Confidence interval on SEIR predictions

11/04/2024

We observe data $(Y_i)_{i\in[\![1,n]\!]}$, which corresponds to the number of deaths per day.

Assumption :

The number of deaths per day follows a SEIR model :
$Y_i = h_\theta(i) + \epsilon_i$

With $\theta = (\beta, \gamma, d) \in \mathbb{R}^3$ the parameters of the SEIR model and $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$, i.i.d with σ unknown.

We want to find θ and to compute confidence intervals on the predictions of the SEIR model.

Let $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$

We find $\hat{\theta}$, an estimator of θ by minimizing the difference between h_θ and the observed data.

$\hat{\theta} = \underset{\tilde{\theta}\in\mathbb{R}^3}{\operatorname{argmin}} \left(\sum_{i=1}^{n} (h_{\tilde{\theta}}(i) - Y_i)^2\right)$
As θ is fixed with $(Y_i)_{i\in[\![1,n]\!]}$ fixed, we note $\hat{\theta} = f(Y)$

Thanks to the Δ-method, we have :

$$Var(\hat{\theta}) \simeq \nabla^T f(Y) Cov(Y) \nabla f(Y)^T \tag{1}$$

with $\nabla f(Y)$ the gradient of f evaluated in Y.

The goal is now to find a way to appriximate $Cov(Y)$.

As the ϵ_i are independante gaussian variables, we have $Cov(Y) = \sigma^2 \mathbf{I}_n$

We could estimate σ^2 with its unbiaised estimator :

$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n} \epsilon_i^2 = \frac{1}{n-1}\sum_{i=1}^{n} (Y_i - h_{\hat{\theta}}(i))^2,$

and plug it into the formula given by the Δ-method (1), but it would be very sensible to the error of approximation of the model (bias error).

We can instead approximate $Cov(Y)$ with the hessian of the objective function. This approximation is exact when the objective function is the log-likehood of the observation of independant gaussian variables, which corresponds to the least mean square method (for the gaussian case). It is not stricty the case for our data.

Indeed, let $J(\theta)$ be the log likehood of the observation Y.

We have $Cov(Y)^{-1} = \nabla^2 J(\theta) \Leftrightarrow Y \sim \mathcal{N}(\begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix}, \sigma^2 I_n)$

In our case we have :

$$Y \sim \mathcal{N}(\begin{pmatrix} h_\theta(1) \\ h_\theta(2) \\ \vdots \\ h_\theta(n) \end{pmatrix}, \sigma^2 I_n),$$

Which justifies the approximation. We can finally compute $Var(\hat{\theta})$ with : $Var(\hat{\theta}) = \nabla^T f(Y) Cov(Y) \nabla f(Y)^T$