

Wreath products of groups acting with bounded orbits

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Abstract

If S is a structure over metric spaces, we say that a group G has property BS if any action on a S -space has bounded orbits. Examples of such structures include metric spaces, Hilbert spaces, connected median graphs, trees or ultra-metric spaces. They correspond respectively to Bergman's property, property FH (which, for countable groups, is equivalent to the celebrated Kazhdan's property (T)), property FW, property FA and $\text{cof} \neq \omega$.

Our main result is that for a large class of structures S , the wreath product $G \wr_X H$ has property BS if and only if both G and H have property BS and X is finite. On one hand, this encompasses in a general setting previously known results for properties FH and FW. On the other hand, this also applies to the Bergman property. Finally, we also obtain that $G \wr_X H$ has $\text{cof} \neq \omega$ if and only if both G and H have $\text{cof} \neq \omega$ and H acts on X with finitely many orbits.

Question 1 : que peut-on dire des groupes topologiques ? Les mêmes preuves doivent marcher mutatis mutandis. Ça serait bien de regarder cela avant publications (éventuellement avant de mettre sur arXiv).

Question 2 : que peut-on dire de la propriété **FR** (n.b, **FR** est strictement plus forte que FA) ? Et des autres dérivés du type FA ? À priori ni plus ni moins que FA. Mais on peut au moins en parler vite fait quelque part.

Question 3 : que peut-on dire des extensions $1 \rightarrow N \rightarrow G \rightarrow H$? Cf sous-section 3.1.

1 Introduction

When working with group properties, it is natural to ask if they are stable under “natural” group operations. For example, one may wonder when a property is stable by subgroups, quotients, direct products or semi-direct products. A slightly less common operation, but still of great use in geometric group theory, is the wreath product which stands in-between the direct and the semi-direct product, see Section 2 for all the relevant definitions.

In this context, the following result on the celebrated Kazhdan's property (T) was obtained in the mid's 2000:

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Theorem 1.1 ([5, 14]). *Let G and H be two groups with G non-trivial and let X be a set on which H acts. The wreath product $G \wr_X H$ has property (T) if and only if G and H have property (T) and X is finite.*

For countable groups (and more generally for σ -compact locally-compact topological groups), property (T) is equivalent, by the Delorme-Guichardet's Theorem, to property FH, see [3, Thm. 2.12.4]. Hence, Theorem 1.1 can also be viewed, for countable groups, as a result on property FH.

The corresponding result for property FA is a little more convoluted and was obtained a few years later by Cornuier and Kar.

Theorem 1.2 ([7]). *Let G and H be two groups with G non-trivial and X a set on which H acts. Suppose that H acts on X with finitely many orbits and without fixed points. Then $G \wr_X H$ has property FA if and only if H has property FA, $G/[G : G]$ is finite and G has $\text{cof} \neq \omega$.*

Finally, in a recent note, the authors proved an analogous of Theorem 1.1 for property FW:

Theorem 1.3 ([12]). *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Suppose that all three of G , H and $G \wr_X H$ are finitely generated. Then the wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

The above results on property FH, FW and FA were obtained with distinct methods even if the final results share a common flavor. On the other hand, all three of properties FH, FW and FA can be characterized by the fact that any action on a suitable metric space (respectively Hilbert space, connected median graph and tree) has bounded orbits. But more group properties can be characterized in terms of actions with bounded orbits. This is for example the case of the Bergman's property (actions on metric spaces) or of $\text{cof} \neq \omega$ (actions on ultrametric spaces).

By adopting the point of view of actions with bounded orbits, we obtain an unified proof of the following.

Theorem 1.4. *Let BS be any one of the following properties: Bergman's property, FH or FW. Let G and H be two groups with G non-trivial and let X a set on which H acts. Then the wreath product $G \wr_X H$ has property BS if and only if G and H have property BS and X is finite.*

With a little twist, we also obtain a similar result for groups with $\text{cof} \neq \omega$:

Theorem 1.5. *Let G and H be two groups with G non-trivial and let X a set on which H acts. Then the wreath product $G \wr_X H$ has $\text{cof} \neq \omega$ if and only if G and H have $\text{cof} \neq \omega$ and H acts on X with finitely many orbits.*

A crucial ingredient of our proofs, is that the spaces under consideration admit a natural notion of Cartesian products. In particular, some of our results do not work for trees and property FA. Nevertheless, we are still able to show that if $G \wr_X H$ has property FA, then H acts on X with finitely many orbits. Combining this with Theorem 1.2 we obtain

Theorem 1.6. *Let G and H be two groups with G non-trivial and X a set on which H acts. Suppose that H acts on X without fixed points. Then $G \wr_X H$ has property FA if and only if H has property FA, H acts on X with finitely many orbits, $G/[G : G]$ is finite and G has $\text{cof} \neq \omega$.*

Organization of the paper The next section contains all the definitions as well as some examples. Section 3 is devoted to the proof of Theorems 1.4 and 1.5 as well as to some related results.

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2 Definitions and examples

This section contains all the definitions, as well as some useful preliminaries facts and some examples.

2.1 Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi: X \rightarrow G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken componentwise.

If H is a group acting on X , then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition

Definition 2.1. Let G and H be groups and X be a set on which H acts. The (retracted) wreath product $G \wr_X H$ is the group $(\bigoplus_X G) \rtimes H$.

A prominent source of examples of wreath products are the ones of the form $G \wr_H H$, where H acts on itself by left multiplication. They are sometimes called *standard wreath products* or simply *wreaths products*, while general $G \wr_X H$ are sometimes called *permutational wreath products*. The probably most well-known examples of wreath product is the so called *lamplighter group* $(\mathbb{Z}/2\mathbb{Z}) \wr_{\mathbb{Z}} \mathbb{Z}$. Another (trivial) examples of wreath products are direct products $G \times H$ which correspond to wreath products over a singleton $G \wr_{\{*\}} H$.

The following decomposition directly follows from the definition and we will use it in the rest of the article without mentioning it.

$$G \wr_X H \cong \bigoplus_{Y \text{ is an } H\text{-orbit}} (G \wr_Y H).$$

2.2 Actions with bounded orbits

Remind that a metric space (X, d) is *ultrametric* if and only if for any x, y and z in X we have $d(x, y) \leq \max\{d(x, z), d(z, y)\}$.

For u and v two vertices of a connected¹ graph \mathcal{G} , we define the total interval $[u, v]$ as the set of vertices that lie on some shortest path between u and v . A connected graph \mathcal{G} is *median* if for any three vertices u, v, w , the intersection $[u, v] \cap [v, w] \cap [u, w]$ consists of a unique vertex, denoted $m(u, v, w)$. A graph is *median* if each of its connected components is median. For surveys on median graphs see [11, 2].

¹We will always assume that our connected graphs are non-empty. This is coherent with the definition that a connected graph is a graph with exactly one connected component.

Maybe add refs with respect to FW.

Add refs

Definition 2.2. Let G be a group. It is said to have

- *property SB* if any action on a metric space has bounded orbits,
- *property FH* if any action on a real Hilbert space has bounded orbits,
- *property FW* if any action on a connected median graph has bounded orbits,
- *property FA* if any action on a tree has bounded orbits,
- $\text{cof} \neq \omega$ if any action on an ultrametric space has bounded orbits.

In the above, *actions* are supposed to preserve the structure. In particular, actions on (ultra)metric spaces are by isometries, actions on graphs (including trees) are by graph isomorphisms and actions on Hilbert spaces are by linear isometries.

For countable groups (and more generally for σ -compact locally compact groups), property FH is equivalent to the celebrated Kazhdan's property (T) by the Delorme-Guichardet theorem, see for example [?], but this is not true in general [?].

Maybe expand a little more.

The names FH, FW and FA come from the fact that these properties admit a description in terms of existence of a Fixed point for actions on Hilbert spaces, on spaces with Walls and on trees (*Arbres* in french). On the other hand, SB stands for Strongly Bounded and is also called the *Bergman's property*. Finally, a group has $\text{cof} \neq \omega$ (that is *not countable cofinality*) if and only if it cannot be written has a countable increasing union of proper subgroups, see Lemma 2.4.

We have the following strict implications between the properties of Definition 2.2 [?, ?, ?, ?]

$$\text{SB} \implies \text{FH} \implies \text{FW} \implies \text{FA} \implies \text{cof} \neq \omega. \quad (\dagger)$$

Trees are the simplest examples of median graphs and a simple verification shows that if X and Y are both (connected) median graphs, then their cartesian product is also a (connected) median graph. On the other hand, the following example will be fundamental for us.

