Sharygin Triangles & Cubic Curves

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What Are Sharygin Triangles

To understand what a Sharygin triangle is, we let ΔABC be a triangle in $\mathbb{R}^2$ (Shown in Fig. 1) and let AD, BE and CF be its angle bisectors. This will yield another triangle ΔDEF. We then define ΔABC to be a Sharygin triangle if ΔDEF is isosceles and ΔABC is non-isosceles. Then let $AC = x$, $BC = y$ and $z = AB$, and through application of the angle bisector theorem:

$$DF - EF = (x-y)\frac{xyz}{(x+y)(x+z)^2(y+z)^2} f_3(x,y,z).$$

Where

$$f_3(x,y,z) = x^3 + x^2y + xy^2 + y^3 + z\left(y^2 + xy + x^2\right) - z^2(x+y) - z^3.$$

This equation ensures that ΔABC is isosceles while also showing that ΔABC is a Sharygin triangle strictly if $f_3(x,y,z) = 0$ and $x \neq y$.

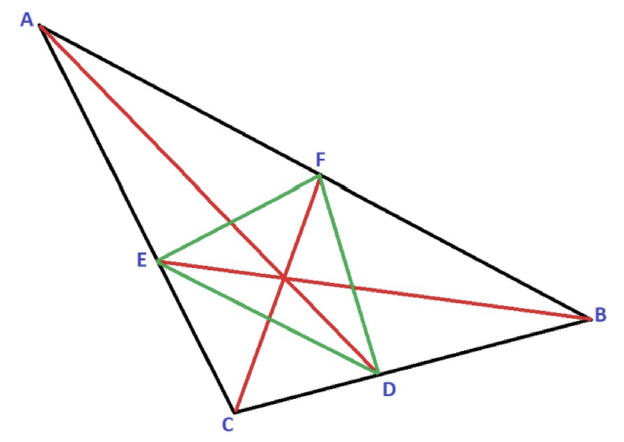

Fig. 1: Labelled Triangle

The History

Sharygin triangles evolved from the Steiner-Lehmus Theorem and its geometric implications. It is a great example of the complex, interwoven nature of modern mathematics.

Sharygin triangles owe their name to Igor Sharygin, a prolific Russian mathematician and mathematical educator, who investigated them in "Problems in Plane Geometry"[3]. Unfortunately, despite his contributions to mathematics, he was forced to resign from Moscow State University due to his anti-Soviet political beliefs. The results of algebraic geometric explored in this poster are largerly a product of the search for Sharygin triangles with have integer-length sides. As will be shown in this poster, such triangles exist (and there are infinitely many of them). Sergei Markelov aimed to create a computer algorithm that would find integer-length Sharygin triangles with side length not exceeding a million but after two months of computations, the algorithm failed to find any. The search for Sharygin triangles led to many novel results in projective geometry.

How Many Sharygin Triangles Are There?

As indicated in "Plotting the Cubic Curve" each point in the subset of the cubic curve C defined by the triangle inequalities corresponds to a unique Sharygin triangle. Therefore, determining the number of pairwise non-similar Sharygin triangles with integer sides would be equivalent to determining the number of rational points in the considered subset of the cubic curve (because they can be scaled up as to have integer sides). It turns out that there are infinitely many of them. The proof of this statement can be found in [1] and it requires some advanced mathematical results that we could not illustrate here.

However, for a better intuitive understanding of the topic we describe an intermediate result, namely that there are infinitely many rational points on the elliptic curve in question (although that does not directly imply that there are infinitely many rational points in any specific subset of the curve).

The first step is noting that point $A = (1:0:1)$ has infinite order. By computing $A, 2A, \ldots, 12A$ we can verify that neither of those points is equal to the chosen identity point $O = (1:-1:0)$, therefore, by Mazur's Theorem, point A has to have infinite order. This result and the fact that the point A is rational, allows us to construct infinitely many rational points lying in the cubic curve C. This is owing to the fact that adding two rational points on an elliptic curve will always result in another rational point, and it is not difficult to see why.

Consider the equation of the elliptic curve E in U_z formed from the cubic equation C by a projective transformation. Firstly, every line passing through 2 rational points in $\mathbb{R}^2$ must have rational coefficients. Also, a line that is tangent to the curve E at a rational point must have a rational gradient, and hence rational coefficients. Finding intersection points of a line with rational coefficients and E would correspond to solving a cubic equation with rational coefficients. Since, we know that 2 of its roots (counting multiplicities) are rational, from Vieta's formulas it follows that the 3rd root must also be rational (if a sum of 2 numbers of which one is rational is rational, the second one must also be rational). Keeping in mind how addition on cubic curves is defined, the above implies that adding two rational points on an elliptic curve will always result in another rational point. Since point A has infinite order and is rational, for all $n \in \mathbb{N}$, nA is a unique rational point. Therefore, there are infinitely many rational points on C. As shown in [1] these points are dense on C, thus, there are infinitely many of them in any subset of C.

