The Property FW for the wreath products

Paul-Henry Leemann, Grégoire Schneeberger

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1 Introduction

The property FW was introduced by de Cornullier. It is a fixed point property for the action on wall spaces (for a detailed treatment of this property see [2]). For discrete groups, this property is implied by the Kazhdan property (T). The behavior of the Kazhdan Property (T) with the wreath product is well known:

Theorem 1.1 ([1, 3])**.** *Let G, H be two discrete groups and X a set on which H acts. The wreath product $G \wr_X H$ has the property (T) if and only if G and H have the property (T) and if X is finite.*

The same kind of result is true for the property FW

Theorem 1.2. *Let G, H be two discrete groups and X a set on which H acts transitively. The wreath product $G \wr_X H$ does not have the property FW if at least one of the following conditions is satisfied:*

1. *The group G does not have the property FW.*
2. *The group H does not have the property FW.*
3. *The set X is infinite.*

An action of a group on a CAT(0) cube complex is *essential* if all the orbits of vertices are unbounded and the action is transitive on the set of hyperplanes.

Corollary 1.3. *Let G, H be two discrete groups and X a set on which H acts transitively. If there exists an essential action of G or H on a CAT(0) cube complex or if X is infinite, then there exists an essential action of $G \wr_X H$ on a CAT(0) cube complex.*

2 Definitions

2.1 The Property FW

For the definition of the property FW, we will follow the survey of Y. de Cornullier [2].

Definition 2.1. Let G be a discrete group and X a discrete set on which G acts. A subset $M \subset X$ is *commensurated* by the G-action if

$$|gM \Delta M| < \infty$$

for all g in G.

An invariant G-subset is automatically commensurated. Moreover, for a subset M such that there exists an invariant G-subset N with $|M \Delta N| < \infty$ then M is commensurated. Such a set is called *transfixed*.

Definition 2.2. A group G has the property FW if all commensurable G-set are transfixed.

There are lot of equivalent characterizations of this property. We will give us without all the details and the precise definitions.

Proposition 2.3. *The following are equivalent:*

1. *G has the property FW;*
2. *every cardinal definite function on G is bounded;*
3. *every cellular action on a CAT(0) cube complex has bounded orbits for the ℓ^1-metric (the complexes can be infinite dimensional);*
4. *every cellular action on a CAT(0) cube complex has a fixed point;*
5. *every action on a connected median graph has bounded orbits;*
6. *every action on a nonempty connected median graph has a fixes point;*
7. *(if G is finitely generated) every Schreier graph of G has at most 1 end;*
8. *For every set Y endowed with a walling structure and compatible action on Y and on the index of the walling, the action on Y has bounded orbits for the wall distance;*
9. *every isometric action on an "integral Hilbert space" $\ell^2(X, \mathbf{Z})$ (X any discrete set), or equivalently on $\ell^2(X, \mathbf{R})$ preserving the integral points, has bounded orbits;*
10. *for every G-set X we have $H^1(G, \mathbf{Z}X) = 0$.*

Note that the name FW comes from the property of "fixed point" for the actions on the walling spaces. We will see in the following that a semi-splittable group does not have the property FW (see corollary **??**).

The property FW has links with other well known properties. For example, the property FH implies the characterisation 9. For discrete groups (and even for countable groups) the property FH is equivalent to the Kazhdan's property (T) by Delorme-Guichardet's Theorem. As trees are CAT(0) cube complexes, the property FW implies Serre's property FA.

2.2 Graphe de Schreier

2.3 Bouts

3 Proof of the Theorem

References

[1] P. A. Cherix, F. Martin, and A. Valette. Spaces with measured walls, the Haagerup property and property (T). *Ergod. Theory Dyn. Syst.*, 24(6):1895–1908, dec 2004.

[2] Y. Cornulier. Group actions with commensurated subsets, wallings and cubings. page 58, 2013.

[3] M. Neuhauser. Relative property (T) and related properties of wreath products. *Math. Zeitschrift*, 251(1):167–177, sep 2005.