<u>**Mathematical Exploration of Temperature within a Thermoflask**</u>

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Math Analysis and Approaches SL

Exam Session: May 2024

Page Number: 20

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Introduction and Rationale:

Over the summer, I played on two ultimate frisbee teams where I constantly felt sweating and dehydrated from exhaustion and the heat, which had reached an all-time high this summer. This was especially detrimental when I played in really important tournaments, playing 7 games over 2 days, in which my physical health was essential to endure the whole tournament. Being dehydrated resulted in me often experiencing muscle cramps and heat exhaustion during games with one time going so far as to make me faint in the middle of a point at Nationals. Therefore, to ensure I was well hydrated, I needed to find a way to keep my water cool for a whole day which led me to buy a Thermoflask. I was intrigued by the branding of a Thermoflask, explaining how it would "keep your drink cold for up to 24 hours [with its] double-wall vacuum insulated stainless steel bottles [which] prevent[s] outside temperatures from affecting the inside beverage temperature so your drink stays just the way you like it – all day long" (mythermoflask.com). However, upon usage, I noticed that the water would become much warmer than I expected, considering the Thermoflask's branding. Thus, I decided to explore this idea in this investigation, by mathematically exploring temperature as a function of time within my Thermoflask. This would help me determine the optimal period in which the water inside a Thermoflask remains cool enough to satisfy my thirst. Satisfying my thirst meant that the water temperature needed to be below 16°C, since, according to a study by the National Library of Medicine, the optimal water temperature to minimize sweating for athletes was 16°C or lower. (Hosseinlou et al, 2013)

Aim and Methodology:

My aim of this investigation is to mathematically determine how temperature can change as a function of time within a Thermoflask. By considering the dimensions of my Thermoflask in relation to my knowledge of heat transfer I explored in Physics, I hope to use these mathematical concepts to derive equations to optimize when I want to refill my water throughout the day so that it remains a temperature to satisfy my thirst. Specifically, I will use the equation of heat transfer by conduction, with my knowledge of the cylindrical dimensions of my Thermoflask and relate these to temperature using the general heat capacity equation to derive a plausible relationship. Then I will apply this to real-life scenarios I faced of how water heats up in my Thermoflask using a graph to determine the optimal period in which I should be drinking and refilling my water bottle.

Investigating Heat Transfer:

To understand temperature as a function of time, I first need to understand heat transfer. Heat is the energy transferred between systems when they are at different temperatures. Heat typically travels from higher temperatures to lower temperatures which is the case of this investigation. A system can transfer heat to another system in three different ways: conduction (heat transfer through contact), convection (heat transfer by liquid or gas), and radiation (heat transfer without contact with anything). (Douglas College) For this IA, I will solely focus on conduction as the water warming inside the bottle is mostly caused by the heat from the insulating material/the bottle and the other forms of heat transfer are negligible. The change in heat per change in time can be calculated using the equation below and each variable will be defined on the following page:

$$\frac{\Delta Q}{\Delta t} = \frac{kA\Delta T}{L}$$

As I want to represent the infinitesimal rate of change in each point of both heat and time, I will denote the equation as such. Definitions of variables are on the following page.

$$\frac{dQ}{dt} = \frac{kA\Delta T}{L} \tag{1}$$

Table 1: Key variables associated with the thermal conductivity equation defined and showing units used

Variable	Definition	Units
dQ	The change in heat	Joules (J)
dt	The change in time	Seconds (s)
k	The conductivity constant, dependent on the material	$(\frac{J}{s \cdot cm \cdot °C})$
A	Area of the material in contact which both systems	Centimeters squared (cm^2)
ΔT	Difference in temperature of the two systems	Degrees Celsius (°C)
L	Length of the material through which heat is transferred	Centimeters (cm)

The units of this equation will be used throughout this investigation, so I will use unit conversion if any value does not match the values shown in Table 1. The factors influencing the rate of heat flow can be more easily understood visually in Figure 1 below:

