed, as the Γ is finitely generated, we know by the Lemma 5.3 that the number of orbits of the action $H \curvearrowright X$ is finite. Let X' be such an infinite orbit and x_0 an element of X'. We define $Y = G \times X'$. The group Γ acts on Y via

$$(\varphi, h) \cdot (g, x) = (\varphi(hx)g, hx)$$

for (φ, h) in Γ and (g, x) in Y. The action is transitive. Indeed, let (g_1, x_1) and (g_2, x_2) be two elements of Y. By transitivity of the action of H on X', there exists h in H such that $hx_1 = x_2$ and φ in $\bigoplus_X G$ such that $\varphi(hx_1) = g_2 g_1^{-1}$. Then,

$$(\varphi, h)(g_1, x_1) = (\varphi(hx_1)g, hx_1) = (g_2, x_2).$$

The graph of the action of Γ on Y is isomorphic to the Schreier graph $\mathrm{Sch}(\Gamma, \mathrm{Stab}(e_G, x_0), \mathcal{S})$. We decompose the graph into leaves of the form $Y_g = \{g\} \times X'$. There are two types of edges in this graph which are coming from the two sets of generators. The first one, of the form $(0, s')$, give us on each leaf a copy of the graph of the action $H \curvearrowright X'$. Indeed,

$$(0, s')(g, x) = (g, s'x).$$

The second one, of the form $(\delta^s_{x_0}, 0)$, give us loops everywhere excepting on vertices of the form (g, x_0). By direct computation, we see that the vertices (g, x_0) and (sg, x_0) connect the leaves Y_g and Y_{sg},

$$(\delta^s_{x_0}, 0)(g, x) = \begin{cases} (g, x) & x \neq x_0 \\ (sg, x) & x = x_0 \end{cases}$$

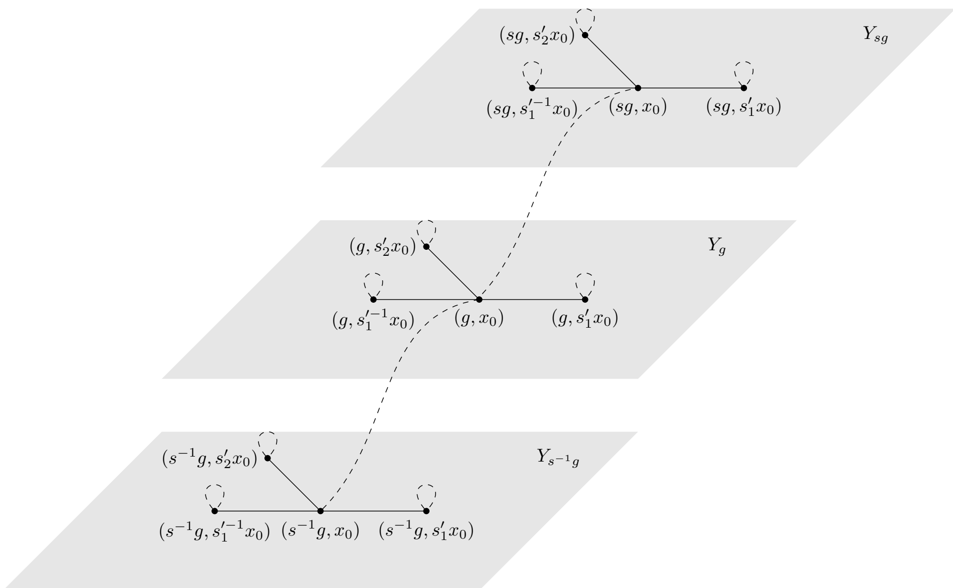

Figure 2: The leaf structure of the graph of the action $\Gamma \curvearrowright Y$.

If we remove a vertex (g, x_0) we disconnect the leaf Y_g to the graph. As X' is infinite, the number of ends is strictly greater than 1.

Suppose that G does not have the property FW. There exists a subgroup K of G such that $\mathrm{Sch}(G, K, S)$ has more than one end. The choice of x_0 is arbitrary in this case. The group Γ acts on $G/K \times X'$ via

$$(\varphi, h)(gK, x) = (\varphi(hx)gK, hx).$$

where X' is the orbit of x_0. As above, the action is transitive and the graph of this action is isomorphic to a Schreier graph. We decompose this graph into leaves in the same way. Now we look at the subgraph made up of vertices (g, x_0) and edges $(\delta^s_{x_0}, 0)$ and we remark that it is isomorphic the Schreier graph $\mathrm{Sch}(G, K, S)$ which has more than one end. Then our graph has also more than one end. $\square$

The following proposition is a direct application of Corollary 5.5 and Lemma 5.6.

Proposition 5.7. *Let G be a non trivial finitely generated group, H be a finitely generated group and X a set on which H acts with a finite number of orbit. The wreath product $G \wr_X H$ has the property FW if and only if G and H have the property FW and X is finite.*

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