Example 2.3. Let X be a set and let $\mathcal{P}(X) = 2^X$ be the set of all subsets of X . Define a graph structure on $\mathcal{P}(X)$ by putting an edge between E and F if and only if $\#(E \Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $E \Delta F$ and the connected component of E is the set of all subsets F with $E \Delta F$ finite. For E and F in the same connected component, $[E, F]$ consist of all subsets of X that both contain $E \cap F$ and are contained in $E \cup F$. In particular, $\mathcal{P}(X)$ is a median graph, with $m(D, E, F)$ being the set of all elements belonging to at least two of D , E and F . In other words, $m(D, E, F) = (D \cap E) \cup (D \cap F) \cup (E \cap F)$.

We denote by $\mathcal{P}_f(X)$, respectively $\mathcal{P}_{\text{cof}}(X)$ the set of all finite, respectively cofinite, subsets of X . They are connected components of $\mathcal{P}(X)$, which coincide if and only if X is finite. More generally, the connected components of $\mathcal{P}(X)$

Give refs

are hypercubes and it turns out that every connected median graph is a retract of a connected component of some $\mathcal{P}(X)$, see [1].

These graphs will be fundamental for us due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}(X)$ by graph homomorphisms: $g.\{x_1, \dots, x_n\} = \{g.x_1, \dots, g.x_n\}$. Be careful that the action of G on $\mathcal{P}(X)$ may exchange the connected components. In fact, the connected component of $E \subset X$ is stabilized by G if and only if E is *commensurated* by G , that is if for every $g \in G$ the set $E \Delta gE$ is finite. For example, both $\mathcal{P}_f(X)$ and $\mathcal{P}_{\text{cof}}(X)$ are always preserved by the action of G .

On the other hand, we will also use the following result on groups with $\text{cof} \neq \omega$:

Lemma 2.4. *Let G be a group. Then the following are equivalent:*

1. *There exists a G -invariant (for the action by left multiplication) ultrametric d on G such that $G \curvearrowright G$ has an unbounded orbit,*
2. *There exists an ultrametric space X on which G acts with an unbounded orbit,*
3. *G can be written as a countable increasing union of proper subgroups.*

Proof. It is clear that the first item implies the second.

Suppose that (X, d) is a ultrametric space on which G acts with an unbounded orbit and let x_0 be an element of X . For any $n \in \mathbb{N}$ let H_n be the subset of G defined by

$$H_n := \{g \in G \mid d(x_0, h.x_0) \leq n\}.$$

It is clear that G is the increasing union of the (countably many) H_n . On the other hand, the H_n are subgroups of G . Indeed, H_n is trivially closed under taking the inverse, and it is also closed under taking products since $d(x_0, gh.x_0) \leq \max\{d(x_0, g.x_0), d(g.x_0, gh.x_0)\} = \max\{d(x_0, g.x_0), d(x_0, h.x_0)\}$.

Finally, suppose that $G = \bigcup_{n \in \mathbb{N}} H_n$ where the H_n form an increasing sequence of proper subgroups. It is always possible to suppose that $H_1 = \{1\}$. Define d on G by $d(g, h) := \min\{n \mid g^{-1}h \in H_n\}$. One easily verify that d is a G -invariant ultrametric. Moreover, the orbit of 1 contains all of G and is hence unbounded. \square

A slight variation of the above lemma gives us

Example 2.5. Let $(G_i)_{i \geq 1}$ be non-trivial groups and let $G := \bigoplus_{i \geq 1} G_i$ be their direct sum. Then $d_\infty(f, g) := \max\{i \mid f(i) \neq g(i)\}$ is an ultra-distance on G which is G -invariant for the action of G on itself by left multiplication.

Finally, it is possible to define other properties in the spirit of Definition 2.2.

Definition 2.6. An *additional structure on metric spaces* is, a category **SMet** together with a faithful functor $F_S: \mathbf{SMet} \rightarrow \mathbf{Met}$, where **Met** is the category of metric spaces with short maps. The objects of **SMet** are called *S-spaces*. A *G-action on a S-space X* is simply an homomorphism $\alpha: G \rightarrow \text{Aut}_{\mathbf{SMet}}(X)$. It has *bounded orbits* if $F_S \circ \alpha: G \rightarrow \text{Aut}_{\mathbf{Met}}(X)$ has bounded orbits.

We can now, formally define the groupe property *BS* as: *Every G -action on a S-space has bounded orbits.*

In other words, an additional structure on metric spaces, is concrete category over **Met**. Since **Met** itself is concrete, that is we have a faithful functor $F: \mathbf{Met} \rightarrow \mathbf{Set}$, the category **SMet** is also concrete and its objects can be thought as set with “extra structure”.

Obvious examples of additional structures on metric spaces include, metric spaces, Hilbert spaces and ultrametric spaces. For connected graphs (and hence for connected median graphs and for trees), one look at the category **Graph** where objects are connected simple graphs $G = (V, E)$ and where a morphism $f: (V, E) \rightarrow (V', E')$ is a function between the vertex sets such that if (x, y) is an edge then either $f(x) = f(y)$ or $(f(x), f(y))$ is an edge. The functor $F_S: \mathbf{Graph} \rightarrow \mathbf{Met}$ sends a connected graph to its vertex set together with the graph distance: $d(x, y)$ is the minimum number of edges on a path between x and y .

An example of an uninteresting property BS is given by taking **SMet** to be the metric spaces of bounded diameter. Indeed, in this case, any group has BS.

While we will be able to obtain some results for a general **SMet**, we will sometimes need to restrict ourself to structure with a suitable notion of cartesian product.

Definition 2.7. A structure S on metric spaces has *compatible cartesian powers* if for any S -space X and any integer n , the set X^n admits an S -structure, called the *cartesian power of X* such that:

1. The projections $\pi_i: X^n \rightarrow X$ are S -homomorphisms,
2. The underlying metric space $F_S(X^n)$ is a product metric space, that is its underlying topology is the product topology on X^n ,
3. For any unbounded set $E \subset X$, the diagonal $\text{diag}(E) \subset X^n$ is unbounded,
4. For any S -automorphisms $\varphi \in \text{Aut}_{\mathbf{SMet}}(X)$ the bijective map $\varphi \times \text{id} \times \cdots \times \text{id}: X^n \rightarrow X^n$ is an S -automorphism,
5. The action by permutation of $\text{Sym}(n)$ on X^n is by S -automorphisms.

That is, we want our cartesian power to be compatible with the S -structure, with the topology, with the bornology and with group actions.

For (ultra)metric spaces, the categorical product (corresponding to the metric $d_\infty = \max\{d_X, d_Y\}$) works fine, but any product metric $d_p = (d_X^p + d_Y^p)^{\frac{1}{p}}$ for $p \in [1, \infty]$ works as well. For Hilbert spaces, we take the usual cartesian product (which is also the categorical product), which corresponds to the metric $d_2 = \sqrt{d_X^2 + d_Y^2}$. For connected median graph, the usual cartesian product (which is not the categorical product!²) with $d_1 = d_X + d_Y$ does the job. On the other hand, trees do not have compatible cartesian powers

We conclude this section by a remark on a variation of Definition 2.2. One might wonder what happens if in Definition 2.2 we replace the requirement of having bounded orbits by having uniformly bounded orbits. It turns out that this is rather uninteresting as a group G is trivial if and only if any G -action on a metric space (respectively on an Hilbert space, on a connected median graph, on a tree or on an ultrametric space) has uniformly bounded orbits. Indeed, if G is non-trivial, then for the action of G on the Hilbert space $l^2(G)$ the orbit

²The categorical product in **Graph** is the strong product.

of $n \cdot \delta_g$ as diameter $n\sqrt{2}$. For a tree (and hence also for a connected median graph), one may look at the tree T obtained by taking a root r on which we glue an infinite ray for each elements of G . Then G naturally acts on T by permuting the rays. The orbits for this action are the $\mathcal{L}_n = \{v \mid \text{at distance } n \text{ of } r\}$ which have diameter $2n$. Finally, it is possible to put an ultradistance on the vertices of T by $d_\infty(x, y) := \max\{d(x, r), d(y, r)\}$ if $x \neq y$. Then the orbits are still the \mathcal{L}_n , but this time with diameter n .

3 Proofs of the main results

We begin this section by insisting on the fact that when speaking of a property BS, it will always stand for one of the five group properties of Definition 2.2, that is SB, FH, FW, FA or $\text{cof} \neq \omega$. Similarly, a structure S will always stand for one of the five structures appearing in Definition 2.2, that is metric space, real Hilbert space, connected median graph, tree or ultrametric space. Indeed, we only guarantee the validity of our results in this restricted context, even if there might be considered as good heuristics in a more general setting.

