Wreath products of groups acting with bounded orbits

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Abstract

If **S** is a subcategory of metric spaces, we say that a group G has property B**S** if any isometric action on an **S**-space has bounded orbits. Examples of such subcategories include metric spaces, affine real Hilbert spaces, CAT(0) cube complexes, connected median graphs, trees or ultra-metric spaces. The corresponding properties B**S** are respectively Bergman's property, property FH (which, for countable groups, is equivalent to the celebrated Kazhdan's property (T)), property FW (both for CAT(0) cube complexes and for connected median graphs), property FA and uncountable cofinality. Historically many of these properties were defined using the existence of fixed points.

Our main result is that for many subcategories **S**, the wreath product $G \wr_X H$ has property B**S** if and only if both G and H have property B**S** and X is finite. On one hand, this encompasses in a general setting previously known results for properties FH and FW. On the other hand, this also applies to the Bergman's property. Finally, we also obtain that $G \wr_X H$ has uncountable cofinality if and only if both G and H have uncountable cofinality and H acts on X with finitely many orbits.

1 Introduction

When working with group properties, it is natural to ask if they are stable under "natural" group operations. One such operation, of great use in geometric group theory, is the wreath product, see Section 2 for all the relevant definitions.

An **S**-space is a metric space with an "additional structure" and we will say that a group G has *property B**S*** if every action by isometries which preserves

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the structure on an **S**-space has bounded orbits. Formally, this means that **S** is a subcategory of the category of metric spaces, and that the actions are by **S**-automorphisms. We note that having one bounded orbit implies that all the orbits are bounded.

In the context of properties defined by actions with bounded orbits, the first result concerning wreath products, due to Cherix, Martin and Valette and later refined by Neuhauser, concerns Kazhdan's property (T).

Theorem 1.1 ([6, 17])**.** *Let G and H be two discrete groups with G non-trivial and let X be a set on which H acts. The wreath product $G \wr_X H$ has property (T) if and only if G and H have property (T) and X is finite.*

For countable groups (and more generally for σ-compact locally-compact topological groups), property (T) is equivalent, by the Delorme-Guichardet's Theorem, to property FH (every action on an affine real Hilbert space has bounded orbits), see [2, Thm. 2.12.4]. Hence, Theorem 1.1 can also be viewed, for countable groups, as a result on property FH.

The corresponding result for property FA (every action on a tree has bounded orbits) is a little more convoluted and was obtained a few years later by Cornulier and Kar.

Theorem 1.2 ([10])**.** *Let G and H be two groups with G non-trivial and let X be a set on which H acts with finitely many orbits and without fixed points. Then $G \wr_X H$ has property FA (respectively property F$\mathbf{R}$) if and only if H has property FA (respectively property F$\mathbf{R}$), G has no quotient isomorphic to $\mathbf{Z}$ and can not be written as a countable increasing union of proper subgroups.*

Observe that our statement of Theorem 1.2 differs of the original statement of [10]. Indeed, where we ask G to have uncountable cofinality and no quotient isomorphic to $\mathbf{Z}$, the authors of [10] ask G to have uncountable cofinality and finite abelianization. However, these two sets of conditions are equivalent. One implication is trivial, as finite Abelian groups do not project onto $\mathbf{Z}$. The other implication follows for countable Abelian groups from the structure theorem of finitely generated Abelian groups. For the general case, Y. Cornulier kindly reminded us that any infinite Abelian group has a countable quotient; the conclusion hence follows from the countable case.

Finally, we have an analogous of Theorem 1.1 for property FW (every action on wall spaces has bounded orbits):

Theorem 1.3 ([8, 15])**.** *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Suppose that all three of G, H and $G \wr_X H$ are finitely generated. Then the wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

It is straightforward to prove that the wreath product $G \wr_X H$ is finitely generated if and only if both G and H are finitely generated and the number of orbits of the action of H on X is finite.

Theorem 1.3 was first proved, using *cardinal definite functions*, for arbitrary groups by Cornulier [8], but without the implication "if $G \wr_X H$ has property FW,

then G has property FW". The authors then gave an elementary proof of it via *Schreier graphs* for the specific case of finitely generated groups [15]. Y. Stalder has let us know (private communication) that, using *space with walls* instead of Schreier graphs, the arguments of [15] can be adapted to replace the finite generation hypothesis of Theorem 1.3 by the condition that all three of G, H and X are at most countable. Finally, A. Genevois published in [11] a proof of Theorem 1.3 for wreath products of the form $G \wr_H H$, based on his *diadem product of spaces*.

The above results on properties FH, FW and FA were obtained with distinct methods even if the final results have a common flavor. In the same time, all three properties FH, FW and FA can be characterized by the fact that any isometric action on a suitable metric space (respectively affine real Hilbert space, connected median graph and tree) has bounded orbits, see Definition 2.4. But more group properties can be characterized in terms of actions with bounded orbits. This is for example the case of the Bergman's property (actions on metric spaces), property FB$_r$ (actions on reflexive real Banach spaces) or of uncountable cofinality (actions on ultrametric spaces).

By adopting the point of view of actions with bounded orbits, we obtain a unified proof of the following result; see also Theorem 3.13 for the general (and more technical) statement.

Theorem A. *Let BS be any one of the following properties: Bergman's property, property FB$_r$, property FH or property FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. Then the wreath product $G \wr_X H$ has property BS if and only if G and H have property BS and X is finite.*

With a little twist, we also obtain a similar result for groups with uncountable cofinality:

Proposition B. *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Then the wreath product $G \wr_X H$ has uncountable cofinality if and only if G and H have uncountable cofinality and H acts on X with finitely many orbits.*

A crucial ingredient of our proofs is that the spaces under consideration admit a natural notion of Cartesian product. In particular, some of our results do not work for trees and property FA, neither for real trees and the corresponding property F$\mathbf{R}$. Nevertheless, we are still able to show that if $G \wr_X H$ has property FA, then H acts on X with finitely many orbits. Combining this with Theorem 1.2 we obtain

Theorem C. *Let G and H be two groups with G non-trivial and X a set on which H acts. Suppose that H acts on X without fixed points. Then $G \wr_X H$ has property FA (respectively has property F$\mathbf{R}$) if and only if H has property FA (respectively has property F$\mathbf{R}$), H acts on X with finitely many orbits, G has no quotient isomorphic to $\mathbf{Z}$ and can not be written as a countable increasing union of proper subgroups.*

2 Definitions and examples

This section contains all the definitions, as well as some useful preliminary facts and some examples.

2.1 Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi \colon X \to G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken componentwise.

If H is a group acting on X, then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition

Definition 2.1. Let G and H be groups and X be a set on which H acts. The *(restricted [1]) wreath product $G \wr_X H$* is the group $(\bigoplus_X G) \rtimes H$.

A prominent particular case of wreath products is of the form $G \wr_H H$, where H acts on itself by left multiplication. They are sometimes called *standard wreath products* or simply *wreath products*, while general $G \wr_X H$ are sometimes called *permutational wreath products*. Best known example of wreath product is the so called *lamplighter group* $(\mathbf{Z}/2\mathbf{Z}) \wr_{\mathbf{Z}} \mathbf{Z}$. Other (trivial) examples of wreath products are direct products $G \times H$ which correspond to wreath products over a singleton $G \wr_{\{*\}} H$.

2.2 Classical actions with bounded orbits

In this subsection we discuss some classical group properties which are defined by actions with bounded orbits on various metric spaces.