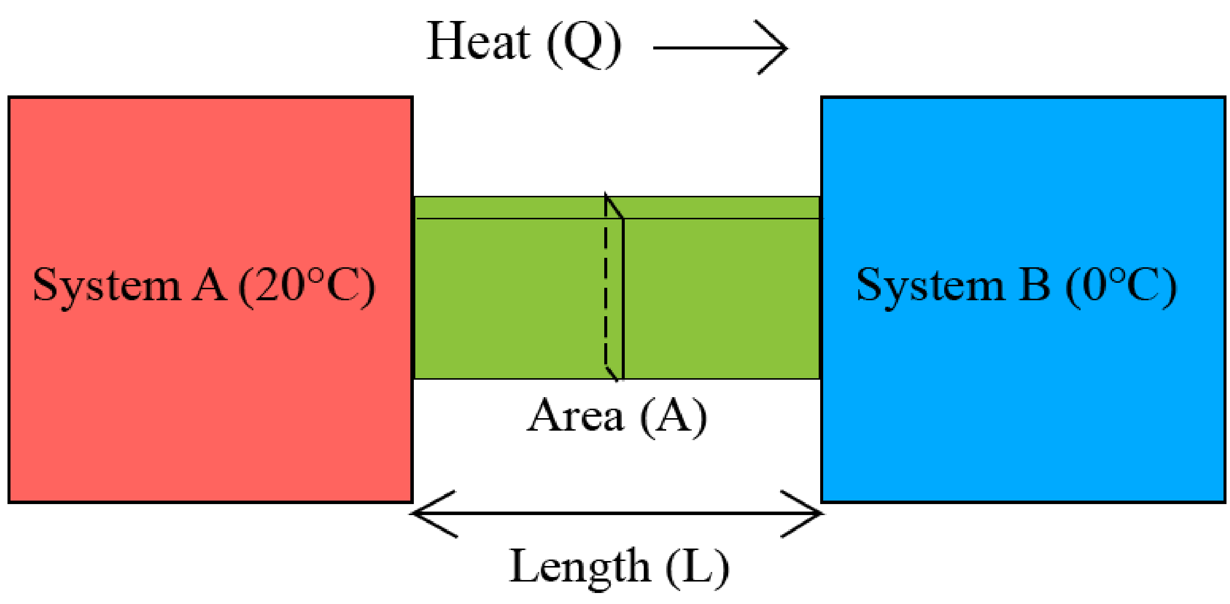

Figure 1: Visual representation of factors relating heat to time (Created by me)

Assessing the Cylindrical Shape:

Due to the nature of the cylindrical shape of the Thermoflask, I will modify the heat transfer by conduction equation to account for this. The area (A) represents the area in contact with both systems. Since my Thermoflask is cylindrical, this area would represent the surface area of the inside of the Thermoflask in contact with water.

Figure 2: Image of my Thermoflask

Notice the irregular shape of the Thermoflask at the bottom and the top of the Thermoflask. For the purpose of this exploration, I will need to make some assumptions. Firstly, I will not be considering the top portion of the Thermoflask, including the lid and the rounded portion at the top as I usually don't fill up water up to those points so the water will not be in contact. Additionally, the area reduced by the rounded edge at the bottom will be neglected to avoid overcomplicating this investigation. Thus, I can determine the surface area by taking the area of the circular portion at the bottom of the Thermoflask, denoted by πr_i^2, and adding the surface area of the sides of the Thermoflask, denoted by $2\pi r_i h$. This can be represented in the equation below.

$$A = 2\pi r_i h + \pi r_i^2 \tag{2}$$

These variables are visually represented in Figure 3 and defined in Table 2 below:

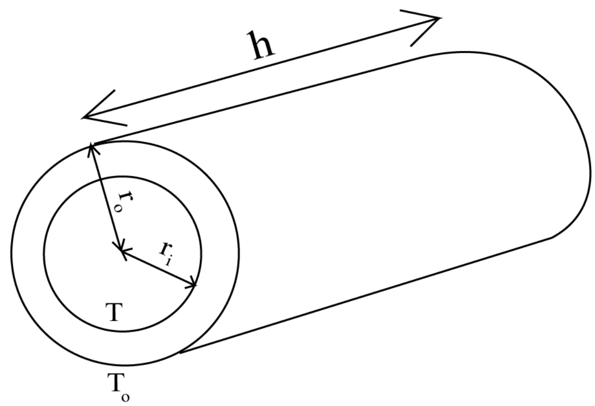

Figure 3: Key variables associated with the following steps visually represented (Created by me)

Table 2: Key variables associated with the cylinder surface area equation and other notable variables defined and showing units used

