

Comparing forecast performance on different synthetic pandemics

4/06/2024

Résumé

My abstract

1 Introduction

Here is the introduction. The simulations are available on this [GitHub Repository](#).

Results

1.1 The data

The models were trained and tested on different points of 324 different pandemics generated with Covasim [5], a complex agent-based model. To generate those pandemics, four parameters that maximized the diversity of the set of pandemics generated were selected (see 2.0.3). Those parameters were varied on the three values 0.5, 1 and 2. These values correspond to scaling factors of the default values of the model. For each pandemic, the transmission probabilities were time dependant and followed one of the four different mobility patterns (see FIG.10). Thanks to those 4 parameters, each varied on three different values and to the four different mobilities, we obtained 324 simulations of pandemics. We decided to focus on the number of hospitalized individuals as the target variable. Indeed, it is a key variable to monitor, as the hospital occupancy correspond to the capacity of the state to treat the patients.

The models

In this paper, we implemented 13 individual models and an ensemble model. Each model was trained on the data from the beginning of a pandemic and made predictions 7 or 14 days ahead (on user's choice). They output a point prediction and a set of confidence intervals on the prediction. Models of two types were implemented. The first type models were only trained on the time-series of the number of hospitalized of the past few days. The second type of models were trained on the time series of the number of hospitalized, but also on the mobility data and the number of infected.

1.1.1 SIRH

The SIRH (Susceptible, Infected, Recovered, Hospitalized) (FIG.11) is a variation of the classical SIR model, with another compartment : the Hospitalized compartment. The parameters of this model correspond to the rates of evolution from a compartment to another one. β correspond to the transmission rate, γ_i and γ_h correspond to the recovery rate from both Infected and Hospitalized compartments, and h correspond to the hospitalization rate. The value of S, I, R and H are linked through a system of differential equations (1). As the curve of the pandemic of a SIRH is deterministic, it is possible to generate it from the values of the parameters $\beta, \gamma_i, \gamma_h$ and h , and to compare it to the value of the number of hospitalized observed from the beginning of the pandemic. The optimal parameters are found through minimization of the least square between the SIRH curve and the real curve. During the predicting phase, a 7 (or 14) days SIRH is run, with initial value estimated from the data and the

fit of the training phase. The confidence interval is estimated through linearization of the regression (see 2.1.7)

Variations of the SIRH model were implemented, with γ_i and γ_h kept constant.

A SIRH model of the second type was also constructed. Following the same idea as the first type SIRH, this model has a time varying transmission rate, which is a linear combination of the mobility : $\beta_t = a \times m_t + b$. The parameters to optimize are now a, b, γ_i, γ_h and h and both curve of the number of hospitalized and the number of infected are fitted to the data.

1.1.2 ARIMA

The ARIMA is a model used for time series forecasting. It is the sum of an AR and a MA model (2.1.2) It is trained by maximizing the likelihood of the observed data.

A VAR model, which is a multi-dimensional AR model was also implemented. It is also optimized through maximum likelihood.

1.1.3 Moving Average

The moving average model is a baseline model that is used as a reference. It returns the value of the average of the past seven days and a confidence interval based on the variance of the values of the past seven days.

1.1.4 Exponential regression

The exponential regression is a model that fits an exponential regression ($x \rightarrow ae^{bx} + c$) to the values of the number of hospitalized. The values of a, b and c are found by minimizing the least square difference between the prediction and the real values. The confidence interval is estimated through linearization of the regression (see 2.1.7).

An exponential regression of the second type was also implemented. The number of hospitalized is then an exponential regression of both mobility data, number of infected and number of hospitalized shifted.

1.1.5 Regressors

A Bayesian and a linear regressor were implemented. They use the last 20 data points to predict the next one (see 2.1.5).

Evaluation of the models

The models described above were tested on 14 points of each of the 324 pandemics generated. For each point, a 7 and 14 days-ahead prediction was made. Each prediction was evaluated thanks to WIS and RMSE (see 2.1.7), and then ranked according to the performances of other models. The points of evaluation were also classified in one of the following categories : 'big increase', 'increase', 'stable', 'inflexion', 'decrease' and 'big decrease'. It is then possible to get global information on the rankings of the models. For instance, if the loss and the reach of prediction is fixed, we can look at the distribution of rankings of all the models for a type of point (to see the best model for a type of point), as in Fig.1. This distribution of rankings of the model can be summed up in one single value : the expected value of the rank of a model (see Fig.2), which enables to get the idea of the best model for this type of point, this loss, and this prediction range on a more compact figure.

This new number loses information (for instance on bimodal rankings distribution) but enables to enlight the expected performance of a model. For each loss and range of prediction, the expected rank of the models for each type of point can be visualized on the same figure, which enables to have a global look of the performances of the models.(Fig.3)

The other heatmaps corresponding to RMSE and WIS for 7 and 14 days ahead predictions can be found in the appendix (3) Some general interpretations can be made on these results. We noticed that regressors performances drop from 7-days ahead to 14-days ahead. The family of the SIRH model does slightly better for long terms predictions than for short term ones. The ARIMA performance drops from 7-days ahead to 14-days ahead predictions. The exponential models are very bad and almost always the last ones.

Distribution of the ranks of each model for big increase points for RMSE and 7 days ahead prediction.

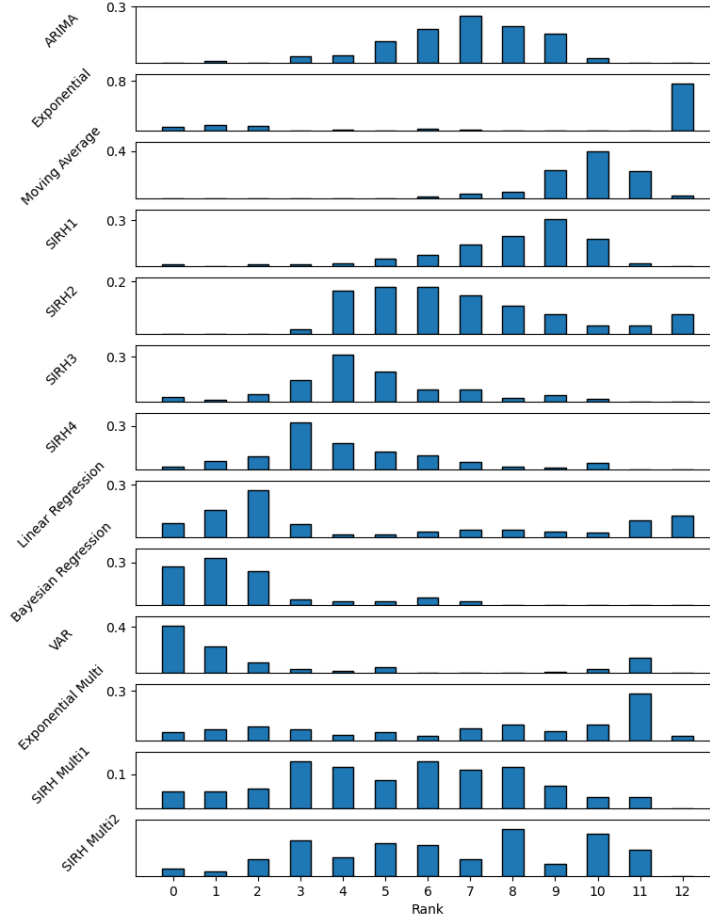


FIGURE 1 – Distribution of rankings of the models for big increase points for 7-days ahead predictions and RMSE loss

1.2 The Ensemble model

It has been shown in the literature ([2], [9]) that ensemble models, which are models taking into account the predictions of many individual models and output their mean, median, or any other function that combines the predictions, such as stacking (as it is done in [9]). We decided to implement our own ensemble models. They consist in a linear combination of the predictions of the 13 individual models (without exponential models) for 7 and 14 days-ahead predictions. The performances of these two models are shown in the two figures below (Fig.4 and Fig.5)

The distribution of the ranks of the ensemble model is almost always on the left side of the x-axis, except for the 'stable' points. The fact that the 'stable' points are overrepresented (see 2.2) leads to a general poor performance of the ensemble model. However, on the other type of points, the ensemble model is very consistent, and almost always in the top models. The heatmaps of expected ranks on all type of points enables to see how consistent is the ensemble model (Fig.7 and Fig.6) compared to the other models.

From these figures, we can see that the ensemble model is rarely the best, but never the last model, which is a result already observed in [2]. Its consistency allows for accurate predictions and helps avoid outlier predicted values.