Median graphs For u and v two vertices of a connected[2] graph $\mathcal{G}$, we define the total interval $[u, v]$ as the set of vertices that lie on some shortest path between u and v. A connected graph $\mathcal{G}$ is *median* if for any three vertices u, v, w, the intersection $[u, v] \cap [v, w] \cap [u, w]$ consists of a unique vertex, denoted $m(u, v, w)$. A graph is *median* if each of its connected components is median. For more on median graphs and spaces see [1, 5, 14]. If X and Y are both (connected) median graphs, then their Cartesian product is also a (connected) median graph. The class

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[1]There exists an unrestricted version of this product where the direct sum is remplaced by a direct product.

[2]We will always assume that our connected graphs are non-empty. This is coherent with the definition that a connected graph is a graph with exactly one connected component.

of median graphs was intruduced by Nebeský in 1971 [**?**] and Gerasimov [**?**, 12], Roller [**?**] and Chepoï [**?**] realized independently that this class coincides with the class of 1-skeleta of CAT(0) cube complexes. Trees are the simplest examples of connected median graphs, while the ensuing classical example will be important for Lemma 3.9.

Example 2.2. Let X be a set and let $\mathcal{P}(X) = 2^X$ be the set of all subsets of X. Define a graph structure on $\mathcal{P}(X)$ by putting an edge between E and F if and only if $\#(E\Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $\#(E\Delta F)$ and the connected component of E is the set of all subsets F with $E\Delta F$ finite. For E and F in the same connected component, $[E, F]$ consists of all subsets of X that both contain $E \cap F$ and are contained in $E \cup F$. In particular, $\mathcal{P}(X)$ is a median graph, with $m(D, E, F)$ being the set of all elements belonging to at least two of D, E and F. In other words, $m(D, E, F) = (D \cap E) \cup (D \cap F) \cup (E \cap F)$.

These graphs will be fundamental for us due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}(X)$ by graph homomorphisms: $g.Y := \{g.y \mid y \in Y\}$ for $Y \subset X$. Note that the action of G on $\mathcal{P}(X)$ may exchange the connected components. In fact, the connected component of $E \in \mathcal{P}(X)$ is stabilized by G if and only if E is *commensurated* by G, that is if for every $g \in G$ the set $E\Delta gE$ is finite.

Uncountable cofinality Recall that a metric space (X, d) is *ultrametric* if d satisfies the strong triangular inequality: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for any x, y and z in X. A group G has *uncountable cofinality* if every action on ultrametric spaces has bounded orbits. The following characterization of groups of countable cofinality can be extracted from [7] and we include a proof only for the sake of completeness. It implies in particular that a countable group has uncountable cofinality if and only if it is finitely generated.

Lemma 2.3. *Let G be a group. Then the following are equivalent:*

1. *G can be written as a countable increasing union of proper subgroups,*

2. *G does not have uncountable cofinality, i.e. there exists an ultrametric space X on which G acts with an unbounded orbit,*

3. *There exists a G-invariant (for the action by left multiplication) ultrametric d on G such that $G \curvearrowright G$ has an unbounded orbit.*

Proof. It is clear that the third item implies the second.

Let (X, d) be an ultrametric space on which G acts with an unbounded orbit $G.x_0$. For any $n \in \mathbf{N}$ let H_n be the subset of G defined by

$$H_n := \{g \in G \mid d(x_0, g.x_0) \leq n\}.$$

Then G is the union of the (countably many) H_n, which are subgroups of G. Indeed, H_n is trivially closed under taking the inverse, and is also closed under taking products as we have $d(x_0, gh.x_0) \leq \max\{d(x_0, g.x_0), d(g.x_0, gh.x_0)\} =$

$\max\{d(x_0, g.x_0), d(x_0, h.x_0)\}$. As $G.x_0$ is unbounded, they are proper subgroups. Since they are proper subgroups and $H_n \leq H_{n+1}$, we can extract an increasing subsequence $(H_{r_n})_n$ that still satisfies $G = \bigcup_n H_{r_n}$.

Finally, suppose that $G = \bigcup_{n \in \mathbf{N}} H_n$, where the H_n form an increasing sequence of proper subgroups. It is always possible to suppose that $H_0 = \{1\}$. Define d on G by $d(g, h) := \min\{n \mid g^{-1}h \in H_n\}$. One easily verifies that d is a G-invariant ultrametric. Moreover, the orbit of 1 contains all of G and is hence unbounded. $\quad\square$

Some classical group properties   We now discuss the bounded orbits properties for actions on various classes of metric spaces.

Definition 2.4. Let G be a group. It is said to have

- *Bergman's property* if any isometric action on a metric space has bounded orbits,

- *property FB_r* if any action by affine isometries on a reflexive real Banach space has bounded orbits,

- *property FH* if any action by affine isometries on a real Hilbert space has bounded orbits,

- *property FW* if any action by graph isomorphisms on a connected median graph has bounded orbits,

- *property F$\mathbf{R}$* if any isometric action on a real tree has bounded orbits,

- *property FA* if any action by graph isomorphisms on a tree has bounded orbits,

- *uncountable cofinality* if any action by isometries on an ultrametric space has bounded orbits.

In the above, we insisted on the fact that actions are supposed to preserve the structure of the metric space under consideration, or in other words to be by automorphisms of the considered category. This is sometimes automatic, as for example any isometry of a real Banach or Hilbert space is affine by the Mazur-Ulam theorem. However, for cube complexes, for example, this is not the case; a 2-regular tree has automorphism group $\mathbf{Z} \rtimes (\mathbf{Z}/2\mathbf{Z})$, while the automorphism group of its geometric realization (the line $\mathbf{R}$) is $\mathbf{R} \rtimes (\mathbf{Z}/2\mathbf{Z})$. In other words, an isometry of a cube complex is not necessarily a cube complex isomorphism. See also Example 2.8.

In the following, we will often do a slight abuse of notation and simply speak of a *group action* on a space X, without always specifying by which kinds of maps the group acts, which should always be clear from the context.

The names FB_r, FH, FW, F$\mathbf{R}$ and FA come from the fact that these properties admit a description in terms of (and were fist studied in the context of) the existence of a Fixed point for actions on reflexive Banach spaces, on Hilbert spaces, on spaces with Walls (or equivalently on CAT(0) cube complexes), on Real trees

and on trees (*Arbres* in french). However this is equivalent with the bounded orbit property, see Proposition 2.12 and the discussion below it.
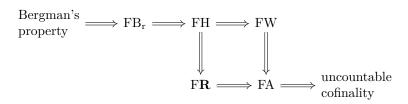
The Bergman's property can also be characterized via length function, see for example the beginning of [21].

For a survey on property FB_r, see [19] and the references therein.

For countable groups (and more generally for σ-compact locally compact groups), property FH is equivalent, by the Delorme-Guichardet theorem, to the celebrated Kazhdan's property (T), but this is not true in general. Indeed by **??**, symmetric groups over infinite sets are uncountable discrete group with Bergman's property which, as we will see just below, implies property FH. Such groups cannot have property (T) as, for discrete groups, it implies finite generation.

A classical result of the Bass-Serre theory of groups acting on trees [20], is that a group G has property FA if and only if it satisfies the following three conditions: G has uncountable cofinality, G has no quotient isomorphic to $\mathbf{Z}$ and G is not a non-trivial amalgam. In view of this characterization, Theorem 1.2 says that property FA almost behaves well under wreath products.

