Assignment 2 (ML for TS) - MVA

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November 28, 2024

1 Introduction

Objective. The goal is to better understand the properties of AR and MA processes and do signal denoising with sparse coding.

Warning and advice.

- Use code from the tutorials as well as from other sources. Do not code yourself well-known procedures (e.g., cross-validation or k-means); use an existing implementation.
- The associated notebook contains some hints and several helper functions.
- Be concise. Answers are not expected to be longer than a few sentences (omitting calculations).

Instructions.

- Fill in your names and emails at the top of the document.
- Hand in your report (one per pair of students) by Monday 2^{nd} December 11:59 PM.
- Rename your report and notebook as follows:
  `FirstnameLastname1_FirstnameLastname1.pdf` and
  `FirstnameLastname2_FirstnameLastname2.ipynb`.
  For instance, `LaurentOudre_CharlesTruong.pdf`.
- Upload your report (PDF file) and notebook (IPYNB file) using this link:
  https://docs.google.com/forms/d/e/1FAIpQLSfCqMXSDU9jZJbYUMmeLCXbVeckZYNiDpPl4hRUwcJ2

2 General questions

A time series $\{y_t\}_t$ is a single realisation of a random process $\{Y_t\}_t$ defined on the probability space $(\Omega, \mathcal{F}, P)$, i.e. $y_t = Y_t(w)$ for a given $w \in \Omega$. In classical statistics, several independent realizations are often needed to obtain a “good” estimate (meaning consistent) of the parameters of the process. However, thanks to a stationarity hypothesis and a "short-memory" hypothesis, it is still possible to make “good” estimates. The following question illustrates this fact.

Question 1

An estimator $\hat{\theta}_n$ is consistent if it converges in probability when the number n of samples grows to ∞ to the true value $\theta \in \mathbb{R}$ of a parameter, i.e. $\hat{\theta}_n \xrightarrow{\mathcal{D}} \theta$.

- Recall the rate of convergence of the sample mean for i.i.d. random variables with finite variance.
- Let $\{Y_t\}_{t\geq 1}$ a wide-sense stationary process such that $\sum_k |\gamma(k)| < +\infty$. Show that the sample mean $\bar{Y}_n = (Y_1 + \cdots + Y_n)/n$ is consistent and enjoys the same rate of convergence as the i.i.d. case. (Hint: bound $\mathbb{E}[(\bar{Y}_n - \mu)^2]$ with the $\gamma(k)$ and recall that convergence in L_2 implies convergence in probability.)

Answer 1

Let $X_1, \ldots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. The sample mean is $\bar{X}_n = \frac{(X_1 + \cdots + X_n)}{n}$. Thanks to Bienaymé-Tchebychev inequality, we have that $\mathbb{P}(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{Var(\bar{X}_n)}{\epsilon} \leq \frac{\sigma^2}{n\varepsilon^2}$. The convergence rate is determined by the standard deviation of the sample mean, which is $\sigma/\sqrt{n}$.

$$
\begin{aligned}
\mathbb{E}[(\bar{Y}_n - \mu)^2] &= \mathbb{E}[(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu))^2] && (1)\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[(Y_i - \mu)(Y_j - \mu)] && (2)\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbb{E}(Y_iY_j) - \mu^2) && (3)\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbb{E}(Y_iY_j) - \mathbb{E}(Y_i)\mathbb{E}(Y_j)) && (4)\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma(i-j) && (5)\\
&= \frac{1}{n^2}\sum_{k=-(n-1)}^{n-1}(n - |k|)\gamma(k) && (6)\\
&\leq \frac{1}{n^2}\sum_{k=-(n-1)}^{n-1}n\gamma(k) && (7)\\
&= \frac{1}{n}\sum_{k=-(n-1)}^{n-1}\gamma(k) && (8)\\
&\leq \frac{1}{n}\sum_{k=-\infty}^{\infty}\gamma(k) \xrightarrow{n\to\infty} 0 && (9)
\end{aligned}
$$

We have proved that $\bar{Y}_n \xrightarrow{L^2} \mu$.

Moreover as the convergence in L_2 implies the convergence in probability, we have that $\bar{Y}_n \xrightarrow{P} \mu$. Finally, $\bar{Y}_n$ converges in proability with the same rate of convergence : $\sqrt{n}$.

3 AR and MA processes

Question 2 *Infinite order moving average MA(∞)*

Let $\{Y_t\}_{t\geq 0}$ be a random process defined bye

$$Y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots = \sum_{k=0}^{\infty} \psi_k \varepsilon_{t-k} \tag{10}$$

where $(\psi_k)_{k\geq 0} \subset \mathbb{R}$ ($\psi = 1$) are square summable, i.e. $\sum_k \psi_k^2 < \infty$ and $\{\varepsilon_t\}_t$ is a zero mean white noise of variance σ_ε^2. (Here, the infinite sum of random variables is the limit in L_2 of the partial sums.)