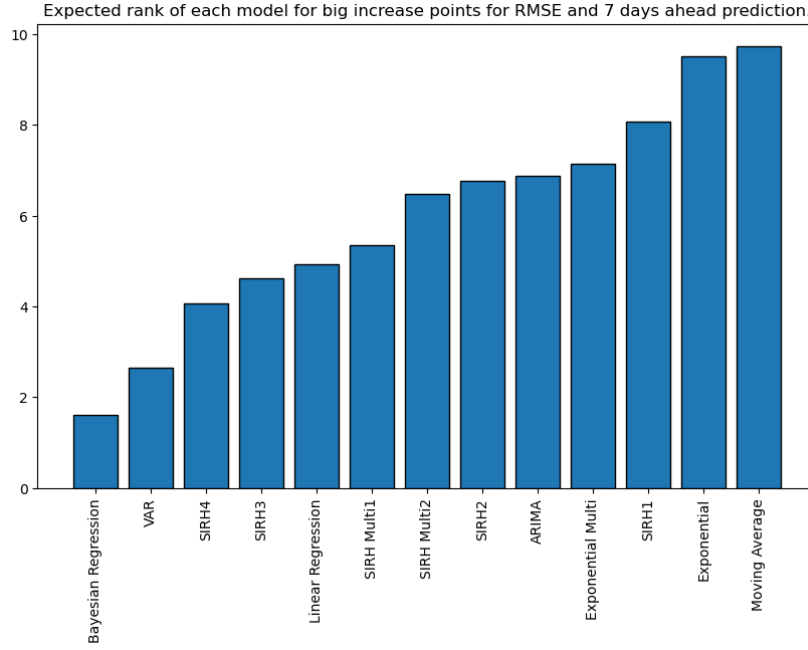


FIGURE 2 – Expected rank of the models for big increase points for 7-days ahead predictions and RMSE loss

Discussion

See noisy data (not perfect data exact on covasim) and its influence on the predictions.

Conclusion

2 Methods

The data

As the goal of this study is to compare some forecasts on many different pandemics, a large set of synthetic pandemics need to be generated, with a particular attention on the diversity of these pandemics. All the simulations in this paper are made on a fixed population of 10^6 agents.

2.0.1 Covasim

To generate the pandemics, Covasim [5], a python library that can simulate the evolution of a pandemic was used. Covasim is an agent-based model that can simulate many different pandemics and has a high diversity of outputs. This model takes as an input many parameters such as the population type, the population size, the age repartition... and outputs a complete description of the pandemic, with real-time values of each relevant information, such as the number of severe, of asymptomatic... but also physical values such as the value of the reproduction number. Covasim enables to generate a huge diversity of pandemics, thanks to the plurality of parameters that can be given as the input of the model, but also with interventions that can be planned by the users. These interventions can help to assess the impact of a vaccination campaign, with changes in the probability transmission.

2.0.2 First pandemics

For the implementation and the first test of the models, two pandemics were generated. The first one focusing on the new deaths count and the second one focusing on the number of hospitalized count. Those pandemics will be referred to as pandemic 1 and pandemic 2.

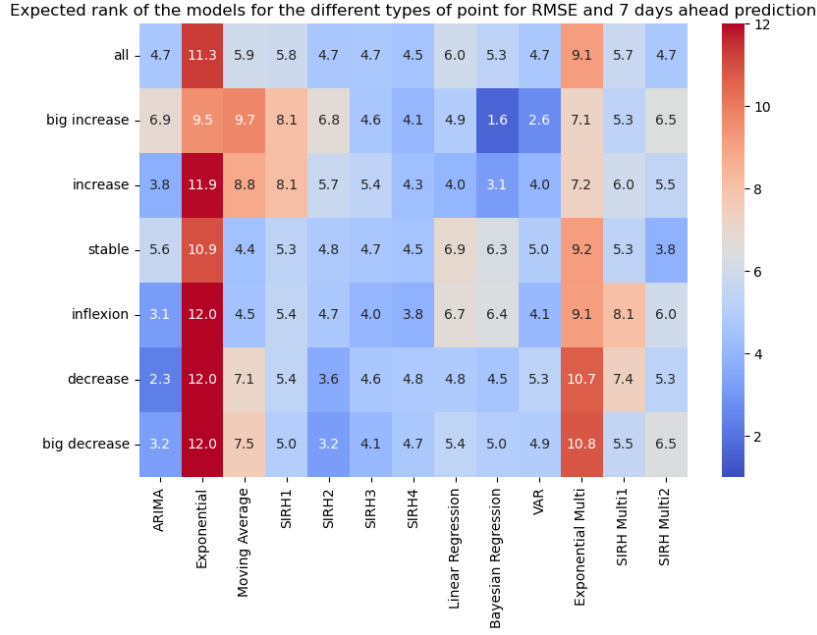


FIGURE 3 – Expected rank of the models for each type of point for 7-days ahead predictions and RMSE loss

TABLE 1 – Table of the parameters used to simulate the first two pandemics.

Parameter	Pandemic 1	Pandemic 2
start day 1	2020-03-02	2020-03-02
end day	2020-07-01	2021-01-01
Population size	1000000	1000000
Interventions	interventions1	interventions2
population type	hybrid	hybrid
β initial	0.015	0.015
location	Sweden	Sweden
n infected initial	20	100

The parameters used to generate these pandemics are described in the table 1. The parameters that are not specified are the default parameters of the Covasim library.

The different interventions were based on mobility reports from Västtrafik (Fig.8), the public transport company of the city of Gothenburg. They were reported during the Covid 19 pandemic and have been retrieved in [4]. These interventions correspond to 53 relative weekly variations of the mobility, with a reference value of 1 for the first week of the report, which correspond to the 9-th week of 2020.

2.0.3 Generating diverse pandemics

In order to evaluate the performances of our models on a wide range of pandemics, a training set of pandemics was generated. A huge diversity of pandemics is needed to determine the most consistent model. It is so relevant to identify the key parameters that generate this diversity. As Covasim has a very huge set of inputs parameters, a first subset of key parameters was identified : the spread parameters and the severity parameters. The severity parameters are the 4 parameters that correspond to the probability for an agent to get from a compartment to another. The spread parameters are 9 parameters that represent the distribution of probability of the time spend by an agent in a compartment (such as infected, critical...) once he entered it. This set of 13 parameters is the noted as S . This distribution is a log-normal distribution, but the spread-parameters correspond to the mean of this log-normal distribution. All the parameters have a default value of 1, which correspond

Distribution of the ranks of the Ensemble model for different type of points for 7 days ahead prediction.

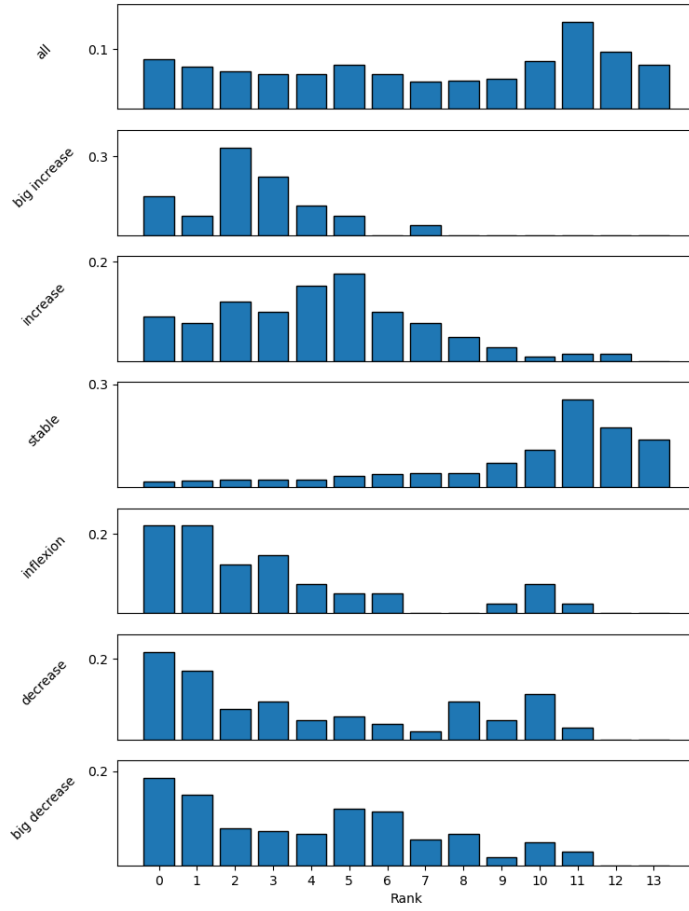


FIGURE 4 – Expected rank of the ensemble model for each type of point for 7-days ahead predictions and RMSE loss

Distribution of the ranks of the Ensemble model for different type of points for 14 days ahead prediction.

