Notes on Evans & Didelez (2023)

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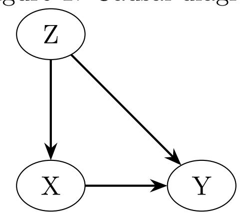
This document collects my notes on Evans and Didelez (2023). Consider the causal structure in Figure 1. They propose a new parameterization, termed **frugal parameterization**, which consists of three pieces: the joint distribution of the treatment and confounders $p_{ZX}(z,x)$ (the 'past'), the causal distribution of interest $p^*_{Y|X}(y|x)$, and a dependence measure between the outcome and the confounders conditional on the treatment $\phi^*_{YZ|X}$ (could be a conditional odds-ratio or a copula). In sequential treatment models, this parameterization circumvents the so-called **g-null paradox** (Robins and Wasserman, 1997). Their main result shows that a frugal parametrization $\theta = (\theta_{ZX}, \theta_{Y|X}, \phi_{YZ|X})$ of the observational distribution induces a corresponding parameterization $\theta^* = (\theta_{ZX}, \theta^*_{Y|X}, \phi^*_{YZ|X})$ that is also frugal. Replacing θ_{ZX} in θ^* with $\eta_{ZX}(\theta_{ZX})$, where η_{ZX} is a twice differentiable function with a Jacobian of constant rank, yields a parameterization of the causal joint distribution p^*_{ZXY}. Using this, they propose a rejection sampling algorithm to sample from p_{ZXY} (implemented in the R-package causl). Furthermore, they show that under certain assumptions we can obtain consistent parameter estimates for the model $p^*_{Y|X}$ by maximizing the likelihood with respect to the observational data from p_{ZXY}.

Figure 1: Causal diagram



My understanding is that the main advantage of the frugal parameterization is that it provides a convenient characterization of the joint distribution p_{ZXY}, which is usually unknown or difficult to handle. Instead, we can specify a model for p_{ZX} and $p^*_{Y|X}$ and choose a dependence measure of Y and Z conditional on X. Then, we can use these three to perform likelihood-based inference or simulate from the joint distribution. The joint distribution can be factorized into the frugal parameterization:

$$
\begin{aligned}
p_{ZXY}(z,x,y|\theta^*) &= p_{ZX}(z,x|\theta_{ZX}) p^*_{Y|ZX}(y|z,x;\theta^*_{Y|X}, \phi^*_{YZ|X}) && (1)\\
&= p_{ZX}(z,x|\theta_{ZX}) p^*_{Y|X}(y|x;\theta^*_{Y|X}) c(y,z|x), && (2)
\end{aligned}
$$

where $c(y,z|x) = \phi^*_{YZ|X}$ is a copula density for continuous Y and Z given X. Given observations for (Y,X,Z), we can estimate θ^* by maximizing the expression above.

In their abstract, they mention that this can be used to conduct Bayesian inference, but do not discuss this in the main text. I assume this would work as follows: We can obtain a joint posterior distribution

$$p(\theta^*|z,x,y) \propto p_{ZXY}(z,x,y|\theta^*) p(\theta^*),$$

where $p(\theta^*)$ is some prior distribution. Then, we can find the posterior distribution for the parameters of the marginal model $p^*_{Y|X}$ by integrating out θ_{ZX} and $\phi^*_{YZ|X}$,

$$p(\theta^*_{Y|X}|z,x,y) = \int\int p(\theta^*|z,x,y) d\theta_{ZX} d\phi^*_{YZ|X}.$$

Comments/Questions:

- In the words of Engle et al. (1983), (1) implies that $[(y|z, x; \theta^*_{Y|X}, \phi^*_{YZ|X}), (z, x; \theta_{ZX})]$ operates a sequential cut on their joint distribution (if $(\theta^*_{Y|X}, \phi^*_{YZ|X})$ and θ_{ZX} are variation free). Therefore, if we can find a frugal paramterization, (x, z) are weakly exogenous for any function of $\theta^*_{Y|X}$. Did I get this right?
- What is the role of η_{ZX}?
- Why is it useful to simulate from the observational distribution p_{ZXY}?

References

Engle, R. F., Hendry, D. F., and Richard, J.-F. (1983). Exogeneity. *Econometrica: Journal of the Econometric Society*, pages 277–304.

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Robins, J. M. and Wasserman, L. (1997). Estimation of effects of sequential treatments by reparameterizing directed acyclic graphs. In *Proceedings of the Thirteenth Conference on Uncertainty in Artificial Intelligence*, UAI'97, page 409–420, San Francisco, CA, USA. Morgan Kaufmann Publishers Inc.