Deformations of $\mathbb{Z}_2$-Harmonic Spinors on 3-Manifolds

Gregory J. Parker

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Abstract

A $\mathbb{Z}_2$-harmonic spinor on a 3-manifold Y is a solution of the Dirac equation on a bundle that is twisted around a submanifold $\mathcal{Z}$ of codimension 2 called the singular set. This article investigates the local structure of the universal moduli space of $\mathbb{Z}_2$-harmonic spinors over the space of parameters (g, B) consisting of a metric and perturbation to the spin connection. The main result states that near a $\mathbb{Z}_2$-harmonic spinor with $\mathcal{Z}$ smooth, the universal moduli space projects to a codimension 1 submanifold in the space of parameters. The analysis is complicated by the presence of an infinite-dimensional obstruction bundle and a loss of regularity in the first variation of the Dirac operator with respect to deformations of the singular set $\mathcal{Z}$, necessitating the use of the Nash-Moser Implicit Function Theorem.

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1 Introduction

The notion of a $\mathbb{Z}_2$-harmonic spinor was introduced by C. Taubes to describe the limits of renormalized sequences of solutions to generalized Seiberg-Witten equations. $\mathbb{Z}_2$-harmonic spinors are also the simplest type of Fueter section, and are therefore of interest in the study of gauge theories and enumerative theories on manifolds with special holonomy. Beyond these connections, $\mathbb{Z}_2$-harmonic spinors are intrinsic objects on low-dimensional manifolds and can be studied without reference to any of these theories.

This article investigates the local structure of the universal moduli space of $\mathbb{Z}_2$-harmonic spinors over the space of parameters on a compact 3-manifold. The main result states that this universal moduli space locally projects to a codimension 1 submanifold, i.e. a "wall", in the space of parameters. This provides a key step toward confirming expectations that $\mathbb{Z}_2$-harmonic spinors should enter into the above theories via wall-crossing formulas. Results in this direction have also been obtained by R. Takahashi using different techniques [49]. The present work grew out of attempts to develop a more robust analytic framework for these results, with an eye towards applications to gluing problems [42] and other deformation problems. As observed by S. Donaldson [9], the same analytic issues appear in many distinct geometric contexts, most of which remain unexplored [26].

1.1 Main Results

Let (Y, g) be a closed, oriented, Riemannian 3-manifold, and fix a spin structure with spinor bundle $S \to Y$. Given a closed submanifold $\mathcal{Z} \subset Y$ of codimension 2, choose a real line bundle $\ell \to Y - \mathcal{Z}$. The spinor bundle $S \otimes_{\mathbb{R}} \ell$ carries a Dirac operator denoted $\not{D}_{\mathcal{Z}}$ formed from the spin connection and the unique flat connection on ℓ with holonomy in $\mathbb{Z}_2$.

A $\mathbb{Z}_2$**-harmonic spinor** is a solution $\Phi \in \Gamma(S \otimes_{\mathbb{R}} \ell)$ of the twisted Dirac equation on $Y - \mathcal{Z}$ satisfying

$$\not{D}_{\mathcal{Z}}\Phi = 0 \qquad \text{and} \qquad \nabla\Phi \in L^2. \tag{1.1}$$

The submanifold $\mathcal{Z}$ is called the **singular set**. When $\mathcal{Z}$ has sufficient regularity, the latter requirement implies that $|\Phi|$ extends continuously to the closed manifold Y with $\mathcal{Z} \subseteq |\Phi|^{-1}(0)$. The existence (and abundance) of $\mathbb{Z}_2$-harmonic spinors with $\mathcal{Z} \neq \emptyset$ on closed 3-manifolds with $b_1 > 1$ was established by Doan–Walpuski in [8].

