Contents

1 Introduction

American options form a family of derivatives well known in finance. They give to the owner a right to exercise at any time before the maturity T, in opposition to, for instance, European options where the owner can only exercise at maturity. The general form of the price at time 0 of such option is the following:

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau)],$$

where $\mathcal{T}$ is the set of $[0,T]$-valued stopping times with $t_0 = 0$ and $t_N = T$, r is the risk free rate (which is supposed to be constant) and g is the function of payoff. To simplify notations, let us define:

$$Y_t = e^{-rt} g(X_t), \ \forall t \in [0,T].$$

To solve the above maximisation problem, the natural idea is to discretized in time the problem: we replace $\mathcal{T}$ by $\bar{\mathcal{T}}$ the $\{t_0, \cdots, t_N\}$-valued stopping times, with $0 = t_0 < t_1 < \cdots < t_N = T$. This leads to a discretized optimal stochastic control problem. Those problems are well-known and the Snell envelope $(V_i)_{i \in \{0,\cdots,N\}}$ gives us a representation of the value function of the problem:

$$\begin{aligned} V_N(X_{t_N}) &= Y_{t_N} \\ V_k(X_{t_k}) &= \max\Big(Y_{t_k}, \mathbb{E}[V_{k+1}(X_{t_{k+1}})|X_{t_k}]\Big). \end{aligned}$$

Thus, the dynamic principle gives us that:

$$V_0(X_{t_0}) = \sup_{\tau \in \bar{\mathcal{T}}} \mathbb{E}[Y_\tau] = \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau)],$$

which is an good approximation of the solution to our initial problem.

We will apply our results on American Puts for the numerical simulations. The payoff of an American Put is $g : x \mapsto (K-x)_+ := \max(0, K-x)$ where K is the strike of the option. We have seen on the course how to apply the Tsitsiklis & Van Roy and Longstaff & Schwartz to compute a lower bond for the price. In this project we will explain the Duality method which gives us a good upper bound.

2 Duality

In this section, we present a duality result that will be helpful for finding an upper bound on the price of an american option. The duality result comes from [Rogers, 2002]. In his paper, he presented the result in continuous time. Here, we will present [Glasserman, 2004]'s approach, so we will study the problem in discrete time. We want to approximate

$$V_0(X_{t_0}) = \sup_{\tau \in \bar{\mathcal{T}}} \mathbb{E}[Y_\tau].$$

For each martingale $(M_i)_{i \in \{0,\cdots,N\}}$ starting from 0 at time 0, we have $\mathbb{E}[M_i] = 0$ for all $i \in \{0, \cdots, N\}$. So for all $\tau \in \bar{\mathcal{T}}$,

$$\mathbb{E}[Y_\tau] = \mathbb{E}[Y_\tau - M_\tau] \leq \mathbb{E}[\max_{k \in \{0,\cdots,N\}} \Big(Y_{t_k} - M_k\Big)].$$

Taking the supremum on all $\tau \in \bar{\mathcal{T}}$ on the right side and the infimum on all $(M_i)_{i \in \{0,\cdots,N\}}$ martingales starting from 0, we get

$$\sup_{\tau \in \bar{\mathcal{T}}} \mathbb{E}[Y_\tau] \leq \inf_{M \in \mathcal{M}_0} \mathbb{E}[\max_{k \in \{0,\cdots,N\}} \Big(Y_{t_k} - M_k\Big)],$$

where $\mathcal{M}_0$ is the set of martingale starting from 0. In fact, we can show the reverse inequality: Let us define the martingale $M_0 = 0$ and $M_k = \Delta_1 + \cdots + \Delta_k$ for all $k \in \{1, \cdots, N\}$, with

$$\Delta_k = V_k(X_{t_k}) - \mathbb{E}[V_k(X_{t_k})|X_{t_{k-1}}].$$

Such a process is indeed a martingale with respect to its natural filtration: for all $k \in \{1, \cdots, N\}$,

$$\mathbb{E}[M_k - M_{k-1}|\mathcal{F}_{t_{k-1}}] = \mathbb{E}[\Delta_k|\mathcal{F}_{t_{k-1}}] = \mathbb{E}[V_k(X_{t_k})|\mathcal{F}_{t_{k-1}}] - \mathbb{E}[V_k(X_{t_k})|X_{t_{k-1}}] = 0,$$

as the process is also a Markov chain. By definition, the martingale starts from 0. We now need to check that

$$V_0(X_{t_0}) = \mathbb{E}[\max_{k\in\{0,\cdots,N\}} \Big(Y_{t_k} - M_k\Big)].$$

In fact we will prove more: for all $k \in \{0, \cdots, N\}$,

$$\begin{aligned} V_k(X_{t_k}) &= \max\Big\{Y_{t_k},\ Y_{t_{k+1}} - \Delta_{k+1},\ \cdots,\ Y_{t_N} - \Delta_N - \cdots - \Delta_{k+1}\Big\} \\ &= \max_{k\in\{0,\cdots,N\}} \Big(Y_{t_k} - M_k\Big). \end{aligned}$$

For $k = m$, the above formula is obious as

$$V_N(X_{t_N}) = Y_{t_N}.$$

By backward induction, let us suppose that the formula is verified for $k+1 \in \{1, \cdots, N\}$ fixed. Then,

$$\begin{aligned} V_k(X_{t_k}) &= \max\Big(Y_{t_k}, \mathbb{E}[V_{k+1}(X_{t_{k+1}})|X_{t_k}]\Big) \\ &= \max\Big(Y_{t_k}, V_{k+1}(X_{t_{k+1}}) - \Delta_{k+1}\Big) \\ &= \max\Big(Y_{t_k}, \max\Big\{Y_{t_{k+1}},\ \cdots,\ Y_{t_N} - \Delta_N - \cdots - \Delta_{k+2}\Big\} - \Delta_{k+1}\Big) \\ &= \max\Big(Y_{t_k}, Y_{t_{k+1}} - \Delta_{k+1},\ \cdots,\ Y_{t_N} - \Delta_N - \cdots - \Delta_{k+1}\Big). \end{aligned}$$

Thus, the formula is satisfied for all $k \in \{1, \cdots, N\}$ and with $k = 0$, we get:

$$V_0(X_{t_0}) = \max_{k\in\{0,\cdots,N\}} \Big\{Y_{t_k} - M_k\Big\}$$

So instead of our initial maximisation problem over stopping times, we now have a minimisation problem over $\mathcal{M}_0$. Moreover, we know what the optimal martingal M^* for this new minimisation problem looks like. By approximating M^*, we will find a good upper bound on $V_0(X_{t_0})$. In fact, we are just studying a special case of the Doob-Meyer decomposition for super-martingale.