Lemma 3.1. *Let G be a group and H be a quotient. If G has property BS, then so does H .*

Proof. We have $H \cong G/N$. If H acts on some S -space X with an unbounded orbit, then the G action on X defined by $g.x := gN.x$ has also an unbounded orbit. \square

We also have the following lemma on semi-direct products:

Lemma 3.2. *Let $N \rtimes H$ be a semidirect product. Then*

1. *If $N \rtimes H$ has property BS, then so does H .*
2. *If both N and H have property BS, then $N \rtimes H$ also has property BS.*

Proof. The first part is Lemma 3.1.

On the other hand, suppose that N and H have BS and let X be a S -space on which G acts. Then both N and H acts on X with bounded orbits. Let x be an element of X , D_1 be the diameter of $H.x$ and D_2 be the diameter of $N.x$. Since G acts by isometries, for every h in H the set $N.(h.x) = Nh.x = hN.x = h.(N.x)$ has also diameter D_2 . Therefore, every element of $G.x = NH.x$ is at distance at most $D_1 + D_2$ of x , which implies that the orbit $G.x$ is bounded. \square

As a direct corollary, we have

Corollary 3.3. *Let G and H be two groups. Then $G \times H$ has property BS if and only if both G and H have property BS.*

By iterating Lemma 3.2, we obtain

Corollary 3.4. *Let G and H be two groups and X a set on which H acts with finitely many orbits. Then,*

1. *If $G \wr_X H$ has property BS, then so does H ,*

2. If both G and H have property BS and X is finite, then $G \wr_X H$ has property BS.

On the other hand, we have the following result on infinite direct sums

Lemma 3.5. *Suppose that BS implies $\text{cof} \neq \omega$. Then*

1. *An infinite direct sum of non-trivial groups does not have BS,*
2. *If $G \neq \{1\}$, then $\bigoplus_X G$ has BS if and only if G has BS and X is finite,*
3. *If $G \wr_X H$ has BS, then H acts on X with finitely many orbits.*

Proof. By Corollary 3.3, it is enough to prove the first assertion for countable direct sums of groups. So let $G := \bigoplus_{i \geq 1} G_i$ and for each i , choose $g_i \neq 1$ in G_i . Let $d_\infty(f, g) := \max\{i \mid f(i) \neq g(i)\}$ be the G -invariant ultrametric of Example 2.5. Then for every integer n , the orbit $G \cdot 1_G$ contains $\{g_1, \dots, g_n, 1, \dots\}$ which is at distance n of 1_G for d_∞ . In particular, an infinite direct sum of non-trivial groups does not have $\text{cof} \neq \omega$, nor does it have BS.

The second and third assertions follows of the first assertion combined with Corollary 3.3. \square

Lemma 3.6. *Suppose that BS implies FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. If $G \wr_X H$ has BS, then X is finite.*

Proof. It is enough to prove the assertion for property FW. Suppose that X is infinite. The group $\bigoplus_X G$ acts coordinatewise on $\bigsqcup_X G$: the group G_x acting by left multiplication on G_x and trivially on G_y for $y \neq x$. On the other hand, H acts on $\bigsqcup_X G$ by permutation of the factors. Altogether we have an action of $G \wr_X H$ on $\bigsqcup_X G$ and hence on the median graph $\mathcal{P}(\bigsqcup_X G)$. Let $\mathbf{1} := \bigcup_{x \in X} 1_G$ be the subset of $\mathcal{P}(\bigsqcup_X G)$ consisting of the identity elements of all the copies of G . Since every element of $\bigoplus_X G$ has only a finite number of non-trivial coordinates, the action of $G \wr_X H$ preserves the connected components of $\mathbf{1}$ (and in fact every connected component of $\mathcal{P}(\bigsqcup_X G)$).

Let $I = \{i_1, i_2, \dots\}$ be a countable subset of X and for every $i \in I$, choose a non-trivial $g_i \in G_i$. Then the orbit of the vertex $\mathbf{1}$ contains the point $\{g_{i_1}, \dots, g_{i_n}\} \cup (\bigcup_{j > n} 1_{G_{j_i}}) \cup (\bigcup_{x \notin I} 1_{G_x})$ which is at distance $2n$ of $\mathbf{1}$. That is the action of $G \wr_X H$ on the connected component of $\mathbf{1}$ has an unbounded orbit. \square

It follows from Theorems 1.2 and 1.5 than in the assumptions of Lemma 3.6 it is not possible to replace property FW by property FA or by $\text{cof} \neq \omega$.

Remark 3.7. A reader familiar with wreath products might have recognized that we used the primitive action of the wreath product in the proof of Lemma 3.6.

Indeed, G acts on itself by left multiplication. It hence acts on the set $G' := G \sqcup \{\varepsilon\}$ by fixing ε , and we have the primitive action of $G \wr_X H$ on G'^X . Now, the set $\bigsqcup_X G$ naturally embeds as the subset of G'^X consisting of all functions $\varphi: X \rightarrow G'$ such that $\varphi(x) = \varepsilon$ for all but one $x \in X$. This subset is $G \wr_X H$ invariant, which gives us the desired action of $G \wr_X H$ on $\bigsqcup_X G$.

We now turn our attention to properties that behave well under products in the sense of Definition 2.7. We first describe the comportement of property BS under finite index subgroups.

Lemma 3.8. *Let G be a group and let H be a finite index subgroup.*

1. *If H has property BS, then so does G ,*
2. *If S has compatible cartesian powers and G has property BS, then H has property BS.*

Proof. Suppose that G does not have BS and let X be a S-space on which G acts with an unbounded orbit \mathcal{O} . Then H acts on X and \mathcal{O} is a union of at most $[G : H]$ orbits. This directly implies that H has an unbounded orbit and therefore does not have BS.

On the other hand, suppose that $H \leq G$ is a finite index subgroup of G without property BS. Let $\alpha: H \curvearrowright X$ be an action of H on a S-space (X, d_X) such that there is an unbounded orbit \mathcal{O} . Similarly to the classical theory of representation of finite groups, we have the induced action $\text{Ind}_H^G(\alpha): G \curvearrowright X^{G/H}$ on the set $X^{G/H}$. Since H has finite index, $X^{G/H}$ is a S-space and the action is by S-automorphisms. On the other hand, the subgroup $H \leq G$ acts diagonally on $X^{G/H}$, which implies that $\text{diag}(\mathcal{O})$ is contained in a G -orbit. Since $\text{diag}(\mathcal{O})$ is unbounded, G does not have property BS.

For readers that are not familiar with representations of finite groups, here is the above argument in more details. Let $(f_i)_{i=1}^n$ be a transversal for G/H . The natural action of G on G/H gives rise to an action of G on $\{1, \dots, n\}$. Hence, for any g in G and i in $\{1, \dots, n\}$ there exists a unique $h_{g,i}$ in H such that $gf_i = f_{g,i}h_{g,i}$. That is, $h_{g,i} = gf_i f_{g,i}^{-1}$. We then define $g.(x_1, \dots, x_n) := (h_{g,g^{-1}.1}.x_{g^{-1}.1}, \dots, h_{g,g^{-1}.n}.x_{g^{-1}.n})$. This is indeed an action, which, by Conditions 4 and 5 of Definition 2.7, is by S-automorphisms of $X^{G/H}$. Moreover, every element $h \in H$ acts diagonally by $h.(x_1, \dots, x_n) = (h.x_1, \dots, h.x_n)$. In particular, this G action has an unbounded orbit. □

check that

We now prove one last lemma that will be necessary for the proof of Theorem 1.4.

Lemma 3.9. *Let S be a structure that has compatible cartesian powers. If X is finite and $G \wr_X H$ has property BS, then G has property BS.*

Proof. Suppose that G does not have BS and let (Y, d_Y) be a S-space on which G acts with an unbounded orbit $G.y$. Then (Y^X, d) is a S-space and we have the primitive action of the wreath product $G \wr_X H$ on Y^X :

$$((\varphi, h).\psi)(x) = \varphi(h^{-1}.x)\psi(h^{-1}.x).$$

By Conditions 4 and 5 of Definition 2.7, this action is by S-automorphisms. The orbit $G.y$ embeds diagonally and hence $\text{diag}(G.y)$ is an unbounded subset of some $G \wr_X H$ -orbit, which implies that $G \wr_X H$ does not have property BS. □

check that

By combining Corollary 3.4 and Lemmas 3.6 and 3.9 we obtain the following result which implies Theorem 1.4.

