The Property FW for the wreath products

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Monday 30th November, 2020 17:10

1 Introduction

The property FW was introduced by Barnhill and Chatterji. It is a fixed point property for the action on wall spaces (for a detailed treatment of this property see [4]). For discrete groups, this property is implied by the Kazhdan property (T). The behavior of the Kazhdan Property (T) with the wreath product is well known:

Give ref (cf article de Cornullier)

Theorem 1.1 ([3, 6])**.** *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property (T) if and only if G and H have the property (T) and X is finite.*

The same kind of result is true for the property FW

Theorem 1.2. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property FW if and only if G and H have the property FW and X is finite.*

An action of a group on a CAT(0) cube complex is *essential* if all the orbits of vertices are unbounded and the action is transitive on the set of hyperplanes.

Corollary 1.3. *Let G, H be two discrete groups and X a set on which H acts transitively. If there exists an essential action of G or H on a CAT(0) cube complex or if X is infinite, then there exists an essential action of $G \wr_X H$ on a CAT(0) cube complex.*

2 Definitions

2.1 The Property FW

For the definition of the property FW, we will follow the survey of Y. de Cornullier [4].

Definition 2.1. Let G be a discrete group and X a discrete set on which G acts. A subset $M \subset X$ is *commensurated* by the G-action if

$$|gM \Delta M| < \infty$$

for all g in G.

An invariant G-subset is automatically commensurated. Moreover, for a subset M such that there exists an invariant G-subset N with $|M \Delta N| < \infty$ then M is commensurated. Such a set is called *transfixed*.

Definition 2.2. A group G has the property FW if all commensurable G-set are transfixed.

There are lot of equivalent characterizations of this property. We will give us without all the details and the precise definitions.

Proposition 2.3. *The following are equivalent:*

1. *G has the property FW;*
2. *every cardinal definite function on G is bounded;*
3. *every cellular action on a CAT(0) cube complex has bounded orbits for the ℓ^1-metric (the complexes can be infinite dimensional);*
4. *every cellular action on a CAT(0) cube complex has a fixed point;*
5. *every action on a connected median graph has bounded orbits;*
6. *every action on a nonempty connected median graph has a finite orbit;*
7. *(if G is finitely generated) every Schreier graph of G has at most 1 end;*
8. *For every set Y endowed with a walling structure and compatible action on Y and on the index of the walling, the action on Y has bounded orbits for the wall distance;*
9. *every isometric action on an "integral Hilbert space" $\ell^2(X, \mathbf{Z})$ (X any discrete set), or equivalently on $\ell^2(X, \mathbf{R})$ preserving the integral points, has bounded orbits;*
10. *for every G-set X we have $H^1(G, \mathbf{Z}X) = 0$.*

Note that the name FW comes from the property of "fixed point" for the actions on the walling spaces. We will see in the following that a semi-splittable group does not have the property FW (see corollary **??**).

The property FW has links with other well known properties. For example, the property FH implies the characterisation 9. For discrete groups (and even for countable groups) the property FH is equivalent to the Kazhdan's property (T) by Delorme-Guichardet's Theorem. As trees are CAT(0) cube complexes, the property FW implies Serre's property FA.

3 Ends of Schreier graph

Ajouter les petits lemmes sur les sous-groupes

For finitely generated groups, the point 7 of the Proposition 2.3 gives us a nice geometrical characteriation of the property FW. We will present more explicit and constructive proofs of the Proposition 4.8 in this contex.

We will begin by a short recall on Schreier graph. .

Utile de redéfinir les Schreier ?

Definition 3.1. Let G be a finitely generated group, H a subgroup of G and S a finite generating set. The *Cayley graph* of G with respect to H and S is the graph where the vertex are the cosets gH and two vertices gH and $g'H$ are adjacent if there exists a generator s such that $g'H = sgH$.

If a group G acts on a set X, we can define the graph of the action of G on the orbit of an element x as the graph where the vertex are the element of the orbit of x and two vertices are adjacent if they are linked by the action of a generator. The Schreier graphs are intimitely linked with group actions by the classical following lemma.

Lemma 3.2. *Let G be a finitely generated groups, S be a finite generating set and X be a G-set. Then for each element x of X, the graph of the action of G on the orbit of x is isomorphic to the Schreier graph* $\mathrm{Sch}(G, \mathrm{Stab}(x), S)$.

In fact all the Schreier graph can be viewed as graphs of actions, by considering the action of G on this Schreier graph.

As we want to work with finitely generated groupm the folowing lemma is useful.

