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Course: Numerical Analysis
Algorithm: Gauss-Seidel Iterative Method for Linear Systems
Programming Language: Maple (Computer Algebra System)
Purpose: Solve a system of linear equations using an iterative approach that converges to the solution

Definition of Gauss-Seidel Method: The Gauss-Seidel method is an iterative technique for solving a system of linear equations Ax = b, using the most recently computed values to update each variable in sequence

Algorithm Implementation

```
restart;
with(ListTools); with(ArrayTools);
```

System Setup

Number of variables:

$$\mathrm{number\_of\_variables} := 3 \tag{1}$$

Linear System Definition

The system of linear equations is defined with coefficients and solutions:

Equation 1:

$$\mathrm{eq}_1 := x \cdot \mathrm{coefx}_{1,1} + y \cdot \mathrm{coefy}_{1,1} + z \cdot \mathrm{coefz}_{1,1} \tag{2}$$

$$\mathrm{coefx}_{1,1} := 16 \tag{3}$$

$$\mathrm{coefy}_{1,1} := -4 \tag{4}$$

$$\mathrm{coefz}_{1,1} := 0 \tag{5}$$

$$\mathrm{solutioneq}_{1,1} := 2 \tag{6}$$

Equation 2:

$$\mathrm{eq}_2 := -4x + 5y - 2z \tag{7}$$
$$\mathrm{coefx}_{2,2} := -4 \tag{8}$$
$$\mathrm{coefy}_{2,2} := 5 \tag{9}$$
$$\mathrm{coefz}_{2,2} := -2 \tag{10}$$
$$\mathrm{solutioneq}_{2,2} := 4 \tag{11}$$

Equation 3:

$$\mathrm{eq}_3 := -2y + 10z \tag{12}$$
$$\mathrm{coefx}_{3,3} := 0 \tag{13}$$
$$\mathrm{coefy}_{3,3} := -2 \tag{14}$$
$$\mathrm{coefz}_{3,3} := 10 \tag{15}$$
$$\mathrm{solutioneq}_{3,3} := 5 \tag{16}$$

Variable Ordering and Iteration Formulas

The algorithm finds the equations with the largest coefficients for each variable:

X-variable sorting:

$$\mathrm{new\_first\_eq} := 1 \tag{17}$$
$$x_{n+1} := \frac{1}{4} y_n \tag{18}$$

Y-variable sorting:

$$\mathrm{new\_second\_eq} := 2 \tag{19}$$
$$y_{n+1} := \frac{4}{5} x_{n+1} + \frac{2}{5} z \tag{20}$$

Z-variable sorting:

$$\mathrm{new\_third\_eq} := 3 \tag{21}$$
$$z_{n+1} := \frac{1}{5} y_{n+1} \tag{22}$$

Matrix Decomposition

The coefficient matrix A and solution vector b:

$$A = \begin{pmatrix} 16 & -4 & 0 \\ -4 & 5 & -2 \\ 0 & -2 & 10 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \tag{23}$$

Diagonal matrix D:

$$D = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{pmatrix} \tag{24}$$

Upper triangular matrix (original):

$$U_{\text{orig}} = \begin{pmatrix} 16 & -4 & 0 \\ 0 & 5 & -2 \\ 0 & 0 & 10 \end{pmatrix} \tag{25}$$

Lower triangular matrix (original):

$$L_{\text{orig}} = \begin{pmatrix} 16 & 0 & 0 \\ -4 & 5 & 0 \\ 0 & -2 & 10 \end{pmatrix} \tag{26}$$

Modified matrices for Gauss-Seidel:

$$U = -U_{\text{orig}} + D = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{27}$$

$$L = -L_{\text{orig}} + D = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \tag{28}$$

Gauss-Seidel Iteration Matrix

The method uses the iteration matrix equation:

$$T_G = (D - L)^{-1} \cdot U = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{1}{25} & \frac{2}{25} \end{pmatrix} \tag{29}$$

$$c_G = (D - L)^{-1} \cdot b = \begin{pmatrix} \frac{1}{8} \\ \frac{9}{10} \\ \frac{17}{25} \end{pmatrix} \tag{30}$$

Iterative Solution

Starting with initial guess:

$$X_0 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{31}$$

The iterations proceed as follows:

Iteration 1:

$$X_1 = \begin{pmatrix} 0.6250000000 \\ 1.700000000 \\ 0.8400000000 \end{pmatrix} \tag{32}$$

Iteration 2:

$$X_2 = \begin{pmatrix} 0.55 \\ 1.576 \\ 0.8152 \end{pmatrix} \tag{33}$$

Iteration 3:

$$X_3 = \begin{pmatrix} 0.519 \\ 1.54128 \\ 0.8082560000000001 \end{pmatrix} \tag{34}$$

Iteration 4:

$$X_4 = \begin{pmatrix} 0.51032 \\ 1.5315584000000002 \\ 0.8063116800000001 \end{pmatrix} \tag{35}$$

Final Results and Verification

The Gauss-Seidel iterative method has been successfully completed after 4 iterations.

Final Solution Approximation:

$$x \approx 0.510 \tag{36}$$
$$y \approx 1.532 \tag{37}$$
$$z \approx 0.806 \tag{38}$$

Convergence Analysis: The algorithm demonstrates convergence as the successive iterations show decreasing changes:

- Iteration $0 \to 1$: Large changes in all variables
- Iteration $1 \to 2$: Smaller changes, indicating convergence
- Iteration $2 \to 3$: Further reduction in changes
- Iteration $3 \to 4$: Minimal changes, approaching steady state

System Solved: The original system of linear equations:

$$16x - 4y + 0z = 2 \tag{39}$$
$$-4x + 5y - 2z = 4 \tag{40}$$
$$0x - 2y + 10z = 5 \tag{41}$$

Matrix Decomposition Summary: The coefficient matrix A was successfully decomposed into A = D - L - U, where:

- D: Diagonal matrix containing the main diagonal elements
- L: Lower triangular matrix (below diagonal)
- U: Upper triangular matrix (above diagonal)

Conclusion: The Gauss-Seidel method converged efficiently to the solution, demonstrating the effectiveness of iterative methods for solving linear systems. The method is particularly useful for large sparse matrices and provides a computationally efficient alternative to direct methods like Gaussian elimination.

Algorithm Status: COMPLETED SUCCESSFULLY