Theorem 3.10. *Let S be a structure that has compatible cartesian powers and such that BS implies FW. Let G and H be two groups with G non-trivial and X a set on which H acts. Then the wreath product $G \wr_X H$ has property BS if and only if G and H have property BS and X is finite.*

We now proceed to prove Theorem 1.5. It is of course possible to prove it using the characterization of $\text{cof} \neq \omega$ in terms of subgroups. However, we find enlightening to prove it using the characterization in terms of actions on ultrametric spaces.

Theorem 3.11. *Let G and H be two groups with G non-trivial and let X a set on which H acts. Then the wreath product $G \wr_X H$ has $\text{cof} \neq \omega$ if and only if G and H have $\text{cof} \neq \omega$ and H acts on X with finitely many orbits.*

Proof. By Corollary 3.4 and Lemma 3.5 we already know that if $G \wr_X H$ has $\text{cof} \neq \omega$, then H has $\text{cof} \neq \omega$ and it acts on X with finitely many orbits. We will now prove that if $G \wr_X H$ has $\text{cof} \neq \omega$ so does G . Let us suppose that G has countable cofinality. Then there exists an ultrametric d on G such that the action of G on itself by left multiplication has an unbounded orbit. But then we have the primitive action of the wreath product $G \wr_X H$ on $G^X \cong \prod_X G$, which preserves $\bigoplus_X G$. It is easy to check that the map $d_\infty: \bigoplus_X G \times \bigoplus_X G \rightarrow \mathbf{R}$ defined by $d_\infty(\psi_1, \psi_2) := \max\{d(\psi_1(x), \psi_2(x)) \mid x \in X\}$ is a $G \wr_X H$ -invariant ultrametric. Finally, let $h \in G$ be an element of unbounded G -orbit for d and let x_0 be any element of X . Then for any g in G we have $(\delta_{x_0}^g, 1) \cdot \delta_{x_0}^h = \delta_{x_0}^{gh}$ and hence $d_\infty(\delta_{x_0}^h, \delta_{x_0}^{gh}) = d(h, gh)$ is unbounded.

Suppose now that both G and H have $\text{cof} \neq \omega$ and that H acts on X with finitely many orbits. We want to prove that $G \wr_X H$ has $\text{cof} \neq \omega$. Since H acts on X with finitely many orbit, by Corollary 3.3 it is enough to show that $G \wr_Y H$ has $\text{cof} \neq \omega$ when Y is one H -orbit. Let (Z, d) be an ultrametric space on which $G \wr_Y H$ acts. Then H and all the G_y act on Z with bounded orbits. Fix an element y_0 of Y and let z be an element of Z . Then $H.z$ has finite diameter D_1 while G_{y_0} has finite diameter D_2 . For any $y \in Y$, there exists $h \in H$ such that $y = h.y_0$. We have

$$\begin{aligned} d((\delta_{y_0}^g, h^{-1}).z, z) &\leq \max\{d((\delta_{y_0}^g, 1).z, z), d((\delta_{y_0}^g, h^{-1}).z, (\delta_{y_0}^g, 1).z)\} \\ &= \max\{d((\delta_{y_0}^g, 1).z, z), d((1, h^{-1}).z, z)\} \\ &\leq \max\{D_1, D_2\}, \end{aligned}$$

which implies that $G_y.z = G_{h.y_0}.z = hG_{y_0}h^{-1}.z$ has the same diameter as $G_{y_0}h^{-1}.z$, which is bounded by $\max\{D_1, D_2\}$. On the other hand, the diameter of $\bigoplus_Y G.z$ is bounded by the supremum of the diameters of the $G_y.z$, and hence also bounded by $\max\{D_1, D_2\}$. Finally, for (φ, h) in $G \wr_Y H$ we have

$$\begin{aligned} d(z, (\varphi, h).z) &\leq \max\{d(z, (\varphi, 1).z), d((\varphi, 1).z, (\varphi, h).z)\} \\ &= \max\{d(z, (\varphi, 1).z), d(z, (1, h).z)\} \\ &\leq \max\{\max\{D_1, D_2\}, D_1\}. \end{aligned}$$

That is, the diameter of $G \wr_Y H.z$ is itself bounded by $\max\{D_1, D_2\}$, which finishes the proof. \square

While the fact that being a tree is not compatible with products is an obstacle to our methods, we still have the following weak version of Theorem 3.10 for property FA.

Proposition 3.12. *Let G and H be two groups with G non-trivial and X a set on which H acts. Then*

1. If $G \wr_X H$ has property FA, then H has property FA and H acts on X with finitely many orbits,
2. If both G and H have property FA and X is finite, then $G \wr_X H$ has property FA.

By Proposition 3.12, it is possible to get ride of the “finitely many orbits” hypothesis in Theorem 1.2 to obtain.

Theorem 3.13. *Let G and H be two groups with G non-trivial and X a set on which H acts without fixed points. Then $G \wr_X H$ has property FA if and only if H has property FA, H acts on X with finitely many orbits, $G/[G : G]$ is finite and G has $\text{cof} \neq \omega$.*

Observe that, by the characterization of Serre [15], a group G has property FA if and only if it has no morphism onto \mathbf{Z} , it has $\text{cof} \neq \omega$ and it is not a non-trivial amalgam. In particular, a finitely generated group G has property FA if and only if $G/[G : G]$ is finite, G has $\text{cof} \neq \omega$ and G is not a non-trivial amalgam. We hence obtain

Proposition 3.14. *Let G and H be two groups with G non-trivial and finitely generated and X a set on which H acts. If $G \wr_X H$ has property FA, then H has property FA, H acts on X with finitely many orbits, $G/[G : G]$ is finite and G has $\text{cof} \neq \omega$.*

Proof. We already know that H acts with finitely many orbits. By Corollary 3.3, the group $G \wr_Y H$ has property FA for every H -orbit Y . Since the result holds if H has no fixed points, we can hence suppose that there is one orbit reduced to a point x_0 . But in this case $G \wr_{\{x_0\}} H \cong G \times H$ and by applying once again Corollary 3.3 we obtain that G has property FA. Since G is finitely generated, this implies the desired result. \square

3.1 On group extensions

Ce qui suit est une note interne. Si on arrive à en faire quelque chose, on le poussera dans le corps du texte.

Definition 3.15. Let G be a group and H be a subgroup. The pair (G, H) has the *relative property BS* if for any G -action on a S -space, the H -orbits are bounded.

Observe that G has property BS if and only if (G, G) has relative property BS. On the other hand, if G has property BS, then for any $H \leq G \leq L$ both (L, G) and (G, H) have relative property BS.

Lemma 3.16. *Let $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ be a group extension. If G has property BS, then H has property BS and (G, N) has relative property BS.*

Proof. One part is Lemma 3.1, while the other part is trivial. \square

Question 3.17. *For which properties BS, the converse of Lemma 3.16 holds?*

By [6], this is the case for property FW. A crude idea would be the following: let X be a S-space on which G acts. Then the action of N on X has bounded orbits. Moreover, $H = G/N$ acts on X/N

C'est là qu'il y a un problème. En effet, rien ne garantit que X/N soit un P-espace. Il faudrait déjà que ce soit un espace métrique (en général un quotient d'un espace métrique est seulement pseudo-métrique). Et même si on regarde le quotient métrique (on identifie les points à distance 0) de l'espace pseudo-métrique X/N ce n'est à priori pas un P-espace. Par exemple, on peut regarder $G := \mathbf{Z}$ agissant sur lui-même, vu comme un arbre (et donc un graphe médian), par translation. Si on quotiente par $3\mathbf{Z}$, on obtient un 3-cycle qui n'est donc ni un arbre, ni un graphe médian :-/

References

- [1] H. J. Bandelt. Retracts of hypercubes. *J. Graph Theory*, 8(4):501–510, 1984.
- [2] Hans-Jürgen Bandelt and Victor Chepoi. Metric graph theory and geometry: a survey. In *Surv. Discret. Comput. Geom.*, volume 453 of *Contemp. Math.*, pages 49–86. Amer. Math. Soc., Providence, RI, 2008.
- [3] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property ($\{T\}$)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.
- [4] Stephen G Brick. Quasi-isometries and ends of groups. *J. Pure Appl. Algebr.*, 86(1):23–33, 1993.
- [5] Pierre-Alain Cherix, Florian Martin, and Alain Valette. Spaces with measured walls, the Haagerup property and property (T). *Ergodic Theory Dynam. Systems*, 24(6):1895–1908, 2004.
- [6] Yves Cornulier. Group actions with commensurated subsets, wallings and cubings. *arXiv e-prints*, page arXiv:1302.5982, February 2013.
- [7] Yves Cornulier and Aditi Kar. On property (FA) for wreath products. *J. Group Theory*, 14(1):165–174, 2011.
- [8] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.
- [9] Reinhard Diestel and Daniela Kühn. Graph-theoretical versus topological ends of graphs. *J. Combin. Theory Ser. B*, 87(1):197–206, 2003. Dedicated to Crispin St. J. A. Nash-Williams.
- [10] Jonathan L. Gross. Every connected regular graph of even degree is a Schreier coset graph. *J. Combinatorial Theory Ser. B*, 22(3):227–232, 1977.
- [11] Sandi Klavžar and Henry Martyn Mulder. Median graphs: characterizations, location theory and related structures. *J. Comb. Math. Comb. Comput.*, 30:103–127, 1999.

