Control Problems

We can formulate our control problem generally as the following system of differential equations. We can express the state variables that compose $\boldsymbol{x}(t)$ in compact form that expresses the state vectors as functions of the current state, time, and the control:

$$\boldsymbol{x} = f(\boldsymbol{x}(t), \boldsymbol{u}(t))$$

or

$$\frac{d\boldsymbol{x}(t)}{dt} = f(\boldsymbol{x}, t, \boldsymbol{u}(t)).$$

where the function(s) $f(\boldsymbol{x}, t, u)$ are continuous, differentiable functions of vectors $\boldsymbol{x}$ and $\boldsymbol{u}$ and variable t. The state variables $\boldsymbol{x} = \boldsymbol{x}(t)$ describe the state of the system as a function of time t. The system of state equations describes how the system changes from its initial states $bmx(0)$ under the controls $\boldsymbol{u} = \boldsymbol{u}(x)$. The final time T is called the time horizon.

For a given control $\boldsymbol{u}(t)$, $0 \leq t \leq T$, the solution $\boldsymbol{x}(t)$ is the response. We may want to consider the functions $\boldsymbol{u}(t)$ that are admissible controls. This class of functions U is defined to be the class of all piecewise-continuous real functions $u(t)$ defined for $0 \leq t \leq T$ and satisfying $u(t) \in U_t$ where U_t is a given interval called the control set.

Many problems may also involve a terminal condition

$$x(T) = x_T$$

A feasible control is then any admissible control such that the response satisfies this terminal condition as well as the initial conditions.

The basic issue in our optimal control problem is to find a feasible control $\boldsymbol{u}(t)$ that maximizes $J[\boldsymbol{u}]$. This would be the optimal control.

$$J[u] = \int_{t_0}^{t_f} g[\boldsymbol{x}(t), t, \boldsymbol{u}(t)]dt$$

where $g[\boldsymbol{x}(t), t, \boldsymbol{u}(t)]$ is a given, continuously differentiable function that represents the response to $u(t)$. The *maximum principle* gives necessary conditions that must be met by an optimal control.

The maximum principle can be expressed in terms of the Hamiltonian:

$$H[\boldsymbol{x}(t), t, \boldsymbol{u}(t); \boldsymbol{\lambda}(t)] = g[\boldsymbol{x}(t), t, \boldsymbol{u}(t)] + \boldsymbol{\lambda}(t) f[\boldsymbol{x}(t), t, \boldsymbol{u}(t)] \tag{1}$$

$$H = g + \lambda^T f \tag{2}$$

[Q: is the t in the f(...) a derivative?; p. 91 of Clark] where $\boldsymbol{\lambda}(t)$ are unknown functions called adjoint variables. For a system where $\boldsymbol{u}(t)$ is the optimal control and $\boldsymbol{x}(t)$ is the associated response, the maximum principle asserts existence of $\boldsymbol{\lambda}(t)$ such that these equations are satisfied for all t, $0 \leq t \leq T$: [not sure if these are vectors or not?]

$$\frac{d\boldsymbol{\lambda}}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}} = -\frac{\partial g}{\partial \boldsymbol{x}} - \boldsymbol{\lambda}(t)\frac{\partial f}{\partial \boldsymbol{x}}, \tag{3}$$

$$H[\boldsymbol{x}(t), t, \boldsymbol{u}(t); \boldsymbol{\lambda}(t)] = \max_{\{u \in U_t\}} H[\boldsymbol{x}(t), t, \boldsymbol{u}(t); \boldsymbol{\lambda}(t)], \tag{4}$$

The first of these requirements are the adjoint equations. These are equivalent to equation (6) in King and Roughgarden (1982), $\dot{\boldsymbol{\lambda}}^T = -\frac{\partial H}{\partial \boldsymbol{x}}$. King and Roughgarden take $\boldsymbol{\lambda}(t_f) = 0$ as an additional point to define the adjoint variables when there are no final values for the state variables. This is a key assumption that states the marginal fitness benefit of investing in growth or reproduction at the time horizon t_f is zero.

The expression for $J[\boldsymbol{u}(t)]$ is maximized by the $\boldsymbol{u}(t)$ that maximize H at every point in time, as is stated in the second of the requirements above.

In one dimension, the optimal control problem has three unknown functions $x(t)$, $u(t)$, and $\lambda(t)$. For these three functions, there are three equations: a state equation, an adjoint equation, and the maximum principle. The state and adjoint equations are first-order, differential equations. Solutions to these equations require initial or terminal conditions. [these are thus key; in Clark p. 92 they are the initial value $x(0)$ and terminal value $x(T)$]

In our version of the problem, each unknown function is a vector of length four. This means that the optimal control problem now has 12 unknown functions.

