Wreath products of groups acting with bounded orbits

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February 15, 2021

Abstract

If **S** is a structure over (a concrete category over) metric spaces, we say that a group G has property B**S** if any action on a **S**-space has bounded orbits. Examples of such structures include metric spaces, Hilbert spaces, CAT(0) cube complexes, connected median graphs, trees or ultra-metric spaces. The corresponding properties B**S** are respectively Bergman's property, property FH (which, for countable groups, is equivalent to the celebrated Kazhdan's property (T)), property FW (both for CAT(0) cube complexes and for connected median graphs), property FA and uncountable cofinality (cof $\neq \omega$).

Our main result is that for a large class of structures **S**, the wreath product $G \wr_X H$ has property B**S** if and only if both G and H have property B**S** and X is finite. On one hand, this encompasses in a general setting previously known results for properties FH and FW. On the other hand, this also applies to the Bergman's property. Finally, we also obtain that $G \wr_X H$ has cof $\neq \omega$ if and only if both G and H have cof $\neq \omega$ and H acts on X with finitely many orbits.

1 Introduction

When working with group properties, it is natural to ask if they are stable under "natural" group operations. One such operation, of great use in geometric group theory, is the wreath product, which stands between the direct product and semi-direct product of groups, see Section 2 for all the relevant definitions.

A **S**-space is a metric space with an "additional structure" and we will said that a group G has *property B***S** if every action by isometries which preserve the **structure** on a **S**-space has bounded orbits, see Definitions 2.8 and 2.9 for formal statements. We note that having one bounded orbit implies that all the orbits are bounded.

In the context of properties defined by actions with bounded orbits, the first result concerning wreath products, due to Cherix, Martin and Valette and later refined by Neuhauser, concerns Kazhdan's property (T).

Theorem 1.1 ([5, 16])**.** *Let G and H be two discrete groups with G non-trivial and let X be a set on which H acts. The wreath product $G \wr_X H$ has property (T) if and only if G and H have property (T) and X is finite.*

[*]Supported by grant 200021_188578 of the Swiss National Fund for Scientific Research.

For countable groups (and more generally for σ-compact locally-compact topological groups), property (T) is equivalent, by the Delorme-Guichardet's Theorem, to property FH (every action on Hilbert space has bounded orbits), see [2, Thm. 2.12.4]. Hence, Theorem 1.1 can also be viewed, for countable groups, as a result on property FH.

The corresponding result for property FA (every action on tree has bounded orbits) is a little more convoluted and was obtained a few years later by Cornulier and Kar.

Theorem 1.2 ([8])**.** *Let G and H be two groups with G non-trivial and let X be a set on which H acts with finitely many orbits and without fixed points. Then $G \wr_X H$ has property FA if and only if H has property FA, G has no quotient isomorphic to $\mathbf{Z}$ and can not be written as a countable increasing union of proper subgroups.*

Finally, in a recent note, the authors proved an analogous of Theorem 1.1 for property FW (every action on wall spaces has bounded orbits):

Theorem 1.3 ([14])**.** *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Suppose that all three of G, H and $G \wr_X H$ are finitely generated. Then the wreath product $G \wr_X H$ has property FW if and only if G and H have property FW and X is finite.*

Since the publication of Theorem 1.3, Y. Stalder has let us know (private communication) that the arguments of [14] can be adapted to replace the finite generation hypothesis of Theorem 1.3 by the condition that all three of G, H and $G \wr_X H$ are at most countable. In particular, Theorem 1.3 covers the result of A. Genevois published in [11].

The above results on properties FH, FW and FA were obtained with distinct methods even if the final results have a common flavor. In the same time, all three properties FH, FW and FA can be characterized by the fact that any isometric action on a suitable metric space (respectively Hilbert space, connected median graph and tree) has bounded orbits, see Definition 2.5. But more group properties can be characterized in terms of actions with bounded orbits. This is for example the case of the Bergman's property (actions on metric spaces), property $\mathrm{FB_r}$ (actions on reflexive Banach spaces) or of uncountable cofinality, denoted by $\mathrm{cof} \neq \omega$, (actions on ultrametric spaces).

By adopting the point of view of actions with bounded orbits, we obtain a unified proof of the following result; see also Theorem 3.12 for the general (and more technical) statement.

Theorem A. *Let $B\mathbf{S}$ be any one of the following properties: Bergman's property, property FB_r, property FH or property FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. Then the wreath product $G \wr_X H$ has property $B\mathbf{S}$ if and only if G and H have property $B\mathbf{S}$ and X is finite.*

With a little twist, we also obtain a similar result for groups with $\mathrm{cof} \neq \omega$:

Theorem B. *Let G and H be two groups with G non-trivial and let X be a set on which H acts. Then the wreath product $G \wr_X H$ has $cof \neq \omega$ if and only if G and H have $cof \neq \omega$ and H acts on X with finitely many orbits.*

A crucial ingredient of our proofs is that the spaces under consideration admit a natural notion of Cartesian product. In particular, some of our results do not work for trees and property FA. Nevertheless, we are still able to show that if $G \wr_X H$ has property FA, then H acts on X with finitely many orbits. Combining this with Theorem 1.2 we obtain

Theorem C. *Let G and H be two groups with G non-trivial and X a set on which H acts. Suppose that H acts on X without fixed points. Then $G \wr_X H$ has property FA if and only if H has property FA, H acts on X with finitely many orbits, G has no quotient isomorphic to $\mathbf{Z}$ and can not be written as a countable increasing union of proper subgroups.*

Acknowledgment The authors are thankful to A. Genevois and T. Nagnibeda for helpful comments and to the NSF for its support.

2 Definitions and examples

This section contains all the definitions, as well as some useful preliminary facts and some examples.

2.1 Wreath products

Let X be a set and G a group. We view $\bigoplus_X G$ as the set of functions from X to G with finite support:

$$\bigoplus_X G = \{\varphi\colon X \to G \mid \varphi(x) = 1 \text{ for all but finitely many } x\}.$$

This is naturally a group, where multiplication is taken componentwise.

If H is a group acting on X, then it naturally acts on $\bigoplus_X G$ by $(h.\varphi)(x) = \varphi(h^{-1}.x)$. This leads to the following standard definition

Definition 2.1. Let G and H be groups and X be a set on which H acts. The *(restricted [1]) wreath product* $G \wr_X H$ is the group $(\bigoplus_X G) \rtimes H$.

For g in G and x in X, we define the following analogs of Kronecker's delta functions

$$\delta_x^g(y) \coloneqq \begin{cases} g & y = x \\ 1 & y \neq x. \end{cases}$$

A prominent particular case of wreath products is of the form $G \wr_H H$, where H acts on itself by left multiplication. They are sometimes called *standard wreath products* or simply *wreath products*, while general $G \wr_X H$ are sometimes called *permutational wreath products*. Best known example of wreath product is the so called *lamplighter group* $(\mathbf{Z}/2\mathbf{Z}) \wr_{\mathbf{Z}} \mathbf{Z}$. Other (trivial) examples of wreath products are direct products $G \times H$ which correspond to wreath products over a singleton $G \wr_{\{*\}} H$.