Lemma 3.3. *Let G and H be groups and X be a H-set. The wreath product $G \wr_X H$ is finitely generated if and only if G and H are finitely generated and if the number of orbits of the action $H \curvearrowright X$ is finite.*

Proof. Suppose that G and H are finitely generated and that the number of orbits is finite. Let S be a finite generating set of G, S' a finite generating set of H and $\{x_0, \ldots, x_n\}$ a representative system of the orbits. For s in S and x in X, we define δ_x^s as

$$\delta_x^s(y) = \begin{cases} e_G & y \neq x \\ s & y = x. \end{cases}$$

It is straightforward to prove that the set

$$\{(\delta_x^s, e_H) : s \in S, x \in \{x_0, \ldots, x_n\}\} \ \cup \{(0, s') : s' \in S'\}$$

is a finite generating set of $G \wr_X H$, where $0(x) = e_g$ for all x in X.

If G or H are not finitely generated, it is clear that $G \wr_X H$ is not either. Suppose that there are infinitely many orbits. If φ is a function of $\bigoplus_X G$ whose support is include in $X_1 \sqcup \ldots X_n$, then for every h in H, the support of $h.\varphi$ is also contain in $X_1 \sqcup \ldots X_n$, where $X = X_1 \sqcup X_2 \sqcup \ldots$ is a decomposition on disjoint orbits. As all the elements of $\bigoplus_X G$ have finite support, it is necessary to have infintely many such element to generate the whole group $G \wr_X H$. $\square$

Lemma 3.4. *Let $G = N \rtimes H$ be a finitely generated semidirect product. Then*

1. *If G has FW, then so does H.*

2. *If both N and H have FW, then G also has FW.*

Proof. It is well known that if N is generated by a set S and H by a set S', then G is generated by $S \cup S'$.

Suppose that H does not have the property FW. There exists a Schreier graph X of H with more than one end. The group G acts on X via

$$(n, h).x = hx$$

for a vertex x of X. This action $G \curvearrowright \Gamma$ is transitive because the action $H \curvearrowright \Gamma$ is. The graph of the action $G \curvearrowright X$ is exactly the graph X with some additional loops for generators of S'. As adding loops does not change the number of ends, this Schreier graph has more than one ends and the G does not have the property FW.

Suppose now that N and H have the property FW. We want to show that every Schreier graphs of G have at most one end. Let X be an infinite Schreier graph of G with respect to the generating set $S \cup S'$. The groups N and H acts on X via

$$n.x = (n, 0)x$$
$$h.x = (0, h)x$$

for a vertex x of X. For each such vertex x we define X_x^H (and respectively X_x^N) the Schreier graph obtained from the action of N (resp. H) on the orbit of x. This is subgraphs of X. As N and H have the property FW, the graphs X_x^H and X_x^N are finite or one-ended. We want to prove that in this case, X has at most one end.

Let K be a finite set of vertices of X. We will construct a path between every pair of vertices which are in infinite connected components. First, for every x in K, if X_x^H is finite, we add all the vertices of this subgraph in K. Adding vertices in K can not decrease the number of ends. Moreover we add a finite number of vertices in K. With this new set K, we have that for every x in $X \setminus K$, the subgraph $X_x^H \setminus K$ is equal to X_x^H or has one end.

Let x and y be two vertices of an infinite component of $X \setminus K$. The action of G on X is transitive, then there exists an element (n, h) of $N \rtimes H$ such that $(n, h).x = y$. We choose a vertex z in X_x^H such that X_z^N does not contain an element of K or is one-ended and such that $z' = (h'h^{-1}.n, 0)z$ is not in K, where h' is an element of H and $h'.x = z$. Such an element exists as X_x^H and X_y^H are infinite and K is finite. The vertex z' is in X_y^H because

$$(0, hh'^{-1})z' = (0, hh'^{-1})(h'h^{-1}.n, 0)(0, h')x = (n, h)x = y.$$

We will construct a path on $X \setminus K$ between x and y as follows. The subgraph X_x^H is one-ended, then there exists a path between x and z. In the same way, X_z^N has one end or has no vertex in K, then there is a path which join z and z'. There exists a path between z' and y in X_y^H which is one-ended. Then x and y are path-connected and then X has one end. We proved that all the Schreier graph of G

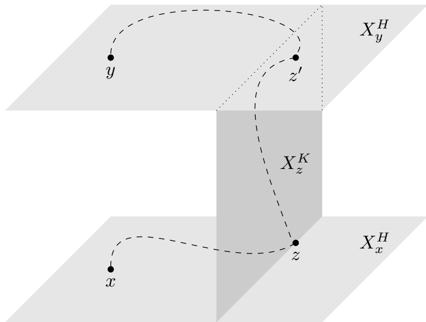

Figure 1: The path between x and y.