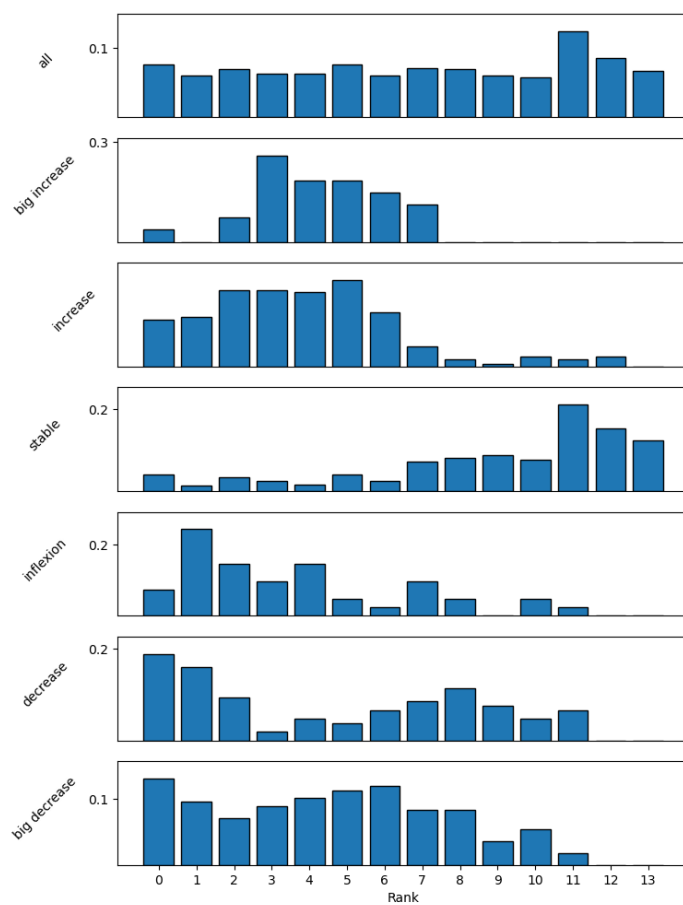


FIGURE 5 – Expected rank of the ensemble model for each type of point for 14-days ahead predictions and RMSE loss

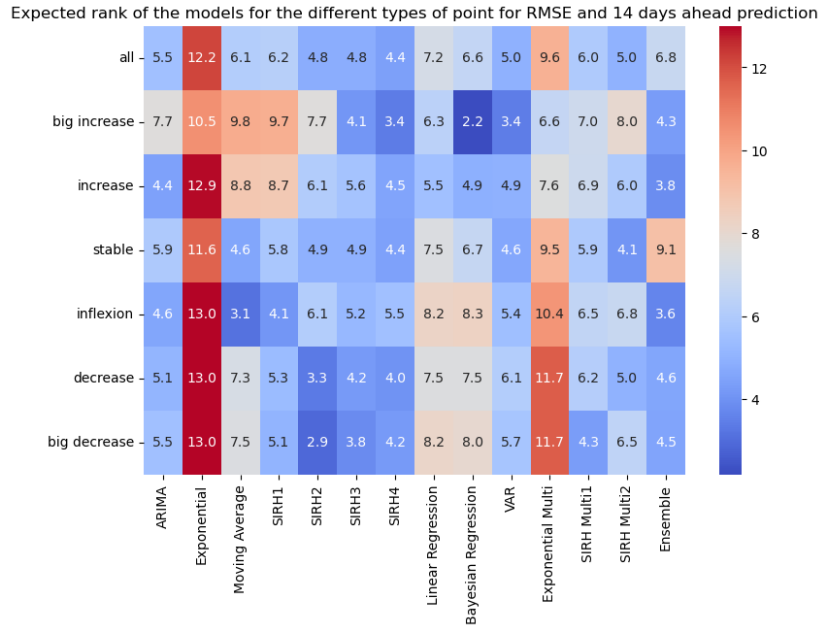


FIGURE 6 – Expected rank of all the models for each type of point for 14-days ahead predictions and RMSE loss on the test set

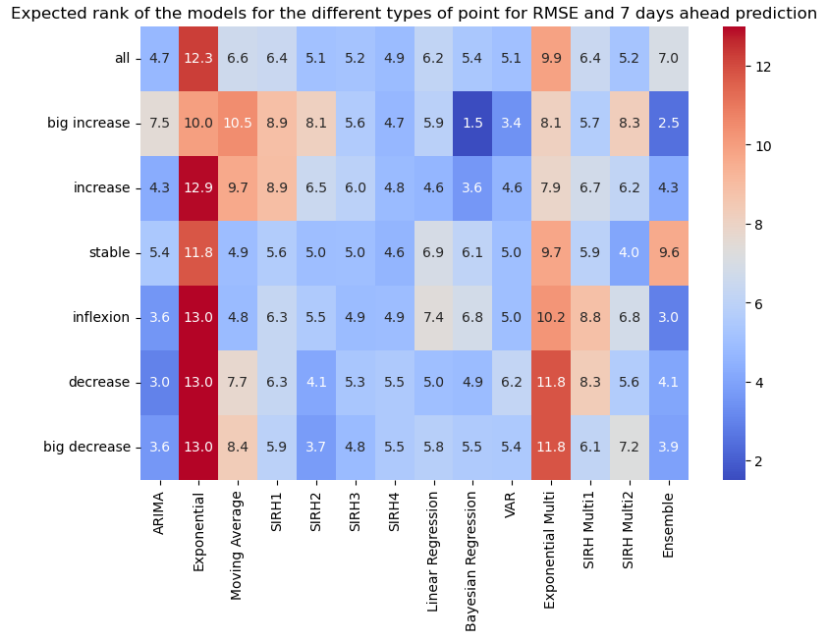


FIGURE 7 – Expected rank of all the models for each type of point for 7-days ahead predictions and RMSE loss on the test set

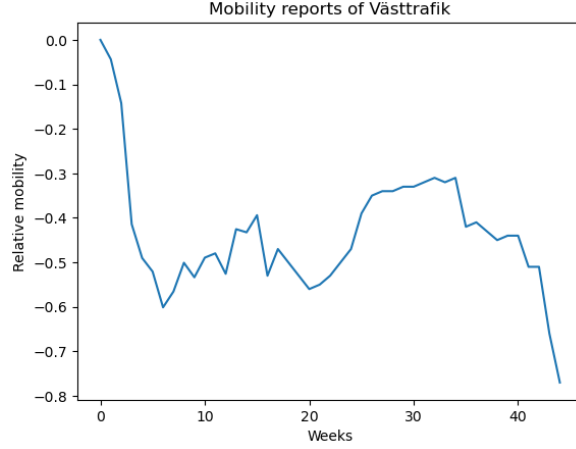


FIGURE 8 – Mobility reports from Västtrafik.

to keeping the reference value. We decided to select 4 parameters and to make them vary in $[0.5, 1, 2]$, leading to a set of 81 pandemics. To select the 4 parameters that generated the most diversity, different diversity metrics were computed.

Let $Y_1, Y_2 \in \mathbb{R}^n$ be two time series of n days representing the number of hospitalized in two pandemics.

Let :

$$\mathcal{L}_1(Y_1, Y_2) = \|Y_1 - Y_2\|_{L_1}$$

$$\mathcal{L}_2(Y_1, Y_2) = \left\| \left(\frac{\max(Y_1)}{\max(Y_2)}; \frac{\max(Y_1')}{\max(Y_2')}; \frac{\max(Y_1'')}{\max(Y_2'')} \right), \|\tilde{Y}_1 - \tilde{Y}_2\|_{L_1}, \|\tilde{Y}_1' - \tilde{Y}_2'\|_{L_1}, \|\tilde{Y}_1'' - \tilde{Y}_2''\|_{L_1} \right\|_{L_2}$$

with Y' and Y'' the first and second derivatives of Y

$$\mathcal{L}_3 = \mathcal{W}(\tilde{Y}_1 - \tilde{Y}_2), \text{ with } \mathcal{W} \text{ the Wasserstein distance.}$$

$$\mathcal{L}_4(Y_1, Y_2) = \left\| \left(\frac{\max(Y_1)}{\max(Y_2)}; \frac{\max(Y_1')}{\max(Y_2')}; \frac{\max(Y_1'')}{\max(Y_2'')} \right), \mathcal{W}(\tilde{Y}_1 - \tilde{Y}_2), \mathcal{W}(\tilde{Y}_1' - \tilde{Y}_2'), \mathcal{W}(\tilde{Y}_1'' - \tilde{Y}_2'') \right\|_{L_2}$$

with Y' and Y'' the first and second derivatives of Y

It can be noted that \mathcal{L}_2 looks like the Sobolev norm $\|\tilde{Y}_1 - \tilde{Y}_2\|_{W^{2,1}}$ with squared terms and with additional terms taking into account the amplitude.

To determine which measure to use, we generated 14 pandemics. Each pandemic but the last one has default parameters except one of them which was doubled. The last pandemic has only default parameters.