Proposition 2.5. *There are the following implications between the properties of Definition 2.4:*

$$
\begin{array}{ccccc}
\text{Bergman's} & & & & \\
\text{property} & \Longrightarrow & FB_r \Longrightarrow & FH \Longrightarrow & FW \\
& & & \Big\Downarrow & \Big\Downarrow \\
& & & F\mathbf{R} \Longrightarrow & FA \Longrightarrow \begin{array}{l}\text{uncountable} \\ \text{cofinality}\end{array}
\end{array}
$$

Moreover, except maybe for the implication [Bergman's property $\implies FB_r$], *all implications are strict.*

Proof. The implications [Bergman's property $\implies FB_r \implies FH$] and [FW $\implies$ FA $\impliedby$ F$\mathbf{R}$] trivially follow from the fact that Hilbert spaces are reflexive Banach spaces, which are themselves metric spaces and that trees are both real trees and connected median graphs. The implication [FH $\implies$ FW] follows from the fact that a group G has property FW if and only if any affine action on a real Hilbert space which preserves integral points has bounded orbits [8]. The implication [FH $\implies$ F$\mathbf{R}$] follows from the fact that real trees are median metric space, and that such spaces can be embedded into L^1-spaces (see for instance [**?**, Theorem V.2.4]). Finally, the implication [FA $\implies$ uncountable cofinality] is due to Serre [20]: if G is an increasing union of subgroups G_i, then $\bigsqcup G/G_i$ admits a tree structure by joining any $gG_i \in G/G_i$ to $gG_{i+1} \in G/G_{i+1}$. The action of G by multiplication on $\bigsqcup G/G_i$ is by graph isomorphisms and with unbounded orbits.

We now present some examples demonstrating the strictness of the implications. Countable groups with property FB_r are finite by [4], while infinite finitely generated groups with property (T), e.g. $SL_3(\mathbf{Z})$, have property FH. The group $SL_2(\mathbf{Z}[\sqrt{2}])$ has property FW but not FA, see [9]. If G is a non-trivial finite group and H is an infinite group with property FA (respectively property F$\mathbf{R}$), then

$G \wr_H H$ has property FA (respectively property F**R**) by Theorem C, but does not have property FW by Theorem A. The group **Z** has uncountable cofinality, while it acts by translations and with unbounded orbits on the infinite 2-regular tree. Finally, Minasyan constructed examples of groups with FA but without F**R** in [16]. $\square$

The reader familiar with triangles groups $\Delta(m, l, n) = \langle a, b, c \,|\, a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ca)^n = 1 \rangle$ with $l, m, n \in \{1, 2, \dots\} \cup \{\infty\}$ will be happy to observe that they provide explicit examples of groups with property FA but not property FW. Indeed, if l, m and n are all three integers, then $\Delta(m, l, n)$ has property FA by Serre [20, Section 6.5, Corollaire 2]. And if $\sigma(l, m, n) := \frac{1}{l} + \frac{1}{m} + \frac{1}{n} \leq 1$, then $\Delta(m, l, n)$ is the infinite symmetric group of a tilling of the Euclidean plane (if $\sigma(m, l, n) = 1$) or of the hyperbolic plane (if $\sigma(m, l, n) < 1$) and hence acts on a space with walls without fixed points.

In view of Proposition 2.5, two questions remain open: is the implication [Bergman's property $\implies$ FB$_r$] strict, and does property FW implies property F**R**.

It is possible to consider relative versions of the properties appearing in Definition 2.4. Let **S** be any classes of metric spaces considered in Definition 2.4 and let B**S** be the corresponding group property. If G is a group and H a subgroup of G, we say that the pair (G, H) has *relative property B**S*** if for every G action on an **S**-space, the H orbits are bounded. A group G has property B**S** if and only if for every subgroup H the pair (G, H) has relative property B**S**, and if and only if for every overgroup L the pair (L, G) has relative property B**S**.

2.3   Groups acting with bounded orbits on S-spaces

It is possible to define other properties in the spirit of Definition 2.4 for any "subclass of metric spaces", or more precisely for any subcategory of pseudo-metric spaces. A reader not familiar with category theory and interested only in one specific subclass of metric spaces may forget all these general considerations and only verify that the arguments of Section 3 apply for their favorite subclass of metric spaces.

Definition 2.6. A *pseudo-metric space* is a set X with a map $d \colon X \times X \to \mathbf{R}_{\geq 0}$, called a *pseudo-distance*, such that

1. $d(x, x) = 0$ for all $x \in X$,

2. $d(x, y) = d(y, x)$ for all $x, y \in X$,

3. $d(x, z) \leq d(x, y) + d(y, z)$.

If moreover $d(x, y) \neq 0$ for $x \neq y$, the map d is a *distance* and (X, d) is a metric space. On the other hand, an *ultra-pseudo-metric space* is a pseudo-metric space (X, d) such that d satisfies the strong triangular inequality: $d(x, z) \leq \max\{d(x, y), d(y, z)\}$. A *morphism* (or *short map*) between two pseudo-metric spaces (X_1, d_1) and (X_2, d_2) is a distance non-increasing map $f \colon X_1 \to X_2$, that

is $d_2(f(x), f(y)) \leq d_1(x, y)$ for any x and y in X_1. If f is bijective and distance preserving, then it is an *isomorphism* (or *isometry*). Pseudo-metric spaces with short maps form a category **PMet**, of which the category of metric spaces (with short maps) **Met** is a full subcategory.

If (X, d) is a pseudo-metric space, we have a natural notion of the *diameter* of a subset $Y \subset X$ with value in $[0, \infty]$, defined by $\mathrm{diam}(Y) := \sup\{d(x, y) \mid x, y \in Y\}$, and we say that Y is *bounded* if it has finite diameter.

Remind that a *subcategory of **PMet***, is a category **S** whose objects are pseudo-metric spaces, and whose morphisms are short maps. The subcategory **S** is *full* if given two **S**-objects X and Y, any short map from X to Y is a **S**-morphism. A *G action on an **S**-space* X is simply a group homomorphism $\alpha \colon G \to \mathrm{Aut}_{\mathbf{S}}(X)$.

In practice, a lot of examples of (full) subcategories of **PMet** are already subcategories of **Met**. Obvious examples of full subcategories of **Met** include metric spaces and ultrametric spaces (with short maps). Affine real Hilbert spaces, real Banach spaces and more generally normed vector spaces are also full subcategories of **Met** if we restrict ourself to morphisms that do not increase the norm (that is such that $\|f(x)\| \leq \|x\|$). In particular, for us isomorphisms of Hilbert and Banach spaces will always be affine isometries.

For connected graphs (and hence for its full subcategories of connected median graphs and of trees), one looks at the category **Graph** where objects are connected simple graphs $G = (V, E)$ and where a morphism $f \colon (V, E) \to (V', E')$ is a function between the vertex sets such that if (x, y) is an edge then either $f(x) = f(y)$ or $(f(x), f(y))$ is an edge. There are two natural ways to see **Graph** as a subcategory of **Met**. The first one consists to look at the vertex set V endowed with the shortest-path metric: $d(v, w)$ is the minimal number of edges on a path between v and w. The second one, consists at looking at the so called geometric realization of (V, E), where each edge is seen as an isometric copy of the segment $[0, 1]$. Similarly to what happens for cube complexes (see the discussion after Definition 2.4), the geometric realization of a graph gives an embedding **Graph** $\hookrightarrow$ **Met** which is not full. Nevertheless, for our purpose, the particular choice of one of the above two embeddings **Graph** $\hookrightarrow$ **Met** will make no difference.