$\square$

Corollary 3.5. *Let G and H be two finitely generated groups and X a H-set such that the number of orbits is finite. Then,*

1. *If $G \wr_X H$ has the property FW, then so does H.*
2. *If G and H have the property FW and X is finite, then $G \wr_X H$ has FW.*

The proof is an iterated application of the previous lemma.

Lemma 3.6. *Let $\Gamma = G \wr_X H$ be a finitely generated wreath product and G be a non-trivial group. If Γ has the property FW, then*

1. *G has the property FW.*
2. *X is finite.*

Proof. We will prove the contrapositives. The idea is to construct, for each of the three cases, a Schreier graph of Γ with more than one end by using actions of Γ. We will suppose, as in the Lemma 3.2 that $G = \langle S \rangle$, $H = \langle S' \rangle$ and if there is no more condition x_0 is an arbitrary point of X.

Suppose that X is an infinite set. There exsits an infinite orbit. Ideed, as the Γ is finitely generated, we know by the Lemma 3.3 that the number of orbits of the action $H \curvearrowright X$ is finite. Let X' be such an infinite orbit and x_0 an element of X'. We define $Y = G \times X'$. The group Γ acts on Y via

$$(\varphi, h) \cdot (g, x) = (\varphi(hx)g, hx)$$

for (φ, h) in Γ and (g, x) in Y. The action is transitive. Indeed, let (g_1, x_1) and (g_2, x_2) be two elements of Y. By transitivity of the action of H on X', there exists h in H such that $hx_1 = x_2$ and φ in $\bigoplus_X G$ such that $\varphi(hx_1) = g_2 g_1^{-1}$. Then,

$$(\varphi, h)(g_1, x_1) = (\varphi(hx_1)g, hx_1) = (g_2, x_2).$$

The graph of the action of Γ on Y is isomorphic to the Schreier graph $\mathrm{Sch}(\Gamma, \mathrm{Stab}(e_G, x_0), \mathcal{S})$. We decompose the graph into leaves of the form $Y_g = \{g\} \times X'$. There are two types of edges in this graph which are coming from the two sets of generators.

The first one, of the form $(0, s')$, give us on each leaf a copy of the graph of the action $H \curvearrowright X'$. Indeed,

$$(0, s')(g, x) = (g, s'x).$$

The second one, of the form $(\delta^s_{x_0}, 0)$, give us loops everywhere excepting on vertices of the form (g, x_0). By direct computation, we see that the vertices (g, x_0) and (sg, x_0) connect the leaves Y_g and Y_{sg},

$$(\delta^s_{x_0}, 0)(g, x) = \begin{cases} (g, x) & x \neq x_0 \\ (sg, x) & x = x_0 \end{cases}$$

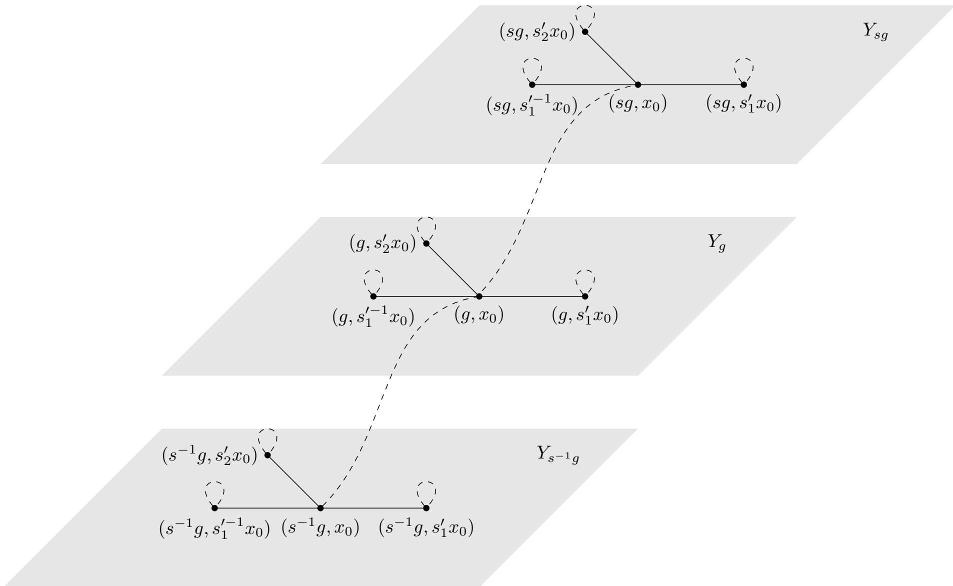

Figure 2: The leaf structure of the graph of the action $\Gamma \curvearrowright Y$.