3 Approximation of Continuation Value Functions

In this section, we will present several tools that will be usefull in the next sections (4 and 5). The main goal of the presented tools will be to approximate the quantities:

$$C_k : x \mapsto \mathbb{E}[V_{k+1}(X_{t_{k+1}})|X_{t_k} = x], \quad \forall k \in \{0, \cdots, N-1\},$$

which are called the continuation functions. First, let us notice that by definition of the conditionnal expectation in $\mathbb{L}^2$,

$$\begin{aligned} \mathbb{E}[V_{k+1}(X_{t_{k+1}})|X_{t_k}] &= \operatorname*{arg\,min}_{Z\in\mathbb{L}^2(\Omega,\sigma(X_{t_k}),\mathbb{P})} \mathbb{E}\Big[\Big(V_{k+1}(X_{t_{k+1}}) - Z\Big)^2\Big] \\ &= \operatorname*{arg\,min}_{\Phi(X_{t_k})\in\mathbb{L}^2} \mathbb{E}\Big[\Big(V_{k+1}(X_{t_{k+1}}) - \Phi(X_{t_k})\Big)^2\Big]. \end{aligned}$$

3.1 Non-parametric regression

A first idea is to approximate this new (arg-)minimisation problem with a Monte-Carlo method:

- at time t_{N-1},

$$\mathbb{E}[V_{k+1}(X_{t_{k+1}})|X_{t_k}] \approx \underset{\Phi(X_{t_k})\in\mathbb{L}^2}{\arg\min} \frac{1}{n}\sum_{j=1}^{n}\left(Y^j_{t_{k+1}} - \Phi(X^k_{t_k})\right)^2,$$

- at time t_k with $k \in \{0, \cdots, N-1\}$,

$$\mathbb{E}[V_{k+1}(X_{t_{k+1}})|X_{t_k}] \approx \underset{\Phi(X_{t_k})\in\mathbb{L}^2}{\arg\min} \frac{1}{n}\sum_{j=1}^{n}\left(V_{k+1}(X^j_{t_{k+1}}) - \Phi(X^k_{t_k})\right)^2,$$

  with $(X^j_{t_k})_{k\in\{0,\cdots,N\}}$ following the law of $(X_{t_k})_{k\in\{0,\cdots,N\}}$ for each $j \in \{1, \cdots, n\}$.

Choosing the right Φ is an infinite dimensional problem. In order to simplify it, we can look for a "good" Φ in a parametrised class of functions:

$$\begin{aligned}\mathcal{A} &= \left\{x \mapsto \sum_{i=1}^{l}\alpha_i\phi_i(x), \text{ with } \alpha_1, \cdots, \alpha_l \in \mathbb{R}\right\}\\ &= \text{Span}(\phi_1, \cdots, \phi_l)\end{aligned}$$

with $l \in \mathbb{N}^*$ and $\phi_1, \cdots, \phi_l$ functions (called base functions) choosen beforehand. Those base functions should be taken so they can easily approximate the real continuation functions. With this in mind, we will try in the following sections the start of several basis of polynoms and we will sometimes add the payoff function (with some power). Here are the base functions that we will try:
With the canonical basis of $P[X]$:

- $\mathcal{A}_{1,1} = \text{Span}(1, X, X^2, X^3)$
- $\mathcal{A}_{1,2} = \text{Span}(1, X, X^2, X^3, g)$
- $\mathcal{A}_{1,3} = \text{Span}(1, X, X^2, X^3, X^4, g)$
- $\mathcal{A}_{1,4} = \text{Span}(1, X, X^2, X^3, g, g^2)$
- $\mathcal{A}_{1,5} = \text{Span}(1, X, X^2, X^3, g, g^2, g^3)$

With Legendre's basis of $P[X]$:

- $\mathcal{A}_{2,1} = \text{Span}(1, X, \frac{1}{2}(3X^2-1), \frac{1}{2}(5X^3-3X))$
- $\mathcal{A}_{2,2} = \text{Span}(1, X, \frac{1}{2}(3X^2-1), \frac{1}{2}(5X^3-3X), g)$
- $\mathcal{A}_{2,3} = \text{Span}(1, X, \frac{1}{2}(3X^2-1), \frac{1}{2}(5X^3-3X), \frac{1}{8}(35X^4-30X^2+3), g)$
- $\mathcal{A}_{2,4} = \text{Span}(1, X, \frac{1}{2}(3X^2-1), \frac{1}{2}(5X^3-3X), g, g^2)$
- $\mathcal{A}_{2,5} = \text{Span}(1, X, \frac{1}{2}(3X^2-1), \frac{1}{2}(5X^3-3X), g, g^2, g^3)$

With Hermite's basis of $P[X]$:

- $\mathcal{A}_{3,1} = \text{Span}(1, X, X^2-1, X^3-3X)$
- $\mathcal{A}_{3,2} = \text{Span}(1, X, X^2-1, X^3-3X, g)$

- $\mathcal{A}_{3,3} = \text{Span}(1, X, X^2 - 1, X^3 - 3X, X^4 - 6X^2 + 3, g)$
- $\mathcal{A}_{3,4} = \text{Span}(1, X, X^2 - 1, X^3 - 3X, g, g^2)$
- $\mathcal{A}_{3,5} = \text{Span}(1, X, X^2 - 1, X^3 - 3X, g, g^2, g^3)$

With Laguerre's basis of $P[X]$:

- $\mathcal{A}_{4,1} = \text{Span}(1, 1 - X, \frac{1}{2}(X^2 - 4X + 2), \frac{1}{6}(-X^3 + 9X^2 - 18X + 6))$
- $\mathcal{A}_{4,2} = \text{Span}(1, 1 - X, \frac{1}{2}(X^2 - 4X + 2), \frac{1}{6}(-X^3 + 9X^2 - 18X + 6), g)$
- $\mathcal{A}_{4,3} = \text{Span}(1, 1 - X, \frac{1}{2}(X^2 - 4X + 2), \frac{1}{6}(-X^3 + 9X^2 - 18X + 6), \frac{1}{24}(X^4 - 16X^3 + 72X^2 - 96X + 24), g)$
- $\mathcal{A}_{4,4} = \text{Span}(1, 1 - X, \frac{1}{2}(X^2 - 4X + 2), \frac{1}{6}(-X^3 + 9X^2 - 18X + 6), g, g^2)$
- $\mathcal{A}_{4,5} = \text{Span}(1, 1 - X, \frac{1}{2}(X^2 - 4X + 2), \frac{1}{6}(-X^3 + 9X^2 - 18X + 6), g, g^2, g^3)$

To conclude this section, we will finally use in the algorithm presented in sections 4 and 5 the approximate value function:

$$\tilde{C}_k : x \mapsto \sum_{i=1}^{l} \alpha_i \phi_i(x),$$

for all $k \in \{0, \cdots, N-1\}$ in the case of the regression.