Meristem and biomass allocation: explanation

In our model, we partition photosynthate among four pools. These pools are meristems committed to vegetative growth (G), vegetative biomass (V), meristems committed to reproduction (F), and reproductive biomass (R).

Growth in a season ($0 < t < T$) is represented by the following system of differential equations:

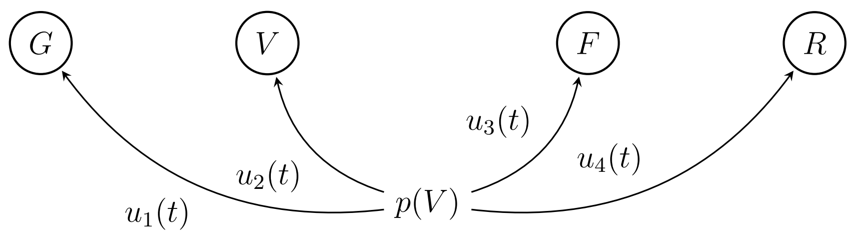
$$\frac{dG}{dt} = u_1(t)p(t) \quad (5)$$

$$\frac{dV}{dt} = u_2(t)p(t) \quad (6)$$

$$\frac{dF}{dt} = u_3(t)p(t) \quad (7)$$

$$\frac{dR}{dt} = u_4(t)p(t) \quad (8)$$

where the u_is represent rates and $P(t)$ is the production function. We assume that all energy produced by photosynthesis is allocated to vegetative meristems, to vegetative growth, to reproductive meristems, or to reproduction. Meristems are allocated to either growth or flowering; no meristems remain undifferentiated. Initial reproductive biomass is zero. We represent this with the following diagram:



Daily photosynthesis ($p(V)$) is a function of vegetative biomass. The plant starts from initial size $p(V(0))$ set by seed resources. Following Iwasa (200), we assume that daily biomass production (photosynthesis production, respiration cost) daily net production, is an increasing, saturating function of the vegetative biomass written as

$$p(V) = \frac{aV}{1 + hV}. \quad (9)$$

Photosynthate is allocated between vegetative meristem differentiation ($u_1(t)$), vegetative biomass ($u_2(t)$), reproductive meristem differentiation ($u_3(t)$) and reproductive biomass ($u_5(t)$). The model jump starts photosynthesis by setting an initial seed size but constraints on u_2 and u_4 (see below) limit production of vegetative biomass unless there is also allocation to meristems for growth and reproduction. Plants are modular and meristem differentiation and subsequent determination are necessary to add vegetative modules (phytomers: internode+node with leaf+axillary meristem bud). The arrow from meristem state (M) to meristems for growth (G) corresponds to determination and growth of vegetative organs (e.g. leaf expansion) and requires energy that limits accumulation in V. For example, a low rate of allocation to meristems for growth (low u_1) limits the allocation of photosynthate to V.

We impose the following constraints

$$u_1 + u_2 + u_3 + u_4 = 1 \tag{10}$$
$$u_i \geq 0 \tag{11}$$
$$u_2 \leq cu_1 \tag{12}$$
$$u_4 \leq cu_3 \tag{13}$$

which can be interpreted as follows. All energy that gets produced is allocated to one of the four pools —the u_is sum to one (Equation 23). Energy flows in a one direction, and either energy is flowing in one of the paths or is not —allocation rates (u_is) are zero or positive (Equation 24). The rate of energy allocation to vegetative meristems constrains the rate of energy allocation to vegetative biomass —u_2 is less than or equal to u_1 multiplied by a constant c that acts as a constraint (Equation 25). Similarly, the rate of energy allocation to vegetative meristems constrains the rate of energy allocation to vegetative biomass —u_4 is less than or equal to u_3 multiplied by a constant d that acts as a constraint (Equation 26). These condition says that to allocate energy to biomass the plant must concurrently allocate energy to meristems.