[1] There exists an unrestricted version of this product where the direct sum is remplaced by a direct product.

2.2 Classical actions with bounded orbits

We will discuss some classical group properties which are defined by actions with bounded orbits on various metric spaces.

Median graphs For u and v two vertices of a connected[2] graph $\mathcal{G}$, we define the total interval $[u, v]$ as the set of vertices that lie on some shortest path between u and v. A connected graph $\mathcal{G}$ is *median* if for any three vertices u, v, w, the intersection $[u, v] \cap [v, w] \cap [u, w]$ consists of a unique vertex, denoted $m(u, v, w)$. A graph is *median* if each of its connected components is median. For more on median graphs and spaces see [1, 4, 13]. If X and Y are both (connected) median graphs, then their cartesian product is also a (connected) median graph. Trees are the simplest examples of such graphs. The ensuing example is important for the following.

Example 2.2. Let X be a set and let $\mathcal{P}(X) = 2^X$ be the set of all subsets of X. Define a graph structure on $\mathcal{P}(X)$ by putting an edge between E and F if and only if $\#(E\Delta F) = 1$, where Δ is the symmetric difference. Therefore, the distance between two subsets E and F is $E\Delta F$ and the connected component of E is the set of all subsets F with $E\Delta F$ finite. For E and F in the same connected component, $[E, F]$ consists of all subsets of X that both contain $E \cap F$ and are contained in $E \cup F$. In particular, $\mathcal{P}(X)$ is a median graph, with $m(D, E, F)$ being the set of all elements belonging to at least two of D, E and F. In other words, $m(D, E, F) = (D \cap E) \cup (D \cap F) \cup (E \cap F)$.

These graphs will be fundamental for us due to the following fact. Any action of a group G on a set X naturally extends to an action of G on $\mathcal{P}(X)$ by graph homomorphisms: $g.\{x_1, \dots, x_n\} = \{g.x_1, \dots, g.x_n\}$. Note that the action of G on $\mathcal{P}(X)$ may exchange the connected components. In fact, the connected component of $E \subset X$ is stabilized by G if and only if E is *commensurated* by G, that is if for every $g \in G$ the set $E\Delta gE$ is finite.

Uncountable cofinality Recall that a metric space (X, d) is *ultrametric* if d satisfies the strong triangular inequality: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for any x, y and z in X. A group G has *uncountable cofinality*, or cof $\neq \omega$, if every action on ultrametric spaces has bounded orbits. The following characterization of groups without cof $\neq \omega$ is well-known and we include a proof only for the sake of completeness. It implies in particular that a countable group has cof $\neq \omega$ if and only if it is finitely generated.

Lemma 2.3. *Let G be a group. Then the following are equivalent:*

1. *G can be written as a countable increasing union of proper subgroups,*
2. *G does not have cof $\neq \omega$, i.e. there exists an ultrametric space X on which G acts with an unbounded orbit,*
3. *There exists a G-invariant (for the action by left multiplication) ultrametric d on G such that $G \curvearrowright G$ has an unbounded orbit.*

[2] We will always assume that our connected graphs are non-empty. This is coherent with the definition that a connected graph is a graph with exactly one connected component.

Proof. It is clear that the third item implies the second.

Suppose that (X, d) is an ultrametric space on which G acts with an unbounded orbit and let x_0 be an element of X such that $G.x_0$ is unbounded. For any $n \in \mathbf{N}$ let H_n be the subset of G defined by

$$H_n \coloneqq \{g \in G \mid d(x_0, g.x_0) \leq n\}.$$

It is clear that G is the increasing union of the (countably many) H_n. On the other hand, the H_n are subgroups of G. Indeed, H_n is trivially closed under taking the inverse, and is also closed under taking products since $d(x_0, gh.x_0) \leq \max\{d(x_0, g.x_0), d(g.x_0, gh.x_0)\} = \max\{d(x_0, g.x_0), d(x_0, h.x_0)\}$. As $G.x_0$ is unbounded, they are proper subgroups.

Finally, suppose that $G = \bigcup_{n \in \mathbf{N}} H_n$ where the H_n form an increasing sequence of proper subgroups. It is always possible to suppose that $H_0 = \{1\}$. Define d on G by $d(g, h) \coloneqq \min\{n \mid g^{-1}h \in H_n\}$. One easily verifies that d is a G-invariant ultrametric. Moreover, the orbit of 1 contains all of G and is hence unbounded. ☐

A slight variation of the above lemma gives us

Example 2.4. Let $(G_i)_{i \geq 1}$ be non-trivial groups and let $G \coloneqq \bigoplus_{i \geq 1} G_i$ be their direct sum. Then $d_\infty(f, g) \coloneqq \max\{i \mid f(i) \neq g(i)\}$ is an ultrametric on G which is G-invariant for the action of G on itself by left multiplication.

Some classical group properties We now discuss the bounded orbits properties introduced above for actions on various classes of metric spaces.

Definition 2.5. Let G be a group. It is said to have

- *Bergman's property* if any action on a metric space has bounded orbits,
- *property FB$_r$* if any action on a reflexive real Banach space has bounded orbits,
- *property FH* if any action on a real Hilbert space has bounded orbits,
- *property FW* if any action on a connected median graph has bounded orbits,
- *property FA* if any action on a tree has bounded orbits,
- *cof* $\neq \omega$ if any action on an ultrametric space has bounded orbits.

In the above, *actions* are supposed to preserve the structure. In particular, actions on (ultra)metric spaces are by isometries, actions on graphs (including trees) are by graph isomorphisms and actions on Hilbert spaces are by scalar product preserving isometries.

The names $\mathrm{FB_r}$, FH, FW and FA come from the fact that these properties admit a description in terms of (and were fist studied in the context of) the existence of a Fixed point for actions on reflexive Banach spaces, on Hilbert spaces, on spaces with Walls (or equivalently on CAT(0) cube complexes) and on trees (*Arbres* in french). For a survey on property $\mathrm{FB_r}$, see [17] and the references therein.

For countable groups (and more generally for σ-compact locally compact groups), property FH is equivalent to the celebrated Kazhdan's property (T) by the Delorme-Guichardet theorem, but this is not true in general. Indeed in [9], Cornulier constructed an uncountable discrete group G with Bergman's property which, as we will see just below, implies property FH. Such a group cannot have property (T) as, for discrete groups, it implies finite generation.

A classical result of the Bass-Serre theory of groups acting on trees [18], is that a group G has property FA if and only if it satisfies the following three conditions: G has cof $\neq \omega$, G has no quotient isomorphic to $\mathbf{Z}$ and G is not a non-trivial amalgam. In view of this characterization, Theorem 1.2 says that property FA almost behave well under wreath products.

Proposition 2.6. *There are the following implications between the properties of Definition 2.5:*

$$\text{Bergman's property} \implies \mathrm{FB_r} \implies \mathrm{FH} \implies \mathrm{FW} \implies \mathrm{FA} \implies \mathrm{cof} \neq \omega. \quad (\dagger)$$

Moreover, except maybe for the leftmost one, all implications are strict.