3.2 Neural networks

An other idea to approximate Φ is to train a neaural network. For the numerical simulations part, we will train a fully connected neural network with 4 hidden layers which have respective width 3, 5, 5 and 3, as the functions we want to approximate are note to complicated.

4 Martingales from Approximate Value Functions

As seen before, the objective of the duality method is to transform the American option pricing problem into a problem of martingale maximization, i.e

$$V_0(X_{t_0})) = \inf_{M \in \mathcal{M}_0} \mathbb{E}[\max_{k \in \{0, \cdots, N\}} (Y_{t_k} - M_k)]$$

We know that the infimum is attained with the martingale $(M_k = \Delta_1 + \cdots + \Delta_k)_{k \in \{0, \cdots, N\}}$ where $\Delta_k = V_k(X_{t_k}) - \mathbb{E}[V_k(X_{t_k})|X_{t_{k-1}}]$ for all $k \in \{1, \cdots, N\}$. As in section 3, we denote by $C_{k-1} : x \mapsto \mathbb{E}[V_k(X_{t_k})|X_{t_{k-1}} = x]$ the continuation value. Thus we can rewrite Δ_k as $V_k(X_{t_k}) - C_{k-1}(X_{t_{k-1}})$.

So, the Snell envelope can be approximated by:

$$\tilde{V}_k(x) = \max(g(x), \tilde{C}_k(x))$$

where $\tilde{C}_k$ is an approximation of the continuation value C_k for $k = 1, \cdots, N-1$, as saw in section 3.

An issue with the approximation is that $\tilde{\Delta}_k = \tilde{V}_k(X_{t_k}) - \tilde{C}_{k-1}(X_{t_{k-1}})$ is no longer necessarily a martingale increment: in general we do not have the equality $\mathbb{E}[\tilde{\Delta}_k|X_{t_{k-1}}] = 0$. Thus, $\tilde{M}_k = \tilde{\Delta}_1 + \cdots + \tilde{\Delta}_k$ has no reason to be a martingale and we will not get a valid upper bound.

The Martingales from Approximate Value Functions method suggests to extract martingale differences from an approximation of value function, so we should estimate directly

$$\tilde{\Delta}_k = \tilde{V}_k(X_{t_k}) - \mathbb{E}[\tilde{V}_k(X_{t_k})|X_{t_{k-1}}],$$

with

$$\tilde{V}_k(X_{t_k}) = \max(g(X_{t_k}), \tilde{C}_k(X_{t_k})).$$

The implementation of this method is as follows:

1. Simulate a path $X_{t_0}, \cdots, X_{t_N}$ for the underlying Markov chain
2. At each step X_{t_k}, $k = 1, \cdots, N-1$ of the Markov chain:
   - Compute $\tilde{V}_k(X_{t_k}) = \max(Y_{t_k}, \tilde{C}_k(X_{t_k})$ (the function $\tilde{C}_k$ has been evaluated beforehand, for instance during the estimation of the lower bound in Longstaff & Schwartz's algorithm)
   - Generate m successors $X_{t_k}^1, \cdots, X_{t_k}^m$ of $X_{t_{k-1}}$
   - Compute $\frac{1}{n}\sum_{j=1}^m \tilde{V}_k(X_{t_k}^j)$ to approximate $\mathbb{E}[\tilde{V}_k(X_{t_k})|X_{t_{k-1}}]$
   - Set $\tilde{\Delta}_k = \tilde{V}_k(X_{t_k}) - \frac{1}{n}\sum_{j=1}^m \tilde{V}_k(X_{t_k}^j)$
3. Do (2) again for step X_{t_N} but with $\tilde{V}_N(X_{t_N}) = Y_{t_N}$ and $\tilde{V}_N(X_{t_N}^j) = Y_{t_N}^j$.
4. Sum to obtain the martingale $(\tilde{M}_k = \tilde{\Delta}_1 + \cdots + \tilde{\Delta}_k)_{k\in\{0,\cdots,N\}}$
5. Evaluate $\max_{k\in\{0,\cdots,N\}}(Y_{t_k} - \tilde{M}_k)$

If we could evaluate perfectly the optimal martingale, as prooven in section 2, we would need to repeat this only once. But, as $\tilde{M}$ is only an approximation of the optimal martingale, we have to repeat the algorithm several times and take the mean on all the results of (6).

5 Martingales from Stopping Rules

In comparison with previous method, we will make our reasoning in terms of stopping times as in Longstaff & Schwartz's algorithm.
Here, instead of approximate Δ_k by $\tilde{\Delta}_k = \tilde{V}_i(X_{t_i}) - \frac{1}{n}\sum_{j=1}^m \tilde{V}_k(X_{t_k}^j)$, we will use that the Δ_k in the optimal martingale re-writes as:

$$\begin{aligned}\Delta_k &= V_k(X_{t_k}) - \mathbb{E}[V_k(X_{t_k})|X_{t_{k-1}}] \\ &= \mathbb{E}[Y_{\tau_k}|X_{t_k}] - \mathbb{E}[Y_{\tau_k}|X_{t_{k-1}}]\end{aligned}$$

with the optimal stopping policy in discrete time given by the dynamical programming principle:

$$\begin{aligned}\tau_i &= \min\Big\{t_k \in \{t_i, \cdots, t_N\} \text{ such that } Y_{t_k} \geq \mathbb{E}[V_{k+1}(X_{t_{k+1}})|X_{t_k}]\Big\} \\ &= \min\Big\{t_k \in \{t_i, \cdots, t_N\} \text{ such that } Y_{t_k} \geq C_k(X_{t_k})\Big\}.\end{aligned}$$

As in the previous method (or in the Tsitsiklis & Van Roy's algorithm or the Longstaff & Schwartz's algorithm, see [Claisse, 2021]), we will use an approximate continuation value function $\tilde{C}_k$ (see section 3) and therefore an approximation of the optimal stopping rule:

$$\tilde{\tau}_i = \min\Big\{t_k \in \{t_i, \cdots, t_N\} \text{ such that } Y_{t_k} \geq \tilde{C}_k(X_{t_k})\Big\},$$

and approximate Δ_k:

$$\tilde{\Delta}_k = \mathbb{E}[Y_{\tilde{\tau}_k}|X_{t_k}] - \mathbb{E}[Y_{\tilde{\tau}_k}|X_{t_{k-1}}].$$