Plotting the Cubic Curve

We can identify U_z with $\mathbb{C}^2$ with coordinates $\bar{x} = \frac{x}{z}$ and $\bar{y} = \frac{y}{z}$. Then $\mathbb{C} \cap U_z$ is a cubic curve in $\mathbb{C}^2$ that is given by

$$\bar{x}^3 + \bar{x}^2\bar{y} + \bar{x}\bar{y}^2 + \bar{y}^3 + \bar{y}^2 + \bar{x}\bar{y} + \bar{x}^2 - \bar{x} - \bar{y} - 1 = 0.$$

Using Python, we can plot its real part:

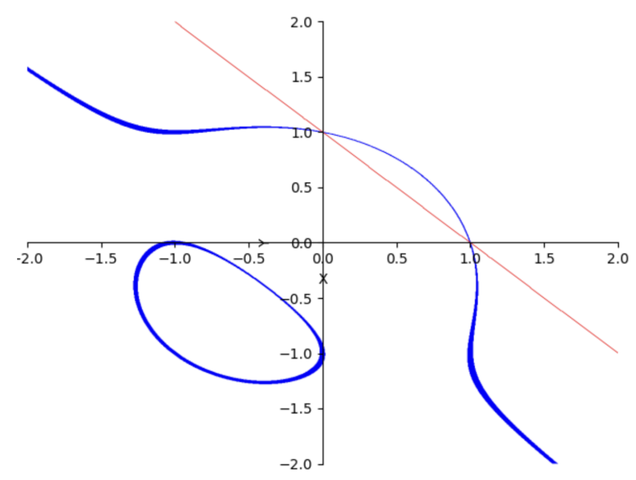

Fig. 2: The Cubic Curve at z=1 overlapping with the Triangle Inequality.

In this figure, the triangle inequality is shown as all points above the red line. Because every triangle has to satisfy the triangle inequalities, Sharygin triangles (up to similarity) are points in the subset of the cubic curve defined by the triangle inequalities, which is the part curve lying above the red line in figure above.

How To Generate A Triangle With Integer Sides

```
x, y, z = sym.symbols('x, y, z')
F = x**3 + x**2 * y + x * y**2 + y**3 + z*(x**2 + x * y + y**2) \
    - z**2*(x + y) - z**3
O = [1, -1, 0]

# Returns whether two points are equal down to scaling
def points_equal(P1, P2):
    return x in sym.solve([p1 - x * p2 for p1, p2 in zip(P1, P2)], x)

# Filters out the two points we know and returns the new point found
def get_new_point(points, given):
    return [p for p in points if not
any(points_equal(p, g) for g in given)][0]

# Differentiates F and returns the tangent line eq at point P
def tangent_at_point(P):
    a = sym.diff(F, x).subs(x, P[0]).subs(y, P[1]).subs(z, P[2])
    b = sym.diff(F, y).subs(x, P[0]).subs(y, P[1]).subs(z, P[2])
    c = sym.diff(F, z).subs(x, P[0]).subs(y, P[1]).subs(z, P[2])
    return a * x + b * y + c * z

# Takes in two points and performs the algorithm defined in [2]
def add_points(P1, P2):
    L = tangent_at_point(P1) if points_equal(P1, P2) else \
        sym.Matrix([[P1, P2, [x, y, z]]).det()
    points_on_L = sym.solve((F, L, x - 1), x, y, z)

    Q = get_new_point(points_on_L, [P1, P2])
    L2 = tangent_at_point(Q) if points_equal(Q, O) else \
        sym.Matrix([Q, O, [x, y, z]]).det()
    points_on_L2 = sym.solve((F, L2, x - 1), x, y, z)

    return integerise(get_new_point(points_on_L2, [Q, O]))


# Take point [1 : 0 : 1] and add it to itself 3 times, creating 8[1 : 0 : 1]
P1 = (1, 0, 1)
_2xP1 = add_points(P1, P1)
_4xP1 = add_points(_2xP1, _2xP1)
print(add_points(_4xP1, _4xP1))
```

Using the algorithm defined in [2], this code yields the triangle generated by $8[1:0:1]$:

$$[301361533449900458837600 : 49105016933436320224063 : 316629033253501281102807]$$

References

References

[1] I. Netay, A. Savvateev, *Sharygin triangles and elliptic curves*. Magadan Volume, Bulletin of the Korean Mathematical Society 54 (2017) 1597–1617

[2] I. Cheltsov, *Algebraic Geometry for sophomores*. Learn, University of Edinburgh, 2019

[3] I. Sharygin, *Problems in Plane Geometry*. Mir Publishers, 1988