Proof. The implications Bergman's property $\implies$ $\mathrm{FB_r}$ $\implies$ FH and FW $\implies$ FA trivially follow from the fact that Hilbert spaces are reflexive Banach spaces, which are themselves metric spaces and that trees are connected median graphs. The implication FA $\implies$ cof $\neq \omega$ is due to Serre [18]: if G is an increasing union of subgroups G_i, then $\bigsqcup G/G_i$ admits a tree structure by joining any $gG_i \in G/G_i$ to $gG_{i+1} \in G/G_{i+1}$. The action of G by multiplication on $\bigsqcup G/G_i$ is by graph isomorphisms and with unbounded orbits. Finally, the implication FH $\implies$ FW follows from the fact that a group G has property FW if and only if any action on a real Hilbert space which preserves integral points has bounded orbits [6].

On the other hand, here are some examples for the strictness of the implications of ($\dagger$). Countable groups with property $\mathrm{FB_r}$ are finite by [3], while infinite finitely generated groups with property (T), e.g. $\mathrm{SL}_3(\mathbf{Z})$, have property FH. The group $\mathrm{SL}_2(\mathbf{Z}[\sqrt{2}])$ has property FW but not FH, see [7]. If G is a non-trivial finite group and H is an infinite group with property FA, then $G \wr_H H$ has property FA by Theorem C, but does not have property FW by Theorem A. Finally, $\mathbf{Z}$ has cof $\neq \omega$, while it acts by translations and with unbounded orbits on the infinite 2-regular tree. ☐

It is possible to consider relative versions of the properties appearing in Definition 2.5. If G is a group and H a subgroup of G, we say that the pair (G, H) has *relative property B*$\mathbf{S}$ if for every G action on a $\mathbf{S}$-space, the H orbits are bounded. A group G has property B$\mathbf{S}$ if and only if for every subgroup H the pair (G, H) has relative property B$\mathbf{S}$, and if and only if for every overgroup L the pair (L, G) has relative property B$\mathbf{S}$.

2.3 Groups acting with bounded orbits on S-spaces

It is possible to define other properties in the spirit of Definition 2.5 for any "additional structure on metric spaces". In order to give a uniform treatment of all of them, we will use the notion of quasipseudo-metric spaces and the language of category theory. A reader not familiar with category theory and interested only in one specific structure may forget all these general considerations and only verify that the arguments of Section 3 apply for their favorite structure.

Definition 2.7. A *quasipseudo-metric space* is a set X with a map $d\colon X \times X \to \mathbf{R}_{\geq 0}$, called a *quasipseudo-distance*, such that

1. $d(x,x) = 0$ for all $x \in X$,
2. $d(x,z) \leq d(x,y) + d(y,z)$.

If moreover $d(x,y) = d(y,x)$ and $d(x,y) \neq 0$ for $x \neq y$, the map d is a *distance* and (X,d) is a metric space. On the other hand, an *ultra-quasipseudo-metric space* is a quasipseudo-metric space (X,d) such that d satisfies the strong triangular inequality. A *morphism* (or *short map*) between two quasipseudo-metric spaces (X_1,d_1) and (X_2,d_2) is a distance non-increasing map $f\colon X_1 \to X_2$, that is $d_2(f(x),f(y)) \leq d_1(x,y)$ for any x and y in X_1. If f is bijective and distance preserving, then it is an *isomorphism* (or *isometry*). Quasipseudo-metric spaces with short maps form a category **QPMet** of which the category of metric spaces (with short maps) **Met** is a full subcategory.

If (X,d) is a quasipseudo-metric space, we have a natural notion of the *diameter* of a subset $Y \subset X$ with value in $[0,\infty]$, defined by $\operatorname{diam}(Y) \coloneqq \sup\{d(x,y) \mid x,y \in Y\}$.

Definition 2.8. An *additional structure on quasipseudo-metric spaces*, or a *qp-metric structure* for short, is a concrete category $(\mathbf{S},F_\mathbf{S})$ over **QPMet**. That is, it is a category $\mathbf{S}$ together with a faithful functor $F_\mathbf{S}\colon \mathbf{S} \to \mathbf{QPMet}$. The objects of $\mathbf{S}$ are called $\mathbf{S}$*-spaces* and the morphisms $\mathbf{S}$*-morphisms*.

A G*-action on a* $\mathbf{S}$*-space* X is simply an homomorphism $\alpha\colon G \to \operatorname{Aut}_\mathbf{S}(X)$. It has *bounded orbits* if $F_\mathbf{S} \circ \alpha\colon G \to \operatorname{Aut}_{\mathbf{QPMet}}(X)$ has bounded orbits.

In practice, we will often simply write $\mathbf{S}$ for the pair $(\mathbf{S},F_\mathbf{S})$. Since **QPMet** itself is concrete, that is we have a faithful functor $F\colon \mathbf{QPMet} \to \mathbf{Set}$, the category $\mathbf{S}$ is also concrete (via $F \circ F_S$) and its objects can be thoughts as sets with “extra structure”.

In practice, a lot of examples of concrete categories over **QPMet** factor through the category **Met**. Obvious examples of concrete categories over **Met** include metric spaces and ultrametric spaces (with short maps). Hilbert spaces, Banach spaces and more generally (semi-) normed spaces are also concrete over **Met** if we restrict ourself to morphisms that do not increase the distance (that is such that $\langle f(x) \mid f(y)\rangle \leq \langle x \mid y\rangle$, respectively $\|f(x)\| \leq \|x\|$). In particular, for us isomorphisms of Hilbert and Banach spaces will always be isometries.

For connected graphs (and hence for connected median graphs and for trees), one looks at the category **Graph** where objects are connected simple graphs $G = (V,E)$ and where a morphism $f\colon (V,E) \to (V',E')$ is a function between the vertex sets such that if (x,y) is an edge then either $f(x) = f(y)$ or $(f(x),f(y))$ is an edge. The functor $F_\mathbf{S}\colon \mathbf{Graph} \to \mathbf{Met}$ sends a connected graph to its vertex set together with the graph distance: $d(x,y)$ is the minimum number of edges on a path between x and y. We can hence identify **Graph** with the full subcategory of **Met** consisting of metric spaces (X,d) such that d has values in $\mathbf{N}$.

We can now formally define the group property B$\mathbf{S}$ as:

Definition 2.9. Let $(\mathbf{S}, F_{\mathbf{S}})$ be a qp-metric structure. A group G has *property* $B\mathbf{S}$ if every G-action on a $\mathbf{S}$-space has bounded orbits. A pair (G, H) of a group and a subgroup has *relative property* $B\mathbf{S}$ if for every G-action on a $\mathbf{S}$-space, the H orbits are bounded.

All the properties of Definition 2.5 are of the form B$\mathbf{S}$. Examples of other interesting properties include $\mathbf{R}$-trees (also called *real trees*) or (some specific subclass of) Banach spaces. The property F$\mathbf{R}$ of having bounded actions on $\mathbf{R}$-trees is known to be strictly stronger than FA [15]. Theorem C holds whenever property FA is replaced by property F$\mathbf{R}$, with a similar proof.