If we remove a vertex (g, x_0) we disconnect the leaf Y_g to the graph. As X' is infinite, the number of ends is strictly greater than 1.

Suppose that G does not have the property FW. There exists a subgroup K of G such that $\mathrm{Sch}(G, K, S)$ has more than one end. The choice of x_0 is arbitrary in this case. The group Γ acts on $G/K \times X'$ via

$$(\varphi, h)(gK, x) = (\varphi(hx)gK, hx).$$

where X' is the orbit of x_0. As above, the action is transitive and the graph of this action is isomorphic to a Schreier graph. We decompose this graph into leaves in the same way. Now we look at the subgraph made up of vertices (g, x_0) and edges $(\delta^s_{x_0}, 0)$ and we remark that it is isomorphic the Schreier graph $\mathrm{Sch}(G, K, S)$ which has more than one end. Then our graph has also more than one end. $\square$

The following proposition is a direct application of Corollary 3.5 and Lemma 3.6.

Proposition 3.7. *Let G be a non trivial finitely generated group, H be a finitely generated group and X a set on which H acts with a finite number of orbit. The wreath product $G \wr_X H$ has the property FW if and only if G and H have the property FW and X is finite.*

4 Median graphs

In this section, we will investigate property FW via the action of groups on median graphs. In order to ease readability, we will always assume our connected graphs to be non-empty.[1]

For u and v two vertices of a connected graph $\mathcal{G}$, we define the total interval $[u, v]$ as the set of vertices that lies on some shortest path between u and v. A connected graph $\mathcal{G}$ is *median* if for any three vertices u, v, w, the intersection $[u, v] \cap [v, w] \cap [u, w]$ consists of a unique vertex, denoted $m(u, v, w)$. A graph is *median* if each of its connected components is median.

Recall that a group G has property FW if and only if every G action on a connected median graph has bounded orbits.[2]

We begin this section by proving a series of results that will be generalized to a broader context in Section 5. The following easy result is folklore, and we provide a proof of it only for the sake of completeness.

Lemma 4.1. *Let G be a group and H be a finite index subgroup. If H has property FW, then so does G.*

Proof. Suppose that G does not have FW and let X be a connected median graph on which G acts with an unbounded orbit O. Then H acts on X and O is a union of at most $[G : H]$ orbits. This directly implies that H has an unbounded orbit and therefore does not have FW. □

We also have the following lemma on semi-direct products:

Lemma 4.2. *Let $G = N \rtimes H$ be a semidirect product. Then*

1. *If G has property FW, then so does H.*
2. *If both N and H have property FW, then G also has property FW.*

Proof. Suppose that G has FW and let X be a non-empty connected median graph on which H acts. Then G acts on X by $g.x := h.x$ where $g = nh$ with $n \in N$ and $h \in H$. By assumption, the action of G on X has bounded orbits and so does the action of H.

On the other hand, suppose that N and H have FW and let X be a non-empty connected median graph on which G acts. Then both N and H acts on X with bounded orbits. Let x be an element of X, d_1 the diameter of $H.x$ and d_2 the diameter of $N.x$. Since G acts by isometries, for every h in H the

[1] This is coherent with the definition that a connected graph is a graph with exactly one connected component.

[2] On the other hand, a group G is finite, if and only if every G action on a connected median graph has *uniformly* bounded orbits. See Example 4.4 and the discussion below it.

set $N.(h.x) = Nh.x = hN.x = h.(N.x)$ has also diameter d_2. Therefore, every element of $G.x = NH.x$ is at distance at most $d_1 + d_2$ of x, which implies that the orbit $G.x$ is bounded, with diameter at most $2(d_1 + d_2)$. $\square$

Multiple applications of Lemma 4.2 give us

Corollary 4.3. *Let $G \wr_X H$ be the wreath product of G and $H \curvearrowright X$. Then*

1. *If $G \wr_X H$ has property FW, then so does H.*
2. *If X is finite and both G and H have property FW, then $G \wr_X H$ has property FW.*

We now turn our attention on results that will rely more on the median structure.

Trees are the simplest examples of median graphs and a simple verification shows that if X and Y are both (connected) median graphs, then their cartesian product is also a (connected) median graph. On the other hand, the following example will be fundamental for us.

Example 4.4. Let X be a set and let $\mathcal{P}(X) = 2^X$ be the set of all subsets of X. Define a graph structure on $\mathcal{P}(X)$ by putting an edge between E and F if and only if $\#(E\Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $E\Delta F$ and the connected component of E is the set of all subsets F with $E\Delta F$ finite. For E and F in the same connected component, $[E, F]$ consist of all subsets of X that both contain $E \cap F$ and are contained in $E \cup F$. In particular, $\mathcal{P}(X)$ is a median graph, with $m(D, E, F)$ being the set of all elements belonging to at least two of D, E and F. In other words, $m(D, E, F) = (D \cap E) \cup (D \cap F) \cup (E \cap F)$.