For each norm \mathcal{L}_k , we determined s , the subset of 4 pandemics that maximized the following quantity :

$$\mathcal{L}_k(S) = \sum_{i,j \in s, i \neq j} \mathcal{L}_k(Y_i, Y_j)$$

The 4 most diverse pandemics (in the subset of 14 pandemics generated) according to each norm are shown in the Fig.9. We decided according to this figure, that \mathcal{L}_2 norm was the most relevant to determine the diversity of the pandemics. But, keeping the parameters $[0, 5, 10, 12]$ would not be accurate, as the parameters were changed independantly, and the diversity did not take into account the correlation between some of them.

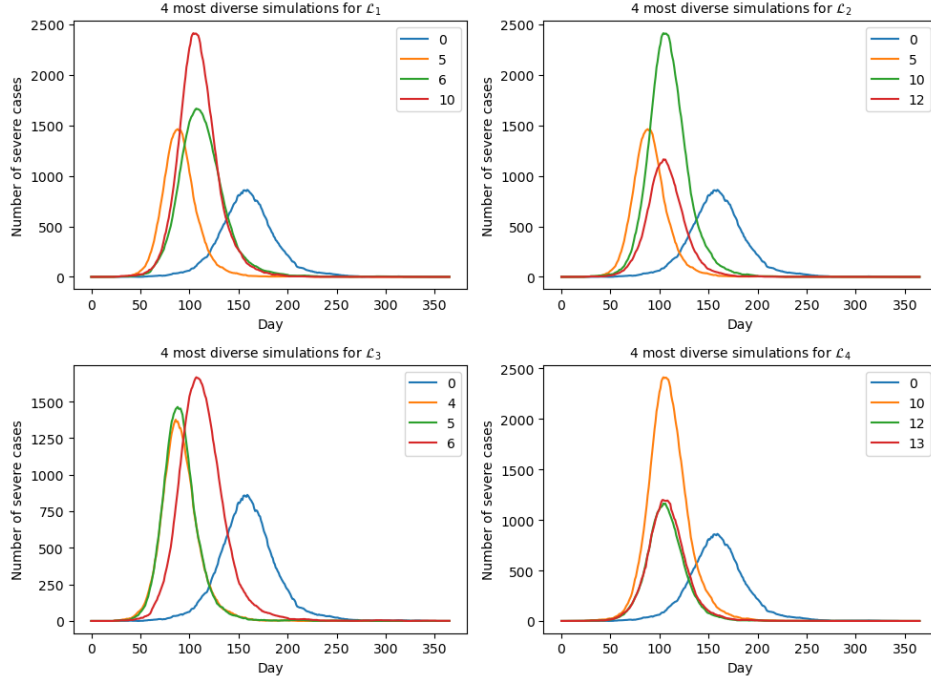


FIGURE 9 – 4 most diverse pandemics according to each norm.

Finding the parameters that maximise the \mathcal{L}_2 diversity is equivalent to solve the following problem :

$s_{opt} = \underset{s \in S, |s|=4}{argmax} \mathcal{L}(s)$, with $\mathcal{L}(s) = \sum_{p_1, p_2 \in \mathcal{P}_g(s), s \neq t} \mathcal{L}_2(p_1, p_2)$, and $\mathcal{P}_g(s)$ the set of the 81 pandemics generated with the 4 parameters of s .

However, generating a pandemic with **Covasim** is time consuming, and it is not possible to compute the diversity of each set of 4 parameters s included in S (the set of all 13 parameters).

A MCMC algorithm [3] was then implemented, to perform a clever grid search on the different subsets $s \subset S$ of parameters. The MCMC algorithm is a method which is used to sample from a distribution that can't be directly sampled. The main idea is to construct a Markov Chain whose stationary distribution is the objective distribution.

Let $\mathcal{S} = \{s \subset S; |s| = 4\}$ be the support of the target distribution, which is, in our case, the set of all the 715 combinations of the 4 parameters among the 13 different possible, and let π be the target distribution on \mathcal{S} . $\forall s \in \mathcal{S}, \pi(s) = \frac{\mathcal{L}(s)}{\sum_{s \in \mathcal{S}} \mathcal{L}(s)}$. π is not directly computable as it is too time consuming to compute the denominator.

For each $s = [a, b, c, d] \in \mathcal{S}$, let $ne(s)$ be the set of the neighbours of s , i.e the set of all the elements of \mathcal{S} who have only one parameter different from s . For instance, $[0, 3, 9, 12] \in ne([0, 3, 10, 12])$, but $[0, 3, 9, 12] \notin ne([0, 3, 8, 10])$.

Let U_n be a sequence of independent uniform random variables on $[0, 1]$ and $\forall s \in \mathcal{S}$, let $U_n^{ne(s)}$ be a sequence of independant uniform random variables on $ne(s)$. Let $s_0 \in \mathcal{S}$ and let S_n be the random sequence defined as follow :

$$\begin{cases} S_0 = s_0 \\ \forall n \in \mathbb{N}, \alpha_n = \frac{\mathcal{L}(U_n^{ne(S_n)})}{\mathcal{L}(S_n)} \\ \forall n \in \mathbb{N}, S_{n+1} = U_n^{ne(S_n)} \mathbb{1}_{\{U_n < \alpha_n\}} + X_n \mathbb{1}_{\{U_n > \alpha_n\}} \end{cases}$$

As S_{n+1} is a function of S_n and of other independant random variables, the sequence S_n is a homogenous Markov Chain. This formula means that at each iteration, a neighbour of S_n is uniformly selected among all the neighbours of S_n (it is $U_n^{ne(S_n)}$). The Markov Chain moves to this neighbour if the value of $\mathcal{L}(U_n^{ne(S_n)})$ is higher than the value of the function $\mathcal{L}(S_n)$ at the current state. If the new value of \mathcal{L} is smaller, the Markov Chain moves with a probability that is equal to the ratio of the two values. This way of moving on the different subsets prevents to be stucked in a local maxima but avoids exploring dummies areas, in which the diversity is very small.

The transition matrix of this Markov Chain is the following :

$$K(s, s') : \begin{cases} 0 & \text{if } s' \notin ne(s) \text{ and } s' \neq s \\ \frac{1}{Card(ne(s))} = \frac{1}{36} & \text{if } s' \in ne(s) \text{ and } \frac{\mathcal{L}(s')}{\mathcal{L}(s)} > 1 \text{ and } s' \neq s \\ \frac{1}{36} \times \frac{\mathcal{L}(s')}{\mathcal{L}(s)} & \text{if } s' \in ne(s) \text{ and } \frac{\mathcal{L}(s')}{\mathcal{L}(s)} \leq 1 \text{ and } s' \neq s \\ 1 - \sum_{s' \in \mathcal{S}, s' \neq s} K(s, s') & \text{if } s' = s \end{cases}$$

Let $(s, s') \in \mathcal{S}^2$. Let us suppose that $s' \neq s$, that $s' \in ne(s)$, and that $\mathcal{L}(s) < \mathcal{L}(s')$ (the other case is symmetric).

$$\begin{aligned} \pi(s)K(s, s') &= \frac{\mathcal{L}(s)}{\sum_{s \in \mathcal{S}} \mathcal{L}(s)} \times \frac{1}{36} \quad \text{as } \mathcal{L}(s) < \mathcal{L}(s') \\ &= \frac{\mathcal{L}(s)}{\sum_{s \in \mathcal{S}} \mathcal{L}(s)} \times \frac{1}{36} \times \frac{\mathcal{L}(s')}{\mathcal{L}(s')} \\ &= \frac{\mathcal{L}(s')}{\sum_{s \in \mathcal{S}} \mathcal{L}(s)} \times \frac{1}{36} \times \frac{\mathcal{L}(s)}{\mathcal{L}(s')} \\ &= \pi(s')K(s', s) \end{aligned}$$

Indeed, each subset s has 36 neighbour, as there are 13 parameters and one can replace each parameter of s by any of the 9 others.

If $s' \notin ne(s)$, then $s \notin ne(s')$ and $K(s, s') = 0$. We directly have $\pi(s)K(s, s') = 0 = \pi(s')K(s', s)$.

Thus, π is **reversible** for K .

