Report on the article “Wreath products of groups acting with bounded orbits” by P.-H. Leemann and G. Schneeberger

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The article under review focuses on fixed-point properties of (permutational) wreath products of groups. In other words, the question is the following: given a wreath product $G \wr_X H$ and a category of metric spaces $\mathbf{S}$, when does every action of $G \wr_X H$ on every metric space in $\mathbf{S}$ have bounded orbits? Here, actions are by isometries that possibly preserve a given structure. (For instance, if $\mathbf{S}$ is a category of cellular complexes, the isometries can be required to preserve the cell structure.)

Several results in this direction are already available in the literature, including the fixed-point properties FA (for trees), FH (for Hilbert spaces), and FW (for median graphs, or equivalently CAT(0) cube complexes). In this article, the authors propose an axiomatic framework that encompasses the properties FH and FW, but also Bergman's property (for arbitrary metric spaces) and the property FB_r (for reflexive Banach spaces). Formally, the main result of the article is the following:

Theorem: *Let* $\mathbf{S}$ *be a category of metric spaces that has unbounded Cartesian powers. Assume that* $\mathrm{B}\mathbf{S}$ *implies FW. Given two groups* G, H *with* G *non-trivial and an* H*-set* X*, the wreath product* $G \wr_X H$ *has property* $\mathrm{B}\mathbf{S}$ *if and only if* X *is finite and* G, H *have property* $\mathrm{B}\mathbf{S}$.

Here, a group A has property $\mathrm{B}\mathbf{S}$ if every action of A on every metric space in $\mathbf{S}$ has bounded orbits. The category $\mathbf{S}$ has unbounded Cartesian prowers if, loosely speaking, there is a good notion of Cartesian product in $\mathbf{S}$.

So the main theorem recovers several known results and obtains properties that have not been considered before.

The article is well-written, pleasant to read; and the proofs are clear and detailed. The proof of the main theorem is rather elementary, and does not bring new ideas. The original contribution of the article comes from the axiomatic framework. However, I do not think that the axioms are optimal: it seems that the assumptions used in the proofs are much weaker, so the axiomatic framework should be optimised. (Some of the comments below go in this direction.)

1 Specific comments

1. Page 1, last line: “which preserves”.

2. Theorem B: The result is rather elementary, maybe a proposition instead of a theorem?

3. Page 4, Median graphs: It could be interesting to mention explicitly the connection with CAT(0) cube complexes (Gerasimov, Chepoi, Roller).

4. Page 5, line -4: It is not clear that the sequence is increasing. I guess the sequence $(n)_n$ should be replaced with a well-chosen increasing subsequence $(r_n)_n$.
5. Page 6, line -15: I do not find that the example about graphs is convincing: describing graphs as discrete metric spaces, isometries are automatically automorphisms. But the same example makes sense for cube complexes.
6. Page 7, line 4: Isn't it already the case for Bergman's original example $\mathrm{Sym}(\mathbb{N})$ (much before Cornulier)?
7. Page 7, Proposition 2.5: In my copy of the article, there are TeX problems with the arrows. Is FH $\Rightarrow$ F$\mathbb{R}$ missing?
8. Page 7, line -8: A triangle Coxeter group would be a more explicit example of a group with FA but not FW.
9. Page 9, line 2: "one looks".
10. Page 9, line 5: "there are two".
11. Page 9, line 6: "The first one consists".
12. Page 9, line -9: This is not true: the infinite dihedral group does not surject onto $\mathbb{Z}$ since it is generated by elements of order two. However, looking at the subcategory $\{\mathbb{Z}\}$ with orientation-preserving isometries, the property B**S** amounts to not having $\mathbb{Z}$ as a quotient.
13. Page 9, line -6: No need of Bass-Serre theory: $\mathbb{Z}$ is a tree, so the implication is obvious.
14. Page 10, Lemma 2.10: There is a more direct argument: Let X be a metric space. Let $G(X)$ denote the graph obtained from a vertex-set X by applying the following process: for any two $x, y \in X$, add a path of length $\lfloor d(x,y) \rfloor + 1$ between x and y. $G(X)$ is connected and the obvious inclusion $\iota : X \to G(X)$ is a quasi-isometric embedding. Moreover, the construction is canonical, so every group action on X extends to a group action on $G(X)$, making ι equivariant. So if a group satisfying the bounded orbit property on connected graphs acts on a metric space X, then its induced action on $G(X)$ has bounded orbits, which implies that its orbits in X are bounded.
15. Page 10, Definition 2.11: Is it a good terminology? The subcategory of bounded metric spaces has unbounded Cartesian products...
16. Page 10, Definition 2.11, item 2: Not clear. Does it mean that the obvious image of $\mathrm{Aut}(X)^n \rtimes \mathrm{Sym}(n)$ in $\mathrm{Bij}(X^n)$ lies in $\mathrm{Aut}(X^n)$ or that $\mathrm{Aut}(X^n)$ contains a subgroup isomorphic to $\mathrm{Aut}(X)^n \rtimes \mathrm{Sym}(n)$? The meaning is clear from the proofs, but it should be precise already in the definition.
17. Page 12, line 6: Roller proved the general bounded orbit property much before Cornulier.
18. Page 12, second paragraph: It could be worth mentioning that one recovers a global fixed point in the category of CAT(0) cube complexes.
19. Page 13, Lemma 3.1: The first sentence of the proof is not used, so it can be removed.
20. Page 14, line 19: The notation $H \simeq \{g_i N\}$ is not clear.