We denote by $\mathcal{P}_{\mathrm{f}}(X)$, respectively $\mathcal{P}_{\mathrm{cof}}(X)$ the set of all finite, respectively cofinite, subsets of X. They are connected components of $\mathcal{P}(X)$, which coincide if and only if X is finite

More generally, the connected components of $\mathcal{P}(X)$ are exactly the hypercubes and it turns out that every connected median graph is a retract of some $\mathcal{P}_{\mathrm{f}}(X)$, see [1].

These graphs will be fundamental for us due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}(X)$ by graph homomorphisms: $g.\{x_1, \dots, x_n\} = \{g.x_1, \dots, g.x_n\}$. Be careful that the action of G on $\mathcal{P}(X)$ may exchange the connected components. In fact, the connected component of $E \subset X$ is stabilized by G if and only if E is *commensurated* by G, that is if for every $g \in G$ the set $E\Delta gE$ is finite. For example, both $\mathcal{P}_{\mathrm{f}}(X)$ and $\mathcal{P}_{\mathrm{cof}}(X)$ are always preserved by the G-action.

If G is an infinite group, then the action of G on itself by left multiplication extends to an action of G on the connected median graphs $\mathcal{P}_{\mathrm{f}}(G)$ with bounded but not uniformly-bounded orbits. Indeed, given distinct g_1 to g_n in G, there exists g such that $g.\{g_1, \dots, g_n\}$ does not intersect $\{g_1, \dots, g_n\}$ and is therefore at distance $2n$ of it.

Building on Example 4.4, we obtain that no infinite sum of groups has the property FW.

Lemma 4.5. *An infinite direct sum of non-trivial groups does not have FW.*

Proof. Let I be an infinite set of indices, $(G_i)_{i\in I}$ be non-trivial groups and $G = \bigoplus_{i\in I} G_i$. Let $X \coloneqq \bigsqcup_{i\in I} G_i$. There is a natural action of G on X: G_i acts by left multiplication on G_i and trivially on G_j for $j \neq i$. Therefore, we have an action of G on the median graph $\mathcal{P}(X)$. Let $\mathbf{1} \coloneqq \bigcup 1_{G_i}$ be the subset of X consisting of the identity elements of all the G_i. Since every element of G has only a finite number of non-trivial coordinates, the action of G preserves the connected components of $\mathbf{1}$ (and in fact every connected component of $\mathcal{P}(X)$).

Since I is infinite, it contains an infinite countable subset $I_c = \{i_1, i_2, \dots\}$. For every $j \in \mathbf{N}$, choose a non-trivial $g_j \in G_{i_j}$. Then the orbit of the vertex $\mathbf{1}$ contains the point $\{g_1, \dots, g_n\} \cup \bigcup_{i\notin\{i_1,\dots,i_n\}} 1_{G_i}$ which is at distance $2n$ of $\mathbf{1}$. That is the action of G on the connected component of $\mathbf{1}$ has an unbounded orbit. □

As a direct corollary of Lemmas 4.5 and 4.2, we obtain

Corollary 4.6. *Let G be a non-trivial group. The group $\otimes_X G$ has property FW if and only if X is finite and G has property FW.*

We also have a converse of Lemma 4.1.

Lemma 4.7. *Let G be a group and H be a finite index subgroup. If G has property FW, then so does H.*

Proof. Suppose that H does not have FW and let $\alpha\colon H \curvearrowright X$ be an action of H on a connected median space such that there is an unbounded orbit O. Similarly to the classical theory of representation of finite groups, we have the induced action $\mathrm{Ind}_H^G(\alpha)\colon G \curvearrowright X^{G/H}$ on the connected graph $X^{G/H}$. Since H has finite index, the graph $X^{G/H}$ is median. On the other hand, the subgroup $H \leq G$ acts diagonaly on $X^{G/H}$, which gives us a H-equivariant isometric embedding from $H \curvearrowright X$ to $H \curvearrowright X^{G/H}$. This embedding sends O to an unbounded H-orbit on $X^{G/H}$, which implies that G does not have FW.