- [12] Paul-Henry Leemann and Grégoire Schneeberger. Property FW and wreath products of groups: a simple approach using Schreier graphs. *arXiv e-prints*, page arXiv:2101.03817, January 2021.
- [13] Alexander Lubotzky. Cayley graphs: eigenvalues, expanders and random walks. In *Surveys in combinatorics, 1995 (Stirling)*, volume 218 of *London Math. Soc. Lecture Note Ser.*, pages 155–189. Cambridge Univ. Press, Cambridge, 1995.
- [14] Markus Neuhauser. Relative property (T) and related properties of wreath products. *Math. Z.*, 251(1):167–177, 2005.
- [15] Jean-Pierre Serre. *Arbres, amalgames, SL_2* . Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.
- [16] John Stallings. *Group theory and three-dimensional manifolds*. Yale University Press, New Haven, Conn.-London, 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969, Yale Mathematical Monographs, 4.

Vieux trucs pour les avoirs sous la main. À enlever de la version finale.

A Definitions and examples

This section contains all the definitions, as well as some useful preliminaries facts and some examples. All the results cited in this section are standard.

A.1 Ends of Schreier graphs and property FW

In what follows, we will always assume that generating sets of groups are *symmetric*, that is we will look at $S \subset G$ such that $s \in S$ if and only if $s^{-1} \in S$. Our graphs will be undirected and we will sometimes identify a graph with its vertex set.

Definition A.1. Let G be a group, H a subgroup of G and S a symmetric generating set. The (left) *Schreier graph* $\text{Sch}(G, H; S)$ is the graph with vertices the left cosets $gH = \{gh \mid h \in H\}$ and for every $\{s, s^{-1}\} \subset S$ an edge between gH and sgH .

Observe that with the above definition, a Schreier graph may have loops as well as multiple edges. Moreover, in $\text{Sch}(G, H; S)$ every vertex has degree $|S|$.

If X is a set with a left action $G \curvearrowright X$ and S is a symmetric generating set for G , the corresponding (left) *orbital graph* $\text{Sch}_\mathcal{O}(G, X; S)$ is the graph with vertex set X and with an edge between x and y for every s in S such that $s.x = y$. As the notation suggest, these two definitions are the two faces of the same coin. Indeed, we have $\text{Sch}(G, H; S) = \text{Sch}_\mathcal{O}(G, G/H; S)$ for the action by left multiplication of G on G/H , while for every $x \in X$, the graph $\text{Sch}(G, \text{Stab}_G(x); S)$ is equal to the connected component of x in $\text{Sch}_\mathcal{O}(G, X; S)$.

Schreier graphs are generalizations of the well-known *Cayley graphs*, with $\text{Cayl}(G; S) = \text{Sch}(G, \{1\}; S)$, see Figures ?? and ?? for some examples.

If S and T are two generating sets of G , the graphs $\text{Sch}(G, H; S)$ and $\text{Sch}(G, H; T)$ need not to be isomorphic. However, if both S and T are finite, then $\text{Sch}(G, H; S)$ and $\text{Sch}(G, H; T)$ are quasi-isometric, see [8, IV.B.21.iii] for a proof and Figure ?? for an example. The only fact we will use about quasi-isometries is that they preserve “large-scale properties” of the graph, as for example the number of ends. Observe that the requirement that both S and T are finite is crucial for the existence of a quasi-isometry between the corresponding Schreier graphs. Indeed, for every group $\text{Cayl}(G; G \setminus \{1\}) = \text{Sch}(G, \{1\}; G \setminus \{1\})$ is always a complete graph on $|G|$ vertices, in particular $\text{Cayl}(\mathbf{Z}; \{1\})$ and $\text{Cayl}(\mathbf{Z}; \mathbf{Z} \setminus \{0\})$ are not quasi-isometric.

Let Γ be a graph and K a finite subset of vertices. The graph $\Gamma \setminus K$ is the subgraph of Γ obtained by deleting all vertices in K and all edges adjacent to them. This graph is not necessarily connected.

Definition A.2. Let Γ be a graph. The *number of ends* of Γ is the supremum, taken over all finite K , of the number of infinite connected components of $\Gamma \setminus K$.

There exists other characterization of the number of ends in graphs, see [9] and the references therein, but Definition A.2 is the one that best suits our purpose. A locally finite graph (i.e. such that every vertex has finite degree) is finite if and only if it has 0 end.

An important fact about the number of ends of a graph, is that it is an invariant of quasi-isometry, see [4]. In particular, if G is a finitely generated group it is possible to speak about the number of ends of the Schreier graph $\text{Sch}(G, H; S)$ without specifying a particular finite generating set S . By a celebrated result of Stallings [16], the number of ends of a Cayley graph of a finitely generated group can only be 0, 1, 2 or infinite (in which case it is uncountable), see Figures ?? and ?? for some examples. On the other hand, Schreier graphs may have any number of ends in $\mathbf{N} \cup \{\infty\}$, see Figure ?? for an example of a graph with 4 ends. In fact, every regular graph of even degree is isomorphic to a Schreier graph, [10, 13].

We are now finally able to introduce property FW. The following characterization is due to Cornuier [6].

Definition A.3. A finitely generated group G has *property FW* if all its Schreier graphs have at most one end.

It directly follows from the definition that all finite groups have property FW, but that \mathbf{Z} does not have it. In fact, if G is a finitely generated group with an homomorphism onto \mathbf{Z} , then it does not have FW. Indeed, in this case $G \cong H \rtimes \mathbf{Z}$ for some H and the Schreier graph $\text{Sch}(G, H; S)$ will be isomorphic to a Cayley graph of $\mathbf{Z} \cong G/H$ and hence has 2 ends.

Property FW admits many distinct characterizations that allow to define it for groups that are non-necessarily finitely generated and even for topological groups. We refer the reader to [6] for a survey of these characterizations.

A.2 Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi: X \rightarrow G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken componentwise.

If H is a group acting on X , then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition

Definition A.4. Let G and H be groups and X be a set on which H acts. The (restricted) wreath product $G \wr_X H$ is the group $(\bigoplus_X G) \rtimes H$.

A prominent source of examples of wreath products are the ones of the form $G \wr_H H$, where H acts on itself by left multiplication. In particular, the group $(\mathbf{Z}/2\mathbf{Z}) \wr_{\mathbf{Z}} \mathbf{Z}$ has become well-known under the name of the *lamplighter group*. Another (trivial) examples of wreath products are direct products $G \times H$ which corresponds to wreath products over a singleton $G \wr_{\{*\}} H$.

Let S be a generating set of G and T a generating set of H . Suppose that H acts transitively on X and let $x \in X$ be any point. Finally, let δ_x^s be the element of $\bigoplus_X G$ defined by $\delta_x^s(x) = s$ and $\delta_x^s(y) = 1_G$ if $y \neq x$ and let $\mathbf{1}$ be the constant function with value 1_G . It is then standard that

$$\{(\delta_x^s, 1_H) \mid s \in S\} \cup \{(\mathbf{1}, t) \mid t \in T\}$$

is a generating set for $G \wr_X H$.

On the other hand, it follows from the definition of the wreath product that we have

$$G \wr_X H \cong \bigoplus_{Y \text{ is an } H\text{-orbit}} (G \wr_Y H).$$

We hence obtain

Lemma A.5. *The group $G \wr_X H$ is finitely generated if and only if both G and H are finitely generated and H acts on X with finitely many orbits.*

Using the above lemma, we could reformulate Theorem 1.4 in the following way:

Proposition A.6. *Let G, H be two groups with G non-trivial and X a set on which H acts. Suppose that all three of G, H and $G \wr_X H$ are finitely generated. Then the wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

In this section, we will investigate property FW via the action of groups on median graphs. In order to ease readability, we will always assume our connected graphs to be non-empty.³

For u and v two vertices of a connected graph \mathcal{G} , we define the total interval $[u, v]$ as the set of vertices that lie on some shortest path between u and v . A connected graph \mathcal{G} is *median* if for any three vertices u, v, w , the intersection $[u, v] \cap [v, w] \cap [u, w]$ consists of a unique vertex, denoted $m(u, v, w)$. A graph is *median* if each of its connected components is median.