A first remark is that the process defined by $(\tilde{M}_k = \tilde{\Delta}_1 + \cdots + \tilde{\Delta}_k)_{k\in\{0,\cdots,N\}}$ is indeed a martingale, as $\mathbb{E}[\tilde{\Delta}_k|X_{t_{k-1}}] = \mathbb{E}[\mathbb{E}[Y_{\tilde{\tau}_k}|X_{t_k}]|X_{t_{k-1}}] - \mathbb{E}[Y_{\tilde{\tau}_k}|X_{t_{k-1}}] = 0$. Thus, the approximate optimal martingale will produce a valid upper bound. A second remark is that the first term in the definition of $\tilde{\Delta}_k$ can be re-written as:

$$\mathbb{E}[Y_{\tilde{\tau}_k}|X_{t_k}] = \begin{cases} Y_{t_k} & \text{if } Y_{t_k} \geq \tilde{C}_k(X_{t_k}) \\ \mathbb{E}[Y_{\tilde{\tau}_{k+1}}|X_{t_k}] & \text{otherwise} \end{cases}$$

and thus, we only have to estimate $\mathbb{E}[Y_{\tilde{\tau}_{k+1}}|X_{t_k}]$ for $k \in \{0,\cdots,N-1\}$. As in Longstaff & Schwartz's algorithm (see [Claisse, 2021]), we do not actually need

The implementation of this method is as follows:

1. Simulate a path $X_{t_0},\cdots,X_{t_N}$ for the underlying Markov chain
2. At each step X_{t_k}, for $k = 0,1,\cdots,N-1$:
   - Simulate m subpaths $(X^1_{t_{k+j}})_{j\in\{1,\cdots,m-i\}},\cdots,(X^m_{t_{k+j}})_{j\in\{1,\cdots,m-i\}}$ starting from X_{t_k}
   - Evaluate $Y^j_{\tilde{\tau}_{k+1}}$ for $j = 1,\cdots,m$, and use $\frac{1}{m}\sum_{j=1}^m Y^j_{\tilde{\tau}_{k+1}}$ as an estimator of $\mathbb{E}[Y_{\tilde{\tau}_{k+1}}|X_{t_k}]$
   - Evaluate the quantity

     $$W_k := \begin{cases} Y_{t_k} & \text{if } Y_{t_k} \geq \tilde{C}_k(X_{t_k}) \\ \frac{1}{m}\sum_{j=1}^m Y^j_{\tilde{\tau}_{k+1}} & \text{otherwise} \end{cases}$$

     and use it as an estimator of $\mathbb{E}[Y_{\tilde{\tau}_k}|X_{t_k}]$
   - Set $\tilde{\Delta}_k := W_k - \frac{1}{m}\sum_{j=1}^m Y^j_{\tilde{\tau}_k}$ (with the $(Y^j_{\tilde{\tau}_k})_{j\in\{1,\cdots,m\}}$ simulated from $X_{t_{k-1}}$ in the previous step)
3. Sum to obtain the martingale $(\tilde{M}_k = \tilde{\Delta}_1 + \cdots + \tilde{\Delta}_k)_{k\in\{0,\cdots,N\}}$
4. Evaluate $\max_{k\in\{0,\cdots,N\}}(Y_{t_k} - \tilde{M}_k)$

As in the section 4, if we could evaluate perfectly the optimal martingale, we would need to repeat this only once. But, as $\tilde{M}$ is only an approximation of the optimal martingale, we have to repeat the algorithm several times and take the mean on all the results of (4).

6 Numerical simulations of the Monte-Carlo methods

6.1 With non-parametric regression

In this section, we are going to study the influence of all the parameters of the algorithms from sections 4 and 5 on the results. The code of this part can be found in the file `main.ipynb`. For the result to be reproductible, we have set the seed of the generator to 0. First, let us recap all the parameters.

First, we will study the influence of the base functions, with the other parameters fixed: `n = 10000, m = 12, n2 = 1000, n_subpaths = 500`. We will estimate the price of an American put option with the parameters of the model: $K = 100, r = 0.06, T = 0.5, \sigma = 0.4$ and $X_{t_0} = 80$. The reel price of the option is ≈ 21.6059 (see [Rogers, 2002], Table 4.1).

From the table 1, we can see that the stopping rule algorithm is not impacted too much by the familly of base fonctions used. The upper bound stays approximately 0.1 above the reel

Algorithms	Approximate Value Function	Stopping Rule
Base functions $\phi_1, \cdots, \phi_l$	–	–
Number of time $t_0, \cdots, t_N$	`m`	`m`
Number of paths simulated	`n`	`n`
Number of subpaths simulated	`n2`	`n_subpaths`

Table 1: Table of all the parameters in the algorithms and their name in the code.

-	Approximate Value Functions		Stopping Rule	
-	Upper bound	STD	Upper bound	STD
$\mathcal{A}_{1,1}$	21.78096	0.00920	21.70837	0.00660
$\mathcal{A}_{1,2}$	21.71084	0.00762	21.72652	0.00628
$\mathcal{A}_{1,3}$	21.64477	0.00526	21.69761	0.00610
$\mathcal{A}_{1,4}$	21.70915	0.00778	21.73352	0.00651
$\mathcal{A}_{1,5}$	21.63482	0.00644	21.69339	0.00613
$\mathcal{A}_{2,1}$	21.81691	0.00898	21.70759	0.00653
$\mathcal{A}_{2,2}$	21.71684	0.00750	21.73379	0.00625
$\mathcal{A}_{2,3}$	21.66794	0.00583	21.71256	0.00621
$\mathcal{A}_{2,4}$	21.65649	0.00702	21.72711	0.00648
$\mathcal{A}_{2,5}$	21.61867	0.00515	21.70950	0.00618
$\mathcal{A}_{3,1}$	21.80167	0.00937	21.72663	0.00694
$\mathcal{A}_{3,2}$	21.69938	0.00736	21.73092	0.00625
$\mathcal{A}_{3,3}$	21.65745	0.00614	21.70833	0.00614
$\mathcal{A}_{3,4}$	21.66151	0.00663	21.71644	0.00624
$\mathcal{A}_{3,5}$	21.63071	0.00601	21.70496	0.00607
$\mathcal{A}_{4,1}$	21.79459	0.00919	21.71253	0.00658
$\mathcal{A}_{4,2}$	21.69621	0.00827	21.71581	0.00624
$\mathcal{A}_{4,3}$	21.66707	0.00571	21.71840	0.00624
$\mathcal{A}_{4,4}$	21.68950	0.00746	21.71606	0.00629
$\mathcal{A}_{4,5}$	21.63450	0.00622	21.69387	0.00641

Figure 1: Table of all the upper bounds and standard deviations with respect to the family of base functions.

value. On the other hand, the approximate value function algorithm gets closer and closer. It is approximatively 0.02 above the real value with the $\mathcal{A}_{i,5}$-families.