For readers that are not familiar with representations of finite groups, here is the above argument in more details. Let $(f_i)_{i=1}^n$ be a transversal for G/H. The natural action of G on G/H gives rise to an action of G on $\{1, \dots, n\}$. Hence, for any g in G and i in $\{1, \dots, n\}$ there exists a unique $h_{g,i}$ in H such that $gf_i = f_{g.i}h_{g,i}$. That is, $h_{g,i} = gf_i f_{g.i}^{-1}$. We then define $g.(x_1, \dots, x_n) \coloneqq (h_{g,g^{-1}.1}.x_{g^{-1}.1}, \dots, h_{g,g^{-1}.n}.x_{g^{-1}.n})$. This is indeed an action, which preserves the graph structure of $X^{G/H}$ and such that every element $h \in H$ acts diagonally by $h.(x_1, \dots, x_n) = (h.x_1, \dots, h.x_n)$. In particular, this G action has an unbounded orbit. □

We finally characterize which wreath products do have property FW, and hence provide a proof of Theorem 1.2.

Proposition 4.8. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

Proof. In view of Corollary 4.3 it remains to show that if $G \wr_X H$ has FW, then G has FW and X is finite.

Firstly, suppose that G does not have FW and let Y be a connected median graph on which G acts with an unbounded orbit $G.y$. Then $G \wr_X H$ acts on the connected median graph $Y \times \mathcal{P}_{\mathrm{f}}(X)$ by

$$(\varphi, h).(y, E) = \begin{cases} (\varphi(h.t).y, h.\{t\}) & \text{if } E = \{t\} \\ (y, h.E) & \text{if } E \text{ is not a singleton.} \end{cases}$$

Since $\{g.y\}_{g\in G}$ is unbounded, so is $\{g.y, \{x\}\}_{g\in G}$. In particular, the orbit of $(y, \{x\})$, which contains

$$\{(\delta_x^g, 1).(y, \{x\}) = (g.y, \{x\}) \mid g \in G\},$$

is unbounded and $G \wr_X H$ does not have FW.

Finally, suppose that X is infinite. As in Lemma 4.5, the group $\bigoplus_X G$ acts coordinatewise on $\bigsqcup_X G$. On the other hand, H acts on $\bigsqcup_X G$ by permutation of the factors. Altogether we have an action of $G \wr_X H$ on $\bigsqcup_X G$ and hence on the median graph $\mathcal{P}(\bigsqcup_X G)$. Let $\mathbf{1} \coloneqq \bigcup_{x\in X} 1_G$ be the subset of $\mathcal{P}(\bigsqcup_X G)$ consisting of the identity elements of all the copies of G. As in Lemma 4.5 we have that the action of $G \wr_X H$ on $\mathcal{P}(\bigsqcup_X G)$ preserves the connected component of $\mathbf{1}$ and that the orbit of $\mathbf{1}$ is unbounded. □

Remark 4.9. A reader familiar with wreath products might have recognize that we used both the classical primitive and imprimitive actions of it in the proof of Proposition 4.8.

On one hand, since H acts on X, it acts on $\mathcal{P}_{\mathrm{f}}(X)$. Hence, for every G action on Y we have the imprimitive action of $G \wr_{\mathcal{P}_{\mathrm{f}}(X)} H$ on $Y \times \mathcal{P}_{\mathrm{f}}(X)$, which can be restricted to an action of the subgroup $G \wr_X H \leq G \wr_{\mathcal{P}_{\mathrm{f}}(X)} H$.

On the other hand, G acts on itself by left multiplication. It hence acts on the set $G' \coloneqq G \sqcup \{\varepsilon\}$ by fixing ε. We hence have the primitive action of $G \wr_X H$ on G'^X. Now, the set $\bigsqcup_X G$ naturally embeds as the subset of G'^X consisting of all functions $\varphi\colon X \to G'$ such that $\varphi(x) = \varepsilon$ for all but one $x \in X$. This subset is $G \wr_X H$ invariant, which gives us the desired action of $G \wr_X H$ on $\bigsqcup_X G$.

5 Related concepts and generalizations

The aim of this section is to give a glimpse of some properties related to the FW property as well as to provide a general framework to study them. We will sometimes only sketch the arguments and let the reader verify all the details for its favorite property.

Property FW is not isolated in the realm of group properties and other similar properties of great interest can also be defined in terms of action with bounded orbits.

Definition 5.1. Let G be a group. It is said to have *property SB* if any action on a metric space has bounded orbits. It has *property FH* if any action on a real Hilbert space has bounded orbits and *property FA* if any action on a tree has bounded orbits. Finally, G is said to have (cofinality $\neq \omega$) if any action on an ultrametric space has bounded orbits.

For countable groups (and more generally for σ-compact locally compact groups), property FH is equivalent to the celebrated Kazdhan's property (T) by the Delorme-Guichardet theorem, see for example [**?**], but this is not true in general [**?**].

Maybe expand a little more.