The definition of the Dirac operator relies on a background choice of a Riemannian metric g on Y and possibly a perturbation B to the spin-connection. Let $\mathcal{P} = \{(g, B)\}$ denote the parameter space of possible choices. Given a pair (g_0, B_0) and a $\mathbb{Z}_2$-harmonic spinor $(\mathcal{Z}_0, \ell_0, \Phi_0)$ with respect to this pair, the goal of the present work is to study the local deformation problem, i.e. to describe the structure of the set of nearby pairs $(g, B) \in \mathcal{P}$ for which there exists a $\mathbb{Z}_2$-harmonic spinor.

This problem cannot be addressed with the standard elliptic theory used for classical harmonic spinors [27, 32]. If ℓ has a non-trivial twist around $\mathcal{Z}_0$, the Dirac operator $\not{D}_{\mathcal{Z}_0}$ is degenerate along the singular set $\mathcal{Z}_0$ and is therefore not a uniformly elliptic operator on a closed manifold. Instead, it is an **elliptic edge operator** – a class of operators well-studied in microlocal analysis [34, 37, 45]. For such operators, elliptic regularity fails and the extension to Sobolev spaces need not be Fredholm. In particular, for natural function spaces $\not{D}_{\mathcal{Z}_0}$ possesses an infinite-dimensional cokernel. As a result, the problem of deforming a solution to a solution for a nearby parameter seemingly carries an infinite-dimensional obstruction. The following key idea, first described by Takahshi in [49], addresses this issue.

Key Idea: the infinite-dimensional obstruction is cancelled by deformations of the singular set $\mathcal{Z}$.

Since the Dirac equation $\not{D}_{\mathcal{Z}}$ depends on $\mathcal{Z}$, but $\mathcal{Z}$ is in turn determined by the vanishing of the norm $|\Phi|$ of a spinor solving (1.1), the singular set and the spinor are coupled and must be solved for simultaneously. The problem thus has a similar character to a free-boundary-value problem, where the domain and solution must be found concurrently, though the "boundary" here has codimension 2. In particular, this analysis requires an understanding of the derivative of the Dirac operator with respect to deformations of the singular set $\mathcal{Z}$.

Upgrading the singular set $\mathcal{Z}$ to a variable, we define the **universal Dirac operator** to be the operator acting on pairs $(\mathcal{Z}, \Phi)$ of a singular set and spinor with reference to a background parameter $p \in \mathcal{P}$ by

$$\not{\mathbb{D}}_p(\mathcal{Z}, \Phi) := \not{D}_{\mathcal{Z}}\Phi$$

where the choice of parameter $p = (g, B)$ is implicit on the right-hand side.

Definition 1.1. Given a parameter pair $p = (g, B) \in \mathcal{P}$ the **moduli space of $\mathbb{Z}_2$-harmonic spinors** is the space

$$\mathscr{M}_{\mathbb{Z}_2}(p) := \left\{ (\mathcal{Z}, \ell, \Phi) \;\middle|\; \not{\mathbb{D}}_p(\mathcal{Z}, \Phi) = 0 \;\;,\;\; w_1(\ell) \in H^1(Y - \mathcal{Z}; \mathbb{Z}_2) \;\;,\;\; \|\Phi\|_{L^2} = 1 \right\} \Big/ \mathbb{Z}_2 \tag{1.2}$$

and the **universal moduli space of $\mathbb{Z}_2$-harmonic spinors** is the union

$$\widetilde{\mathscr{M}}_{\mathbb{Z}_2} := \bigcup_{p \in \mathcal{P}} \mathscr{M}_{\mathbb{Z}_2}(p).$$

The middle condition in (1.2) means that the real line bundle $\ell \to Y - \mathcal{Z}_0$ is considered up to its topological isomorphism class. Because $\not{D}_{\mathcal{Z}}$ is $\mathbb{R}$-linear and $\mathbb{Z}_2$ acts by $\Phi \mapsto -\Phi$, the fiber of the moduli space $\mathscr{M}_{\mathbb{Z}_2}(p)$ is a real projective space for each choice of $(\mathcal{Z}, \ell)$. Here, the singular set $\mathcal{Z} \subset Y$ is assumed to be closed and rectifiable subset of (Hausdorff) codimension 2. It is conjectured that the singular set is generically a differentiable submanifold of codimension 2 when the metric g is smooth, though at present is only known to be rectifiable in general [23, 56, 71]. Taubes and Wu have constructed examples for which the singular set $\mathcal{Z}$ is modeled by a collection of rays from the origin, which are expected to model non-generic behavior [60]. We do not attempt to address these regularity issues here.