Now, let us fix the family $\mathcal{A}_{1,5}$ and the parameters: `n = 10000, n2 = 1000, n_subpaths = 500`. We will try several values for `m`.

We can remark that when `m` is to small (≤ 10 for instance), the price is underestimated. `m` $\in \{11, \cdots, 14\}$ seems to be a good choice.

Let us now study the impact of the number of subpaths in each algorithm. We will always use smaller value for `n_subpaths` than for `n2` because in the stopping rule algorithm, we need to simulate the whole subpath starting from X_{t_k} and this is more costly than only simulating the next step (as in the approximate value function algorithm). We fix the family of base functions $\mathcal{A}_{1,5}$ and the parameters: `n = 10000, m = 12`.

We can see that those parameters definitly have the bigger impact on the upper bound. It is also the parameters with `m` which increase the most the time of computation.

To conclude this section, let us change the parameter `n` to observe its impact. Let us fix the family of base functions $\mathcal{A}_{1,5}$ and the parameters: `n2 = 1000, n_subpaths = 500, m = 12`.

-	Approximate Value Functions		Stopping Rule	
-	Upper bound	STD	Upper bound	STD
m = 3	21.22756	0.00460	21.26823	0.00645
m = 5	21.43922	0.00469	21.49547	0.00593
m = 10	21.60830	0.00544	21.66441	0.00603
m = 11	21.61883	0.00561	21.69167	0.00649
m = 12	21.63501	0.00530	21.71376	0.00638
m = 15	21.67385	0.00674	21.72666	0.00594
m = 20	21.72330	0.00633	21.80384	0.00629

Figure 2: Table of all the upper bounds and standard deviations with respect to m.

-	Approximate Value Functions		Stopping Rule	
-	Upper bound	STD	Upper bound	STD
n2 = 200 and n_subpaths = 100	21.85505	0.00953	22.16101	0.01299
n2 = 400 and n_subpaths = 200	21.71403	0.00686	21.89255	0.00942
n2 = 600 and n_subpaths = 300	21.67304	0.00629	21.79183	0.00775
n2 = 800 and n_subpaths = 400	21.63865	0.00588	21.75591	0.00697
n2 = 1000 and n_subpaths = 500	21.62798	0.00545	21.69681	0.00590
n2 = 1500 and n_subpaths = 600	21.60133	0.00524	21.68349	0.00572

Figure 3: Table of all the upper bounds and standard deviations with respect to n2 and n_subpaths.

-	Approximate Value Functions		Stopping Rule	
-	Upper bound	STD	Upper bound	STD
n = 1000	21.81825	0.03244	21.76691	0.02214
n = 3000	21.68098	0.01617	21.72294	0.01244
n = 6000	21.68553	0.00859	21.73337	0.00823
n = 10000	21.61806	0.00511	21.69384	0.00611
n = 15000	21.61530	0.00416	21.69966	0.00485
n = 20000	21.61805	0.00348	21.69434	0.00422

Figure 4: Table of all the upper bounds and standard deviations with respect to n.

We can remark that from 1000, the value stabilises. Moreover, this parameter is definitly the one that have the bigger impact on the standard deviation.

6.2 With neural network approximation

As presented in the section 3, there are several ways to approximate the continuation value functions. We had the idea to replace the non-parametric regression by a small neural network (small as the function we want to approximate are quite simple). The code can be found in the file `main_nn_sklearn.ipynb`. The structure choosen is a neural network with 4 hidden layers with widths 3, 5, 5 and 3. It is probably not an optimal structure but we still find back the result we had with a non-parametric regression, with an upper bound of and with the algorithm "Martingale from Approximate Value Functions" (4) of 21.73689623 with the algorithm "Martingale from stopping rule" (5).

7 Finite differences

7.1 Presentation of the method

We have to come back to a more general theory to explain the finite differences method. One more time, we focus on pricing an American put option in a finite time $[0, T]$.

We proceed like in [Lamberton and Lapeyre, 2008]. Assume that $\mathbb{Q}$ is an equivalent martingale measure (EMM), $(W_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-Brownian Motion and $S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}$ solves the Black-Scholes SDE which takes the following form under the probability $\mathbb{Q}$

$$\begin{cases} dS_t = S_t(rdt + \sigma dW_t) \\ S_0 = x \end{cases} \tag{1}$$

Denote X_t the log of the solution, i.e. $X_t = log(S_0)+(r-\frac{\sigma^2}{2})t+\sigma W_t$ and $dX_t = \left(r-\frac{\sigma^2}{2}\right)dt+\sigma dW_t$. We will apply the finite differences method to this process.

Like for the previous methods, we approximate the continuous time case by a discrete time problem: the number of exercise dates is finite ($t \in [0, T]$). The American payout is vanilla, i.e. the payoff is the same function Φ for all times. Here, as we consider X_t and not S_t, the price of the American option at time t is function of (t, X_t) and we use $\Phi(x) = (K - e^x)_+$ instead of g introduced in previous sections. So denote $u(t, X_t)$ the price of the option at time t and $u(T, x) = \Phi(x)$.

The principle of this kind of American option is, because we are in finite number of exercise dates (no exercise between time t_i and t_{i+1}), at each time, the holder chooses to exercise the option and get $\Phi(X_{t_i})$ or to hold it and he owns $u(t_i^+, X_{t_i})$ (which is the price of the option on (t_i, t_{i+1})). So we can rewrite $u(t_i, X_{t_i}) = \max(u(t_i^+, X_{t_i}), \Phi(X_{t_i}))$. We can also consider an optimal strategy given by a stopping time $\bar{\tau} = \inf(s \in \{t_1, \cdots, t_N\} \cap [0, T], u(s, X_s) = \Phi(X_s))$ which is the holder should not exercise at times $s < \bar{\tau}$. In mathematical terms, we have the variational inequality

$$\max(\partial_t u(t, x) + \frac{\sigma^2}{2}\partial^2_{xx} u(t, x) + (r - \frac{\sigma^2}{2})\partial_x u(t, x) - ru(t, x), \Phi(x) - u(t, x)) = 0 \tag{2}$$

$$\iff \max(\partial_t u(t, x) + \mathcal{A}u(t, x), \Phi(x) - u(t, x)) = 0$$

where $\mathcal{A}_X f = \frac{\sigma^2}{2}\partial^2_{xx} f + (r - \frac{\sigma^2}{2})\partial_x f$ is the infinitesimal generator of the process X and we consider $\mathcal{A}f = \frac{\sigma^2}{2}\partial^2_{xx} f + (r - \frac{\sigma^2}{2})\partial_x f - rf$ (remark: it is not time dependent).