Variable	Definition	Units
A	The change in heat	Joules (J)
r_i	The radius of the inside of the Thermoflask	Centimeters (cm)
r_o	The radius of the outside of the Thermoflask	Centimeters (cm)
h	The height of the part of the Thermoflask I am measuring, see figure 2	Centimeters (cm)
T_o	The temperature of the outside of the Thermoflask	Degrees Celsius (°C)
T	The temperature of the inside of the Thermoflask, since I am determining this Temperature as a function of time, I will represent this as T without a subscript.	Degrees Celsius (°C)

Now that I have what is needed to determine the required surface area, I can substitute Equation 2 ($A = 2\pi r_i h + \pi r_i^2$) into Equation 1.

$$\frac{dQ}{dt} = \frac{k(2\pi r_i h + \pi r_i^2)\Delta T}{L}$$

Notice in Figure 2 and Table 3 the variables T and T_o. ΔT in the equation is the change of temperature from the hot system to the cold, which in this case is the change of the outside temperature (T_o) to the inside (T) so I can substitute that in as well.

$$\frac{dQ}{dt} = \frac{k(2\pi r_i h + \pi r_i^2)(T_o - T)}{L}$$

Additionally, from Figure 1, the Length in the equation was defined as the length of the material through which heat is transferred. The length, albeit very small, is the difference in the outside radius to the inside:

$$L = r_o - r_i$$

The length represents the thickness of the Thermoflask since that is the material through which heat is transferred. When substituted in, I derive the equation:

$$\frac{dQ}{dt} = \frac{k(2\pi r_i h + \pi r_i^2)(T_o - T)}{(r_o - r_i)} \tag{3}$$

Relating to Temperature:

Temperature directly relates to heat since heat is the total kinetic energy of a system while temperature is the average kinetic energy of the particles within the system. This relationship can be represented using the heat capacity equation below, displaying the amount of heat gained or lost by a system by a change in temperature.

$$\Delta Q = mc\Delta T$$

Since I want to only determine the Heat and Temperature at one point of the function, according to first principles, I will change ΔQ to dQ and ΔT to dT:

$$dQ = mcdT \quad (4)$$

Note that this equation only considers the change in heat of water (dT), which is the inside system, since it is not related to heat transfer. This is not the temperature change of the outside system to the inside system which was denoted in equation 1, $(T_o - T)$.

Table 3: Key variables associated with the heat capacity equation defined and showing units used

Variable	Definition	Units
dQ	The change in heat	Joules (J)
m	The mass of the water	Grams (g)
c	The specific heat capacity of water which is 4.186 (Energy Education)	$\frac{J}{g°C}$
dT	The change in temperature of water. Note that this value is not the same as the value in Table 1 which displays the difference in temperature between systems while this only displays 1 system.	Degrees Celsius (°C)

I can now combine Equation 3 and Equation 4 as follows. The dQ term is identical in both equations, representing the change in heat of the water system, so I can substitute $dQ = mcdT$ into equation 3 to get

$$mc\frac{dT}{dt} = \frac{k(2\pi r_i h + \pi r_i^2)(T_o - T)}{(r_o - r_i)}$$

This equation is a first-order differential equation and I can use the separation of variables method to solve this equation. By isolating temperature variables on one side and time variables on another, I can then integrate both sides to determine a final function. All other variables including surface area, length, mass and specific heat capacity are constant so they do not have an impact when I integrate both sides.

By rearranging the equation, I can isolate temperature variables to look like so:

$$\frac{dT}{(T_o - T)} = \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} dt$$

Now I can integrate both sides to solve for the derivative on both sides of this differential equation noting that the value of $\frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)}$ is a constant therefore it can be taken out of the integral.

$$\int \frac{dT}{(T_o - T)} = \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} \int dt$$

Since I want to produce an equation with a definite integral instead of an indefinite one with constants that I won't be able to find, I will be setting upper and lower limits corresponding to the final and initial time and temperature. The upper limit of the integral of Temperature will be T since that is what I want to find. The lower limit will be set to T_i, representing the initial temperature of the water. The upper limit of the integral of the Time will be set to t since that is the variable I want to find. The lower limit will be set to 0 since the time of temperature change will start at zero. Hence I get the equation:

$$\int_{T_i}^{T} \frac{dT}{(T_o - T)} = \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} \int_0^t dt$$

Notice the integral on the left side of the equation is a reciprocal. However, it is a composite function since $(T_o - T)$ is also a function inside the original reciprocal function. Thus, I will use u-substitution to evaluate this integral by setting one function equal to u:

u = $(T_o - T)$.