Recall that a group G has property FW if and only if every G action on a connected median graph has bounded orbits.

Dire quelque part qu'un groupe G est non-trivial ssi tout action sur un espace métrique (resp, un Hilbert, un arbre, un graphe médian connexe, un espace ultra-métrique) est uniformément bornée.

Pour un Hilbert (et donc un espace métrique), il suffit de prendre $l^2(G)$ est de regarder l'orbite de $n \cdot \delta_g$ qui a diamètre $n\sqrt{2}$. Pour un arbre (et donc graphe médian connexe), on prend une racine r sur laquelle on colle un rayon infini pour chaque $g \in G$. On obtient un arbre T sur lequel G agit par permutations des rayons. Les orbites sont les $\mathcal{L}_n = \{v \mid \text{à distance } n \text{ de la racine}\}$ qui ont diamètre $2n$. Finalement, on peut mettre une ultradistance sur les sommets de T par $d(x, y) = \max(\mathcal{L}(x), \mathcal{L}(y), \cdot)$. Les orbites sont toujours les \mathcal{L}_n avec diamètre n .

We begin this section by proving a series of results that will be generalized to a broader context in Section B. The following easy lemma is folklore, and we provide a proof of it only for the sake of completeness.

Lemma A.7. *Let G be a group and H be a finite index subgroup. If H has property FW, then so does G .*

Proof. Suppose that G does not have FW and let X be a connected median graph on which G acts with an unbounded orbit \mathcal{O} . Then H acts on X and \mathcal{O} is a union of at most $[G : H]$ orbits. This directly implies that H has an unbounded orbit and therefore does not have FW. \square

³This is coherent with the definition that a connected graph is a graph with exactly one connected component.

We also have the following lemma on semi-direct products:

Lemma A.8. *Let $G = N \rtimes H$ be a semidirect product. Then*

1. *If G has property FW, then so does H .*
2. *If both N and H have property FW, then G also has property FW.*

Proof. Suppose that G has FW and let X be a non-empty connected median graph on which H acts. Then G acts on X by $g.x := h.x$ where $g = nh$ with $n \in N$ and $h \in H$. By assumption, the action of G on X has bounded orbits and so does the action of H .

On the other hand, suppose that N and H have FW and let X be a non-empty connected median graph on which G acts. Then both N and H acts on X with bounded orbits. Let x be an element of X , d_1 the diameter of $H.x$ and d_2 the diameter of $N.x$. Since G acts by isometries, for every h in H the set $N.(h.x) = Nh.x = hN.x = h.(N.x)$ has also diameter d_2 . Therefore, every element of $G.x = NH.x$ is at distance at most $d_1 + d_2$ of x , which implies that the orbit $G.x$ is bounded. \square

Multiple applications of Lemma 3.2 give us

Corollary A.9. *Let $G \wr_X H$ be the wreath product of G and $H \curvearrowright X$. Then*

1. *If $G \wr_X H$ has property FW, then so does H .*
2. *If X is finite and both G and H have property FW, then $G \wr_X H$ has property FW.*

We now turn our attention on results that will rely more on the median structure.

Trees are the simplest examples of median graphs and a simple verification shows that if X and Y are both (connected) median graphs, then their cartesian product is also a (connected) median graph. On the other hand, the following example will be fundamental for us.

Example A.10. Let X be a set and let $\mathcal{P}(X) = 2^X$ be the set of all subsets of X . Define a graph structure on $\mathcal{P}(X)$ by putting an edge between E and F if and only if $\#(E \Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $E \Delta F$ and the connected component of E is the set of all subsets F with $E \Delta F$ finite. For E and F in the same connected component, $[E, F]$ consist of all subsets of X that both contain $E \cap F$ and are contained in $E \cup F$. In particular, $\mathcal{P}(X)$ is a median graph, with $m(D, E, F)$ being the set of all elements belonging to at least two of D , E and F . In other words, $m(D, E, F) = (D \cap E) \cup (D \cap F) \cup (E \cap F)$.

We denote by $\mathcal{P}_f(X)$, respectively $\mathcal{P}_{\text{cof}}(X)$ the set of all finite, respectively cofinite, subsets of X . They are connected components of $\mathcal{P}(X)$, which coincide if and only if X is finite

More generally, the connected components of $\mathcal{P}(X)$ are hypercubes and it turns out that every connected median graph is a retract of a connected component of some $\mathcal{P}(X)$, see [1].

These graphs will be fundamental for us due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}(X)$ by graph

homomorphisms: $g.\{x_1, \dots, x_n\} = \{g.x_1, \dots, g.x_n\}$. Be careful that the action of G on $\mathcal{P}(X)$ may exchange the connected components. In fact, the connected component of $E \subset X$ is stabilized by G if and only if E is *commensurated* by G , that is if for every $g \in G$ the set $E\Delta gE$ is finite. For example, both $\mathcal{P}_f(X)$ and $\mathcal{P}_{\text{cof}}(X)$ are always preserved by the action of G .

Building on Example A.10, we obtain that no infinite sum of groups has the property FW.

Lemma A.11. *An infinite direct sum of non-trivial groups does not have FW.*

Proof. By Lemma 3.2 it is enough to prove the assertion for countable sums of groups. Let $(G_i)_{i \geq 1}$ be non-trivial groups and $G = \bigoplus_{i \geq 1} G_i$. Let $X := \bigsqcup_{i \geq 1} G_i$. There is a natural action of G on X : G_i acts by left multiplication on G_i and trivially on G_j for $j \neq i$. Therefore, we have an action of G on the median graph $\mathcal{P}(X)$. Let $\mathbf{1} := \bigcup_{i \geq 1} 1_{G_i}$ be the subset of X consisting of the identity elements of all the G_i . Since every element of G has only a finite number of non-trivial coordinates, the action of G preserves the connected components of $\mathbf{1}$ (and in fact every connected component of $\mathcal{P}(X)$).

For every $i \in \mathbf{N}$, choose a non-trivial $g_i \in G_i$. Then the orbit of the vertex $\mathbf{1}$ contains the point $\{g_1, \dots, g_n\} \cup \bigcup_{i > n} 1_{G_i}$ which is at distance $2n$ of $\mathbf{1}$. That is the action of G on the connected component of $\mathbf{1}$ has an unbounded orbit. \square

As a direct corollary of Lemmas A.11 and 3.2, we obtain

Corollary A.12. *Let G be a non-trivial group. The group $\otimes_X G$ has property FW if and only if X is finite and G has property FW.*

We also have a converse of Lemma A.7.

Lemma A.13. *Let G be a group and H be a finite index subgroup. If G has property FW, then so does H .*

Proof. Suppose that H does not have FW and let $\alpha: H \curvearrowright X$ be an action of H on a connected median space such that there is an unbounded orbit \mathcal{O} . Similarly to the classical theory of representation of finite groups, we have the induced action $\text{Ind}_H^G(\alpha): G \curvearrowright X^{G/H}$ on the connected graph $X^{G/H}$. Since H has finite index, the graph $X^{G/H}$ is median. On the other hand, the subgroup $H \leq G$ acts diagonally on $X^{G/H}$, which gives us a H -equivariant isometric embedding from $H \curvearrowright X$ to $H \curvearrowright X^{G/H}$. This embedding sends \mathcal{O} to an unbounded H -orbit on $X^{G/H}$, which implies that G does not have FW.

For readers that are not familiar with representations of finite groups, here is the above argument in more details. Let $(f_i)_{i=1}^n$ be a transversal for G/H . The natural action of G on G/H gives rise to an action of G on $\{1, \dots, n\}$. Hence, for any g in G and i in $\{1, \dots, n\}$ there exists a unique $h_{g,i}$ in H such that $gf_i = f_{g,i}h_{g,i}$. That is, $h_{g,i} = gf_i f_{g,i}^{-1}$. We then define $g.(x_1, \dots, x_n) := (h_{g,g^{-1}.1}.x_{g^{-1}.1}, \dots, h_{g,g^{-1}.n}.x_{g^{-1}.n})$. This is indeed an action, which preserves the graph structure of $X^{G/H}$ and such that every element $h \in H$ acts diagonally by $h.(x_1, \dots, x_n) = (h.x_1, \dots, h.x_n)$. In particular, this G action has an unbounded orbit. \square

We finally characterize which wreath products do have property FW, and hence provide a proof of Theorem 1.4.