- Derive $\mathbb{E}(Y_t)$ and $\mathbb{E}(Y_t Y_{t-k})$. Is this process weakly stationary?
- Show that the power spectrum of $\{Y_t\}_t$ is $S(f) = \sigma_\varepsilon^2 |\phi(e^{-2\pi i f})|^2$ where $\phi(z) = \sum_j \psi_j z^j$. (Assume a sampling frequency of 1 Hz.)

The process $\{Y_t\}_t$ is a moving average of infinite order. Wold's theorem states that any weakly stationary process can be written as the sum of the deterministic process and a stochastic process which has the form (10).

Answer 2

$\mathbb{E}(Y_t) = \sum_{k=0}^{\infty} \psi_k \mathbb{E}(\varepsilon_{t-k}) = 0$

$$\mathbb{E}(Y_t Y_{t-k}) = \mathbb{E}[(\sum_{k=0}^{\infty} \psi_k \epsilon_{t-k})(\sum_{k'=0}^{\infty} \psi'_k \epsilon_{t-k'})] \tag{11}$$

$$= \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} \psi_k \psi_{k'} \mathbb{E}(\epsilon_{t-k} \epsilon_{t-k'}) \tag{12}$$

$$= \sum_{k=0}^{\infty} \psi_k^2 \sigma_\epsilon^2 \tag{13}$$

$$\tag{14}$$

This quantity exists, as $\sum_k \psi_k^2 < \infty$. Moreover, it only depends on $|k - k'|$, so the process **is weakly stationary**.

$$
\begin{aligned}
S(f) &= \sum_{\tau=-\infty}^{\infty} e^{-2\pi f\tau} && (15)\\
&= \sum_{\tau=-\infty}^{\infty} (\sum_{j=\tau}^{\infty} \psi_j \psi_{j-\tau}) \sigma_\epsilon^2 e^{-2i\pi f\tau} && (16)\\
&= \sum_{\tau=-\infty}^{\infty} (\sum_{j=\tau}^{\infty} \psi_j \psi_{j-\tau}) \sigma_\epsilon^2 e^{-2i\pi f\tau} && (17)\\
&= \sum_{j=-\infty}^{\infty} \psi_j \sum_{\tau=-\infty}^{j} \psi_{j-\tau} \sigma_\epsilon^2 e^{-2i\pi f\tau} && (18)\\
&= \sum_{j=-\infty}^{\infty} \psi_j \sum_{\tau=0}^{\infty} \psi_\tau \sigma_\epsilon^2 e^{-2i\pi f(j-\tau)} && (19)\\
&= \sum_{j,j'} \psi_j \psi_{j'} \sigma_\epsilon^2 e^{-2i\pi f(j-j')} && (20)\\
&= \sum_{j,j'} \psi_j \psi_{j'} \sigma_\epsilon^2 e^{-2i\pi fj} e^{-2i\pi fj'} && (21)\\
&= \sigma_\epsilon^2 \sum_{j} \psi_j e^{-2i\pi fj} \sum_{j'} \psi_{j'} e^{2i\pi fj'} && (22)\\
&= \sigma_\epsilon^2 |\sum_{j} \psi_j e^{-2i\pi fj}|^2 && (23)\\
&= \sigma_\epsilon^2 |\phi(e^{-2\pi if})|^2 && (24)
\end{aligned}
$$

Which is the expected result.

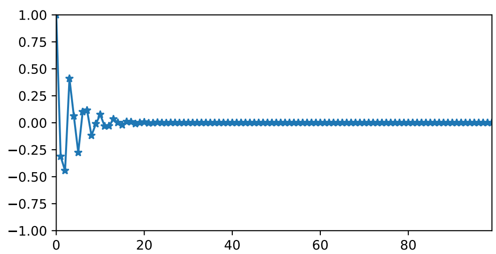
Question 3 *AR(2) process*

Let $\{Y_t\}_{t\geq 1}$ be an AR(2) process, i.e.

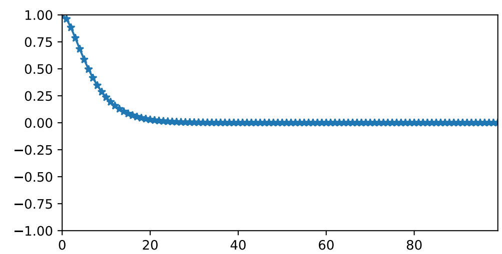
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \tag{25}$$

with $\phi_1, \phi_2 \in \mathbb{R}$. The associated characteristic polynomial is $\phi(z) := 1 - \phi_1 z - \phi_2 z^2$. Assume that ϕ has two distinct roots (possibly complex) r_1 and r_2 such that $|r_i| > 1$. Properties on the roots of this polynomial drive the behavior of this process.

