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BACHELOR OF SCIENCE IN MATHEMATICS

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Course: Numerical Analysis
Algorithm: QR Factorization (Householder Reflections)
Programming Language: Maple (Computer Algebra System)
Purpose: Decompose a matrix A into the product of an orthogonal matrix Q and an upper triangular matrix R

Definition of QR Factorization: A factorization of a matrix A into a product A = QR where Q is an orthogonal matrix and R is an upper triangular matrix

Algorithm Implementation

```
restart;
with(LinearAlgebra);
```

Matrix Input

QR Factorization Problem - Given Matrix, enter the data in the table:
Number of rows/columns:

$$N := 4 \tag{1}$$

Input Matrix Definition

	COLUMN 1	COLUMN 2	COLUMN 3	COLUMN 4
ROW 1	$n_{1,1} := -5$	$n_{1,2} := 2$	$n_{1,3} := -5$	$n_{1,4} := -5$
ROW 2	$n_{2,1} := -3$	$n_{2,2} := 5$	$n_{2,3} := -3$	$n_{2,4} := 2$
ROW 3	$n_{3,1} := -5$	$n_{3,2} := 5$	$n_{3,3} := 1$	$n_{3,4} := 1$
ROW 4	$n_{4,1} := 4$	$n_{4,2} := -2$	$n_{4,3} := 4$	$n_{4,4} := 4$

The initial matrix A is:

$$A_0 = \begin{pmatrix} -5 & 2 & -5 & -5 \\ -3 & 5 & -3 & 2 \\ -5 & 5 & 1 & 1 \\ 4 & -2 & 4 & 4 \end{pmatrix} \tag{2}$$

Householder Reflection Process

The algorithm proceeds by constructing column vectors and computing their norms:

For each column g from 1 to N:

$$v_g = [n_{1,g}, n_{2,g}, n_{3,g}, n_{4,g}] \tag{3}$$

$$\|v_g\| = \sqrt{|n_{1,g}|^2 + |n_{2,g}|^2 + |n_{3,g}|^2 + |n_{4,g}|^2} \tag{4}$$

First Householder Reflection (P)

For the first column: - Determine sign of α based on the first element - If $v_1[1] > 0$, then $\alpha = -\|v_1\|$, else $\alpha = \|v_1\|$ - Construct identity vector h and compute $b = \alpha \cdot h$ - Calculate reflection vector: $u_1 = v_1 - c$ - Form Householder matrix: $P_1 = I - \frac{2(u_1 u_1^T)}{u_1^T u_1}$

The computed P_1 matrix is displayed with 15 significant figures.

P computation completed

Second Householder Reflection (P)

Apply P_1 to get $A_1 = P_1 \cdot A_0$

Extract vector from column 2 of A_1: - For i from 1 to 4: $m_i = A_1[i][2]$ - Compute norm: $\|q\|_2 = \sqrt{\sum_{i=2}^{N} |q_i|^2}$ - Determine α sign based on m_2 - If $m_2 > 0$, then $\alpha = -\|q\|_2$, else $\alpha = \|q\|_2$ - Construct vectors: $q = [m_1, m_2, m_3, m_4]$ and $p = [m_1, \alpha, 0, 0]$ - Calculate reflection: $u = q - p$ - Form Householder matrix: $P_2 = I - \frac{2(u \cdot u^T)}{u^T \cdot u}$

The computed P_2 matrix is displayed with full precision.

P computation completed

Third Householder Reflection (P)

Apply previous transformations: $A_2 = P_2 \cdot P_1 \cdot A_0$

Extract vector from column 3 of A_2: - For i from 1 to 4: $m_i = A_2[i][3]$ - Compute norm: $\|q\|_3 = \sqrt{\sum_{i=3}^{N} |q_i|^2}$ - Determine α sign based on m_3 - If $m_3 > 0$, then $\alpha = -\|q\|_3$, else $\alpha = \|q\|_3$ - Construct vectors: $q = [m_1, m_2, m_3, m_4]$ and $p = [m_1, m_2, \alpha, 0]$ - Calculate reflection: $u = q - p$ - Form Householder matrix: $P_3 = I - \frac{2(u \cdot u^T)}{u^T \cdot u}$

Final transformation: $A_3 = P_3 \cdot P_2 \cdot P_1 \cdot A_0$

The computed P_3 matrix is displayed with full precision.

Numerical Results

The algorithm outputs the following matrices with 15 significant figures:

P Matrix:

$$P_1 = \text{(15 significant figure numerical result from Maple)} \tag{5}$$

P Matrix:

$$P_2 = \text{(15 significant figure numerical result from Maple)} \tag{6}$$

P Matrix:

$$P_3 = \text{(15 significant figure numerical result from Maple)} \tag{7}$$

R Matrix (Upper Triangular):

$$R = \text{(15 significant figure numerical result from Maple)} \tag{8}$$

Note: Small numerical values in R are replaced with 0 for clarity as specified in the original algorithm.

Final Results

The algorithm outputs the following matrices with detailed numerical precision:

P Matrix (15 significant figures): The first Householder reflection matrix computed by the algorithm.

P Matrix (15 significant figures): The second Householder reflection matrix computed by the algorithm.

P Matrix (15 significant figures): The third Householder reflection matrix computed by the algorithm.

R Matrix (15 significant figures): The final upper triangular matrix where small numerical values are replaced with 0 for clarity.

The Maple algorithm displays all matrices using:

```
print("P1 IS ", evalf[15](P[1]));
print("P2 IS ", evalf[15](P[2]));
print("P3 IS ", evalf[15](P[3]));
print("R IS ", evalf[15](R));
```

QR Decomposition Result

The QR factorization is complete where:

$$Q = P_1 \cdot P_2 \cdot P_3 \tag{9}$$
$$A = Q \cdot R \tag{10}$$

Verification

The algorithm computes the Householder reflections sequentially to transform the original matrix into upper triangular form. Each reflection eliminates the elements below the diagonal in the current column while preserving the upper triangular structure of previous columns.

Applications and Uses

QR decomposition has several important applications in numerical linear algebra:

Linear System Solving: For overdetermined systems $Ax = b$, QR decomposition provides a stable method to solve $Rx = Q^T b$ where R is upper triangular.

Least Squares Problems: QR decomposition is the standard method for solving linear least squares problems $\min\|Ax - b\|_2$.

Eigenvalue Computation: The QR algorithm uses repeated QR decompositions to find eigenvalues of matrices.

Gram-Schmidt Orthogonalization: QR decomposition provides a numerically stable alternative to the classical Gram-Schmidt process.

Matrix Rank Determination: The rank of matrix A can be determined by examining the diagonal elements of R.

Conclusion

The QR factorization using Householder reflections has been completed successfully. This method provides a numerically stable way to decompose a matrix into the product of an orthogonal matrix Q and an upper triangular matrix R. Among the many matrix decompositions in Linear Algebra (LU, SVD, Cholesky, etc.), QR decomposition stands out for its numerical stability and wide range of applications, making it essential for solving linear systems, least squares problems, and eigenvalue computations.

Algorithm Status: COMPLETED SUCCESSFULLY