We can now formally define the group property B**S** as:

Definition 2.7. Let **S** be a subcategory of **PMet**. A group G has *property* B**S** if every G action by **S**-automorphisms on an **S**-space has all its orbits bounded. A pair (G, H) of a group and a subgroup has *relative property* B**S** if for every G action by **S**-automorphisms on an **S**-space, the H-orbits are bounded.

Observe that a group G has property B**S** if and only any G action on an **S**-space has at least one bounded orbit.

All the properties of Definition 2.4 are of the form B**S**. Another example of property of the form B**S** can be found in [13, Definition 6.22]: a group has property (FHyp$_\mathbf{C}$) if any action on a complex or real hyperbolic space has bounded orbits. This property is implied by property FH, but does not imply property FA [13, Corollary 6.23 and Example 6.24]. One can also want to look at the category of all Banach spaces (the corresponding property B**S** hence stands between the

Bergman's property and property FB$_r$), or the category of L^p-spaces [3] (in this case B**S** is between FB$_r$ and FH).

Another interesting example of a property of the form B**S** is the fact to have no quotient isomorphic to **Z**, see Example 2.8. The main interest for us of this example is that property FA is the conjonction of three properties, two of them (uncountable cofinality and having no quotient isomorphic to **Z**) still being of the form B**S**.

Example 2.8. Let Z be the 2-regular tree, or in other words the Cayley graph of **Z** for the standard generating set. Then $\mathrm{Aut}_{\mathbf{Graph}}(Z) = \mathbf{Z} \rtimes (\mathbf{Z}/2\mathbf{Z})$ is the infinite dihedral group and its subgroup of orientation preserving isometries is isomorphic to **Z**. Let **S** be the category with one object Z and with morphisms the orientation preserving isometries. Hence, we obtain that a group G has no quotient isomorphic to **Z** if and only if every G action on **S**-space has bounded orbits. Let us denote by B**Z** this property.

Since Z is a tree, property FA implies property B**Z**. This implication is strict as demonstrated by **Q**. In fact, the counterexample **Q** shows that B**Z** does not imply uncountable cofinality, while **Z** demonstrate that uncountable cofinality does not implies B**Z**.

An example of an uninteresting property B**S** is given by taking **S** to be the category of metric spaces of bounded diameter (together with short maps), in which case any group has B**S**. On the opposite, if **S** is the category of extended pseudo-metric spaces (d takes values in $\mathbf{R}_+ \cup \{\infty\}$), only the trivial group has B**S**. Indeed, one can put the extended metric $d(x, y) = \infty$ if $x \neq y$ on G and then the action by left multiplication of G on (G, d) is transitive and with an unbounded orbit as soon as G is non trivial.

The category **PMet** has the advantage (over **Met**) of behaving more nicely with respect to categorical constructions and quotients. However, we have

Lemma 2.9. *A group G has Bergman's property (respectively uncountable cofinality) if and only if any isometric G action on a pseudo-metric (respectively ultra-pseudo-metric) space has bounded orbits.*

Proof. One direction is trivial. For the other direction, let (X, d) be a pseudo-metric space on which G acts by isometries. Let $X' := X/\sim$ be the quotient of X for the relation $x \sim y$ if $d(x, y) = 0$ and let d' be the quotient of d. Then (X', d') is a metric space, the action of G passes to the quotient and $G.x$ is d-bounded if and only if $G.[x]$ is d'-bounded. Finally, if d satisfies the strong triangular inequality, then so does d'. □

On the other hand, the following result is perhaps more surprising.

Lemma 2.10. *A group G has Bergman's property if and only if any G action by graph automorphisms on a connected graph has bounded orbits.*

Proof. The left-to-right implication is clear.

For the other direction, let X be a metric space. Let $G(X)$ denotes the graph obtained from a vertex-set X by applying the following process: for any two

$x, y \in X$ add a path of length $\lfloor d(x, y) \rfloor + 1$ between x and y. $G(X)$ is connected and the obvious inclusion $\iota \colon X \to G(X)$ is a quasi-isometric embedding. Moreover, the construction is canonical, so every group action on X extends to a group action on $G(X)$, making ι equivariant. So if a group satisfying the bounded orbit property on connected graphs acts on a metric space X, then its induced action on $G(X)$ has bounded orbits, which implies that its orbits in X are bounded. $\qquad\square$

Observe that an alternative proof of the above Lemma can be easily deduced from the following characterization of Bergman's property due to Cornulier [7]: a group G has Bergman's property if and only if it has uncountable cofinality and for every generating set T of G the Calyey graph $\mathrm{Cayl}(G; T)$ is bounded. Details are let to the reader.

While we will be able to obtain some results for a general subcategory of **PMet**, we will sometimes need to restrict ourselves to subcategories with a good notion of Cartesian powers.

Definition 2.11. A subcategory **S** of **PMet** has *finite bornological Cartesian powers* if for any **S**-space X and any integer n, there exists an **S**-object, called the nth *Cartesian power of X* and written X^n, such that:

1. As a set, X^n is the nth Cartesian power of X,

2. the canonical image of $\mathrm{Aut}_{\mathbf{S}}(X)^n \rtimes \mathrm{Sym}(n)$ in $\mathrm{Bij}(X^n)$ lies in $\mathrm{Aut}_{\mathbf{S}}(X^n)$,

3. If $E \subset X$ is unbounded, then the diagonal $\mathrm{diag}(E) \subset X^n$ is unbounded.

A good heuristic is that your favorite subcategory of **PMet** would have *finite bornological Cartesian powers* in the above sense if and only if it already has a classical operation which is called *Cartesian product*.

For (pseudo-) metric spaces, the categorical product (corresponding to the metric $d_\infty = \max\{d_X, d_Y\}$) works fine, but any product metric of the form $d_p = (d_X^p + d_Y^p)^{\frac{1}{p}}$ for $p \in [1, \infty]$ works as well. For (pseudo-) ultra-metric spaces, the categorical product d_∞ works fine. For Hilbert and (reflexive) Banach spaces, we take the usual Cartesian product (which is also the categorial product), which corresponds to the metric $d_2 = \sqrt{d_X^2 + d_Y^2}$. For connected median graphs, the usual Cartesian product (which is not the categorical product![3]) with $d_1 = d_X + d_Y$ works well. On the other hand, (real) trees do not have Cartesian powers, even without asking them to respect the bornology.

2.4 Variations and generalizations

This subsection is devoted to variations and generalizations of property B**S**. It is intended as a note for the interested reader, and can be skipped without any harm.

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[3]The categorial product in **Graph** is the strong product, while the category of median graphs does not have categorial products.

Groups acting with fixed point on S-spaces Some of the properties that are of interest for us have been historically defined via the existence of a fixed point for some action. More generally, we say that a group G has *property FS* if any G action on an **S**-space has a fixed point.