Meristem and biomass allocation: control problem

We can formulate our control problem as the following system of differential equations. Growth in a season ($0 < t < T$) is represented by the following system of differential equations:

$$\frac{dx_1}{dt} = \frac{ax_2}{1 + hx_2} \times u_1(t) \tag{14}$$
$$\frac{dx_2}{dt} = \frac{ax_2}{1 + hx_2} \times u_2(t) \tag{15}$$
$$\frac{dx_3}{dt} = \frac{ax_2}{1 + hx_2} \times u_3(t) \tag{16}$$
$$\frac{dx_4}{dt} = \frac{ax_2}{1 + hx_2} \times u_4(t), \tag{17}$$

and

$$0 \leq t \leq T \tag{18}$$

with the following initial conditions

$$x_1(0) = x_0 \tag{19}$$
$$x_2(0) = x_0 \tag{20}$$
$$x_3(0) = x_0 \tag{21}$$
$$x_4(0) = x_0. \tag{22}$$

We impose the following constraints to define the class U of admissible controls

$$u_1 + u_2 + u_3 + u_4 = 1 \tag{23}$$
$$u_i \geq 0 \tag{24}$$
$$u_2 \leq cu_1 \tag{25}$$
$$u_4 \leq cu_3 \tag{26}$$

Optimization: implementation

Optimal control

The state variables in the ODE are G, V, F, and R. The constants a and h are the parameters in the ODE. The values c and d are the constraints in the ODE.

We rewrite these as

$$u_1 = f_1(t) \tag{27}$$
$$u_2 = f_2(t) \tag{28}$$
$$u_3 = f_3(t) \tag{29}$$
$$u_4 = 1 - (u_1 + u_2 + u_3) \tag{30}$$

and build the constraints into the differential equations as

$$u_2 = \min(u_2, c * u_1) \tag{31}$$
$$u_4 = \min(u_1, d * u_3) \tag{32}$$

Constraint matrix

See https://cran.r-project.org/web/views/Optimization.html for various R packages associated with optimization.

We are using linear programming to deal with our problem [this comes from notes to Lecture 1, Math 407, Jim Burke, UW]. Linear programming takes an optimization problem over $\mathbb{R}^n$ where the objective function is a linear function of the form

$$c_1x_1 + c_2x_2 + \cdots + c_nx + n \tag{33}$$

for some $c_i \in \mathbb{R}^n; i = 1, \ldots, n$ and the region is a set of solutions subject to linear inequality and equality constraints

$$a_{i1}x_i + a_{i2}x_2 + \cdots + a_{in}x + n \leq \alpha_i; i = 1, \ldots, s \tag{34}$$
$$\tag{35}$$

and

$$b_{i1}x_i + b_{i2}x_2 + \cdots + b_{in}x + n = \beta_i; i = 1, \ldots, r \tag{36}$$

These can be rewritten in compact form as

$$\text{maximize } c^T x \tag{37}$$
$$\text{subject to } Ax \leq \alpha \text{ and } Bx = \beta \tag{38}$$

We can reformulate the optimization problem as follows. We are implementing a constrained optimization problem, as we have inequality constraints. The constraint matrix comes from linear programming and defines the equations and inequalities that define the set of solutions. For optimization, we then constructed a constraint matrix with four blocks. We split the season $[0, T]$ into t time steps.

We implemented the constraints $u_1 + u_2 + u_3 - 1 \leq 0$ and that u_4 is positive as:

$$A_{11} = \begin{bmatrix} -1 & 0 & \ldots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & -1 & 0 \\ 0 & \ldots & 0 & -1 \end{bmatrix}_{t \times t}$$

$$A_{12} = \begin{bmatrix} -1 & 0 & \ldots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & -1 & 0 \\ 0 & \ldots & 0 & -1 \end{bmatrix}_{t \times t}$$

$$A_{13} = \begin{bmatrix} -1 & 0 & \ldots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & -1 & 0 \\ 0 & \ldots & 0 & -1 \end{bmatrix}_{t \times t}$$

$$A_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \end{bmatrix}_{t \times 3t}$$

$$b_1 = \begin{bmatrix} -1 & -1 & \ldots & -1 \end{bmatrix}_{1 \times t}$$

We implemented the constraint that u_i be positive (??) as:

$$A_2 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}_{3t \times 3t}$$

$$b_2 = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}_{1 \times 3t}$$

Together

$$\boldsymbol{A} = \begin{bmatrix} A_1 & A_2 \end{bmatrix}_{4t \times 3t}$$

and

$$\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}_{1 \times 4t}$$

These matrices define the following

$$A\boldsymbol{\theta} - \boldsymbol{b} \geq 0 \tag{39}$$

Our approach included only inequality constraints given by A and b. (meq=0 in the constrOptim function). We define our constraint as Ax greater than or equal to b, we don't have a second constraint for equality. Our objective function gives the output that we want to maximize/minimize. The variables are the inputs we control (?). The constraints are equations that limit the variables. The variables influence the objective function, and the constraints place limits on the domain of variables. In our case, the constraints limit the u functions which are the variables that go into the objective function where the objective function integrates the system of DE over time.