Another interesting example of a property of the form B$\mathbf{S}$ is the fact to have no quotient isomorphic to $\mathbf{Z}$, see Example 2.10. The main interest for us of this example is that property FA is the conjonction of three properties, two of them (cof $\neq \omega$ and having no quotient isomorphic to $\mathbf{Z}$) still being of the form B$\mathbf{S}$.

Example 2.10. Let Z^n be the integer lattice of dimension n, that is Z^n is the Cayley graph of $\mathbf{Z}^n$ for the standard generating set. Then every infinite subgroup of $\mathrm{Aut}(Z^n)$ projects onto $\mathbf{Z}$. Hence, we obtain that a group G has no quotient isomorphic to $\mathbf{Z}$ if and only if every G action on Z has bounded orbits, if and only if every G action on Z^n has bounded orbits. Let us denote by B$\mathbf{Z}$ this property.

It follows from Bass-Serre theory that FA implies B$\mathbf{Z}$. This implication is strict as demonstrated by $\mathbf{Q}$. In fact, the counterexample $\mathbf{Q}$ shows that B$\mathbf{Z}$ does not implies cof $\neq \omega$. By looking at $\mathbf{Z}$, we see that cof $\neq \omega$ does not implies B$\mathbf{Z}$ neither.

An example of an uninteresting property B$\mathbf{S}$ is given by taking $\mathbf{S}$ to be the category of metric spaces of bounded diameter (together with short maps). Indeed, in this case, any group has B$\mathbf{S}$. On the opposite, if $\mathbf{S}$ is the category of extended quasipseudo-metric spaces (d takes values in $\mathbf{R} \cup \{\infty\}$), only the trivial group has B$\mathbf{S}$.

The category **QPMet** has the advantage (over **Met**) of behaving more nicely with respects to categorical constructions. However, we have

Lemma 2.11. *A group G has Bergman's property (respectively cof $\neq \omega$) if and only if any G action on a quasipseudo-metric (respectively ultra-quasipseudo-metric) space has bounded orbits.*

Proof. One direction is trivial.

For the other direction, let (X, d) be a quasipseudo-metric space on which G acts by isometries. Let $d'(x, y) \coloneqq \frac{1}{2}(d(x, y) + d(y, x))$ be the symmetrization of d. Then the action of G on X is by d'-isometries and a subset $Y \subset X$ is bounded for d if and only if it is bounded for d'. Finally, let $\tilde{X} \coloneqq X/\sim$ be the quotient of X for the relation $x \sim y$ if $d'(x, y) = 0$ and let $\tilde{d}$ be the quotient of d'. Then $(\tilde{X}, \tilde{d})$ is a metric space, the action of G passes to the quotient and $G.x$ is d' bounded (hence d bounded) if and only if $G.[x]$ is $\tilde{d}$ bounded. Finally, if d satisfies the strong triangular inequality, then so do d' and $\tilde{d}$. ☐

On the other hand, the following result is perhaps more surprising.

Lemma 2.12. *A group G has Bergman's property if and only if any G action on a connected graph has bounded orbits.*

Proof. The left-to-right implication is clear.

For the other direction, we will use the following characterization of Bergman's property due to Cornulier [9]. A group G has Bergman's property if and only if it has cof $\neq \omega$ and for every generating set T of G the Calyey graph $\mathrm{Cayl}(G;T)$ is bounded.

Suppose that any G action on a connected graph has bounded orbits. Then G has property FW and hence cof $\neq \omega$. On the other hand, for every generating set T, the group G acts transitively on $\mathrm{Cayl}(G;T)$, which implies that the latter is bounded. By the above characterization, we obtain that G has property SB. □

While we will be able to obtain some results for a general qp-metric structure $\mathbf{S}$, we will sometimes need to restrict ourselves to structures with a suitable notion of cartesian product.

Definition 2.13. A qp-metric structure $(\mathbf{S}, F_{\mathbf{S}})$ has *cartesian powers* if for any $\mathbf{S}$-space X and any integer n, there exists a $\mathbf{S}$-object, called the *n^{th} cartesian power of X* and written X^n, such that:

1. X^n is compatible with the cartesian product of sets. That is $F \circ F_{\mathbf{S}}(X^n)$ is the set cartesian power.
2. If $E \subset X$ is unbounded, then the diagonal $\mathrm{diag}(E) \subset X^n$ is unbounded.
3. $\mathrm{Aut}_{\mathbf{S}}(X)^n \rtimes \mathrm{Sym}(n)$ is a subgroup of $\mathrm{Aut}_{\mathbf{S}}(X^n)$.

For (quasipseudo-/ultra-) metric spaces, the categorical product (corresponding to the metric $d_\infty = \max\{d_X, d_Y\}$) works fine, but any product metric of the form $d_p = (d_X^p + d_Y^p)^{\frac{1}{p}}$ for $p \in [1, \infty]$ works as well. For Hilbert and Banach spaces, we take the usual cartesian product (which is also the categorial product), which corresponds to the metric $d_2 = \sqrt{d_X^2 + d_Y^2}$. For connected median graphs, the usual cartesian product (which is not the categorical product![3]) with $d_1 = d_X + d_Y$ works well. On the other hand, trees do not have cartesian powers.

Remark 2.14. In view of Definition 2.9 and 2.13, the reader might ask why we are working in **QPMet** instead of **Born**, the category of bornological spaces together with bounded maps. The reason behind this is the forthcoming Lemma 3.2 and its corollaries, which fail for general bornological spaces.

Groups acting with fixed point on S-spaces Some of the properties that are of interest for us have been historically defined via the existence of a fixed point for some action. More generally, we say that a group G has *property F*$\mathbf{S}$ if any G action on a $\mathbf{S}$-space has a fixed point.

Since our actions are by isometries, property F$\mathbf{S}$ implies property B$\mathbf{S}$. The other implication holds as soon as we have a suitable notion of the center of a (non-empty) bounded subset X. For a large class of metric spaces, this is provided by the following result of Bruhat and Tits:

Proposition 2.15 ([10, Chapter 3.b]). *Let (X, d) be a complete metric space such that the following two conditions are satisfied:*

[3]The categorial product in **Graph** is the strong product.

1. *For all x and y in X, there exists a unique $m \in X$ (the middle of $[x,y]$) such that $d(x,m) = d(y,m) = \frac{1}{2}d(x,y)$;*
2. *For all x, y and z in X, if m is the middle of $[y,z]$ we have the median's inequality $2d(x,m)^2 + \frac{1}{2}d(y,z)^2 \leq d(x,y)^2 + d(x,z)^2$.*

Then if G is a group acting by isometries on X with a bounded orbit, it has a fixed point.