Proposition A.14. *Let G, H be two groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

Proof. In view of Corollary 3.4 it remains to show that if $G \wr_X H$ has FW, then G has FW and X is finite.

Firstly, suppose that X is infinite. As in Lemma A.11, the group $\bigoplus_X G$ acts coordinatewise on $\bigsqcup_X G$. On the other hand, H acts on $\bigsqcup_X G$ by permutation of the factors. Altogether we have an action of $G \wr_X H$ on $\bigsqcup_X G$ and hence on the median graph $\mathcal{P}(\bigsqcup_X G)$. Let $\mathbf{1} := \bigcup_{x \in X} 1_G$ be the subset of $\mathcal{P}(\bigsqcup_X G)$ consisting of the identity elements of all the copies of G . As in Lemma A.11 we have that the action of $G \wr_X H$ on $\mathcal{P}(\bigsqcup_X G)$ preserves the connected component of $\mathbf{1}$ and that the orbit of $\mathbf{1}$ is unbounded.

It remains to treat the case of X finite and such that G does not have FW. Let Y be a connected median graph on which G acts with an unbounded orbit $G.y$. Then Y^X is also a connected median graph, on which $G \wr_X H$ acts by

$$((\varphi, h). \psi)(x) = \varphi(h^{-1}.x) \psi(h^{-1}.x),$$

which is known as the *primitive wreath action* of $G \wr_X H$. On the other hand, $G.y$ embeds diagonally into Y^X . Indeed, let φ_g be the constant function equal to g and ψ_y the constant function equal to y . Then $(\varphi_g, h). \psi_y = \psi_{g.y}$ for any $h \in H$. By definition of the graph distance, we have $d_{Y^X}(\psi_y, \psi_{g.y}) = d_Y(y, g.y)^{|X|}$, which is unbounded. □

Remark A.15. A reader familiar with wreath products might have recognized that we also used the primitive wreath action in the proof of the first part of Proposition A.14.

Indeed, G acts on itself by left multiplication. It hence acts on the set $G' := G \sqcup \{\varepsilon\}$ by fixing ε . We hence have the primitive action of $G \wr_X H$ on G'^X . Now, the set $\bigsqcup_X G$ naturally embeds as the subset of G'^X consisting of all functions $\phi: X \rightarrow G'$ such that $\phi(x) = \varepsilon$ for all but one $x \in X$. This subset is $G \wr_X H$ invariant, which gives us the desired action of $G \wr_X H$ on $\bigsqcup_X G$.

B Related concepts and generalizations

The aim of this section is to give a glimpse of some properties related to the FW property as well as to provide a general framework to study them. We will sometimes only sketch the arguments and let the reader verify all the details for its favorite property. In all such places, the assertions will be preceded by the prefix “meta”.

Property FW is not isolated in the realm of group properties and other similar properties of great interest can also be defined in terms of action with bounded orbits.

Add refs

Definition B.1. Let G be a group. It is said to have *property SB* if any action on a metric space has bounded orbits. It has *property FH* if any action on a real Hilbert space has bounded orbits and *property FA* if any action on a tree

has bounded orbits. Finally, G is said to have (cofinality $\neq \omega$) if any action on an ultrametric space has bounded orbits.

For countable groups (and more generally for σ -compact locally compact groups), property FH is equivalent to the celebrated Kazhdan's property (T) by the Delorme-Guichardet theorem, see for example [?], but this is not true in general [?].

Maybe expand a little more.

The names FH, FW and FA come from the fact that these properties admit a description in terms of existence of a Fixed point for actions on Hilbert spaces, on spaces with Walls and on trees (*Arbres* in french). On the other hand, SB stands for Strongly Bounded and is sometimes called the Bergman property. Finally, a group has cofinality $\neq \omega$ if and only if it cannot be written as an increasing union of proper subgroups.

Give refs

We have the following strict implications [?, ?, ?, ?]

$$SB \implies FH \implies FW \implies FA \implies (\text{cofinality} \neq \omega).$$

It is possible to define other properties in the spirit of Definition 2.2.

Definition B.2. Let P be a property of metric spaces (for example *be a connected median graph*) and BP be the group property: *Every G -action on a space with P has bounded orbits*, where a G -action is supposed to “preserve the P -structure”.

The proofs of Lemmas A.7 and 3.2 and Corollary 3.4 remain true for groups with property BP and we obtain.

Proposition B.3. *Let P be a property of metric spaces.*

1. *If a group G has a finite index subgroup with property BP, then G has property BP.*
2. *If $N \rtimes H$ has property BP, then so does H .*
3. *If both N and H have property BP, then $N \rtimes H$ also has property BP.*
4. *If $G \wr_X H$ has property BP, then so does H .*
5. *If X is finite and both G and H have property BP, then $G \wr_X H$ has property BP.*

On the other hand, the other results of Section A.2 require a specific construction and do not generalize straightforward to groups with property BP. Nevertheless, it is possible to extract the main ingredients of the proofs given in Section A.2 and to adapt them to some specific cases. We will now give a raw outline of this process, but let the details to the interested reader.

In order to generalize Example A.10, we will need to construct from a G action on a set X a G action on a P -space $K(X)$, such that X embeds G -equivariantly into $K(X)$. For median graphs we took $K(X) = \mathcal{P}(X)$ in Example A.10, and then carefully choose some connected component of it. Since median graphs are example of metric spaces, it is also possible to take $K(X) = \mathcal{P}(X)$ for property SB. For Hilbert spaces, one can take $K(X) = \ell^2(X)$

to be the real Hilbert space generated by X , see Example B.4. In the context of trees and ultrametric spaces, we do not know any interesting⁴ general constructions. However, for direct sums of groups $G = \bigoplus_i G_i$ they are canonical trees and ultrametric spaces on which G acts, see Examples B.5 and B.6.

Example B.4. Let G be a group acting on a set X . Let $\ell^2(X)$ be the real Hilbert space generated by X . That is,

$$\ell^2(X) = \left\{ f: X \rightarrow \mathbf{R} \mid \sum_{x \in X} f(x)^2 < \infty \right\}.$$

Then G acts on $\ell^2(X)$ by permutations of the element of the basis: $g \cdot \delta_x = \delta_{g \cdot x}$, where δ_x is the Dirac mass at x . Moreover, the map $x \mapsto \delta_x$ is a G -equivariant embedding of X into $\ell^2(X)$.

Example B.5. Let $(G_i)_{i \geq 1}$ be non-trivial groups and let ε denotes an element not in any of the G_i . Define

$$F = F((G_i)_{i \geq 1}) := \{f = (f_1, f_2, \dots) \mid \exists n \forall j < n : f_j = \varepsilon \text{ and } \forall k \geq n : f_k \in G_k\}.$$

The integer n in the above definition is called the *level* of the sequence f . We now define a graph structure on the set F , viewed as the set of vertices. We put an edge between two vertices f and g if one of them is of level n , the other is of level $n+1$ and $f_i = g_i$ for $i > n$. If $f \in F$ has level n , then it has one neighbourhood of level $n+1$, $|G_{n-1}|$ neighbourhoods of level $n-1$ and no neighbourhoods of other levels. This implies that F is a forest. The group $G := \bigoplus_{i \geq 1} G_i$ acts on F by $(g_1, g_2, \dots) \cdot (f_1, f_2, \dots) = (g_1 f_1, g_2 f_2, \dots)$ with the convention that $G_i \cdot \varepsilon = \varepsilon$ for any i . It is trivial that this action preserves the graph structure on F .

The space $F = F((G_i)_{i \geq 1})$ is not new to us. Indeed, it can naturally be seen as a subset of $\mathcal{P}(\bigsqcup_{i \geq 1} G_i)$ by $\alpha: (f_1, f_2, \dots) \mapsto \{f_i \mid f_i \neq \varepsilon\}$. Under this identification, which is G -equivariant, F is a subgraph of $\mathcal{P}(\bigsqcup_{i \geq 1} G_i)$ and we have $d(\alpha(f), \alpha(g)) \leq d(f, g)$.

Example B.6. Let $(G_i)_{i \geq 1}$ be non-trivial groups and let $G := \bigoplus_{i \geq 1} G_i$. Define a distance d_∞ on G by $d_\infty(f, g) := \max\{i \mid f(i) \neq g(i)\}$. Since the elements f and g agree on all but finitely many indices, the maximum is attained. On the other hand it is trivial that $d_\infty(f, g) = 0$ if and only if $f = g$ and that d is symmetric. Finally, if f, g, k are in G we have $d_\infty(f, g) \leq \max\{d_\infty(f, k), d_\infty(k, g)\}$ and hence (G, d_∞) is an ultrametric space. The action by left translation of the group G on itself preserves the distance d_∞ .

