

Notes on Evans & Didelez (2023)

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This document collects my notes on [Evans and Didelez \(2023\)](#). They propose a new parameterization for causal problems, termed **frugal parameterization**, which consists of three pieces: the joint distribution of the treatment and covariates $p_{ZX}(z, x)$ (the 'past'), the causal distribution of interest $p_{Y|X}^*(y|x)$, and a dependence measure between the outcome and the covariates conditional on the treatment $\phi_{YZ|X}^*$ (could be a conditional odds-ratio or a copula). In sequential treatment models, this parameterization circumvents the so-called **g-null paradox** ([Robins and Wasserman, 1997](#)). Their main result shows that a frugal parametrization $\theta = (\theta_{ZX}, \theta_{Y|X}, \phi_{YZ|X})$ of the observational distribution induces a corresponding parameterization $\theta^* = (\theta_{ZX}, \theta_{Y|X}^*, \phi_{YZ|X}^*)$ that is also frugal. Replacing θ_{ZX} in θ^* with $\eta_{ZX}(\theta_{ZX})$, where η_{ZX} is a twice differentiable function with a Jacobian of constant rank, yields a parameterization of the causal joint distribution p_{ZXY}^* . Using this, they propose a rejection sampling algorithm to sample from p_{ZXY} (implemented in the R-package [causal](#)). Furthermore, they show that under certain assumptions we can obtain consistent parameter estimates for the model $p_{Y|X}^*$ by maximizing the likelihood with respect to the observational data from p_{ZXY} .

As far as I understand, the main advantage of the frugal parameterization is that it provides a convenient characterization of the joint distribution p_{ZXY} , which is usually unknown or difficult to handle. Instead, we can specify a model for p_{ZX} and $p_{Y|X}^*$ and choose a dependence measure of Y and Z conditional on X . Then, we can use these three to perform likelihood-based inference or simulate from the joint distribution. Consider the following example corresponding to Figure 1: The joint distribution can be factorized into the frugal parameterization:

$$p_{ZXY}(z, x, y|\theta^*) = p_{ZX}(z, x|\theta_{ZX})p_{Y|ZX}^*(y|z, x) \quad (1)$$

$$= p_{ZX}(z, x|\theta_{ZX})p_X^*(y|x; \theta_{Y|X}^*)c(y, z|x), \quad (2)$$

where $c(y, z|x) =: \phi_{YZ|X}^*$ is a copula density for continuous Y and Z given X . Given observations for (Y, X, Z) , we can estimate θ^* by maximizing the expression above. Or we can even obtain a posterior distribution

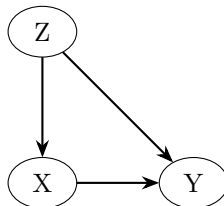
$$p(\theta^*|z, x, y) \propto p_{ZXY}(z, x, y|\theta^*)p(\theta^*), \quad (3)$$

where $p(\theta^*)$ is some prior distribution.

Comments/Questions:

- It's not clear to me what the role of η_{ZX} is and how we obtain it. Is it chosen by the analyst?
- The parameterization is not unique: Given $p_{Y|X}^*$, p_{ZX} and the dependence measure $\phi_{YZ|X}^*$ need to be chosen. How do we choose them? Can this choice introduce uncertainty that is not accounted for, maybe like selective inference type problems?

Figure 1: Causal diagram



- Similarly, what happens if the causal model $p_{Y|X}^*$ is misspecified? Then, the causal and observational joint distributions will be misspecified as well, leading to false simulation results, right?

Further reading:

- [Robins and Wasserman \(1997\)](#) and [McGrath et al. \(2022\)](#) to better understand the g-null paradox. Also, problem 29.1 in [Ding \(2023\)](#) is similar to their example R2.
- [Bergsma and Rudas \(2002\)](#) to better understand the discrete case and for some of the proofs.

References

- Bergsma, W. P. and Rudas, T. (2002). Marginal models for categorical data. *The Annals of Statistics*, 30(1):140 – 159.
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