- Express the autocovariance coefficients $\gamma(\tau)$ using the roots r_1 and r_2.
- Figure 1 shows the correlograms of two different AR(2) processes. Can you tell which one has complex roots and which one has real roots?
- Express the power spectrum $S(f)$ (assume the sampling frequency is 1 Hz) using $\phi(\cdot)$.
- Choose ϕ_1 and ϕ_2 such that the characteristic polynomial has two complex conjugate roots of norm $r = 1.05$ and phase $\theta = 2\pi/6$. Simulate the process $\{Y_t\}_t$ (with $n = 2000$) and display the signal and the periodogram (use a smooth estimator) on Figure 2. What do you observe?



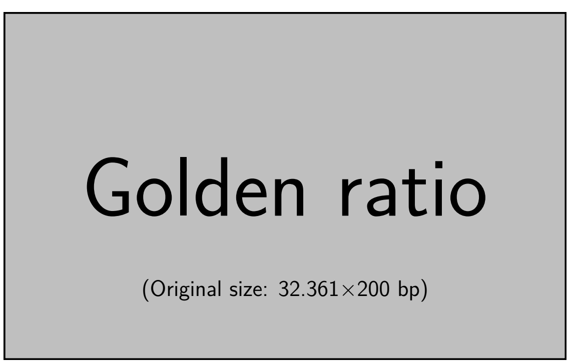
Correlogram of the first AR(2)



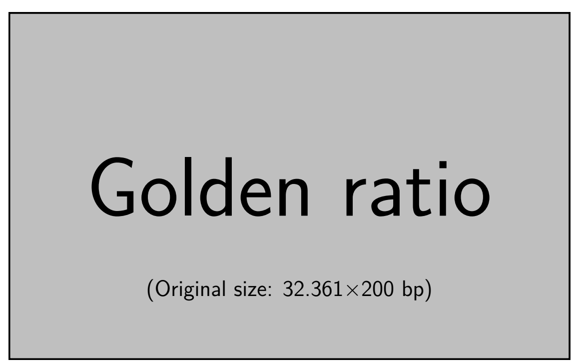
Correlogram of the second AR(2)

Figure 1: Two AR(2) processes

Answer 3



Signal



Periodogram

Figure 2: AR(2) process

4 Sparse coding

The modulated discrete cosine transform (MDCT) is a signal transformation often used in sound processing applications (for instance, to encode an MP3 file). A MDCT atom $\phi_{L,k}$ is defined for a length 2L and a frequency localisation k ($k = 0, \dots, L-1$) by

$$\forall u = 0, \dots, 2L-1, \quad \phi_{L,k}[u] = w_L[u]\sqrt{\frac{2}{L}}\cos[\frac{\pi}{L}\left(u + \frac{L+1}{2}\right)(k+\frac{1}{2})] \tag{26}$$

where w_L is a modulating window given by

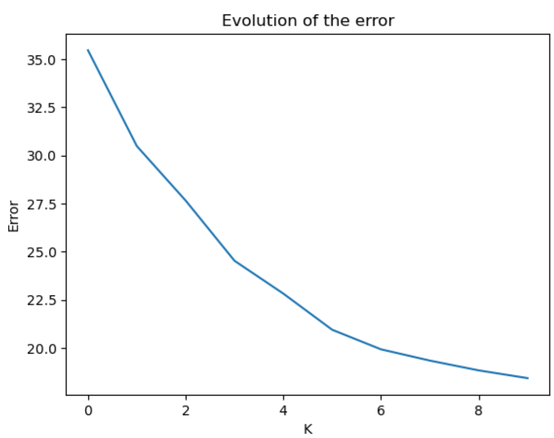
$$w_L[u] = \sin\left[\frac{\pi}{2L}\left(u+\frac{1}{2}\right)\right]. \tag{27}$$

Question 4 *Sparse coding with OMP*

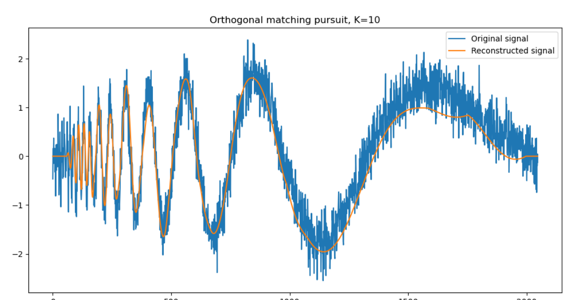
For the signal provided in the notebook, learn a sparse representation with MDCT atoms. The dictionary is defined as the concatenation of all shifted MDCDT atoms for scales L in $[32, 64, 128, 256, 512, 1024]$.

- For the sparse coding, implement the Orthogonal Matching Pursuit (OMP). (Use convolutions to compute the correlation coefficients.)
- Display the norm of the successive residuals and the reconstructed signal with 10 atoms.

Answer 4



Norms of the successive residuals



Reconstruction with 10 atoms

Figure 3: Question 4