The Property FW for the wreath products

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1 Introduction

The property FW was introduced by de Cornullier. It is a fixed point property for the action on wall spaces (for a detailed treatment of this property see [2]). For discrete groups, this property is implied by the Kazhdan property (T). The behavior of the Kazhdan Property (T) with the wreath product is well known:

Theorem 1.1 ([1, 3])**.** *Let G, H be two discrete groups and X a set on which H acts. The wreath product $G \wr_X H$ has the property (T) if and only if G and H have the property (T) and if X is finite.*

The same kind of result is true for the property FW

Theorem 1.2. *Let G, H be two discrete groups and X a set on which H acts transitively. The wreath product $G \wr_X H$ does not have the property FW if at least one of the following conditions is satisfied:*

1. *The group G does not have the property FW.*
2. *The group H does not have the property FW.*
3. *The set X is infinite.*

An action of a group on a CAT(0) cube complex is *essential* if all the orbits of vertices are unbounded and the action is transitive on the set of hyperplanes.

Corollary 1.3. *Let G, H be two discrete groups and X a set on which H acts transitively. If there exists an essential action of G or H on a CAT(0) cube complex or if X is infinite, then there exists an essential action of $G \wr_X H$ on a CAT(0) cube complex.*

2 Definitions

2.1 The Property FW

For the definition of the property FW, we will follow the survey of Y. de Cornullier [2].

Definition 2.1. Let G be a discrete group and X a discrete set on which G acts. A subset $M \subset X$ is *commensurated* by the G-action if

$$|gM \Delta M| < \infty$$

for all g in G.

An invariant G-subset is automatically commensurated. Moreover, for a subset M such that there exists an invariant G-subset N with $|M \Delta N| < \infty$ then M is commensurated. Such a set is called *transfixed*.

Definition 2.2. A group G has the property FW if all commensurable G-set are transfixed.

There are lot of equivalent characterizations of this property. We will give us without all the details and the precise definitions.

Proposition 2.3. *The following are equivalent:*

1. *G has the property FW;*
2. *every cardinal definite function on G is bounded;*
3. *every cellular action on a CAT(0) cube complex has bounded orbits for the ℓ^1-metric (the complexes can be infinite dimensional);*
4. *every cellular action on a CAT(0) cube complex has a fixed point;*
5. *every action on a connected median graph has bounded orbits;*
6. *every action on a nonempty connected median graph has a fixes point;*
7. *(if G is finitely generated) every Schreier graph of G has at most 1 end;*
8. *For every set Y endowed with a walling structure and compatible action on Y and on the index of the walling, the action on Y has bounded orbits for the wall distance;*
9. *every isometric action on an "integral Hilbert space" $\ell^2(X, \mathbf{Z})$ (X any discrete set), or equivalently on $\ell^2(X, \mathbf{R})$ preserving the integral points, has bounded orbits;*
10. *for every G-set X we have $H^1(G, \mathbf{Z}X) = 0$.*

Note that the name FW comes from the property of "fixed point" for the actions on the walling spaces. We will see in the following that a semi-splittable group does not have the property FW (see corollary **??**).

The property FW has links with other well known properties. For example, the property FH implies the characterisation 9. For discrete groups (and even for countable groups) the property FH is equivalent to the Kazhdan's property (T) by Delorme-Guichardet's Theorem. As trees are CAT(0) cube complexes, the property FW implies Serre's property FA.

2.2 Graphe de Schreier

Expliquer lien action de groupe, graphe de Schreier

2.3 Bouts

3 Proof of the Theorem

Proof. Let us begin by fixing the notation. We denote by S (and respectively by S') a finite generating set of G (and respectively of H). We choose an arbitrary point of x_0 of X. The group $\Gamma = G \wr_X H$ is generated by the set

$$\mathcal{S} = \{(\delta^s_{x_0}, e_h) : s \in S\} \cup \{(0, s') : s' \in S'\}$$

where

$$\delta^s_{x_0} = \begin{cases} e_g & x \neq x_0 \\ s & x = x_0 \end{cases} \quad \text{and} \quad 0(x) = e_g \quad \forall x \in X.$$

The idea of the proof is to construct, for each of the three cases, a Schreier graph of Γ with more than one end. To do this we will consider actions of Γ and associated graph's action. We will treat the 3 cases separately.

Suppose that X is an infinite set. We define $Y = G \times X$ and an action of Γ on Y as

$$(\varphi, h) \cdot (g, x) = (\varphi(hx)g, hx)$$

for (φ, h) in Γ and (g, x) in Y. This action is transitive. Indeed, let (g_1, x_1) and (g_2, x_2) be two elements of Y. By transitivity of the action of H on Y, there exists h in H such that $hx_1 = x_2$. We can always find φ in $\bigoplus_X G$ such that $\varphi(hx_1) = g_2 g_1^{-1}$. Then,

$$(\varphi, h)(g_1, x_1) = (\varphi(hx_1)g, hx_1) = (g_2, x_2).$$

The graph of the action of Γ on Y is isomorphic to the Schreier graph $\mathrm{Sch}(\Gamma, \mathrm{Stab}(e_G, x_0), \mathcal{S})$. We decompose the graph into leaves of the form $Y_g = \{g\} \times G$. There are two types of edges on this graph which are coming from the two types of generators. The first one, of the form $(0, s')$, give us on each leaf a copy of the graph of the action of H on X. Indeed, we have

$$(0, s')(g, x) = (g, s'x).$$

Tthe second one, of the form $(\delta^s_{x_0}, 0)$, give us loops everywhere excepting on vertices with x_0 as second coordinate. By direct computation, we see that a leaf Y_g is connected to Y_{sg} by the vertices (g, x_0) and (sg, x_0):

$$(\delta^s_{x_0}, 0)(g, x) = \begin{cases} (g, x) & x \neq x_0 \\ (sg, x) & x = x_0 \end{cases}$$