The names FH, FW and FA come from the fact that these properties admit a description in terms of existence of a Fixed point for actions on Hilbert spaces, on spaces with Walls and on trees (*Arbres* in french). On the other hand, SB stands for Strongly Bounded and is sometimes called the Bergman property, while a group has cofinality $\neq \omega$ if and only if it cannot be written has an increasing union of proper subgroups. Give refs

We have the following strict implications [**?**, **?**, **?**, **?**]

$$\text{SB} \implies \text{FH} \implies \text{FW} \implies \text{FA} \implies (\text{cofinality} \neq \omega).$$

It is possible to define other properties in the spirit of Definition 5.1. Let P be a property of metric spaces (for example *be a connected median graph*) and BP be the group property: *Every G-action on a space with P has bounded orbits*, where a G-action is supposed to "preserve the P-structure". Lemmas 4.1 and 4.2 and Corollary 4.3 as well as their proofs remain true for groups with property BP.

On the other hand, the other results of Section 4 require a specific construction and do not generalize straightforward to groups with property BP. Nevertheless, it is possible to extract the main ingredients of the proofs given in Section 4 and to adapt them to some specific cases. We will now give a raw outline of this process, but let the details to the interested reader.

In order to generalize Example 4.4, we will need to construct from a G action on a set X a G action on a P-space $F(X)$, such that X embeds G-equivariently into $F(X)$. For median graphs we took $F(X) = \mathcal{P}(X)$ in Example 4.4, and then carefully choose some connected component of it. For real Hilbert spaces it is possible to take $F(X)$ to be the real Hilbert space generated by X. That is,

$$F(X) = \ell^2(X) = \left\{ f\colon X \to \mathbf{R} \;\middle|\; \sum_{x\in X} f(x)^2 < \infty \right\},$$

where G acts by permutation. For property SB, we can take $F(X)$ to be $\mathcal{P}(X)$ or $\ell^2(X)$, as well as many other possibilities. In the context of trees see Example 5.2 for a construction that works for some X. Finally, for ultrametric spaces it is always possible to take $F(X) = X$ with the discrete metric, however this will not be useful in practice as any action on it has bounded orbits. Rewrite ?

We will prove the analogous of Lemma 4.5 and Corollary 4.6 for property FA. In order to do that we will rely on the following example.

Example 5.2. Let $(G_i)_{i\geq 1}$ be non-trivial groups and ε denotes an element not in any of the G_i. Define $Y = Y((G_i)_{i\geq1}) \coloneqq \{f = (f_1, f_2, \dots) \mid f_i \in G_i \sqcup \{\varepsilon\}\}$. We say that a sequence f in Y has level n if $f_n \neq \varepsilon$ while $f_i = \varepsilon$ for $i < n$, if no such n exists (that is if f is the constant sequence ε) we say that f has level infinity. We put an edge between f and g in Y if one of them is of level n and the other of level $n+1$ and $f_i = g_i$ for $i > n$. If $f \in Y$ has level n, then it has exactly one neighbourhood of level $n+1$, which implies that Y is a forest. The group $G \coloneqq \bigoplus_{i\geq1} G_i$ acts on Y by $(g_1, g_2, \dots).(f_1, f_2, \dots) = (g_1 f_1, g_2 f_2, \dots)$ with the convention that $G_i.\varepsilon = \varepsilon$ for any i. It is trivial that this action preserves the graph structure on Y.

Lemma 5.3. *An infinite direct sum of non-trivial groups does not have property FA.*

As a consequence, if G is a non-trivial group, then the group $\otimes_X G$ has property FA if and only if X is finite and G has property FA.

Proof. The first assertion implies the second assertion since the analogous of Lemma 4.2 holds for property FA. Moreover, as in the proof of Lemma 4.5 it is enough to prove the first assertion for a countable numbers of groups.

Let $G = \bigoplus_{i\geq 1} G_i$ be a direct sum of non-trivial groups and let $Y = Y((G_i)_{i\geq 1})$ as in Example 5.2. Let $\mathbf{1} \coloneqq (1_{G_1}, 1_{G_2}, \dots)$ be the sequence consisting of the identity elements of all the G_i. Since every element of G has only a finite number of non-trivial coordinates, the action of G preserves the connected components of $\mathbf{1}$ (and in fact every connected component of Y).