Let $(s, s') \in \mathcal{S}^2$. Let us note (a, b, c, d) and (a', b', c', d') the elements of s and s' . We note :

$$\begin{aligned} s_1 &= [a', b, c, d] \\ s_2 &= [a', b', c, d] \\ s_3 &= [a', b', c', d] \end{aligned}$$

$$\begin{aligned}
\mathbb{P}(S_{n+4} = s' | S_n = s) &\geq \mathbb{P}(S_{n+4} = s' \cap S_{n+3} = s_3 \cap S_{n+2} = s_2 \cap S_{n+1} = s_1 | S_n = s) \\
&\geq \mathbb{P}(S_{n+4} = s' | S_{n+3} = s_3 \cap S_{n+2} = s_2 \cap S_{n+1} = s_1 \cap S_n = s) \\
&\quad \times \mathbb{P}(S_{n+3} = s_3 \cap S_{n+2} = s_2 \cap S_{n+1} = s_1 | S_n = s) \text{ (Baye' s Formula)} \\
&\geq \mathbb{P}(S_{n+4} = s' | S_{n+3} = s_3) \\
&\quad \times \mathbb{P}(S_{n+3} = s_3 \cap S_{n+2} = s_2 \cap S_{n+1} = s_1 | S_n = s) \text{ (by Markov's property)} \\
&\vdots \\
&\geq \mathbb{P}(S_{n+4} = s' | S_{n+3} = s_3) \times \mathbb{P}(S_{n+3} = s_3 | S_{n+2} = s_2) \times \mathbb{P}(S_{n+2} = s_2 | S_{n+1} = s_1) \\
&\quad \times \mathbb{P}(S_{n+1} = s_1 | S_n = s) \text{ (by Markov's property)} \\
&\geq \left(\frac{1}{36}\right)^4 \times \min\left(1, \frac{\mathcal{L}(s')}{\mathcal{L}(s)}\right) \times \min\left(1, \frac{\mathcal{L}(s_3)}{\mathcal{L}(s_2)}\right) \times \min\left(1, \frac{\mathcal{L}(s_2)}{\mathcal{L}(s_1)}\right) \times \min\left(1, \frac{\mathcal{L}(s_1)}{\mathcal{L}(s)}\right) \\
&> 0
\end{aligned}$$

Thus, S_n is **irreducible**.

A Markov chain of transition matrix P on the support \mathcal{S} is said to be aperiodic if :
 $\forall s \in \mathcal{S}, \forall s' \in \mathcal{S}, \exists N \in \mathbb{N}, \text{ s.t } \forall n > N, P(s, s')^n > 0$ [1]

First, note that $\forall s \in \mathcal{S}$, s is a local minimum (i.e if $\forall s' \in ne(s), \mathcal{L}(s') > \mathcal{L}(s)$) if and only if $K(s, s) = 0$

Thus, if s is not a local minimum, then $K(s, s) > 0$. Moreover, $\forall s \in \mathcal{S}, \forall s' \in ne(s)$, if $s \neq s'$, then $K(s, s') \neq 0$

Let $(s, s') \in \mathcal{S}^2$.

- If s' is not a local minimum,
 $\forall n > 4, \mathbb{P}(S_n = s' | S_0 = s) \geq \mathbb{P}(S_4 = s' | S_0 = s) \times K(s', s')^{n-4} > 0$
- If s' is a local minimum, $\forall s^* \in ne(s'), s^*$ is not a local minimum and $K(s^*, s^*) \neq 0$.
 $\forall n > 5, \mathbb{P}(S_n = s' | S_0 = s) \geq \mathbb{P}(S_3 = s^* | S_0 = s) \times K(s^*, s^*)^{n-4} \times K(s^*, s') > 0$

Thus S_n is an **aperiodic** Markov Chain.

Finally, according to the **Theorem 5.5** from [1], as S_n is irreducible and aperiodic, as π is the stationary distribution, and as \mathcal{S} is countable, S_n converges in distribution to π .

The most probable set that will be sampled by S_n is so the one that maximises the diversity. We implemented this MCMC algorithm to maximise \mathcal{L}_2 on \mathcal{S} . The first value of the sequence was a clever starting point : the four parameters that maximized \mathcal{L}_2 on the 14 pandemics generated (Fig.9). After 200 iterations, the set of parameters that maximised the diversity was [2, 4, 9, 10], which correspond to the parameters [sym2sev, asym2rec, rel_symp_prob, rel_severe_prob]. The \mathcal{L}_2 -diversity increased from 62353 to 93553.

To create the most diverse set possible, we also created 4 different mobilities reports Fig.10, corresponding to constant mobility, annual variations, lockdown scenario and the reports from Vasträffik from [4]. These time-varying mobilities enabled us to model more complex behaviours of the pandemics. We finally modelled 324 pandemics. Indeed each of the 4 parameters was scaled among 3 values : [0.5, 1, 2] and the 4 mobilities reports were used.

2.1 The models

In this study, we define a model h_θ as a function h defined on \mathbb{N} , with parameters θ and trained on the data \mathcal{D} . In the training phase, $\hat{\theta}$, an estimator of θ is computed from \mathcal{D} , and used for the prediction. We elaborated two types of models : the first type correspond to models which are only trained on the time series we want it to predict (the number of hospitalized in our case), and the second type of

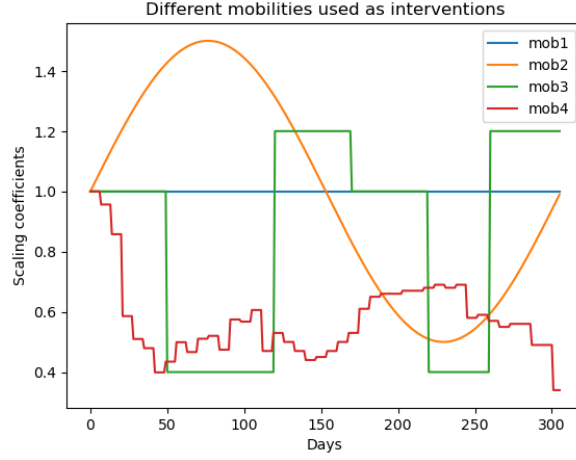


FIGURE 10 – Mobility reports.

models are trained on the time series we want to predict, but also on other time series that can be relevant to predict the number of hospitalized (the mobility and the number of infected). All of these models were implemented in Python, and are available on the github repository provided with this article (1). During the training or predicting phase, the computation sometimes fail (for instance, when the matrix is non-inversible for the linear regression model). The model then outputs the value of the moving average model (see 2.1.3), which can be interpreted as a naive output when the computation fails.

Task of a model :

Each model h is given :

1. A training set \mathcal{D}
2. A reach of prediction r
3. A confidence threshold α

And outputs :

1. A prediction \hat{Y}_r
2. A $(1 - \alpha)$ confidence interval on the prediction, $I_{\alpha,r}$

The model will train on the data \mathcal{D} to compute $\hat{\theta}$ the parameter estimator and the output $\hat{Y}_r = h_{\hat{\theta}}(r)$.

2.1.1 The SIRH model

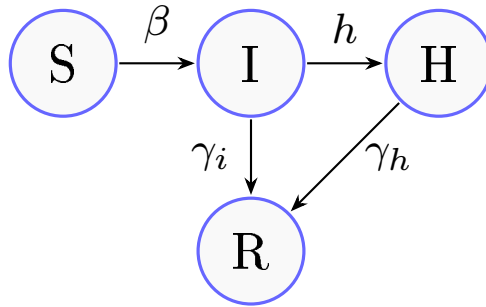


FIGURE 11 – Scheme of the SIRH model

The SIRH model (Fig.11) is an extension of the classic compartmental SIR (Susceptible-Infectious-Recovered) model used to describe the spread of infectious diseases. In the SIRH model, a fourth compartment, "H" for "Hospitalized," is added. Each compartment correspond to the number of person in the state of health of the compartment. The evolution of the number of person in each compartment is described by a system of ordinary differential equations :

$$\begin{cases} \frac{dS}{dt} = -\beta \frac{SI}{N} \\ \frac{dI}{dt} = \beta \frac{SI}{N} - \gamma_i I - hH \\ \frac{dR}{dt} = \gamma_i I + \gamma_h H \\ \frac{dH}{dt} = hI - \gamma_h H \end{cases} \quad (1)$$

At $t = 0$, the values of (S_0, I_0, R_0, H_0) is fixed to $(10^6 - 1, 1, 0, 0)$. As the system of equation can't be directly solved, we use a Euler method to solve it :

$$\begin{cases} S_{t+dt} = S_t + dt \frac{dS}{dt} \\ I_{t+dt} = I_t + dt \frac{dI}{dt} \\ R_{t+dt} = R_t + dt \frac{dR}{dt} \\ H_{t+dt} = H_t + dt \frac{dH}{dt} \end{cases} \quad (2)$$

We chose to fix $dt = 0.001$.

To train this model, we minimize the least square between the curve of the number of hospitalized observed to the curve of the number of hospitalized of the training data with respect to $\theta = (\beta, \gamma_i, \gamma_h, h)$. We implemented some variations of the model in which γ_i , γ_h or both were fixed to the value 0.2 (see the Table.??).