Since our actions are by isometries, property F**S** implies property B**S**. The other implication holds as soon as we have a suitable notion of the center of a (non-empty) bounded subset X. For a large class of metric spaces, this is provided by the following result of Bruhat and Tits:

Proposition 2.12 ([13, Chapter 3.b]). *Let (X, d) be a complete metric space such that the following two conditions are satisfied:*

1. *For all x and y in X, there exists a unique $m \in X$ (the middle of $[x, y]$) such that $d(x, m) = d(y, m) = \frac{1}{2}d(x, y)$;*

2. *For all x, y and z in X, if m is the middle of $[y, z]$ we have the median's inequality $2d(x, m)^2 + \frac{1}{2}d(y, z)^2 \leq d(x, y)^2 + d(x, z)^2$.*

Then if G is a group acting by isometries on X with a bounded orbit, it has a fixed point.

Exemples of complete metric spaces satisfying Proposition 2.12 include among others: Hilbert spaces, Bruhat-Tits Buildings, Hadamard spaces (i.e. complete CAT(0) spaces and in particular CAT(0) cube complexes which are either finite dimensional or locally finite), trees and **R**-trees; with the caveat that for trees and **R**-trees, the fixed point is either a vertex or the middle of an edge. See [13, Chapter 3.b] and the references therein for more on this subject. On the other hand, [2, Lemma 2.2.7] gives a simple proof of the existence of a center for bounded subsets of Hilbert spaces, and more generally of reflexive Banach spaces, but this also directly follows from the Ryll-Nardzewski fixed-point theorem. Finally, properties F**S** and B**S** are equivalent for the category of separable uniformly convex Banach spaces by the existence of the Chebyshev center of a (nonempty) bounded set.

For action on metric spaces or on connected median graphs, F**S** is strictly stronger than B**S**. Indeed, this trivially follows from the action by rotation of $\mathbf{Z}/4\mathbf{Z}$ on the square graph. However, by [?, 12] if a group G acts on a connected median graph with a bounded orbit, then it has a finite orbit. For actions on an ultra-metric spaces F**S** is strictly stronger than B**S** since the finite group $\mathbf{Z}/2\mathbf{Z}$ acts without fixed point on the Cantor space $X \subset [0, 1]$ by $x \mapsto 1 - x$. Property F**S** is also strictly stronger than B**S** for **S** the category of all Banach spaces. Indeed, by [18] any infinite discrete group admits an action without fixed point on some Banach space and hence does not have property F**S**, while there exists infinite groups with the Bergmann's property which implies property B**S**.

Actions with uniformly bounded orbits One might wonder what happens if in Definition 2.4 we replace the requirement of having bounded orbits by having uniformly bounded orbits. It turns out that this is rather uninteresting, as a group G is trivial if and only if any G action on a metric space (respectively on an Hilbert space, on a connected median graph, on a tree or on an ultrametric space) has

uniformly bounded orbits. Indeed, if G is non-trivial, then, for the action of G on the Hilbert space $\ell^2(G)$ the orbit of $n \cdot \delta_g$ has diameter $n\sqrt{2}$. For a tree (and hence also for a connected median graph), one may look at the tree T obtained by taking a root r on which we glue an infinite ray for each element of G. Then G naturally acts on T by permuting the rays. The orbits for this action are the $\mathcal{L}_n = \{v \mid d(v, r) = n\}$ which have diameter $2n$. Finally, it is possible to put an ultradistance on the vertices of T by $d_\infty(x, y) := \max\{d(x, r), d(y, r)\}$ if $x \neq y$. Then the orbits are still the $\mathcal{L}_n$, but this time with diameter n.

Topological groups One can wonder what happens for topological groups. While, the wreath product of topological groups is not in general a topological group, this is the case if G is discrete and X is a discrete set endowed with a continuous H action. In this particular context, Theorem 3.13, as well as its proof, remains true. The details are let to the interested reader.

Categorical generalizations In the above, we defined property B**S** for **S** a subcategory of **PMet**. It is possible to generalize this definition to more general categories. We are not aware of any example of the existence of a group property arising in this general context that is not equivalent to a property B**S** in the sense of Definition 2.7, but still mention it fur the curious reader.

On one hand, we can replace **PMet** with a more general category. For example, one can look at the category **M** of sets X endowed with a map $d\colon X \times X \to \mathbf{R}_{\geq 0}$ satisfying the triangle inequality. That is, d is a pseudo-distance, except that is is not necessary symmetric and $d(x, x)$ may be greater than 0. All the statements and the proofs remain true for **S** a subcategory of **M**.

On the other hand, we can define property B**S** for any category **S** over **PMet**, that is for any category **S** endowed with a faithful functor $F\colon \mathbf{S} \to \mathbf{PMet}$. Such a couple $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{PMet})$ is sometimes called a *structure over **PMet***, and F is said to be *forgetful*. In this context, we need to be careful to define finite bornological Cartesian powers (Definition 2.11) using F, but appart for that all the statements and all the proofs remained unchanged. An example of such an **S** that cannot be expressed as a subcategory of **PMet** is the category of edge-labeled graphs, where the morphisms are graph morphisms that induce a permutation on the set of labels. However, in this case the property B**S** is equivalent to the Bergman's property.

One can also combine the above two examples and look at couples $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{M})$, with F faithful.

Finally, in view of Definition 2.7 and 2.11, the reader might ask why we are working in **PMet** or **M** instead of **Born**, the category of bornological spaces together with bounded maps. The reason behind this is the forthcoming Lemma 3.2 and its corollaries, which fail for general bornological spaces. In fact, all the statements and the proofs remain true for a general $(\mathbf{S}, F\colon \mathbf{S} \to \mathbf{Born})$ as soon as **S** satisfies Lemma 3.2. Here is an example of such an **S** which does not appear as a category over **M**. Take κ to be an infinite cardinal and let $\mathbf{S}_\kappa$ be the subcategory of **Born** where a subset E of a $\mathbf{S}_\kappa$-space is bounded if and only if $|E| < \kappa$. A group G has property B$\mathbf{S}_\kappa$ if and only if $|G| < \kappa$.

3 Proofs of the main results

Throughout this section, **S** will denote subcategory of **PMet** and B**S** the group property *every action by **S**-automorphisms on an **S**-space has bounded orbits.*

We begin this section with two easy but useful results.

Lemma 3.1. *Let G be a group and H be a quotient. If G has property B**S**, then so has H.*

Proof. If H acts on some **S**-space X with an unbounded orbit, then the surjection $G \twoheadrightarrow H$ gives us a G action on X, with the same orbits as the H action. $\square$

Lemma 3.2. *Let G be a group and A an B be two subgroups such that $G = AB$. If both (G, A) and (G, B) have relative property B**S**, then G has property B**S**.*

Proof. Let X be an **S**-space on which G acts and let x be an element of X. Let D_1 be the diameter of $A.x$ and D_2 the diameter of $B.x$. By assumption, they both are finite. Since A acts by isometries, all the $a.Bx$ have diameter D_2. Let y be an element of $G.x$. There exists $a \in A$ such that y belongs to $a.Bx$. Since 1 belongs to B, y is at distance at most D_2 of $a.x$ and hence at distance at most $D_1 + D_2$ of x. Therefore, the diameter of $G.x$ is finite. $\square$

By combining Lemmas 3.1 and 3.2, we obtain the following three corollaries on direct, semi-direct and wreath products.

