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Course: Numerical Analysis
Algorithm: Gauss-Seidel Iterative Method for Linear Systems
Programming Language: Maple (Computer Algebra System)
Purpose: Solve a system of linear equations using an iterative approach that converges to the solution

Definition of Gauss-Seidel Method: The Gauss-Seidel method is an iterative technique for solving a system of linear equations Ax = b, using the most recently computed values to update each variable in sequence

Algorithm Implementation

```
restart;
```

1. S is a collection of points in the domain of the function;
2. The user inputs the independent variables in this collection.
3. For example, this dataset will contain four data points.
4. The values in the domain are referred to as nodes.
5. By convention the index begins at 0, thus we will have $k+1$ nodes.
6. If there are four points, then the beginning index is 0 and thus we have 0, 1, 2, 3 points, resulting in $k=3$, thus $k+1$ nodes
7. In order to offset this, we will assume that our index begins at 1 as in the standard natural counting set, thus in the amount of data points we subtract one (In the next line)

$$S := \{0, 1, 2, 3\} \tag{1}$$

N is the amount of data points, $N = |S| - 1$

$$N := 3 \tag{2}$$

These are the observed values corresponding to the domain:

$$x_0 := -4 \tag{3}$$
$$x_1 := -9 \tag{4}$$
$$x_2 := 0 \tag{5}$$
$$x_3 := 7 \tag{6}$$

These are the values that we desire to check against, ensuring that our polynomial also hits these values as a confirmation of the correctness of the algorithm:

$$y_0 := 2 \tag{7}$$
$$y_1 := -10 \tag{8}$$
$$y_2 := 10 \tag{9}$$
$$y_3 := 6 \tag{10}$$

Calculates which index to exclude so that the $j \neq k$ cases below:

$$A_0 := \{1, 2, 3\} \tag{11}$$
$$A_1 := \{0, 2, 3\} \tag{12}$$
$$A_2 := \{0, 1, 3\} \tag{13}$$
$$A_3 := \{0, 1, 2\} \tag{14}$$

The Lagrange basis requires the following condition:

$$P(x_i) = y_i \text{ for all } 0 \leq i \leq n \tag{15}$$

$$L_k = \prod_{\substack{1 \leq j \leq n \\ j \neq k}} \left(\frac{x - x_j}{x_k - x_j} \right) \tag{16}$$

The Lagrange basis polynomials are:

$$L_0(x) := -\frac{\left(\frac{x}{5} + \frac{9}{5}\right) x \left(-\frac{x}{11} + \frac{7}{11}\right)}{4} \tag{17}$$
$$L_1(x) := -\frac{\left(-\frac{x}{5} - \frac{4}{5}\right) x \left(-\frac{x}{16} + \frac{7}{16}\right)}{9} \tag{18}$$
$$L_2(x) := \left(\frac{x}{4} + 1\right) \left(\frac{x}{9} + 1\right) \left(-\frac{x}{7} + 1\right) \tag{19}$$
$$L_3(x) := \frac{\left(\frac{x}{11} + \frac{4}{11}\right) \left(\frac{x}{16} + \frac{9}{16}\right) x}{7} \tag{20}$$

The Lagrange interpolating polynomial for those nodes through the corresponding values $\{y_0, y_1, \ldots, y_k\}$ is the linear combination:

$$P(x) = \sum_{k=0}^{n} y_k \cdot L_k(x) \tag{21}$$

The individual polynomial components are:

$$P_0 := -\frac{\left(\frac{x}{5} + \frac{9}{5}\right) x \left(-\frac{x}{11} + \frac{7}{11}\right)}{2} \tag{22}$$

$$P_1 := \frac{10 \left(-\frac{x}{5} - \frac{4}{5}\right) x \left(-\frac{x}{16} + \frac{7}{16}\right)}{9} \tag{23}$$

$$P_2 := 10 \left(\frac{x}{4} + 1\right) \left(\frac{x}{9} + 1\right) \left(-\frac{x}{7} + 1\right) \tag{24}$$

$$P_3 := \frac{6 \left(\frac{x}{11} + \frac{4}{11}\right) \left(\frac{x}{16} + \frac{9}{16}\right) x}{7} \tag{25}$$

Recalling the Nth index, which is the same as kth index via the calculation from the second line, now invoke the sum of the polynomials just created above:

$$\sum_{r=0}^{N} P_r = -\frac{\left(\frac{x}{5} + \frac{9}{5}\right) x \left(-\frac{x}{11} + \frac{7}{11}\right)}{2} + \frac{10 \left(-\frac{x}{5} - \frac{4}{5}\right) x \left(-\frac{x}{16} + \frac{7}{16}\right)}{9} + 10 \left(\frac{x}{4} + 1\right) \left(\frac{x}{9} + 1\right) \left(-\frac{x}{7} + 1\right) + \frac{6 \left(\frac{x}{11} + \frac{4}{11}\right) \left(\frac{x}{16} + \frac{9}{16}\right) x}{7} \tag{26}$$

Final Result

FORMAT THE INTERPOLATING POLYNOMIAL:

$$P(x) = 10 - \frac{41}{3465} x^3 - \frac{229}{1155} x^2 + \frac{4838}{3465} x \tag{27}$$

The polynomial in standard form with coefficients converted to decimal approximation of 10 significant figures:

$$P(x) = 10.0 - 0.01183261183 x^3 - 0.1982683983 x^2 + 1.396248196 x \tag{28}$$

Verification

Check if values in our (x, y) pairs are correct by evaluating $P(x_i)$:

$$P(x_0) = P(-4) = 2 \tag{29}$$

$$P(x_1) = P(-9) = -10 \tag{30}$$

$$P(x_2) = P(0) = 10 \tag{31}$$

$$P(x_3) = P(7) = 6 \tag{32}$$

The Lagrange polynomial (Curve Fitting) algorithm has been completed successfully. The interpolating polynomial correctly passes through all given data points.