	γ_i	γ_h
SIRH1	0.2	0.2
SIRH2	0.2	free
SIRH3	free	0.2
SIRH4	free	free

TABLE 2 – Difference between the SIRH models

In the prediction phase, a r day SIRH simulation is launched, with the parameter $\hat{\theta}$ computed during the training phase. The initial value for S and I correspond to the last value of the fit of the training phase. The initial value for H correspond to the last value of \mathcal{D} , the training data. The initial value of R is fixed by the previous values as the equation $S_t + I_t + R_t + H_t = N$ is always true. The confidence interval of the prediction is computed thanks to a linearization and the use of the delta-method (see 2.1.7)

A SIRH model of the second type was implemented. It has the same structure but uses the mobility data and the number of infected to be more precise. The idea is the same, but there are two differences :

- β varies with the time as a linear combination of the mobility : $\beta_t = a + b \times m_t$

- The data is fitted to both the number of hospitalized and the number of infected.

To write it more formally, let $H_\theta(t)$ and $I_\theta(t)$ be the number of hospitalized and infected at time t in the SIRH model with parameters θ . Let $Y_{H,t}$ and $Y_{I,t}$ be the number of hospitalized and infected at time t in the data. We have :

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^5}{\operatorname{argmin}} \sum_{t=1}^n \left(\frac{H_{\theta}(t) - Y_{H,t}}{\max(Y_{H,t})} \right)^2 + \left(\frac{I_{\theta}(t) - Y_{I,t}}{\max(Y_{I,t})} \right)^2$$

With $\theta = (a, b, \gamma_i, \gamma_h, h)$ and m_t the mobility at time t . The normalization factors enables to prevent the optimization to focus on the number of infected, which is bigger than the number of hospitalized. Once again, a variation of the SIRH model was implemented, in which the value of γ_h and γ_i is fixed to 0.2. SIRH multi 1 refer to the model in which γ_h and γ_i are free and SIRH multi 2 refer to the model in which γ_h and γ_i are fixed to 0.2.

2.1.2 ARIMA and VAR models

The ARIMA and VAR models are used for time-series forecasting and have outperformed many models in pandemic prediction (see [6] and [11]). The *ARIMA*(p, d, q) model is the sum of an *AR*(p) and a *MA*(q) model applied on the time series differentiated d times. It follows the equation :

$$Y_t^d = \alpha + \sum_{i=1}^p \beta_{t-i} Y_{t-i}^d + \sum_{j=1}^q \phi_{t-j} \epsilon_{t-j}$$

where Y_t^d is the time series at time t , d is the order of the differentiation, α is a constant, p is the order of the autoregressive part, q is the order of the moving average part and ϵ_{t-j} is the difference between the prediction of the model and the real value at time $t - j$.

The coefficient are estimated through maximum likelihood estimation. This method is implemented in the `statsmodels` library, which directly provides prediction and confidence intervals. We realized a grid search on a single pandemic to identify the combination of parameters that would optimize the prediction accuracy. We found an optimal value for $p = 3, d = 0, q = 3$.

The VAR model is a multi-dimensional *AR* model, in which different variable are predicted. It so corresponds to a second type model. This model exploits the correlation between variables. The value of the parameters of a VAR have physical sense and can be interpreted to find correlations between variables. Let $Y_{1,t}, \dots, Y_{k,t}$ be the times series (in our case, $k = 3$ and they correspond to the number of hospitalized, the number of infected and the mobility data).

$$\begin{aligned} VAR(p) : \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \\ \vdots \\ Y_{k,t} \end{pmatrix} &= \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} + \begin{pmatrix} \phi_{11,1} & \phi_{12,1} & \cdots & \phi_{1k,1} \\ \phi_{21,1} & \phi_{22,1} & \cdots & \phi_{2k,1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1,1} & \phi_{k2,1} & \cdots & \phi_{kk,1} \end{pmatrix} \begin{pmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ \vdots \\ Y_{k,t-1} \end{pmatrix} \\ &+ \cdots + \begin{pmatrix} \phi_{11,p} & \phi_{12,p} & \cdots & \phi_{1k,p} \\ \phi_{21,p} & \phi_{22,p} & \cdots & \phi_{2k,p} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{k1,p} & \phi_{k2,p} & \cdots & \phi_{kk,p} \end{pmatrix} \begin{pmatrix} Y_{1,t-p} \\ Y_{2,t-p} \\ \vdots \\ Y_{k,t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \vdots \\ \epsilon_{k,t} \end{pmatrix} \end{aligned}$$

Again, the $\phi_{i,j,k}$ and c_i are estimated through maximum likelihood estimation with the `statsmodel` library. The confidence intervals are also directly provided by the library.

2.1.3 The moving average model

A mere moving average model was also implemented. It returns a constant prediction that correspond to the mean of the 7 past days. The confidence intervals are computed by assuming that the predictions follow a normal distribution of variance equal to the variance of the 7 past data-points; This model is used as a baseline. A model that does not manage to outperform the moving average model would not be useful.

2.1.4 Exponential regression

An exponential regression model was implemented. It correspond to fitting the data of the number of hospitalized (Y_t) to the function $E_{a,b,c}(t) = a \times e^{bt} + c$. The value of $\theta = (a, b, c)$ is computed through a least square method. The confidence interval on the prediction is estimated with the same method as SIRH (see 2.1.7).

An exponential regression of the second type was also implemented : The data of the number of hospitalized is fitted to the function $E_{a,b,c,d,e}(t) = a \times e^{bm_t - i + ct + d \times \inf_{t-j} + e}$.

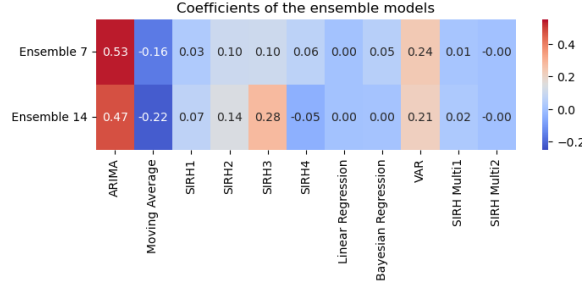


FIGURE 12 – Weights of the ensemble models

m_t is the mobility data at time t and inf_t is the number of infected at time t . The value of $\theta = (a, b, c, d, e)$ is computed through a least square method. The optimal value of the time lag i and j is optimized during the training phase through a grid search among all the values between 0 and 14. The confidence interval on the prediction is estimated with the same method as SIRH (see 2.1.7).

2.1.5 Machine learning models

In order to implement machine learning regressor, we converted the time-series $Y_{t,t \in \{1, \dots, n\}}$ in a training set (X_i, Y_i) such that :
 $\forall i \in \{1, \dots, n\}, X_i = (Y_{i-1}, Y_{i-2}, \dots, Y_{i-20})$.

We then trained and optimized both regressors : the linear and the bayesian regression, which were the only one that did not output absurd results on the **scikit-learn** models among : linear regression, bayesian regression, Ridge, Gradient boosting regressor, Random Forest regressor, Bayesain Ridge and SVR. The confidence interval for the linear regression prediction was computed as follow :
Let us suppose that the data follows a linear regression model : $Y = X\beta + \epsilon$, with $Y \in \mathbb{R}^n$, $X \in \mathbb{R}^{n \times d}$, $\beta \in \mathbb{R}^d$ and $\epsilon \sim \mathcal{N}(0, \sigma^2)$. The least square estimator of β is $\hat{\beta} = (X^T X)^{-1} X^T Y$. If we have new data $\tilde{X} \in \mathbb{R}^{1 \times d}$ that we want to predict, the prediction is :

$$\begin{aligned}
\tilde{Y} &= \tilde{X} \hat{\beta} \\
&= \tilde{X} \hat{\beta} = \tilde{X} (X^T X)^{-1} X^T Y \\
&= \tilde{X} (X^T X)^{-1} X^T (X\beta + \epsilon) \\
&= \tilde{X} \beta + \tilde{X} (X^T X)^{-1} X^T \epsilon.
\end{aligned}$$

$\tilde{X} (X^T X)^{-1} \tilde{X}^T \sigma^2$.

The confidence interval on bayesian regression was directly computed with the variance of the parameters given by the scikit-learn library and the delta method.