Structure du Schreier et des feuilles

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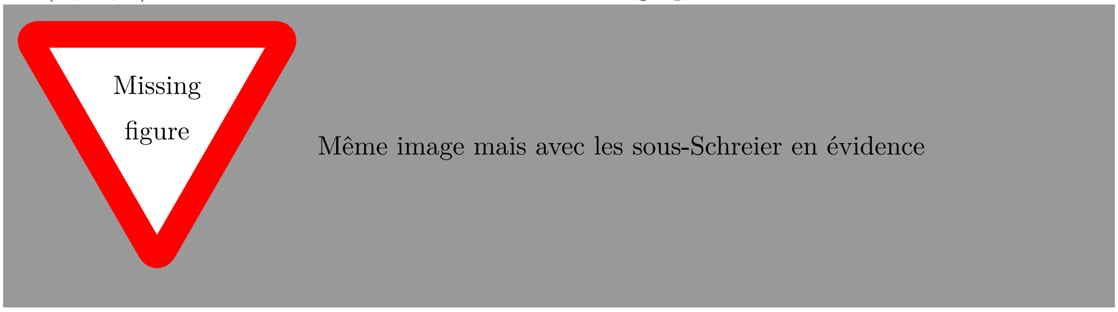
If we remove a vertex (g, x_0) we disconnect the leaf Y_g to the graph. As X is infinite, the number of ends is strictly greater than 1.

Suppose that G does not have the property FW. By the point 7 of the Proposition 2.3 there exists a subgroup K of G such that $\mathrm{Sch}(G,K,S)$ has more than one end. The group Γ acts on $G/K \times X$:

$$(\varphi, h)(gK, x) = (\varphi(hx)gK, hx).$$

As above, the action is transitive and the graph of this action is isomorphic to a Schreier graph. We decompose this graph into leaves in the same way. Now we look at the subgraph made up of vertices (g, x_0) and edges $(\delta_{x_0}^s, 0)$ and we remark that it is isomorphic the Schreier graph $\mathrm{Sch}(G,K,S)$ which has more than one end. Then our graph has also more than one end.



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Suppose that H does not have the property FW. There is a subgroup K of H such that $\mathrm{Sch}(H,K,S')$ has more than one end. We consider the action of Γ on H/K defined as

$$(\varphi, h)h'K = hh'K.$$

This is a transitive action. All the edges of type $(\delta_{x_0}^s, 0)$ are loops and the edges $(0, s')$ are the edges of $\mathrm{Sch}(H,K,S')$ which has more than one end. Then this Schreier graph has also more than one end. $\square$

4 Stuff

Lemma 4.1. *Let $G = N \rtimes H$ be a semidirect product. If both N and H have FW, then G also has FW.*

Proof. Let X be a non-empty connected median graph on which G acts. Then both N and H acts on X, with orbits bounded respectively by d_N and d_H. Now, for every $x \in X$ and $g \in G$, there is $n \in N$ and $h \in H$ such that $g = nh$ ans thus $g.x = n.(h.x)$ is at distance at most $d_N + d_H$ from x. $\square$

Corollary 4.2. *If G and H have FW and H acts on a finite set X, then $G \wr_X H$ has FW.*

Proof. Since X is finite, this directly follows from multiple application of Lemma 4.1 to $G \wr_X H = (\bigoplus_X G) \rtimes H$. $\square$

Lemma 4.3. *Let $G = N \rtimes H$ be a semidirect product. If G has FW, then so does H.*

Proof. Let X be a non-empty connected median graph on which H acts. Then G acts on X by $g.x \coloneqq h.x$ where $g = nh$ with $n \in N$ and $h \in H$. By assumption, the action of G on X has bounded orbits and so does the action of H. $\square$

Corollary 4.4. *If $G \wr_X H$ has FW, then so does H.*

Proposition 4.5 (Cornullier)**.** *If X is infinite, then $G \wr_X H$ does not have FW.*

Cornullier l'énonce pour des groupes topologiques. Regarder que ça correspond bien et éventuellement donner une preuve utilisant les graphes médians.

Theorem 4.6 (L-S)**.** *The group $G \wr_X H$ has FW if and only if both G and H have FW and X is finite.*

Proof. The "only if" part is Corollary 4.2. On the other hand, if $G \wr_X H$ has FW, then H has FW by Corollary 4.4 and X is finite by Proposition 4.5. It remains to show that if $G \wr_X H$ has FW, then so does G.

Suppose that both G and H are finitely generated and that $G \wr_X H$ has FW. Then by [stuff about ends], G has FW.

Est-ce qu'on peut dire qqch en général, en utilisant une autre méthode ?

□

References

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[2] Y. Cornulier. Group actions with commensurated subsets, wallings and cubings. page 58, 2013.

[3] M. Neuhauser. Relative property (T) and related properties of wreath products. *Math. Zeitschrift*, 251(1):167–177, sep 2005.