Corollary 3.3. *Let G and H be two groups. Then $G \times H$ has property B**S** if and only if both G and H have property B**S**.*

Corollary 3.4. *Let $N \rtimes H$ be a semidirect product. Then*

  1. *If $N \rtimes H$ has property B**S**, then so has H.*

  2. *If both N and H have property B**S**, then $N \rtimes H$ also has property B**S**.*

Corollary 3.5. *Let G and H be two groups and X a set on which H acts.Then,*

  1. *If $G \wr_X H$ has property B**S**, then so has H,*

  2. *If both G and H have property B**S** and X is finite, then $G \wr_X H$ has property B**S**.*

When **S** has a suitable notion of quotients (by a group of isometries), it is possible to obtain a strong version of Lemmas 3.1 and 3.2. Here is the corresponding result for Bergman's property and uncountable cofinality.

Proposition 3.6. *Let B**S** be either Bergman's property or the property of having uncountable cofinality. Let $1 \to N \to G \to H \to 1$ be a group extension. Then G has property B**S** if and only if H has property B**S** and the pair (G, N) has the relative B**S** property.*

Proof. One direction is simply Lemma 3.1 and the definition of relative property B**S**.

On the other hand, let (X, d) be a pseudo-metric space on which G acts by isometries and let x be an element of X. Let $\{g_i \mid i \in I\}$ be a transversal for N, that is $H \cong \{g_i N \mid i \in I\}$ with the quotient multiplication. By assumption, $N.x$ is bounded of diameter D_1 and for any $i \in I$ the subset $g_i N.x$ of X has also diameter D_1. Since N is a subgroup of isometries of X, the map $d' \colon X/N \times X/N \to \mathbf{R}$ defined by $d'([x], [y]) := \inf\{d(x', y') \mid x' \in N.x, y' \in N.y\}$ is the quotient pseudo-distance on X/N. Indeed, while the map d' might not satisfies the triangle inequality for a generic quotient $X/\sim$, this is the case if the quotient is by a subgroup of isometries; details are left to the reader. Moreover, if d satisfies the strong triangle inequality, then so does d'. The quotient action of $H \cong G/N$ on X/N is by isometries and the diameter of $H.xN$ is bounded, say by D_2. In particular, for any i and j in I, the distance between the subsets $g_i N.x$ and $g_j N.x$ of X is bounded by D_2. Since this distance is an infimum, there exist actual elements of $g_i N.x$ and $g_j N.x$ at distance less than $D_2 + 1$. Altogether, we obtain that any y in $G.x$ is at distance at most $D_1 + D_2 + 1$ of x. Hence, the orbit $G.x$ is bounded. $\square$

Since the triangle graph, which is not median, is a quotient of the 2-regular infinite tree by a subgroup of isometries, the proof of Proposition 3.6 does not carry over for properties FW and FA. Similarly, the quotient of $\mathbf{R}$ by the action of $\mathbf{Z}/2\mathbf{Z}$ given by $x \mapsto -x$ is not a Banach space and hence the proof of Proposition 3.6 does not apply to properties FH and FB$_r$. However, the statement of Proposition 3.6 (stability under extension) remains true for properties FH, FB$_r$, FW and FA. For properties FH and FB$_r$, this follows from the fixed-point definition and the fact that a non-empty closed subset of an Hilbert space (respectively of a reflexive Banach space) is an Hilbert space (respectively a reflexive Banach space) itself. For property FW and FA, see [8] and [20].

We now state a result on infinite direct sums.

Lemma 3.7. *Let G and $(G_x)_{x \in X}$ be non-trivial groups. Then*

1. *$\bigoplus_{x \in X} G_x$ has uncountable cofinality if and only if all the G_x have uncountable cofinality and X is finite,*

2. *If $G \wr_X H$ has uncountable cofinality, then H acts on X with finitely many orbits.*

It is of course possible to prove Lemma 3.7 using the characterization of uncountable cofinality in terms of subgroups, in which case the proof is a short exercise let to the reader. However, we find enlightening to prove it using the characterization in terms of actions on ultrametric spaces.

Proof of Lemma 3.7. One direction of the first assertion is simply Corollary 3.3 and holds for any property of the form B**S**. For the other direction, for any **S**, if $\bigoplus_{x \in X} G_x$ has property B**S** then all its quotients, and hence all the G_x, have property B**S**. Hence, we have to prove that an infinite direct sum of non-trivial groups

does not have uncountable cofinality. If X is infinite, there exists a countable subset $Y \subset X$. Let $Z := X \setminus Y$, thus we have $X = Y \sqcup Z$. We can decompose the direct sum as $\bigoplus_{x \in X} G_x = (\bigoplus_{x \in Y} G_x) \times (\bigoplus x \in Z G_x)$ and then, by Corollary 3.3, if $\bigoplus_{x \in Y} G_x$ does not have uncountable cofinality, then neither does $\bigoplus_{x \in X} G_x$. So let us fix an enumeration of Y and let $K := \bigoplus_{i \geq 1} G_i$ and for each i, choose $g_i \neq 1$ in G_i. Let $d_\infty(f, g) := \max\{i \mid f(i) \neq g(i)\}$. This is a K-invariant ultra-metric for the action by left multiplication of G on itself. Then for every integer n, the orbit $K.1_K$ contains $\{g_1, \ldots, g_n, 1, \ldots\}$ which is at distance n of 1_K for d_∞ if the g_i are not equal to 1.

The second assertion is a simple variation on the first. Indeed, we have

$$G \wr_X H \cong \big(\bigoplus_{Y \in X/H} L_Y \big) \rtimes H \qquad \text{with} \qquad L_Y \cong \bigoplus_{y \in Y} G_y,$$

where X/H is the set of H-orbits. The important fact for us is that H fixes the decomposition into L_Y factors: for all Y we have $H.L_Y = L_Y$. Up to regrouping some of the L_Y together we hence have $G \wr_X H \cong \big(\bigoplus_{i \geq 1} L_i \big) \rtimes H$ with $H.L_i = L_i$ for all i. Now, we have an ultradistance d_∞ on $L := \bigoplus_{i \geq 1} L_i$ as above and we can put the discrete distance d on H. Then $d'_\infty = \max\{d_\infty, d\}$ is an ultradistance on $\big(\bigoplus_{i \geq 1} L_i \big) \rtimes H$, which is $\big(\bigoplus_{i \geq 1} L_i \big) \rtimes H$-invariant (for the action by left multiplication). From a practical point of view, we have $d'_\infty\big((\varphi, h), (\varphi', h')\big) := \max\{i \mid \varphi(i) \neq \varphi'(i)\}$ if $\varphi \neq \varphi'$ and $d'_\infty\big((\varphi, h), (\varphi, h')\big) = 1$ if $h \neq h'$. Since the action of L on itself has an unbounded orbit for d_∞, the action of $\big(\bigoplus_{i \geq 1} L_i \big) \rtimes H$ on itself has an unbounded orbit for d'_∞. $\qquad\square$

We directly obtain

Corollary 3.8. *Suppose that B**S** implies having uncountable cofinality. Let G and $(G_x)_{x \in X}$ be non-trivial groups. Then*

1. *$\bigoplus_{x \in X} G_x$ has property B**S** if and only if all the G_x have property B**S** and X is finite,*

2. *If $G \wr_X H$ has property B**S**, then H acts on X with finitely many orbits.*

While the statement (and the proof) of Corollary 3.8 is expressed in terms of uncountable cofinality, it is also possible to state it and prove it for a subcategory **S** of **PMet** without a priori knowing if B**S** is stronger than having uncountable cofinality. The main idea is to find a "natural" **S**-space on which $G = \bigoplus_{i \geq 1} G_i$ acts. For example, for Hilbert spaces, one can take $\bigoplus_{i \geq 1} \ell^2(G_i)$. For connected median graphs, one takes the connected component of $\{1_{G_1}, 1_{G_2}, \ldots\}$ in $\mathcal{P}(\bigsqcup_{i \geq 1} G_i)$. For trees, it is possible to put a forest structure on $\mathcal{P}(\bigsqcup_{i \geq 1} G_i)$ in the following way. For $E \in \mathcal{P}(\bigsqcup_{i \geq 1} G_i)$, and for each i such that $E \cap G_j$ is empty for all $j \leq i$, add an edge from E to $E \cup \{g\}$ for each $g \in G_i$. The graph obtained this way is a G-invariant subforest of the median graph on $\mathcal{P}(\bigsqcup_{i \geq 1} G_i)$.