Now to evaluate the dT, I need to find a way to convert dT into du. To do this I can take the derivative of the function u = $(T_o - T)$. Assuming the outside temperature does not change, T_o is a constant and the derivative of T is just the coefficient which is -1. Thus, I can evaluate the derivative of this function as.

$$\frac{du}{dT} = -1$$

I can now rearrange the equations to solve for dT.

$$dT = -du$$

It is important to note that when I substitute the u value for T, the limits also change to represent u and not T. Thus, by inserting the original upper limit (T) and lower limit (T_i) into the $u = (T_o - T)$ equation, I get the upper limit $(T_o - T)$ and the lower limit $(T_o - T_i)$.

Substituting the u value, the du value and the new limits into the original equation I get:

$$\int_{T_o - T_i}^{T_o - T} \frac{-1}{u} du = \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} \int_0^t dt$$

Since -1 is a constant, I can take it out of the integral. Knowing the general formula $\int \frac{1}{x} dx \; = \; ln|x| + \; c$, I can apply this to this portion of the equation to solve for the integral:

$$- \; [ln|u|]_{T_o - T_i}^{T_o - T} \; = \; \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} \int_0^t dt$$

Since there is no time variable in the right-hand portion of the equation, I can use integration rules to integrate this portion.

$$- \; [ln|u|]_{T_o - T_i}^{T_o - T} \; = \; \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} [t]_0^t$$

I will now substitute the upper and lower limit in the integrals to solve the definite integrals.

$$- \left[(ln\left|T_o \; - \; T\right|) \; - \; (ln\left|T_o \; - \; T_i\right|) \right] \; = \; \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} [t \; - \; 0]$$

Now since I am trying to find Temperature as a function of time, I want to isolate T by first multiplying both sides by -1.

$$(ln\left|T_o \; - \; T\right|) \; - \; (ln\left|T_o \; - \; T_i\right|) \; =- \; \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} [t]$$

Now I will move the constant $(ln\left|T_o \; - \; T_i\right|)$ to the right side of the equation.

$$(ln\left|T_o \; - \; T\right|) \; =- \; \frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)} [t] + \; (ln\left|T_o \; - \; T_i\right|)$$

I will now raise e to the power of the right side of the equation to evaluate the natural logarithm on the left side of the equation.

$$T_o - T = e^{-\frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)}[t] + (ln|T_o - T_i|)}$$

Finally, by moving T_o to the right side and multiplying by -1 to remove the negative from the T value, I derive the final equation representing the relationship between Temperature and Time:

$$T = -e^{-\frac{k(2\pi r_i h + \pi r_i^2)}{mc(r_o - r_i)}[t] + (ln|T_o - T_i|)} + T_o \quad (5)$$

Application to Real Life:

I will now apply this equation to a real-life scenario, with values corresponding to the measurements I took on my Thermoflask and Temperature values I took from my experiences over the summer. This will allow me to graph Temperature as a function of time specifically for my Thermoflask.

Table 4: Parameters I set to determine the Temperature and Time function

Variable	Quantity	Justification
Outside Temperature (T_o)	30.0 °C	Outside temperature fluctuates, but I generally played ultimate in the summer at around 30 Degrees Celsius
Initial Inside Temperature (T_i)	0 °C	Theoretically, this should be ice, but for mathematical purposes, I chose to start the water temperature as cold as I could.
Conductivity Constant (k)	0.00005475 $\frac{J}{s \cdot cm \cdot °C}$	The conductivity constant of the Thermoflask (Ajmera et al, 2015)
Outside Radius (r_o)	4.55 cm	Measured using a ruler

Inside Radius (r_i)	4.35 cm	The thickness of the walls is 2mm (Kingstar, 2021) so I subtracted that from the outside radius
Height (h)	20.0 cm	The height of the bottom to the top of the cylindrical portion of the Thermoflask
Mass of Water (m)	1000 g	The mass and volume of water are the same due to density and the water fills to 1 litre so 1000 grams is appropriate
Specific Heat Capacity of Water (c)	4.186 $\frac{J}{g°C}$	The Specific Heat Capacity of water is 4.186 $\frac{J}{g°C}$ (Energy Education)