We want to discretize directly the variational inequality in order to implement it. We localize the inequality to the space $\mathcal{O}_z = (-z, z)$ and add additional restrictions like Neumann condition $\partial_x v(t, \pm z) = 0$.

We use the same notations for the discretization as in [Claisse, 2021]: we introduce a mesh $\{(t_n, x_i) := (nh, -z + i\delta);\ 0 \leq n \leq m,\ 0 \leq i \leq l+1\}$ where $h = \frac{T}{m}$ and $\delta = \frac{2z}{l+1}$, and $(u_i^n)_{0\leq n\leq m,\ 0\leq i\leq l+1}$ is an approximation of $(u(t,x))_{t\in[0,T], x\in(-z,z)}$. We set $u^m = \Phi_\delta = (\Phi(x_i))_{1\leq i\leq l}$ (and denote Φ_i the approximation of $\Phi(x_i)$) and recursively for $n = m-1, \cdots, 0$, given u^{n+1}, we want to find u^n satisfying for all $1 \leq i \leq l$:

$$\max\Big(\frac{u_i^{n+1} - u_i^n}{h} + \frac{\sigma^2}{2}\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\delta^2} + (r - \frac{\sigma^2}{2})\frac{u_{i+1}^n - u_{i-1}^n}{2\delta} - ru_i^n, \Phi_i - u_i^n\Big) = 0. \tag{3}$$

Considering the term on the left, we have :

$$\frac{u^{n+1} - u^n}{h} + A_\delta u^n = 0 \iff u^{n+1} - (I - hA_\delta)u^n = 0$$

where $A_\delta \in \mathbb{R}^{l\times l}$ is a tridiagonal matrix with coefficients $a_{i,i-1} = \frac{\sigma^2}{2\delta^2} - \frac{1}{2\delta}\Big(r - \frac{\sigma^2}{2}\Big)$, $a_{i,i} = -\Big(\frac{\sigma^2}{\delta^2} + r\Big)$, $a_{i,i+1} = \frac{\sigma^2}{2\delta^2} + \frac{1}{2\delta}\Big(r - \frac{\sigma^2}{2}\Big)$.
Because for $h > 0$, $\max(x, y) = 0 \iff \max(hx, y) = 0$, we can rewrite (3) as

$$\max\Big(u^{n+1} - (I - h\mathcal{A}_\delta)u^n, \Phi_\delta - u^n\Big) = 0 \tag{4}$$

Let $0 \leq k \leq \frac{\delta^2}{\sigma^2+\delta^2 r}$ and denote $P_\delta = I + kA_\delta \in \mathbb{R}^{l\times l}$, we obtain :

$$\max\Big(u^{n+1} + \frac{h}{k}P_\delta u^n - (1 + \frac{h}{k})u^n, \Phi_\delta - u^n\Big) = 0$$

$$\iff \max\Big(\frac{k}{k+h}\Big(u^{n+1} + \frac{h}{k}P_\delta u^n\Big) - u^n, \Phi_\delta - u^n\Big) = 0$$

$$\iff \max\Big(\frac{k}{k+h}\Big(u^{n+1} + \frac{h}{k}P_\delta u^n\Big), \Phi_\delta\Big) = u^n$$

Denote $G(v) = \max\Big(\frac{k}{k+h}\Big(u^{n+1} + \frac{h}{k}P_\delta v\Big), \Phi_\delta\Big)$, $v \in \mathbb{R}^l$, we have (4) is equivalent to $u^n = G(u^n)$. Moreover,

$$|G(v) - G(\tilde{v})|_\infty \leq \frac{h}{h+k}|P_\delta(v - \tilde{v}))|_\infty$$

and

$$|P_\delta(v - \tilde{v}))|_\infty \leq \|P_\delta\|_\infty . |v - \tilde{v}|_\infty \leq |v - \tilde{v}|_\infty,$$

where $|\cdot|_\infty$ denotes the infinity norm on $\mathbb{R}^l$.
The third inequality holds if $\|P_\delta\|_\infty = \max_{i=1,\cdots,l}\sum_{j=1}^l |p_{ij}| \leq 1$. The elements of matrix P_δ are $p_{i,i-1} = k(\frac{\sigma^2}{\delta^2} - \frac{1}{2\delta}(r - \frac{\sigma^2}{2}))$, $p_{i,i} = 1 - k(\frac{\sigma^2}{\delta^2} + r)$, $p_{i,i+1} = k(\frac{\sigma^2}{\delta^2} + \frac{1}{2\delta}(r - \frac{\sigma^2}{2}))$. So the condition above is satisfied as $p_{i,j} \geq 0\ \forall (i,j) \in \{0, \cdots, l\}^2$ and $\sum_{j=1}^l p_{i,j} \leq 1\ \forall i \in \{0, \cdots, l\}$ for $\delta|r - \frac{\sigma^2}{2}| \leq \sigma^2$ $\Big(\iff \delta|b| \leq \sigma^2$ under [Claisse, 2021] notations$\Big)$. Thus, G is a contraction and admits an unique fixed-point u^*, and every sequence $(u^n)_{n\in\mathbb{N}}$ defined by $u^n = G(u^n)$ converges to u. So there is uniqueness and existence to this problem. Moreover, as we are in the case of an implicit scheme with condition $\delta|r - \frac{\sigma^2}{2}| \leq \sigma^2$ satisfied, the scheme is unconditionnaly converging and verifies the conditions of consistency and stability.

More generally, we can discretize (2) by a θ-scheme where $\theta \in [0, 1]$. And instead of (4), we obtain

$$\max\Big((I - h\theta A_\delta)u^n - (I + h(1-\theta)A_\delta)u^{n+1}, \Phi_\delta - u^n\Big) = 0$$

For $\theta = 0$, we have an explicit scheme. For $\theta = \frac{1}{2}$ we have the Crank-Nicholson scheme. Finally, for $\theta = 1$, we find back the implicit scheme as in (4). To have properties of monotony, stability and consistency, we have to add the condition $\sigma^2(1-\theta)h \leq \delta^2$. Moreover, if $\theta < 1$, the convergence holds if $\lim_{h\to 0, k\to 0} \frac{h}{\delta^2} = 0$ by [Claisse, 2021].