Lemma 3.9. *Suppose that B**S** implies FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. If $G \wr_X H$ has B**S**, then X is finite.*

Proof. We will prove that if X is infinite, then $G \wr_X H$ does not have property FW. Suppose that X is infinite. The group $\bigoplus_X G$ acts coordinatewise on $\bigsqcup_X G$: the group G_x acting by left multiplication on G_x and trivially on G_y for $y \neq x$. On the other hand, H acts on $\bigsqcup_X G$ by permutation of the factors. Altogether we have an action of $G \wr_X H$ on $\bigsqcup_X G$ and hence on the median graph $\mathcal{P}(\bigsqcup_X G)$. Let $\mathbf{1} := \bigcup_{x \in X} 1_G$ be the subset of $\mathcal{P}(\bigsqcup_X G)$ consisting of the identity elements of all the copies of G. Since every element of $\bigoplus_X G$ has only a finite number of non-trivial coordinates, the action of $G \wr_X H$ preserves the connected components of $\mathbf{1}$ (and in fact every connected component of $\mathcal{P}(\bigsqcup_X G)$).

Let $I = \{i_1, i_2, \dots\}$ be a countable subset of X and for every $i \in I$, choose a non-trivial $g_i \in G_i$. Then the orbit of the vertex $\mathbf{1}$ contains the point $\{g_{i_1}, \dots, g_{i_n}\} \cup \left(\bigcup_{j > n} 1_{G_{i_j}}\right) \cup \left(\bigcup_{x \notin I} 1_{G_x}\right)$ which is at distance $2n$ of $\mathbf{1}$. Therefore, the action of $G \wr_X H$ on the connected component of $\mathbf{1}$ has an unbounded orbit and then $G \wr_X H$ does not have property FW. $\qquad\square$

Once again, given a suitable $\mathbf{S}$, it is sometimes possible to give a direct proof of Lemma 3.9. For example, for Hilbert spaces one can take $\bigoplus_X \ell^2(G)$ with $\bigoplus_X G$ acting coordinatewise and H by permutations. On the other hand, both the forest structure on $\mathcal{P}(\bigsqcup_X G)$ and the ultrametric structure on $\bigoplus_X G$ are in general not invariant under the natural action of H by permutations. In fact, it follows from Theorem 1.2 and Proposition B that in the assumptions of Lemma 3.9 it is not possible to replace property FW by property FA or by having uncountable cofinality.

Remark 3.10. A reader familiar with wreath products might have recognized that we used the primitive action of the wreath product in the proof of Lemma 3.9.

Indeed, G acts on itself by left multiplication. It hence acts on the set $G' := G \sqcup \{\varepsilon\}$ by fixing ε, and we have the primitive action of $G \wr_X H$ on G'^X. Now, the set $\bigsqcup_X G$ naturally embeds as the subset of G'^X consisting of all functions $\varphi \colon X \to G'$ such that $\varphi(x) = \varepsilon$ for all but one $x \in X$. This subset is $G \wr_X H$ invariant, which gives us the desired action of $G \wr_X H$ on $\bigsqcup_X G$.

We now turn our attention to properties that behave well under Cartesian products in the sense of Definition 2.11.

We first describe the comportement of property B$\mathbf{S}$ under finite index subgroups.

Lemma 3.11. *Let G be a group and let H be a finite index subgroup.*

1. *If H has property B$\mathbf{S}$, then so has G,*

2. *If $\mathbf{S}$ has finite bornological Cartesian powers and G has property B$\mathbf{S}$, then H has property B$\mathbf{S}$.*

Proof. Suppose that G does not have B$\mathbf{S}$ and let X be an $\mathbf{S}$-space on which G acts with an unbounded orbit $\mathcal{O}$. Then H acts on X and $\mathcal{O}$ is a union of at most $[G : H]$ orbits. This directly implies that H has an unbounded orbit and therefore does not have B$\mathbf{S}$.

On the other hand, suppose that $H \leq G$ is a finite index subgroup of G without property B$\mathbf{S}$. Let $\alpha\colon H \curvearrowright X$ be an action of H on an $\mathbf{S}$-space (X, d_X) such that there is an unbounded orbit $\mathcal{O}$. Similarly to the classical theory of representations of finite groups, we have the induced action $\mathrm{Ind}_H^G(\alpha)\colon G \curvearrowright X^{G/H}$ on the set $X^{G/H}$. Since H has finite index, $X^{G/H}$ is an $\mathbf{S}$-space and the action is by $\mathbf{S}$-automorphisms. On the other hand, the subgroup $H \leq G$ acts diagonally on $X^{G/H}$, which implies that $\mathrm{diag}(\mathcal{O})$ is contained in a G-orbit. Since $\mathrm{diag}(\mathcal{O})$ is unbounded, G does not have property B$\mathbf{S}$.

For readers that are not familiar with representations of finite groups, here is the above argument in more details. Let $(f_i)_{i=1}^n$ be a transversal for G/H. The natural action of G on G/H gives rise to an action of G on $\{1, \ldots, n\}$. Hence, for any g in G and i in $\{1, \ldots, n\}$ there exists a unique $h_{g,i}$ in H such that $gf_i = f_{g.i}h_{g,i}$. That is, $h_{g,i} = f_{g.i}^{-1}gf_i$. We then define $g.(x_1, \ldots, x_n) :=$ $(h_{g,g^{-1}.1}.x_{g^{-1}.1}, \ldots, h_{g,g^{-1}.n}.x_{g^{-1}.n})$. This is indeed an action by $\mathbf{S}$-automorphisms on $X^{G/H}$ by Condition 2 of Definition 2.11. Moreover, every element $h \in H$ acts diagonally by $h.(x_1, \ldots, x_n) = (h.x_1, \ldots, h.x_n)$. In particular, this G action has an unbounded orbit. $\qquad\square$

We now prove one last lemma that will be necessary fo the proof of Theorem A.