We now state the main results. The first result, Theorem 1.3 describes the linearized deformation theory near a $\mathbb{Z}_2$-harmonic spinor; the next result, Theorem 1.4, address the non-linear version. Throughout, we fix a central parameter $p_0 = (g_0, B_0)$ such that there exists a $\mathbb{Z}_2$-harmonic spinor $(\mathcal{Z}_0, \ell_0, \Phi_0)$ with respect to p_0 meeting the following requirements.

Definition 1.2. A $\mathbb{Z}_2$-harmonic spinor $(\mathcal{Z}_0, \ell_0, \Phi_0)$ is said to be **regular** if the following three assumptions hold:

Assumption 1. The singular set $\mathcal{Z}_0 \subset Y$ is a smooth, embedded link, and the real line bundle ℓ_0 restricts to the möbius bundle on every disk normal to $\mathcal{Z}_0$.

Assumption 2. The spinor Φ_0 has non-vanishing leading-order, i.e. there is a constant c_1 such that

$$|\Phi_0| \geqslant c_1 \text{dist}(-, \mathcal{Z}_0)^{1/2}$$

Assumption 3. Φ_0 is isolated, i.e. it is the unique $\mathbb{Z}_2$-harmonic spinor for the pair $(\mathcal{Z}_0, \ell_0)$ with respect to (g_0, B_0) up to normalization and sign.

With these assumptions, the Dirac operator

$$\not{D}_{\mathcal{Z}_0} : H^1(S \otimes_{\mathbb{R}} \ell) \to L^2(S \otimes_{\mathbb{R}} \ell) \tag{1.3}$$

has closed range and infinite-dimensional cokernel, where H^1 is the Sobolev space of sections whose covariant derivative is L^2. Let Π_0 denote the L^2-orthogonal projection to the orthogonal complement of the range, which is naturally isomorphic to the cokernel. The first result gives a precise manifestation of the key idea explained above:

Theorem 1.3. The projection of the first-variation of the universal Dirac operator with respect to deformations of the singular set $\mathcal{Z}$

$$\Pi_0 \circ \mathrm{d}_{(\mathcal{Z}_0, \Phi_0)} \not{\mathbb{D}} : L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \longrightarrow \text{Coker}(\not{D}_{\mathcal{Z}_0}) \tag{1.4}$$

is an elliptic pseudo-differential operator of order $\frac{1}{2}$ and its Fredholm extension has index -1.

Here, sections of the normal bundle $N\mathcal{Z}_0$ is the tangent space to the space of embeddings of $\mathcal{Z}_0$. In Section 4, it is shown there is an isomorphism $\text{Coker}(\not{D}_{\mathcal{Z}_0}) \simeq L^2(\mathcal{Z}_0; \mathcal{S})$ of the infinite-dimensional cokernel with a space of sections on $\mathcal{Z}_0$; composing with this isomorphism, (1.4) is a map of sections of vector bundles on $\mathcal{Z}_0$ and the meaning of pseudodifferential operator is the standard one.