Substituting the values from Table 4 into Equation 5, I get this equation:

$$T = -e^{-\frac{(0.00005475)[2\pi(4.35)(20.0)+\pi(4.35)^2]}{(1000)(4.186)(4.55-4.35)}[t]+(ln|30.0-0|)} + 30.0$$

I will now evaluate this equation using technology to obtain my final equation:

$$T \approx -e^{-0.0000396t + 3.40} + 30.0 \quad (6)$$

The notation of approximately equal represents that there are more decimal places to some of the values in the equation, but I rounded to 3 significant figures to save space. The exact values will be used when graphing.

Solving and Graphing the Equation:

Linking back to the purpose of this exploration, I wanted to determine the optimal period of time to drink the water in my Thermoflask so that it may satisfy my thirst, which I determined was 16 °C. To solve for time, I will set T equal to 16 °C in equation 6.

$$16.0 \approx -e^{-0.0000396t + 3.40} + 30.0$$

I can then rearrange the variables to isolate the e value on the right side.

$$14.0 \approx e^{-0.0000396t + 3.40}$$

Taking the natural logarithm of both sides, I can take the variable t out of the exponent.

$$ln(14.0) = -\ 0.0000396t\ +\ 3.40$$

Now rearranging the variables again to isolate t, I get the final equation to solve for time.

$$t \approx -\frac{ln(14.0) - 3.40}{0.0000396}$$

Using technology, I can solve this equation and determine this period of time. This solved equation used the precise values instead of the rounded values to be more accurate.

$$t \approx 19228.578\text{ s}$$

To compare my result with what was advertised by the Thermoflask branding, I will use unit conversion to change seconds to hours.

$$19228.578s \cdot \frac{1min}{60s} \cdot \frac{1hr}{60min} \approx 5.34\ hr \text{ (rounded to 3 significant figures)}$$

Thus, the optimal period of drinking water after my water bottle has been filled is around 5.34 hours. Using graphing technology, I can further visualize how the temperature in my Thermoflask changes as a result of time as shown on the following page.