Exemples of complete metric spaces satisfying Proposition 2.15 include among others: Hilbert spaces, Bruhat-Tits Buildings, Hadamard spaces (i.e. complete CAT(0) spaces), trees and $\mathbf{R}$-trees; with the caveat that for ($\mathbf{R}$-)trees, the fixed point is either a vertex or the middle of an edge. See [10, Chapter 3.b] and the references therein for more on this subject. On the other hand, [2, Lemma 2.2.7] gives a simple proof of the existence of a center for bounded subsets of Hilbert spaces, and more generally of reflexive Banach spaces.

For action on (ultra)-metric spaces or on connected median graphs, F$\mathbf{S}$ is strictly stronger than B$\mathbf{S}$. Indeed, this trivially follows from the action by rotation of C_4 on the square graph. However, by [6, 12] if a group G acts on a connected median graph with a bounded orbit, then it has a finite orbit.

Remark 2.16. We conclude this section by a remark on a variation of Definition 2.5. One might wonder what happens if in Definition 2.5 we replace the requirement of having bounded orbits by having uniformly bounded orbits. It turns out that this is rather uninteresting, as a group G is trivial if and only if any G-action on a metric space (respectively on an Hilbert space, on a connected median graph, on a tree or on an ultrametric space) has uniformly bounded orbits. Indeed, if G is non-trivial, then, for the action of G on the Hilbert space $\ell^2(G)$ the orbit of $n \cdot \delta_g$ has diameter $n\sqrt{2}$. For a tree (and hence also for a connected median graph), one may look at the tree T obtained by taking a root r on which we glue an infinite ray for each element of G. Then G naturally acts on T by permuting the rays. The orbits for this action are the $\mathcal{L}_n = \{v \mid d(v,r) = n\}$ which have diameter $2n$. Finally, it is possible to put an ultradistance on the vertices of T by $d_\infty(x,y) \coloneqq \max\{d(x,r), d(y,r)\}$ if $x \neq y$. Then the orbits are still the $\mathcal{L}_n$, but this time with diameter n.

3 Proofs of the main results

Throughout this section, $\mathbf{S}$ will denote a qp-metric structure and B$\mathbf{S}$ the group property "every action on a $\mathbf{S}$-space has bounded orbits". Heuristically, a $\mathbf{S}$-space is a (quasipseudo-) metric space with an additional structure (as for example an Hilbert space). For a precise definition, see Definition 2.9.

We begin this section with two easy but useful results.

Lemma 3.1. *Let G be a group and H be a quotient. If G has property B$\mathbf{S}$, then so has H.*

Proof. We have $H \cong G/N$. If H acts on some $\mathbf{S}$-space X with an unbounded orbit, then the G action on X defined by $g.x \coloneqq gN.x$ has also an unbounded orbit. $\square$

Lemma 3.2. *Let G be a group and A an B be two subgroups such that $G = AB$. If both (G, A) and (G, B) have relative property $B\mathbf{S}$, then G has property $B\mathbf{S}$.*

Proof. Let X be a $\mathbf{S}$-space on which G acts and let x be an element of X. Let D_1 be the diameter of $A.x$ and D_2 the diameter of $B.x$. By assumption, they both are finite. Since A acts by isometries, all the $a.Bx$ have diameter D_2. Let y be an element of $G.x$. There exists $a \in A$ such that y belongs to $a.Bx$. Since 1 belongs to B, y is at distance at most D_2 of $a.x$ and hence at distance at most $D_1 + D_2$ of x. Therefore, the diameter of $G.x$ is finite. □

By combining Lemmas 3.1 and 3.2, we obtain the following three corollaries on direct, semi-direct and wreath products.

Corollary 3.3. *Let G and H be two groups. Then $G \times H$ has property $B\mathbf{S}$ if and only if both G and H have property $B\mathbf{S}$.*

Corollary 3.4. *Let $N \rtimes H$ be a semidirect product. Then*

1. *If $N \rtimes H$ has property $B\mathbf{S}$, then so has H.*
2. *If both N and H have property $B\mathbf{S}$, then $N \rtimes H$ also has property $B\mathbf{S}$.*

Corollary 3.5. *Let G and H be two groups and X a set on which H acts. Then,*

1. *If $G \wr_X H$ has property $B\mathbf{S}$, then so has H,*
2. *If both G and H have property $B\mathbf{S}$ and X is finite, then $G \wr_X H$ has property $B\mathbf{S}$.*

When $\mathbf{S}$ has a suitable notion of quotients (by a group of isometries), it is possible to obtain a strong version of Lemmas 3.1 and 3.2. Here is the corresponding result for Bergman's property and $\mathrm{cof} \neq \omega$.

Proposition 3.6. *Let $B\mathbf{S}$ be either Bergman's property or the property $\mathrm{cof} \neq \omega$. Let $1 \to N \to G \to H \to 1$ be a group extension. Then G has property $B\mathbf{S}$ if and only if H has property $B\mathbf{S}$ and the pair (G, N) has the relative $B\mathbf{S}$ property.*

Proof. One direction is simply Lemma 3.1 and the definition of relative property B$\mathbf{S}$.

On the other hand, let (X, d) be a quasipseudo-metric space on which G acts by isometries and let x be an element of X. Let $\{g_i \mid i \in I\}$ be a transversal for N, that is $H \cong \{g_i N\}$. By assumption, $N.x$ is bounded of diameter D_1 and for any $i \in I$ the subset $g_i N.x$ of X has also diameter D_1. Since N is a subgroup of isometries of X, the map $d' \colon X/N \times X/N \to \mathbf{R}$ defined by $d'([x], [y]) \coloneqq \inf\{d(x', y') \mid x' \in N.x, y' \in N.y\}$ is the quotient quasipseudo-distance on X/N. Indeed, while the map d' might not satisfies the triangle inequality for a generic quotient $X/\sim$, this is the case if the quotient is by a subgroup of isometries; details are left to the reader. Moreover, if d satisfies the strong triangle inequality, then so does d'. The quotient action of $H \cong G/N$ on X/N is by isometries and the diameter of $H.xN$ is bounded, say by D_2. In particular, for any i and j in I, the distance between the subsets $g_i N.x$ and $g_j N.x$ of X is bounded by D_2. Since this distance is an infimum, there exists actual elements of $g_i N.x$ and $g_j N.x$ at distance less than $D_2 + 1$. Altogether, we obtain that any y in $G.x$ is at distance at most $D_1 + D_2 + 1$ of x. Hence, the orbit $G.x$ is bounded. □

Since the square graph, which is not median, is a quotient of the 2-regular infinite tree by a subgroup of isometries, the proof of Proposition 3.6 does not carry over for properties FW and FA. Similarly, the quotient of $\mathbf{R}$ by the action of $\mathbf{Z}/2\mathbf{Z}$ given by $x \mapsto -x$ is not an Hilbert space and hence the proof of Proposition 3.6 does not apply to property FH. However, the statement of Proposition 3.6 (stability under extension) remains true for properties FH, FW and FA. For property FH this is an exercice using the fixed-point definition, for property FW and FA, see [6] and [18].

We now state a result on infinite direct sums.