TODO add more details on the confidence interval + the prior of the bayesian

2.1.6 Ensemble model

It has been showned ([2] and [9]) that ensemble models, which combine the outputs of many models, can outperform by far individual models. We implemented an ensemble model which is a linear combination of the outputs of the 13 models described above, with the exponential models removed. The weights of this model were found by minimizing the least-squared error between the prediction of the ensemble model and the real value of the number of hospitalized on a train set of approximately 80% of the pandemic generated. As the ensemble model only outputs a single value without confidence intervals, it is only evaluated with the RMSE metric. TODO add the coefficient found on a figure with the weights

2.1.7 Computing confidence intervals on the prediction

Assumption :

We suppose that the data of the pandemic observed follows the model h , of parameter $\theta^* \in \mathbb{R}^d$. Let Y_i , $i = 1, \dots, n$ be the number of hospitalized at each day. We suppose that : $Y_i = h_{\theta^*}(i) + \epsilon_i$, with $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$, iid, and independent from all the other variables. The objective is to estimate θ^* . We use $\hat{\theta}$, the least square estimator of θ^* as an estimator of θ^* :

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \sum_{i=1}^n (Y_i - h_{\theta}(i))^2$$

Let :

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$h_{\theta} = \begin{pmatrix} h_{\theta}(1) \\ \vdots \\ h_{\theta}(n) \end{pmatrix}$$

We have :

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \|Y - h_{\theta}\|^2$$

Now, if θ is close enough to θ^* , we can write (from [10]) :

$$\forall i \in \{1, \dots, n\} : h_{\theta}(i) = h_{\theta^*}(i) + (\theta - \theta^*)^T \nabla_{\theta} h_{\theta^*}(i)$$

which leads to :

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \|Y - h_{\theta^*} - (\theta - \theta^*)^T \nabla_{\theta} h_{\theta^*}\|^2$$

Let us define :

$$\tilde{Y} = Y - h_{\theta^*}$$

$$\beta = \theta - \theta^*$$

$$\hat{\beta} = \theta - \hat{\theta}$$

and let us define the matrix $A \in \mathbb{R}^{n \times d}$ such that $\forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, d\}, A_{i,j} = \frac{dh_{\theta^*}}{d\theta_j}(i)$.

The previous problem can be re-written as :

$$\hat{\beta} = \underset{\beta \in \mathbb{R}^d}{\operatorname{argmin}} \|\tilde{Y} - A\beta\|^2$$

This is a regression linear problem.

Let us solve this problem in the general case.

Let (A_i, \tilde{Y}_i) be the observations Let \mathbb{P} be the law from which the A_i are drawn, and let us assume that $Y_i = A_i \beta^* + \epsilon_i$, with $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.

The solution of this problem is explicitly (from [8]) :

$$\hat{\beta} = (A^T A)^{-1} A^T \tilde{Y}$$

This least-square estimator is unbiased :

$$\mathbb{E}[\hat{\beta}] = \beta^*$$

$$\hat{\beta} = \left(\sum_{i=1}^n A_i^T A_i \right)^{-1} \times \left(\sum_{i=1}^n A_i^T \tilde{Y}_i \right)$$

$$\hat{\beta} = \frac{n}{n} \left(\sum_{i=1}^n A_i^T A_i \right)^{-1} \times \left(\sum_{i=1}^n A_i^T \tilde{Y}_i \right)$$

$$\hat{\beta} = \left(\frac{1}{n} \sum_{i=1}^n A_i^T A_i \right)^{-1} \times \left(\frac{1}{n} \sum_{i=1}^n A_i^T \tilde{Y}_i \right)$$

Let us denote :

$$\hat{D} = \frac{1}{n} \sum_{i=1}^n A_i^T A_i, \quad \text{and} \quad \hat{\delta} = \left(\frac{1}{n} \sum_{i=1}^n A_i^T \tilde{Y}_i \right)$$

We have :

$$\hat{\beta} = \hat{D}^{-1} \hat{\delta}$$

$$\hat{D} \xrightarrow{a.s} D = \mathbb{E}[A_i^T A_i]$$

$$\hat{\delta} \xrightarrow{a.s} \delta = \mathbb{E}[A_i^T \tilde{Y}_i]$$

$\hat{\beta} = \hat{D}^{-1} \hat{\delta} \xrightarrow{a.s} D^{-1} \delta$, as the following function ϕ is continuous :

$$\phi : \begin{cases} \mathcal{GL}_n(\mathbb{R}) & \rightarrow \mathcal{GL}_n(\mathbb{R}) \\ A & \mapsto A^{-1} \end{cases}$$

Now, let us show that $\hat{\beta}$ is asymptotically normal :

$$\begin{aligned} \sqrt{n}(\hat{\beta} - \beta^*) &= \sqrt{n}(\hat{D}^{-1} \hat{\delta} - \beta^*) \\ &= \sqrt{n}(\hat{D}^{-1} \hat{\delta} - \hat{D}^{-1} \hat{D} \beta^*) \\ &= \sqrt{n} \hat{D}^{-1} (\hat{\delta} - \hat{D} \beta^*) \\ &= \sqrt{n} \hat{D}^{-1} \left(\frac{1}{n} \sum_{i=1}^n A_i^T \tilde{Y}_i - \frac{1}{n} \sum_{i=1}^n A_i^T A_i \beta^* \right) \\ &= \frac{\sqrt{n}}{n} \hat{D}^{-1} \left(\sum_{i=1}^n A_i^T (\tilde{Y}_i - A_i \beta^*) \right) \\ &= \frac{1}{\sqrt{n}} \hat{D}^{-1} \left(\sum_{i=1}^n A_i^T \epsilon_i \right) \end{aligned}$$

This line is made of two terms. Let's show that each one of them converges in law.

$$\begin{aligned} \frac{1}{\sqrt{n}} \left(\sum_{i=1}^n A_i^T \epsilon'_i \right) &= \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n A_i^T \epsilon'_i \right) \\ &= \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n A_i^T \epsilon'_i - 0 \right) \\ &\xrightarrow{\mathcal{L}} \mathcal{N}(0, \text{Var}(A_i^T \epsilon_i)) \end{aligned}$$

Yet, as ϵ_i and A_i are independant, and $\mathbb{E}[A_i^T \epsilon'_i] = 0$, $\text{Var}(A_i^T \epsilon_i) = \mathbb{E}[A_i A_i^T \epsilon_i^2] = \mathbb{E}[A_i A_i^T] \sigma'^2$.

Finally, $\frac{1}{\sqrt{n}} (\sum_{i=1}^n A_i^T \epsilon'_i) \xrightarrow{\mathcal{L}} \mathcal{N}(0, D\sigma'^2)$.

On the other hand, $\hat{D}^{-1} \xrightarrow{\mathcal{L}} D^{-1}$, which is constant.

Finally, with Slutsky, we obtain that :

$$\begin{aligned} \sqrt{n}(\hat{\beta} - \beta^*) &\xrightarrow{\mathcal{L}} D^{-1} \mathcal{N}(0, D\sigma'^2) \\ &\xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1}(D\sigma'^2)(D^{-1})^T) \\ &\xrightarrow{\mathcal{L}} \mathcal{N}(0, D^{-1}D\sigma'^2(D^{-1})^T) \\ &\xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma'^2 D^{-1}) \\ &\xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma'^2 (A^T A)^{-1}) \end{aligned}$$

Let's get back to the first problem :

As $\beta^* = 0$ and $\hat{\beta} = \hat{\theta} - \theta^*$, we have :

$$\sqrt{n}(\hat{\theta} - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2 (A^T A)^{-1})$$

and,

$$\hat{\theta} \sim \mathcal{N}(\theta^*, \frac{\sigma^2}{n} (A^T A)^{-1})$$

As a first conclusion, we have that $\hat{\theta}$ is asymptotically normal.

Let Σ be the covariance matrix estimated from the computation of $\hat{\theta}$. In our case, $\Sigma = \frac{\sigma^2}{n} (A^T A)^{-1}$.

As $\hat{\theta}$ is asymptotically normal, we can apply the delta-method :

$$\sqrt{n}(\hat{\theta} - \theta^*) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

$$\sqrt{n}(h_{\hat{\theta}} - h_{\theta^*}) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \nabla_{\theta} h_{\theta}^T \Sigma \nabla_{\theta} h_{\theta})$$

And finally :

$$h_{\hat{\theta}} \rightarrow \mathcal{N}(h_{\theta^*}, \frac{1}{n} \nabla_{\theta} h_{\theta}^T \Sigma \nabla_{\theta} h_{\theta})$$

By estimating $\frac{1}{n} \Sigma$ from `curve_fit`, we can compute the confidence interval of the prediction with the quantiles of the normal distribution. The gradient of h_{θ} is approximated through numerical approximation :

$$\nabla_{\theta} h_{\theta}[i] \simeq \frac{h_{\theta+d\theta_i} - h_{\theta}}{dt}, \text{ with } dt = 0.0001.$$

$$d\theta_i = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{i-1} \\ \theta_i + dt \\ \theta_{i+1} \\ \vdots \\ \theta_n \end{pmatrix}$$

Assessing the performance of the models

Metrics

Two metrics were used to assess the performance of the models. The first metric is the Weighted Confidence Interval (WIS), which is a metric commonly used in forecast evaluation (see [2, text] or [7]).