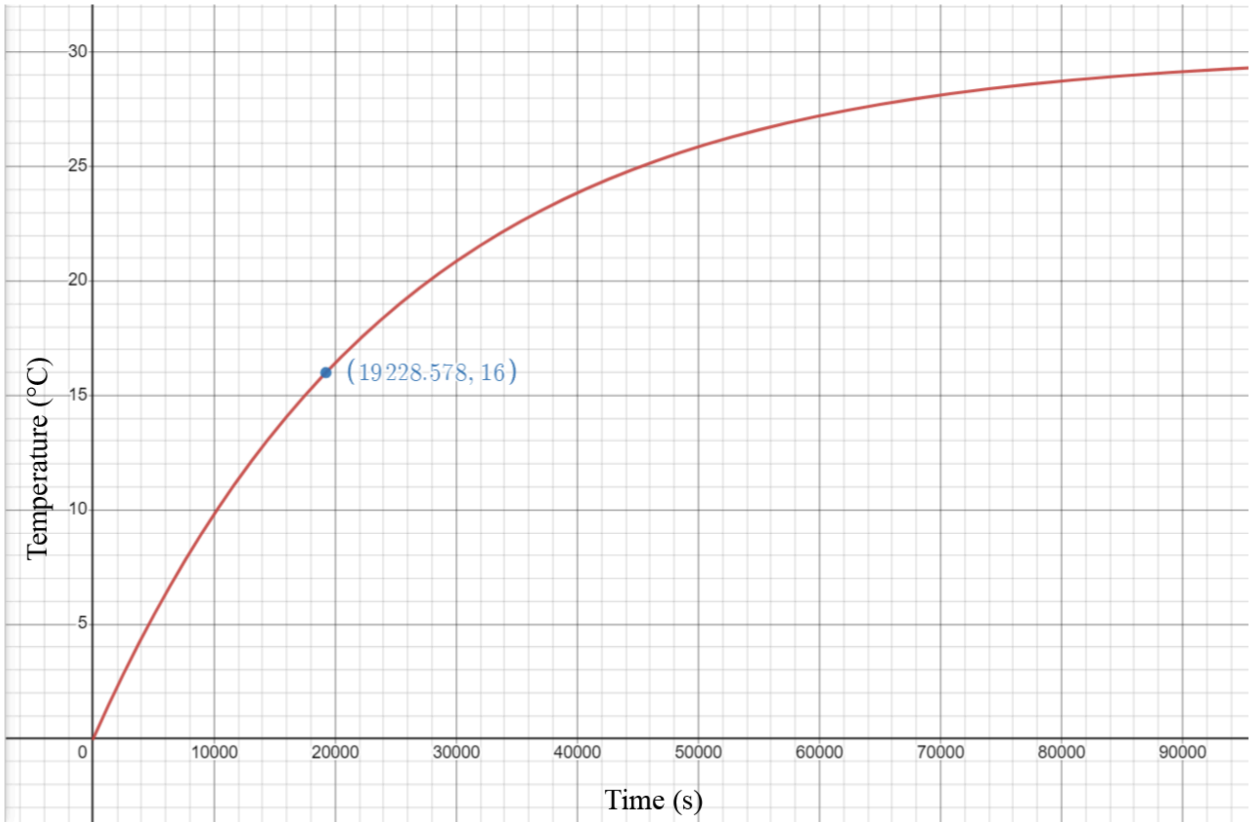

Figure 4: Graph of Temperature as a function of Time (Desmos)

Notice the exponential curve of the graph. This demonstrates how as water gets closer to the outside temperature, the rate of heat flow slows. This also leaves room for interpretation, as Thermoflask's branding is extremely vague, so determining an optimal temperature for water consumption can be extremely different depending on what measures are put in place. Nonetheless, this graph also shows that the rate of getting towards the optimal drinking temperature (16 °C) is still pretty fast as it is near the beginning of the exponential function. Thus, if I set the outside temperature lower than 30 °C, this can slow down the heating rate of the water.

Evaluation and Conclusion:

After evaluating how temperature within my Thermoflask changes as a function of time, I can now say that I was correct when I noticed that the temperature increased much faster than I anticipated given the Thermoflask's branding. The temperature became 16°C after only 5.34 hours which is way faster than the advertised 24-hour period it was supposed to remain cold. There are of course limitations to this mathematical investigation. Notably throughout this exploration, I made the following assumptions:

1. Only heat transfer by conduction is present within the water inside the Thermoflask.
2. The water will not be in contact with the upper portion of the Thermoflask to avoid modelling that surface area.
3. The bevelled edge on the bottom of the Thermoflask would not affect the surface area.
4. The amount of water will not change throughout the heat transfer.
5. The outside temperature would not change throughout the investigation.

These assumptions can cause a drastic change in the results of this investigation. There are alternative mathematical approaches to remove these assumptions such as using Newton's Law of Cooling with data collection to determine a more experimental model, modeling the surface area completely in contact with water or determining temperature as a function of position in the thermoflask to account for the amount of water. However, these approaches are over-complicated for the purpose of this exploration. This investigation proves valuable in gaining a sense of how the temperature of water changes as a function of time which will be helpful to know next time I am preparing for an ultimate frisbee tournament or a hot day in the summer and has ultimately met this aim. So next time I fill my Thermoflask with cold water in preparation for being in the heat for a prolonged period of time, I will note the water is only optimally hydrating for 5.34 hours.

Reflection:

Through this investigation, I developed my understanding of calculus by being able to apply my knowledge of differentiation to real-life scenarios. By being faced with the challenge of solving a differential equation, something not taught in the IB Math AA SL syllabus, I took it upon myself to learn this concept, gaining a more profound understanding of the intricacies behind calculus. Additionally, I learned different ways to integrate functions, learning that I can solve first-order differential equations by integrating both sides of the equation. This also led me to learn that although I am taught that $\frac{dy}{dx}$ is not a fraction, I can separate dx and dy as long as the evaluation of these variables remains in the domain of calculus. Furthermore, I deepened my understanding of integration through my evaluation of the definite integrals when solving my differential equation. I was faced with needing to find the integral of a composite function, something not taught in the syllabus, so I learned how to use u-substitution with definite integrals to determine a solution. Finally, through graphing my final function, I was able to learn how to more effectively use graphing technology, as I needed to set limits on the scaling due to the large time axis. This also allowed me to gain a deeper understanding of exponential functions as when I attempted different configurations of the function, I learned how the graph transformed as a result.

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