Lemma 3.7. *Suppose that B$\mathbf{S}$ implies cof $\neq \omega$. Let $(G_x)_{x\in X}$ be non-trivial groups. Then*

1. *$\bigoplus_{x\in X} G_x$ has property B$\mathbf{S}$ if and only if all the G_x have property B$\mathbf{S}$ and X is finite,*

2. *If $G \wr_X H$ has property B$\mathbf{S}$, then H acts on X with finitely many orbits.*

It is of course possible to prove Lemma 3.7 using the characterization of cof $\neq \omega$ in terms of subgroups. However, we find enlightening to prove it using the characterization in terms of actions on ultrametric spaces.

Proof of Lemma 3.7. One direction of the first assertion is simply Corollary 3.3. For the other direction, if $\bigoplus_{x\in X} G_x$ has property B$\mathbf{S}$ then all its quotients, and hence all the G_x, have property B$\mathbf{S}$. On the other hand, if X is infinite, there exists a countable subset $Y \subset X$. Let $Z := X \setminus Y$, thus we have $X = Y \sqcup Z$. We can decompose the direct sum as $\bigoplus_X G = (\bigoplus_Y G) \times (\bigoplus_Z G)$ and then, by Corollary 3.3, if $\bigoplus_Y G$ does not have cof $\neq \omega$, then neither does $\bigoplus_X G$. So let $G := \bigoplus_{i\geq 1} G_i$ and for each i, choose $g_i \neq 1$ in G_i. Let $d_\infty(f,g) := \max\{i \mid f(i) \neq g(i)\}$ be the G-invariant ultrametric of Example 2.4. Then for every integer n, the orbit $G.1_G$ contains $\{g_1,\dots,g_n,1,\dots\}$ which is at distance n of 1_G for d_∞ if the g_i are not equal to 1. In particular, an infinite direct sum of non-trivial groups does not have cof $\neq \omega$, nor does it have B$\mathbf{S}$.

The second assertion is a simple variation on the first. Indeed, we have

$$G \wr_X H \cong \Big(\bigoplus_{Y\in X/H} L_Y\Big) \rtimes H \qquad \text{with} \qquad L_Y \cong \bigoplus_{y\in Y} G_y,$$

where X/H is the set of H-orbits. The important fact for us is that H fixes the decomposition into L_Y factors: for all Y we have $H.L_Y = L_Y$. Up to regrouping some of the L_Y together we hence have $G \wr_X H \cong \big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$ with $H.L_i = L_i$ for all i. Now, we have an ultradistance d_∞ on $L := \bigoplus_{i\geq 1} L_i$ as above and we can put the discrete distance d on H. Then $d'_\infty = \max\{d_\infty, d\}$ is an ultradistance on $\big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$, which is $\big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$-invariant (for the action by left multiplication). From a practical point of view, we have $d'_\infty\big((\varphi,h),(\varphi',h')\big) := \max\{i \mid \varphi(i) \neq \varphi'(i)\}$ if $\varphi \neq \varphi'$ and $d'_\infty\big((\varphi,h),(\varphi,h') = 1$ if $h \neq h'$. Since the action of L on itself has an unbounded orbit for d_∞, the action of $\big(\bigoplus_{i\geq 1} L_i\big) \rtimes H$ on itself has an unbounded orbit for d'_∞. $\square$

While the statement (and the proof) of Lemma 3.7 is expressed in terms of cof $\neq \omega$, it is also possible to state it and prove it for a qp-metric structure $\mathbf{S}$ without a priori knowing if B$\mathbf{S}$ is stronger than cof $\neq \omega$. The main idea is to find a "natural" **S**-space on which $G = \bigoplus_{i\geq 1} G_i$ acts. For example, for Hilbert spaces, one can take $\bigoplus_{i\geq 1} \ell^2(G_i)$. For connected median graphs, one takes the connected component of $\{1_{G_1}, 1_{G_2}, \dots\}$ in $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$. For trees, it is possible to put a forest structure on $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$ in the following way. For $E \in \mathcal{P}(\bigsqcup_{i\geq 1} G_i)$, and for each i such that $E \cap G_j$ is empty for all $j \leq i$, add an edge from E to $E \cup \{g\}$ for each $g \in G_i$. The graph obtained this way is a G-invariant subforest of the median graph on $\mathcal{P}(\bigsqcup_{i\geq 1} G_i)$.

Lemma 3.8. *Suppose that B***S** *implies FW. Let G and H be two groups with G non-trivial and let X be a set on which H acts. If $G \wr_X H$ has B***S***, then X is finite.*

Proof. We will prove that if X is infinite, then $G \wr_X H$ does not have property FW. Suppose that X is infinite. The group $\bigoplus_X G$ acts coordinatewise on $\bigsqcup_X G$: the group G_x acting by left multiplication on G_x and trivially on G_y for $y \neq x$. On the other hand, H acts on $\bigsqcup_X G$ by permutation of the factors. Altogether we have an action of $G \wr_X H$ on $\bigsqcup_X G$ and hence on the median graph $\mathcal{P}(\bigsqcup_X G)$. Let $\mathbf{1} \coloneqq \bigcup_{x\in X} 1_G$ be the subset of $\mathcal{P}(\bigsqcup_X G)$ consisting of the identity elements of all the copies of G. Since every element of $\bigoplus_X G$ has only a finite number of non-trivial coordinates, the action of $G \wr_X H$ preserves the connected components of $\mathbf{1}$ (and in fact every connected component of $\mathcal{P}(\bigsqcup_X G)$).

Let $I = \{i_1, i_2, \dots\}$ be a countable subset of X and for every $i \in I$, choose a non-trivial $g_i \in G_i$. Then the orbit of the vertex $\mathbf{1}$ contains the point $\{g_{i_1}, \dots, g_{i_n}\} \cup \left(\bigcup_{j>n} 1_{G_{i_j}}\right) \cup \left(\bigcup_{x\notin I} 1_{G_x}\right)$ which is at distance $2n$ of $\mathbf{1}$. Therefore, the action of $G \wr_X H$ on the connected component of $\mathbf{1}$ has an unbounded orbit and then $G \wr_X H$ does not have property FW. $\square$

Once again, given a suitable **S**, it is sometimes possible to give a direct proof of Lemma 3.8. For example, for Hilbert spaces one can take $\bigoplus_X \ell^2(G)$ with $\bigoplus_X G$ acting coordinatewise and H by permutations. On the other hand, both the forest structure on $\mathcal{P}(\bigsqcup_X G)$ and the ultrametric structure on $\bigoplus_X G$ are in general not invariant under the natural action of H by permutations.

In fact, it follows from Theorems 1.2 and B that in the assumptions of Lemma 3.8 it is not possible to replace property FW by property FA or by cof $\neq \omega$.

Remark 3.9. A reader familiar with wreath products might have recognized that we used the primitive action of the wreath product in the proof of Lemma 3.8.