21. Page 14, line -14: “there exist”.
22. Page 15, Lemma 3.7: The group G is not defined.
23. Page 15, lines 13-14: Each G should be replaced with G_x.
24. Page 15, line 15: An enumeration of Y has to be fixed.
25. Page 15, Lemma 3.7: I find weird that what is proved is rather different from what is claimed in the lemma. The proof is really about uncountable cofinality and not about B**S**.

    A suggestion: write your lemma as a corollary of a new lemma stating that, if a semidirect product $A \rtimes B$ has uncountable cofinality, then there exists a finite subset $S \subset A$ such that A is generated by $\bigcup_{b \in B} bSb^{-1}$. (From the characterisation of uncountable cofinality in terms of subgroups, this is pretty obvious.)
26. Page 16, Lemma 3.8: Same remark as before: it would be more natural to state the lemma about FW and to mention the result as a corollary.
27. Page 16, Lemma 3.8: It could be interesting to formulate the lemma in greater generality. If G has a non-trivial action on Z, then $G \wr_X H$ naturally acts on $\bigoplus_X Z$ with unbounded orbits if X is infinite. So you just need your category to have a well-defined infinite product operation and the property that every non-trivial group admits a non-trivial action on one space in the category. (For FW, G acts non-trivially on the tree whose vertices are G, $\{g\}$ $(g \in G)$ and whose edges are given by inclusion.).
28. Page 17: Lemma 3.11 can be seen as a direct corollary of Lemma 3.10(2). Indeed, if H' is the kernel of the action of H on X, the pre-image of H' under the quotient map $G \wr_X H \to H$ is a finite-index subgroup that splits as $H' \oplus \bigoplus_X G$. The latter must have B**S**, but G is a quotient of this group.
29. Page 17, Theorem 3.12: The implication B**S** $\Rightarrow$ FW is a rather artificial assumption. I think you should list the few axioms you are actually using. It seems to me that there are only two: (1) an axiom about products, with possibly infinitely many factors, which could generalise the definition of unbounded Cartesian powers; (2) every non-trivial group G admits a non-trivial action on a space in **S**.

    If $G \wr_X H$ has B**S**, then X must be finite by Lemma 3.8 (deduced from (1) and (2)), H must have B**S** because it is a quotient, and G must have B**S** because of Lemma 3.11 (which is a corollary of Lemma 3.10, itself obtained from (1)). Conversely, if X is finite and G, H have B**S**, then $G \wr_X H$ virtually splits as a direct sum of finitely many copies of G and a finite-index subgroup of H. But stability of B**S** under direct sum is given by (1).
30. Page 19, Lemma 3.13: The assumption of having finitely many orbits can be removed.
31. Page 19: As written, Theorem C is only proved for FA.
32. Proofs of Lemma 3.7 and Theorem B: You should mention that the proofs using the characterisation in terms of subgroups are actually much shorter. In fact, they are almost exercices.
33. In all the article: “an **S**-space” instead of “a **S**-space”?
34. In all the article: There are several references to results from Cornulier’s monograph. This is a rather long paper, so please add precise references.