Three or More Body Problem

HW 5: Wednesday, Oct. 9, 2019
DUE: Monday, Oct. 21, 2019 (NOTE day change; fall break is Oct. 12(Sat.) to 15(Tues.))
READ: *Numerical Recipes in C++*, Section 16.2 pages 910-920

The numerical calculation of many interacting objects is discussed further in: R.W.Hockney, J.W.Eastwood, *Computer Simulation Using Particles*, McGraw-Hill 1981,1989. This may also be a good source of ideas for your final project

PROBLEM (10 points):

This homework will numerically calculate the behavior of four objects (Sun, Earth, Mars and Jupiter) in the solar system interacting via gravity. For simplicity assume that the planets travel in a 2D xy plane with the coordinate system shown in Fig. 1.

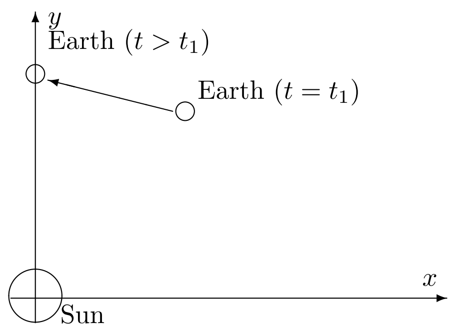

Figure 1: Solar system coordinate system.

The goal will be to test the orbital stability when the orbit of Jupiter is changed. Use a Cartesian coordinate system such that the planets orbit about the Sun in the xy plane (for simplicity, ignore motion in the z direction), with the Sun near the origin. This is approximately true in practice and makes it much easier to graph your results. You should also allow for the center of mass motion of the Sun. The initial conditions for the normal orbits for position (x, y) and velocity (v_x, v_y) in mks units are:

	Earth	Mars	Jupiter	Sun
x	149.598e9	228.0e9	778.298e9	0.0
y	0.0	0.0	0.0	0.0
v_x	0.0	0.0	0.0	0.0
v_y	2.9786e4	2.4127e4	1.30588e4	-3.0e1

The gravitational constant is $G = 6.674 \times 10^{-11}$ Nm^2/kg^2 and other relevant constants of the planets are [1]:

	Earth	Mars	Jupiter	Sun
Mass (kg)	5.9742e24	0.64191e24	1898.8e24	1.9891e30
radius (m)	6.378e6	3.3934e6	7.1398e7	6.96e8

The value for the gravitational constant G has recently been measured to a higher precision[3].

These objects interact under the influence of gravity following the equations below (using mks units and Cartesian coordinates). There are really $2 \times 2 \times N$ coupled equations (where N is the number of interacting objects).

$$\frac{d\vec{v}_i}{dt} = \frac{1}{m_i}\vec{F}_i = G\sum_{j\neq i} m_j \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|^3} \tag{1}$$

$$\frac{d\vec{x}_i}{dt} = \vec{v}_i \tag{2}$$

where i, j=1,2,3,... for each of the objects, and $\vec{x}_i$ and $\vec{v}_i$ are the position and velocity respectively of the i^{th} object. You should map the variables $\vec{v}_i$ and $\vec{x}_i$ into a single array y[i] for use in the Runge-Kutta method. It is best to map them in a systematic way as $x_1, x_2, \cdots, y_1, y_2, \cdots$ (or similar) so that it is easier to program. Try to program this for an arbitrary number (N) of objects, not just this small number of objects. Hint: the main calculation can be done with a pair of embedded for loops and a variable mapping and offset constants such as:

```
y[i+xo]  = x pos. of ith object         xo = 0
y[i+yo]  = y pos. of ith object         yo = N
y[i+vxo] = vx velocity of ith object    vxo = 2*N
y[i+vyo] = vy velocity of ith object    vyo = 3*N
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You should write a general purpose subroutine using the automatic step size Runge-Kutta with the embedded 5th/4th order solution as described in section 17.2 of Numerical Recipes[2] and the Dormand-Prince coefficients (in the table on page 913). To use these coefficient, equation 17.2.5 of the text should really read:

$$y^*_{n+1} = y_n + b^*_1 k_1 + b^*_2 k_2 + \cdots + b^*_6 k_6 + b^*_7 h f(x_n + h, y_{n+1}) + \mathcal{O}(h^5)$$

You may use the function Odeint() and driver StepperDopr5() from Numerical Recipes and the functions it calls or you may write your own version. This code does more than is really necessary and you may find that it is actually easier to write your own than figure out Odeint() including the routines it calls. You should use double precision and you may also want to write the data to a disk file instead of a storing it in a large array. Calculate the trajectories of all objects.

You should first test your program with the normal orbits given above, which should produce closed circular orbits (why?). (You might also test your code with the harmonic oscillator equation to get started). Make an xy graph of these normal orbits for all four objects for a period of 25 years (all four on one graph).

Next reduce the initial orbit of Jupiter by a factor of r=4.8 and increase its velocity by a factor of $\sqrt{r}$ to maintain a circular orbit in a two-body approximation. This is inbetween the Earth's orbit and the orbit of Mars. Produce another xy graph of the trajectories of all four objects. Is this a stable orbit for the Earth?

OPTIONAL-1: try different initial conditions to see what happens.

OPTIONAL-2: Expand the scale near the sun and plot the Sun's motion (mainly due to the pull of Jupiter). This is one way of detecting planets in other solar systems.

References

[1] Jay M. Pasachoff, in: *AIP Physics Desk Reference* E. Richard Cohen, David R. Lide, George J. Trig (editors), (Springer-Verlag, 2003), section 4.1.

[2] W. H. Press, S. A, Teukolsky, W. T. Vetterling and B. P. Flannery *Numerical Recipes, The Art of Scientific Computing, 3rd edit.*, Camb. Univ. Press 2007.

[3] Qing Li, et al, 'Measurements of the gravitational constant using two independent methods', Nature vol. 560 (30 Aug. 2018) p. 582.