Indeed, G acts on itself by left multiplication. It hence acts on the set $G' \coloneqq G \sqcup \{\varepsilon\}$ by fixing ε, and we have the primitive action of $G \wr_X H$ on G'^X. Now, the set $\bigsqcup_X G$ naturally embeds as the subset of G'^X consisting of all functions $\varphi\colon X \to G'$ such that $\varphi(x) = \varepsilon$ for all but one $x \in X$. This subset is $G \wr_X H$ invariant, which gives us the desired action of $G \wr_X H$ on $\bigsqcup_X G$.

We now turn our attention to properties that behave well under cartesian products in the sense of Definition 2.13.

We first describe the comportement of property B**S** under finite index subgroups.

Lemma 3.10. *Let G be a group and let H be a finite index subgroup.*

1. *If H has property B***S***, then so has G,*

2. *If* $\mathbf{S}$ *has cartesian powers and* G *has property B*$\mathbf{S}$*, then* H *has property B*$\mathbf{S}$*.*

Proof. Suppose that G does not have B$\mathbf{S}$ and let X be a $\mathbf{S}$-space on which G acts with an unbounded orbit $\mathcal{O}$. Then H acts on X and $\mathcal{O}$ is a union of at most $[G:H]$ orbits. This directly implies that H has an unbounded orbit and therefore does not have B$\mathbf{S}$.

On the other hand, suppose that $H \leq G$ is a finite index subgroup of G without property B$\mathbf{S}$. Let $\alpha\colon H \curvearrowright X$ be an action of H on a $\mathbf{S}$-space (X, d_X) such that there is an unbounded orbit $\mathcal{O}$. Similarly to the classical theory of representations of finite groups, we have the induced action $\mathrm{Ind}_H^G(\alpha)\colon G \curvearrowright X^{G/H}$ on the set $X^{G/H}$. Since H has finite index, $X^{G/H}$ is a $\mathbf{S}$-space and the action is by $\mathbf{S}$-automorphisms. On the other hand, the subgroup $H \leq G$ acts diagonally on $X^{G/H}$, which implies that $\mathrm{diag}(\mathcal{O})$ is contained in a G-orbit. Since $\mathrm{diag}(\mathcal{O})$ is unbounded, G does not have property B$\mathbf{S}$.

For readers that are not familiar with representations of finite groups, here is the above argument in more details. Let $(f_i)_{i=1}^n$ be a transversal for G/H. The natural action of G on G/H gives rise to an action of G on $\{1, \dots, n\}$. Hence, for any g in G and i in $\{1, \dots, n\}$ there exists a unique $h_{g,i}$ in H such that $gf_i = f_{g.i}h_{g,i}$. That is, $h_{g,i} = f_{g.i}^{-1} g f_i$. We then define $g.(x_1, \dots, x_n) \coloneqq (h_{g,g^{-1}.1}.x_{g^{-1}.1}, \dots, h_{g,g^{-1}.n}.x_{g^{-1}.n})$. This is indeed an action by $\mathbf{S}$-automorphisms on $X^{G/H}$ by Condition 3 of Definition 2.13. Moreover, every element $h \in H$ acts diagonally by $h.(x_1, \dots, x_n) = (h.x_1, \dots, h.x_n)$. In particular, this G action has an unbounded orbit. $\square$

We now prove one last lemma that will be necessary fo the proof of Theorem A.

Lemma 3.11. *Suppose that* $\mathbf{S}$ *has cartesian powers. If* X *is finite and* $G \wr_X H$ *has property B*$\mathbf{S}$*, then* G *has property B*$\mathbf{S}$*.*

Proof. Suppose that G does not have B$\mathbf{S}$ and let (Y, d_Y) be a $\mathbf{S}$-space on which G acts with an unbounded orbit $G.y$. Then (Y^X, d) is a $\mathbf{S}$-space and we have the *primitive action* of the wreath product $G \wr_X H$ on Y^X:

$$\big((\varphi, h).\psi\big)(x) = \varphi(h^{-1}.x).\psi(h^{-1}.x).$$

By Condition 3 of Definition 2.13, this action is by $\mathbf{S}$-automorphisms. The orbit $G.y$ embeds diagonally and hence $\mathrm{diag}(G.y)$ is an unbounded subset of some $G \wr_X H$-orbit, which implies that $G \wr_X H$ does not have property B$\mathbf{S}$. $\square$

By combining Corollary 3.5 and Lemmas 3.8 and 3.11 we obtain the following result which implies Theorem A.

Theorem 3.12. *Suppose that* $\mathbf{S}$ *has cartesian powers and is such that B*$\mathbf{S}$ *implies FW. Let* G *and* H *be two groups with* G *non-trivial and let* X *be a set on which* H *acts. Then the wreath product* $G \wr_X H$ *has property B*$\mathbf{S}$ *if and only if* G *and* H *have property B*$\mathbf{S}$ *and* X *is finite.*

We now proceed to prove Theorem B. As for Lemma 3.7, it is also possible to prove it using the characterization of $\mathrm{cof} \neq \omega$ in terms of subgroups, but we will only give a proof using the characterization in terms of actions on ultrametric spaces.

Proof of Theorem B. By Corollary 3.5 and Lemma 3.7 we already know that if $G \wr_X H$ has cof $\neq \omega$, then H has cof $\neq \omega$ and it acts on X with finitely many orbits. We will now prove that if $G \wr_X H$ has cof $\neq \omega$ so does G. Let us suppose that G has countable cofinality. By Lemma 2.3, there exists an ultrametric d on G such that the action of G on itself by left multiplication has an unbounded orbit. But then we have the primitive action of the wreath product $G \wr_X H$ on $G^X \cong \prod_X G$, which preserves $\bigoplus_X G$. It is easy to check that the map $d_\infty \colon \bigoplus_X G \times \bigoplus_X G \to \mathbf{R}$ defined by $d_\infty(\psi_1, \psi_2) := \max\{d\big(\psi_1(x), \psi_2(x)\big) \mid x \in X\}$ is a $G \wr_X H$-invariant ultrametric. Finally, let $h \in G$ be an element of unbounded G-orbit for d and let x_0 be any element of X. Then for any g in G we have $(\delta^g_{x_0}, 1).\delta^h_{x_0} = \delta^{gh}_{x_0}$ and hence $d_\infty(\delta^h_{x_0}, \delta^{gh}_{x_0}) = d(h, gh)$ is unbounded.

Suppose now that both G and H have cof $\neq \omega$ and that H acts on X with finitely many orbits. We want to prove that $G \wr_X H$ has cof $\neq \omega$.

Let (Y, d) be an ultrametric space on which $G \wr_X H$ acts. Then H and all the G_x act on Y with bounded orbits. Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ be the H-orbits on X and for each $1 \leq i \leq n$ choose an element x_i in $\mathcal{O}_i$. Let y be any element of Y. Then $H.y$ has finite diameter D_0 while $G_{x_i}.y$ has finite diameter D_i. For any $x \in X$, there exists $1 \leq i \leq n$ and $h \in H$ such that $x = h.x_i$. We have

$$\begin{aligned} d\big((\delta^g_{x_i}, h^{-1}).y, y\big) &\leq \max\{d\big((\delta^g_{x_i}, h^{-1}).y, (\delta^g_{x_i}, 1).y\big), d\big((\delta^g_{x_i}, 1).y, y\big)\} \\ &= \max\{d\big((1, h^{-1}).y, y\big), d\big((\delta^g_{x_i}, 1).y, y\big)\} \\ &\leq \max\{D_0, D_i\}, \end{aligned}$$

which implies that the diameter of $G_{x_i} h^{-1}.y$ is bounded by $\max\{D_0, D_i\}$. But $G_{x_i} h^{-1}.y$ has the same diameter as $hG_{x_i}h^{-1}.y = G_{h.x_i}.y = G_x.y$.