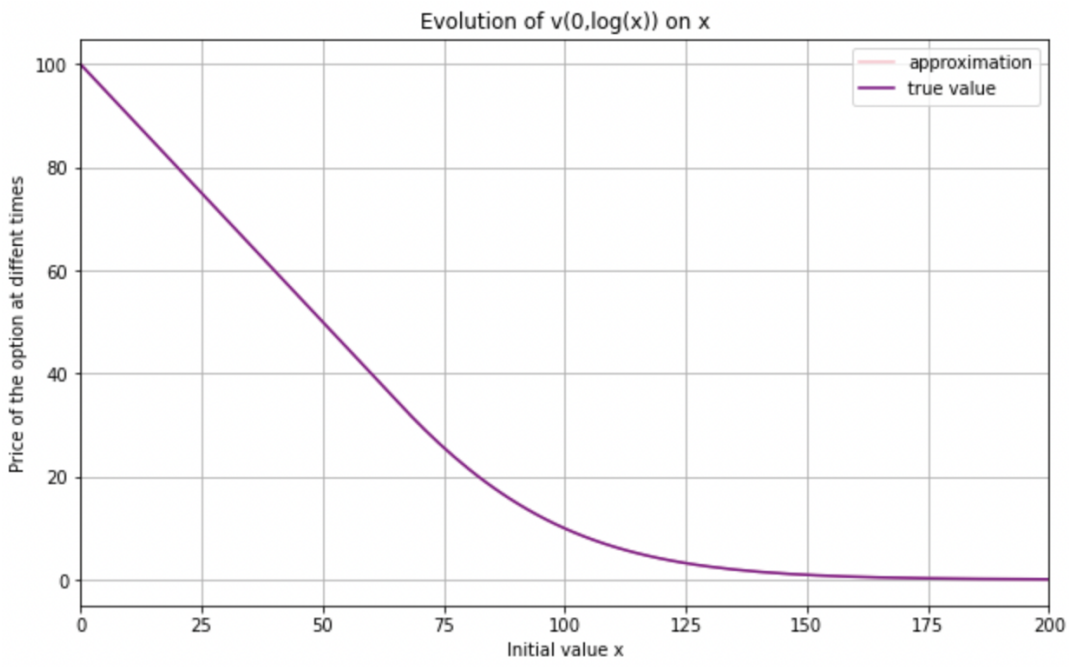
The implementation of this method is as follow:

1. Define discretization parametres.
2. Define matrix A_δ with the coefficient given above, the terminal time $u^m = \Phi_\delta$ and Neumann conditions $u_0^n = u_1^n$ and $u_l^n = u_{l+1}^n$ for each time $n \in \{0, \cdots, m\}$.
3. For each time, knowing u^{n+1}, solve $(I - h\theta A_\delta)u^{n+\frac{1}{2}} - (I + h(1-\theta)A_\delta)u^{n+1} = 0$.
4. The solution we expect is given by $u^n := \max\left(u^{n+\frac{1}{2}}, \Phi_\delta\right)$ at each time (with $u^{n+\frac{1}{2}}$ found at previous step).

7.2 Numerical simulations of the finite difference method

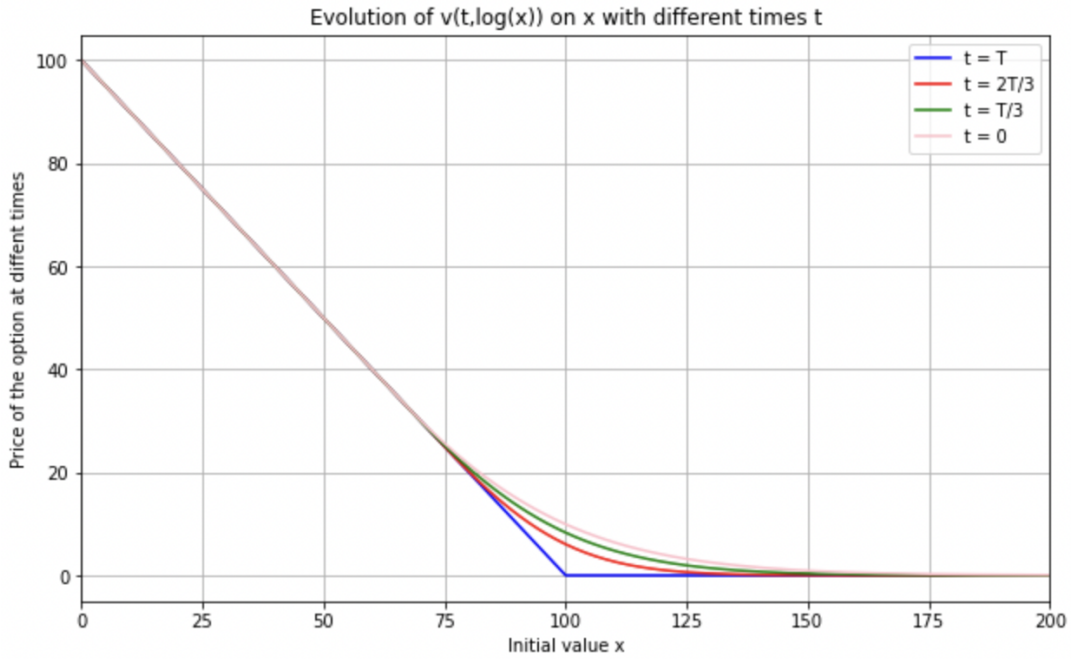
In this section, we will see the implementation of the finite differences method on an American put with parameters of the model: $K = 100, r = 0.06, T = 0.5, \sigma = 0.4$ and $X_{t_0} = 80$ (as in previous cases).

For all θ-schemes mentioned (implicit scheme, explicit scheme and Crank-Nicholson scheme), we obtain the following value function:



This method seems to be a good approximation of the value function of this American put. Remark that we find back the real price (≈ 21.6059).

Watching the evolution of the solution given by this method (for the 3 scheme) at several times, we can observe how the difference finite method converge to the solution when the parameters are well set i.e $\delta|r - \frac{\sigma^2}{2}| \leq \sigma^2$ and $\sigma^2(1-\theta)h \leq \delta^2$:



Now, we will try to change the parameters to see how this method is impacted. First, we will modify the space parameter l.