Once again, the space (G, d_∞) is not new. Indeed, the map $\beta: f \mapsto \{f(i) \mid i \in \mathbf{N}\}$ is a G -equivariant embedding of G into the set of vertices of level 1 of $F((G_i)_{i \geq 1})$ and we have $d(\beta(f), \beta(g)) = 2d_\infty(f, g)$.

We obtain the following “meta”-result

Lemma B.7. *Let P be a property of spaces such that a group with property BP has cofinality $\neq \omega$. Then*

⁴An example of an *uninteresting* construction is the discrete metric which turns any set X into an ultrametric space, but of diameter 1.

Maybe
rename
“Lemma”
into
“Meta-
lemma”
and so on.

1. An infinite direct sum of non-trivial groups does not have property BP.
2. If G is a non-trivial group, then the group $\otimes_X G$ has property BP if and only if X is finite and G has property BP.
3. If $G \wr_X H$ has property BP, then H acts on X with finitely many orbits.

Proof. 1. By the second assertion of Proposition B.3, it only remains to show that a countable sum of non-trivial groups does not have BP. But this directly follows from the fact that a countable sum of non-trivial groups does not have cofinality $\neq \omega$. Let $G = \bigoplus_{i \geq 1} G_i$ be a direct sum of non-trivial groups. It is trivial that G can be written as an increasing union of proper subgroups and hence has countable cofinality. However, we find enlightening to give another proof of this fact using action on ultrametric spaces. Let $1_G = (1_{G_1}, 1_{G_2}, \dots)$ be the identity element of G . For every $i \in \mathbf{N}$, choose a non-trivial $g_i \in G_i$. Then for every n , the orbit of the vertex 1_G contains the point $\{g_1, \dots, g_n, 1_{G_{n+1}}, \dots\}$, which is at distance n of 1_G , and is hence unbounded.

2. This is a direct consequence of 1. and of the third assertion of Proposition B.3.

3. This follows from 1. and from the fact that

$$G \wr_X H = \bigoplus_{Y \text{ is an } H\text{-orbit}} G \wr_Y H.$$

□

As a direct consequence of Proposition A.14, we have

Lemma B.8. *Let P be a property of spaces such that property BP implies property FW and let G be a non-trivial group. If $G \wr_X H$ has property BP, then X is finite.*

In the above lemma, it is not possible to replace property FW by property FA, as demonstrated by Theorem B.14. The reason for this fact is that even if, for countable X , the group $G \wr_X H$ naturally acts on the ultrametric space $(\bigoplus_X G, d_\infty)$ and on connected components of the forest $F((G_x)_{x \in X})$, these actions are not by isometries.

In order to generalize Lemma A.13, we only need that given a P-space (X, d_X) , the square $(X \times X, d_{X \times X})$ is a P-space such that for any unbounded set $\mathcal{O} \subset X$ the set $\{(x, x) \mid x \in \mathcal{O}\}$ is unbounded.

Observe that the distance that we put on $X \times Y$ depend on the property P. For example, for graphs we take $d_1((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$, for Hilbert spaces we take $d_2((x, y), (x', y')) = (d_X(x, x')^2 + d_Y(y, y')^2)^{\frac{1}{2}}$ while for ultrametric spaces we take $d_\infty((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}$. However, all these distances (and more generally the distance $d_p = (d_X^p + d_Y^p)^{\frac{1}{p}}$ for $p \in [1, \infty]$) satisfy the fact that if $\mathcal{O} \subset X$ is unbounded so is $\{(x, x) \mid x \in \mathcal{O}\} \subset X \times X$. We will say that such a distance on $X \times Y$ *preserves unboundedness*.

On one hand, we have that metric spaces, ultra-metric spaces and Hilbert spaces are closed under products (the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ for Hilbert spaces) and hence Lemma A.13 apply to them. On the other hand, for trees there is two natural candidates on which $G_1 \times G_2$ acts: the cartesian product $T_1 \times T_2$

or the tensor product $T_1 \otimes T_2$ ⁵, but none of them are trees ($T_1 \times T_2$ has cycles while $T_1 \otimes T_2$ is not connected).

Lemma B.9. *Let P be a property of spaces. Suppose that for every P space (X, d_X) , the n^{th} power (X^n, d_{X^n}) is a P -space such that d_{X^n} preserves unbound-
edness.*

Let $G \wr_X H$ be the wreath product of G and $H \curvearrowright X$. If X is finite and $G \wr_X H$ has property BP , then so does G .

Proof. The proof is similar to the second part of the proof of Proposition A.14 and is left to the reader. \square

By combining Proposition B.3 and Lemmas B.8 and B.9 we recover the analogous of Proposition A.14 for properties FH and SB .

Proposition B.10. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property FH if and only if G and H have the property FH and X is finite.*

Proposition B.11. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property SB if and only if G and H have the property SB and X is finite.*

With a little more work, we obtain

Theorem B.12. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. Then $G \wr_X H$ has cofinality $\neq \omega$, if and only if G and H have cofinality $\neq \omega$ and H acts on X with finitely many orbits.*

Proof. It only remains to prove two things. First, that if $G \wr_X H$ has cofinality $\neq \omega$, then so does G . And secondly, that if both G and H have cofinality $\neq \omega$ and H acts transitively on X , then $G \wr_X H$ has cofinality $\neq \omega$.

Firstly, let us suppose that G has countable cofinality. We will prove that $G \wr_X H$ also has countable cofinality. Since G has countable cofinality, there exists a G -invariant ultrametric d on G . We have the primitive action of $G \wr_X H$ on $G^X \cong \prod_X G$ which preserves the subgroup $K := \bigotimes_X G$. Define d_∞ on $K \times K$ by putting $d_\infty(\varphi_1, \varphi_2) := \max\{d(\varphi_1(x), \varphi_2(x)) \mid x \in X\}$. Using that d is a G -invariant ultrametric on G , we obtain that d_∞ is an ultrametric on K , which is $G \wr_X H$ invariant. Let x_0 be an element of X . Then we have a G -equivariant isometric embedding of G into K by sending g to $\delta_{x_0}^g$.

Now, suppose that both G and H have cofinality $\neq \omega$ and that H acts transitively on X . Suppose that $G \wr_X H$ can be written as an increasing union $\bigcup_{i \geq 1} K_i$ of subgroups. We have to prove that for i big enough $K_i = G \wr_X H$. Let $H_i := H \cap K_i$. Then $H = \bigcup_{i \geq 1} H_i$ and there exists N such that for $i \geq N$ we have $H = H_i$. Similarly, for $x_0 \in X$ let $F_i := G_{x_0} \cap K_i$, where G_{x_0} is the copy of G in position x_0 . Then there exists M such that for $i \geq M$ we have $F_i = G_{x_0}$. Hence, for $i \geq \max\{M, N\}$ we have $K_i \geq G_{H.x_0} H = G \wr_X H$. \square

Finally, we also have the following partial result

⁵The distance on $T_1 \otimes T_2$ is given by $d_\otimes((x, y), (x', y')) = d_{T_1}(x, x') = d_{T_2}(y, y')$ if the distances are equal and ∞ (that is (x, y) and (x', y') are not in the same connected component) otherwise.

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Proposition B.13. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. Then*

1. *If $G \wr_X H$ has property FA, then H has property FA and acts on X with finitely many orbits.*
2. *If X is finite and both G and H have property FA, then $G \wr_X H$ has property FA.*

The following more precise version of Proposition B.13 was obtained by Cornuier and Kar.

Theorem B.14 ([7]). *Let G, H be two discrete groups with G non-trivial and X a set on which H acts with a finite number of orbits and no fixed point. Then the wreath product $G \wr_X H$ has the property FA if and only if H has the property FA and if G is a group with a finite abelianization, which cannot be expressed as an union of a proper increasing sequence of subgroups.*

Observe that, by the characterization of Serre [15], a finitely generated group with property FA has finite abelianization and cannot be expressed as an union of a proper increasing sequence of subgroups.

Regarder le cas PW, Haagerup,... i.e. action propre. Rappel: une action isométrique de G est propre si pour tout x (de manière équivalente il existe x), pour tout $r \in \mathbf{R}$ l'ensemble $\{g \in G \mid d(x, g.x) \leq r\}$ est fini.