On the other hand, the diameter of $\bigoplus_X G.y$ is bounded by the supremum of the diameters of the $G_{x_i}.y$, and hence bounded by $\max\{D_0, D_1, \ldots, D_n\}$. Finally, for (φ, h) in $G \wr_Y H$ we have

$$\begin{aligned} d\big(y, (\varphi, h).y\big) &\leq \max\{d\big(y, (\varphi, 1).y\big), d\big((\varphi, 1).y, (\varphi, h).y\big)\} \\ &= \max\{d\big(y, (\varphi, 1).y\big), d\big(y, (1, h).y\big)\} \\ &\leq \max\{\max\{D_0, D_1, \ldots, D_n\}, D_0\}. \end{aligned}$$

That is, the diameter of $G \wr_Y H.z$ is itself bounded by $\max\{D_0, D_1, \ldots, D_n\}$, which finishes the proof. □

While the fact that trees do not have cartesian powers is an obstacle to our methods, we still have a weak version of Theorem 3.12 for property FA. Before stating it, remind that we already know, by Theorem B, the behavior of cof $\neq \omega$ under wreath products. On the other hand, we have the following result (which also admits a direct algebraic proof)

Lemma 3.13. *Suppose that H acts on X with finitely many orbits. Then the group $G \wr_X H$ has no quotient isomorphic to $\mathbf{Z}$ if and only if both G and H have no quotient isomorphic to $\mathbf{Z}$.*

Proof. Remind that a group K has no quotient isomorphic to $\mathbf{Z}$ if and only if K^{ab} has no quotient isomorphic to $\mathbf{Z}$. The desired result then follows from $(G \wr_X H)^{\mathrm{ab}} \cong \bigoplus_{X/H}(G^{\mathrm{ab}}) \times H^{\mathrm{ab}}$ and Corollary 3.3. □

By Corollary 3.5, Theorem B and Lemma 3.13, we directly obtain the following partial version of Theorem 1.2.

Proposition 3.14. *Let G and H be two groups with G non-trivial and X a set on which H acts. Then*

1. *If $G \wr_X H$ has property FA, then H has property FA, H acts on X with finitely many orbits, G has no quotient isomorphic to $\mathbf{Z}$ and G has cof $\neq \omega$,*
2. *If both G and H have no quotient isomorphic to $\mathbf{Z}$, have cof $\neq \omega$ and H acts on X with finitely many orbits, then $G \wr_X H$ has no quotient isomorphic to $\mathbf{Z}$ and has cof $\neq \omega$,*
3. *If both G and H have property FA and X is finite, then $G \wr_X H$ has property FA.*

Moreover, by using Lemma 3.7 we can get ride of the "finitely many orbits" hypothesis in Theorem 1.2 in order to obtain Theorem C.

Observe that our statement of Theorem 1.2 differs of the original statement of [8]. Indeed, where we ask G to have cof $\neq \omega$ and no quotient isomorphic to $\mathbf{Z}$, the authors of [8] ask G to have cof $\neq \omega$ and finite abelianization. However, these two sets of conditions are equivalent. One implication is trivial, as finite abelian groups do not projects onto $\mathbf{Z}$. The other implication follows, for countable abelian groups, from the structure theorem of finitely generated abelian groups. For the general case, Y. Cornulier kindly reminded us that any infinite abelian group as a countable quotient.

References

[1] Hans-Jürgen Bandelt and Victor Chepoi. Metric graph theory and geometry: a survey. In *Surv. Discret. Comput. Geom.*, volume 453 of *Contemp. Math.*, pages 49–86. Amer. Math. Soc., Providence, RI, 2008.

[2] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[3] Nathanial Brown and Erik Guentner. Uniform embeddings of bounded geometry spaces into reflexive Banach space. *Proc. Amer. Math. Soc.*, 133(7):2045–2050, 2005.

[4] Indira Chatterji, Cornelia Druţu, and Frédéric Haglund. Kazhdan and Haagerup properties from the median viewpoint. *Adv. Math.*, 225(2):882–921, 2010.

[5] Pierre-Alain Cherix, Florian Martin, and Alain Valette. Spaces with measured walls, the Haagerup property and property (T). *Ergodic Theory Dynam. Systems*, 24(6):1895–1908, 2004.

[6] Yves Cornulier. Group actions with commensurated subsets, wallings and cubings. *arXiv e-prints*, page arXiv:1302.5982, February 2013.

[7] Yves Cornulier. Irreducible lattices, invariant means, and commensurating actions. *Math. Z.*, 279(1-2):1–26, 2015.

[8] Yves Cornulier and Aditi Kar. On property (FA) for wreath products. *J. Group Theory*, 14(1):165–174, 2011.

[9] Yves de Cornulier. Strongly bounded groups and infinite powers of finite groups. *Comm. Algebra*, 34(7):2337–2345, 2006.

[10] Pierre de la Harpe and Alain Valette. La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger). *Astérisque*, (175):158, 1989. With an appendix by M. Burger.

[11] Anthony Genevois. Lamplighter groups, median spaces, and a-T-menability. *arXiv e-prints*, page arXiv:1705.00834, May 2017.

[12] V. Gerasimov. Fixed-point-free actions on cubings [translation of *algebra, geometry, analysis and mathematical physics (russian) (novosibirsk, 1996)*, 91–109, 190, Izdat. Ross. Akad. Nauk Sibirsk. Otdel. Inst. Mat., Novosibirsk, 1997; MR1624115 (99c:20049)]. *Siberian Adv. Math.*, 8(3):36–58, 1998.

[13] Sandi Klavžar and Henry Martyn Mulder. Median graphs: characterizations, location theory and related structures. *J. Comb. Math. Comb. Comput.*, 30:103–127, 1999.

[14] Paul-Henry Leemann and Grégoire Schneeberger. Property FW and wreath products of groups: a simple approach using Schreier graphs. *arXiv e-prints*, page arXiv:2101.03817, January 2021.

[15] Ashot Minasyan. New examples of groups acting on real trees. *J. Topol.*, 9(1):192–214, 2016.

[16] Markus Neuhauser. Relative property (T) and related properties of wreath products. *Math. Z.*, 251(1):167–177, 2005.

[17] Piotr W. Nowak. Group actions on Banach spaces. In *Handbook of group actions. Vol. II*, volume 32 of *Adv. Lect. Math. (ALM)*, pages 121–149. Int. Press, Somerville, MA, 2015.

[18] Jean-Pierre Serre. *Arbres, amalgames,* SL_2. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.