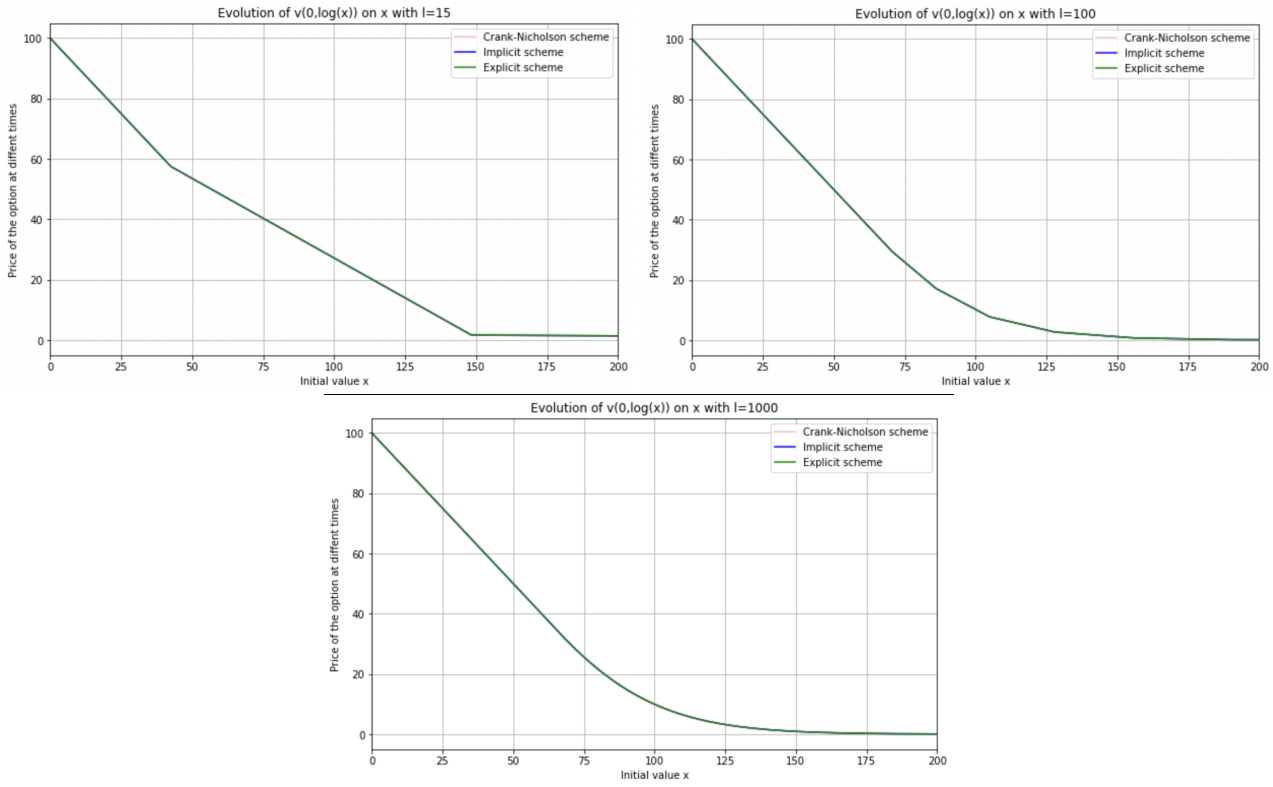


Figure 5

Then, we modify the time parameter m.

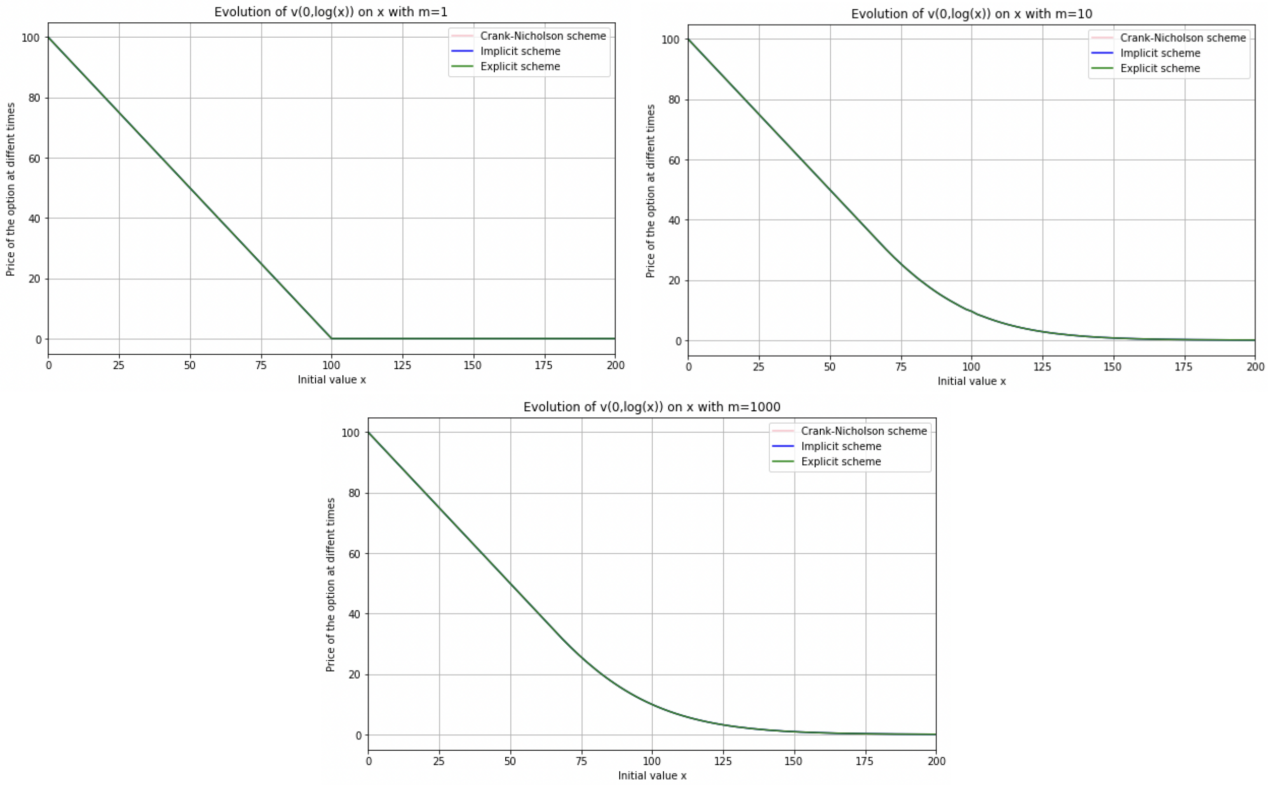


Figure 6

8 Conclusion

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