Lemma 3.12. *Suppose that $\mathbf{S}$ has finite bornological Cartesian powers. If X is finite and $G \wr_X H$ has property B$\mathbf{S}$, then G has property B$\mathbf{S}$.*

Proof. Suppose that G does not have B$\mathbf{S}$ and let (Y, d_Y) be an $\mathbf{S}$-space on which G acts with an unbounded orbit $G.y$. Then (Y^X, d) is an $\mathbf{S}$-space and we have the *primitive action* of the wreath product $G \wr_X H$ on Y^X:

$$\big((\varphi, h).\psi\big)(x) = \varphi(h^{-1}.x).\psi(h^{-1}.x).$$

By Condition 2 of Definition 2.11, this action is by $\mathbf{S}$-automorphisms. The orbit $G.y$ embeds diagonally and hence $\mathrm{diag}(G.y)$ is an unbounded subset of some $G \wr_X H$-orbit, which implies that $G \wr_X H$ does not have property B$\mathbf{S}$. $\qquad\square$

Observe that it is also possible to derive Lemma 3.12 directly from Lemma 3.11 (2), with a more algebraic proof. Indeed, using the notation and hypothesis of Lemma 3.12, let H' be the kernel of the action of H on X and $\pi\colon G \wr_X H \to H$ be the canonical projection. Then $\pi^{-1}(H') \cong \bigoplus_X G \oplus H'$ is a finite index subgroup of $G \wr_X H$ and hence has property B$\mathbf{S}$. Since G is a quotient of $\bigoplus_X G \oplus H'$ we conclude that it also has property B$\mathbf{S}$.

By combining Corollary 3.5 and Lemmas 3.9 and 3.12 we obtain the following result which implies Theorem A.

Theorem 3.13. *Suppose that $\mathbf{S}$ has finite bornological Cartesian powers and is such that B$\mathbf{S}$ implies FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. Then the wreath product $G \wr_X H$ has property B$\mathbf{S}$ if and only if G and H have property B$\mathbf{S}$ and X is finite.*

We now proceed to prove Proposition B. As for Lemma 3.7, it is also possible to prove it using the characterization of uncountable cofinality in terms of subgroups, in which case it is an easy exercise, but we will only give a proof using the characterization in terms of actions on ultrametric spaces.

Proof of Proposition B. By Corollary 3.5 and Lemma 3.7 we already know that if $G \wr_X H$ has uncountable cofinality, then H has uncountable cofinality and it acts on X with finitely many orbits. We will now prove that if $G \wr_X H$ has uncountable cofinality so does G. Let us suppose that G has countable cofinality. By Lemma 2.3, there exists an ultrametric d on G such that the action of G on itself by left multiplication has an unbounded orbit. But then we have the primitive action of the wreath product $G \wr_X H$ on $G^X \cong \prod_X G$, which preserves $\bigoplus_X G$. It is easy to check that the map $d_\infty \colon \bigoplus_X G \times \bigoplus_X G \to \mathbf{R}$ defined by $d_\infty(\psi_1, \psi_2) \coloneqq \max\{d\big(\psi_1(x), \psi_2(x)\big) \mid x \in X\}$ is a $G \wr_X H$-invariant ultrametric. Finally, let $h \in G$ be an element of unbounded G-orbit for d and let x_0 be any element of X. For g in G and x in X, we define the following analogs of Kronecker's delta functions

$$\delta_x^g(y) \coloneqq \begin{cases} g & \text{if } y = x \\ 1 & \text{if } y \neq x. \end{cases}$$

Then for any g in G we have $(\delta_{x_0}^g, 1).\delta_{x_0}^h = \delta_{x_0}^{gh}$ and hence $d_\infty(\delta_{x_0}^h, \delta_{x_0}^{gh}) = d(h, gh)$ is unbounded.

Suppose now that both G and H have uncountable cofinality and that H acts on X with finitely many orbits. We want to prove that $G \wr_X H$ has uncountable cofinality.

Let (Y, d) be an ultrametric space on which $G \wr_X H$ acts. Then H and all the G_x act on Y with bounded orbits. Let $H.x_1, \ldots, H.x_n$ be the H orbits on X and let y be any element of Y. Then $H.y$ has finite diameter D_0 while $G_{x_i}.y$ has finite diameter D_i. For any $x \in X$, there exists $1 \leq i \leq n$ and $h \in H$ such that $x = h.x_i$. We have

$$d\big((\delta_{x_i}^g, h^{-1}).y, y\big) \leq \max\{d\big((\delta_{x_i}^g, h^{-1}).y, (\delta_{x_i}^g, 1).y\big), d\big((\delta_{x_i}^g, 1).y, y\big)\}$$
$$= \max\{d\big((1, h^{-1}).y, y\big), d\big((\delta_{x_i}^g, 1).y, y\big)\}$$
$$\leq \max\{D_0, D_i\},$$

which implies that the diameter of $G_{x_i} h^{-1}.y$ is bounded by $\max\{D_0, D_i\}$. But $G_{x_i} h^{-1}.y$ has the same diameter as $h G_{x_i} h^{-1}.y = G_{h.x_i}.y = G_x.y$.

On the other hand, the diameter of $\bigoplus_X G.y$ is bounded by the supremum of the diameters of the $G_x.y$, and hence bounded by $\max\{D_0, D_1, \ldots, D_n\}$. Finally, for (φ, h) in $G \wr_Y H$ we have

$$d\big(y, (\varphi, h).y\big) \leq \max\{d\big(y, (\varphi, 1).y\big), d\big((\varphi, 1).y, (\varphi, h).y\big)\}$$
$$= \max\{d\big(y, (\varphi, 1).y\big), d\big(y, (1, h).y\big)\}$$
$$\leq \max\{\max\{D_0, D_1, \ldots, D_n\}, D_0\}.$$

That is, the diameter of $G \wr_Y H.z$ is itself bounded by $\max\{D_0, D_1, \ldots, D_n\}$, which finishes the proof. $\qquad \square$

While the fact that trees do not have (bornological) Cartesian powers is an obstacle to our methods, we still have a weak version of Theorem 3.13 for properties FA and F$\boldsymbol{R}$. Before stating it, remind that we already know, by Proposition B, the behavior of uncountable cofinality under wreath products. On the other hand, we have the following result, which also admits a direct algebraic proof.

Lemma 3.14. *Suppose that H acts on X with finitely many orbits. Then the group $G \wr_X H$ has no quotient isomorphic to $\mathbf{Z}$ if and only if both G and H have no quotient isomorphic to $\mathbf{Z}$.*

Proof. Remind that a group K has no quotient isomorphic to $\mathbf{Z}$ if and only if K^{ab} has no quotient isomorphic to $\mathbf{Z}$. The desired result then follows from $(G \wr_X H)^{\mathrm{ab}} \cong \bigoplus_{X/H}(G^{\mathrm{ab}}) \times H^{\mathrm{ab}}$, Example 2.8 and Corollary 3.3. $\qquad\square$

By Corollary 3.5, Proposition B and Lemma 3.14, we directly obtain the following partial version of Theorem 1.2.

Proposition 3.15. *Let G and H be two groups with G non-trivial and X a set on which H acts. Then*

1. *If $G \wr_X H$ has property FA (respectively property F$\boldsymbol{R}$), then H has property FA (respectively property F$\boldsymbol{R}$), H acts on X with finitely many orbits, G has no quotient isomorphic to $\mathbf{Z}$ and G has uncountable cofinality,*

2. *If both G and H have no quotient isomorphic to $\mathbf{Z}$, have uncountable cofinality and H acts on X with finitely many orbits, then $G \wr_X H$ has no quotient isomorphic to $\mathbf{Z}$ and has uncountable cofinality,*

3. *If both G and H have property FA (respectively property F$\boldsymbol{R}$) and X is finite, then $G \wr_X H$ has property FA (respectively property F$\boldsymbol{R}$).*

Moreover, by using Lemma 3.7 we can get ride of the *finitely many orbits* hypothesis in Theorem 1.2 in order to obtain Theorem C.

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