For every $i \in \mathbf{N}$, choose a non-trivial $g_i \in G_i$. Then for every n, the orbit of the vertex $\mathbf{1}$ contains the point $\{g_1, \dots, g_n, 1_{G_{n+1}}, \dots\}$ which is at distance $2n$ of $\mathbf{1}$. That is the action of G on the connected component of $\mathbf{1}$ has an unbounded orbit. □

We also obtain the following "meta-proposition"

Proposition 5.4. *Let P be a property of spaces such that property BP implies property BA. Let $G \wr_X H$ be the wreath product of G and $H \curvearrowright X$ with G non-trivial. Then*

1. *If $G \wr_X H$ has property BP, then H has property BP and X is finite.*
2. *If X is finite and both G and H have property BP, then $G \wr_X H$ has property BP.*

Proof. The second assertion and half of the first assertion are Corollary 4.3 which holds for any property P.

It remains to show that if $G \wr_X H$ has property BP then X is finite. Which is equivalent to prove that for infinite X the group $G \wr_X H$ does not have FA. The proof is similar to the proof of the second part of Proposition 4.8 and rely on the fact that $G \wr_X H$ acts on $Y = (G_{x\in X})$. The action of $\bigoplus_{x\in X} G$ is as in Lemma 5.3, while H acts by permutation. □

In order to generalize Lemma 4.7, we only need that a product of two P-spaces is still a P-space. This property holds for metric spaces, ultra-metric spaces and also for Hilbert spaces where we take the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$. On the other hand, for trees there is two natural candidats on which $G_1 \times G_2$ acts: the cartesian product $T_1 \times T_2$ or the tensor product $T_1 \otimes T_2$, but none of them are tree ($T_1 \times T_2$ has cycles while $T_1 \otimes T_2$ is not connected).

Suppose that in the class of P-spaces the following hold: for any G-space Y and any H-space X we have a natural action of $G \wr_X H$ on $Y \times F(X)$. Then if $G \wr_X H$ has BP, so does G. This is the first part of Proposition 4.8. We now explicit this for Hilbert spaces.

Lemma 5.5. *Let $\mathcal{H}$ be an Hilbert space with an isometric G-action and X be a set with an H-action. Then $G \wr_X H$ acts isometrically on the Hilbert space $\mathcal{H} \oplus \ell^2(X)$ by* Check this

$$(\varphi, h).(y, f) = \begin{cases} (\varphi(h.x).y, \delta_{h.x}) & \text{if } f = \delta_x \\ (y, h.f) & \text{if } f \neq \delta_x. \end{cases}$$

In particular, if $G \curvearrowright \mathcal{H}$ has an unbounded orbit, so does $G \wr_X H \curvearrowright \mathcal{H} \oplus \ell^2(X)$.

We hence recover

Check si on peut enlever discret, + mettre réfs.

Proposition 5.6. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property FH if and only if G and H have the property FH and X is finite.*

It also follows from the above discussion that we also have

Proposition 5.7. *Let G, H be two discrete groups with G non-trivial and X a set on which H acts. The wreath product $G \wr_X H$ has the property SB if and only if G and H have the property SB and X is finite.*

We also have the following partial result for groups with the Bergmann property

Proposition 5.8. *Let $G \wr_X H$ be the wreath product of G and $H \curvearrowright X$. Then*

1. *If $G \wr_X H$ has cofinality $\neq \omega$, then so do G and H.*
2. *If X is finite and both G and H have cofinality $\neq \omega$, then $G \wr_X H$ has cofinality $\neq \omega$.*

Proof. Once again, Corollary 4.3 applies. It thus remains to show that if $G \wr_X H$ has cofinality $\neq \omega$, then so does G. Let (Y, d_y) be an ultrametric space on which G acts with an unbounded orbit $G.y_0$. If we endow X with the discrete metric d_X, the function $d_\infty((y, x), (y', x')) = \max\{d_Y(y, y'), d_X(x, x')\}$ is an ultra-metric on $Y \times X$ and the group $G \wr_X H$ acts isometrically on $(Y \times X, d_\infty)$. Finally, for any $x \in X$ the orbit of (y_0, x) is unbounded. $\square$

Finally, we mention the following result on property FA that was obtained by Cornulier and Kar and which strengthen Proposition 5.4. Observe that, by the characterization of Serre [**?**], a finitely generated group with property FA is a group with a finite abelianization, which cannot be expressed as an union of proper increasing sequence of subgroups

Theorem 5.9 ([5]). *Let G, H be two discrete groups with G non-trivial and X a set on which H acts with a finite number of orbits and no fixed point. Then the wreath product $G \wr_X H$ has the property FA if and only if H has the property FA and if G is a group with a finite abelianization, which cannot be expressed as an union of proper increasing sequence of subgroups.*

Regarder le cas PW, Haagerup,... i.e. action propre. Rappel: une action isométrique de G est propre si pour tout x (de manière équivalente il existe x), pour tout $r \in \mathbf{R}$ l'ensemble $\{g \in G \mid d(x, g.x)\}$ est fini.

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