Let α be in $]0, 1[$. Let \hat{y} be the prediction of the model and y the real value. Let $[l, u]$ be the $(1 - \alpha)$ confidence interval of the prediction. We define the Interval Score (IS) that way :

$IS_\alpha([l, u], \hat{y}, y) = \frac{2}{\alpha} \times (\mathbb{1}_{\{y < l\}}(l - y) + \mathbb{1}_{\{y > u\}}(y - u) + (u - l))$. This metric is made of three terms : a term of overprediction that punishes a model predicting a confidence interval which is above the real value, a term of underprediction that punishes a model whose confidence interval is under the real prediction, and a term of range, that punishes too wide confidence intervals. Let $(\alpha_k)_{k \in \{1, \dots, K\}} \in]0, 1[^K$. The WIS is defined as follow.

$$WIS_\alpha([l, u], \hat{y}, y) = \sum_{k=0}^K w_k IS_{\alpha_k}([l, u], \hat{y}, y), \text{ with } (w_k)_{k \in \{1, \dots, K\}} \in \mathbb{R}_+^K \text{ weights chosen by the user.}$$

According to previous literature ([2]), we decided to set $(\alpha_k)_{k \in \{1, \dots, K\}} = [0.02, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]$.

One can notice that the WIS does not take into account the point prediction, but focuses on confidence interval accuracy.

The second metric chosen is the Root Mean Square Error ($RMSE$). With the same notations as above, we define the $RMSE$ as follow :

$$RMSE([l, u], \hat{y}, y) = \sqrt{(y - \hat{y})^2}$$

This metric focuses on the point prediction, and does not take into account the confidence intervals.

The models were tested on all the 324 pandemics, on 14 data points different (at days 20, 40, 60, ..., 280). For each individual point, the models were trained on the previous days of the pandemic. A 7 and 14 days ahead prediction was asked, and $[0.02, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]$ confidence intervals were computed. The WIS and the RMSE of these predictions were then computed.

2.2 Point classification

In order to compare the performance of the models at different point of the pandemic, we classified the points in six categories : *stable*, *increase*, *big increase*, *inflexion*, *decrease*, *big decrease*. For a pandemic $X_{i, i \in [1, \dots, n]}$, and a day $d \in [1, \dots, n]$, the classification is made according the following rule :

$$\text{define } d = \frac{1}{7} \times \frac{X[d+7] - X[d]}{X[d]}$$

$$\text{and } d' = \frac{1}{49} \times \frac{X[d+7] + X[d-7] - 2 \times X[d]}{X[d]}$$

1. If $X[d] < 100$:
classification = "stable"
2. Elif $d < -0.05$:
classification = "big decrease"
3. Elif $d < -0.03$:
classification = "decrease"
4. Elif $d < 0.03$:
 - if $|d| > 0.003$:
classification = inflexion
 - else : classification = "stable"
5. Elif $d < 0.2$:
classification = "increase"
6. Else :
classification = "big increase"

The values of the thresholds were chosen by hand. As the model are tested on the days [20, 40, 60, ..., 280], we labelled all of thses points with thye function described above. It resulted that among

all those 4536 points, 646 were classified as 'big decrease', 370 as decrease, '152' as inflexion, 2485 as 'stable', 760 as 'increase' and 123 as 'big increase'.

3 Appendix

Here is my appendix

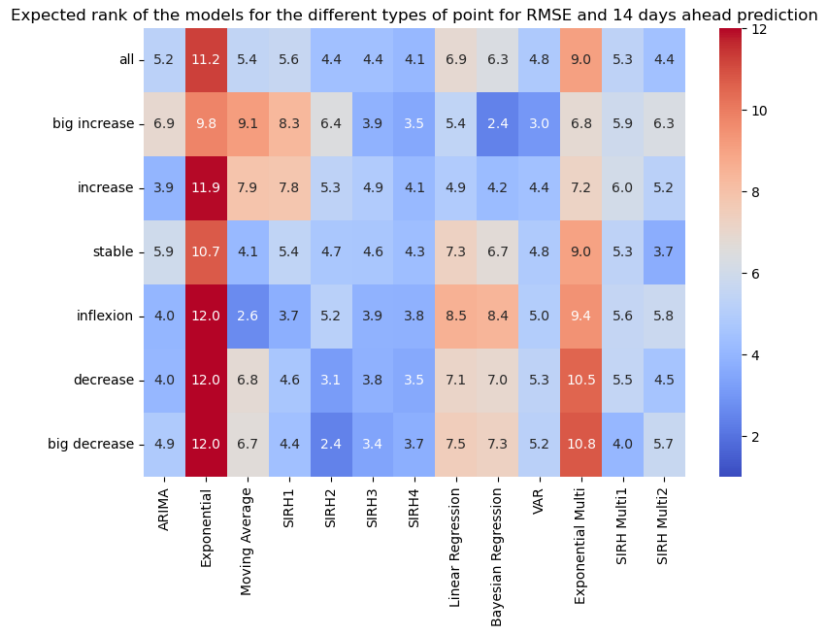


FIGURE 13 – Expected rank of the models for each type of point for 14-days ahead predictions and RMSE loss

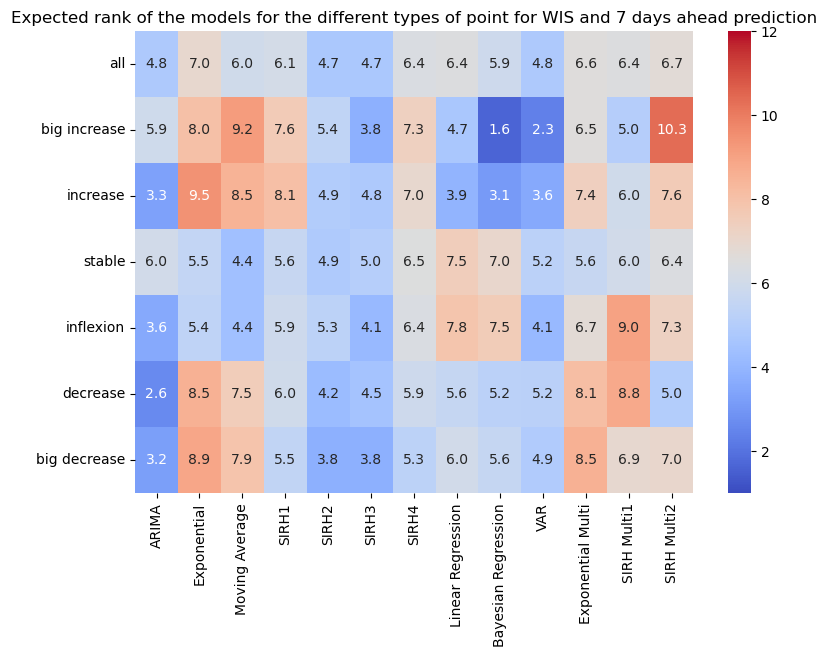


FIGURE 14 – Expected rank of the models for each type of point for 7-days ahead predictions and WIS loss

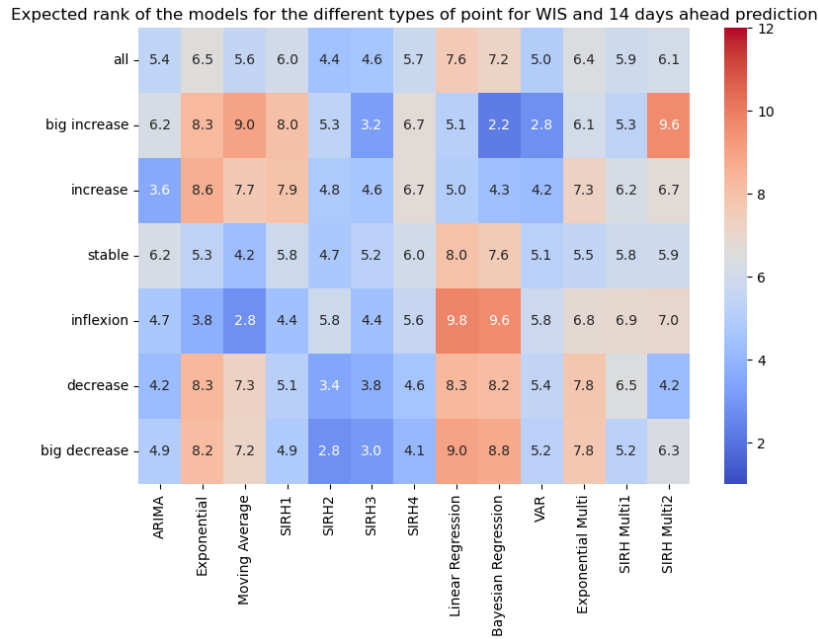


FIGURE 15 – Expected rank of the models for each type of point for 14-days ahead predictions and WIS loss

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