Theorem 1.3 shows that the operator $\not{\mathbb{D}}$ displays a **loss of regularity** of order $\frac{3}{2}$. While the non-linear operator $\not{\mathbb{D}}$ is only bounded into L^2, the fact that (1.4) is elliptic of order $\frac{1}{2}$ means it has the following properties:

(i) the kernel is finite-dimensional, and the closure of the range has finite codimension,

(ii) the range is dense and not closed, but is closed as a map into $\text{Coker}(\not{D}_{\mathcal{Z}_0}) \cap H^{3/2}$ with the $\frac{3}{2}$-norm,

(iii) as a map into $\text{Coker}(\not{D}_{\mathcal{Z}_0}) \cap H^{3/2}$, it is Fredholm with index equal to -1.

Loss of regularity is an intriguing phenomenon intrinsic to many types of PDE [1, 18, 29]. More details are given during the proof, but for our immediate purposes a loss of regularity means that there are no function spaces for which the universal Dirac operator is simultaneously bounded and has Fredholm derivative. For every natural function space $\mathcal{X}$ for the domain, the codomain $\mathcal{Y}$ of the universal Dirac operator $\not{\mathbb{D}} : \mathcal{X} \to \mathcal{Y}$ may be chosen *either* so that the non-linear part of $\not{\mathbb{D}}$ is bounded, in which case the derivative $\mathrm{d}\not{\mathbb{D}}$ does not have closed range, *or* it may be chosen so that the derivative is Fredholm, in which case non-linear part is unbounded.

Deformation problems for equations displaying a loss of regularity cannot be addressed using the standard Implicit Function Theorem on Banach spaces; instead they must be solved using various versions of the Nash-Moser Implicit Function Theorem on tame Fréchet manifolds, denoted in our case by $\mathcal{X}$ and $\mathcal{Y}$. Using the linearized result Theorem 1.3 and the Nash-Moser Implicit Function Theorem leads to our main result:

Theorem 1.4. There exists an open neighborhood $\mathscr{U}_0$ of the universal moduli space $\widetilde{\mathscr{M}}_{\mathbb{Z}_2}$ centered at $(p_0, (\mathcal{Z}_0, \ell_0, \Phi_0))$ such that the projection π to the parameter space

$$\begin{array}{ccc} \widetilde{\mathscr{M}}_{\mathbb{Z}_2} \subseteq \mathcal{P} \times \mathcal{X} & \xrightarrow{\mathbb{D}_p} & \mathcal{Y} \\ \pi \big\downarrow & & \\ \mathcal{P} & & \end{array}$$

restricts to a homeomorphism from $\mathscr{U}_0$ to $\pi(\mathscr{U}_0)$, and the image $\pi(\mathscr{U}_0)$ posseses a Kuranishi chart of virtual codimension 1. The same conclusion holds replacing $\mathcal{P}$ by any tame Fréchet submanifold $\mathcal{P}' \subseteq \mathcal{P}$.

To possess a Kuranishi chart of virtual codimension 1 means that the set is locally modeled by the zero-locus of a smooth map $\kappa : \mathcal{P} \to \mathbb{R}$ (see e.g. Section 3.3 of [7]). In particular, if the map (1.4) has empty kernel, then κ is transverse to 0 and $\pi(\mathscr{U}_0)$ is a smooth Fréchet submanifold of codimension 1. In either case, $\mathscr{U}_0$ also consists of regular $\mathbb{Z}_2$-harmonic spinors.

More generally, the universal eigenvalue problem has a spectral crossing along $\pi(\mathscr{U}_0)$:

Corollary 1.5. There is a set $V_0 \subseteq \mathcal{P}$ centered at p_0 possessing a Kuranishi chart of virtual codimension 0 such that for $p \in V_0$ there exists triples $(\mathcal{Z}_p, \Phi_p, \Lambda_p)$ defined implicitly as smooth functions of p satisfying

$$\not{D}_{\mathcal{Z}_p} \Phi_p = \Lambda_p \Phi_p \tag{1.5}$$

for $\Lambda_p \in \mathbb{R}$ and such that $\pi(\mathscr{U}_0) = \Lambda^{-1}(0)$.

Of course, the triple coincides with $(\mathcal{Z}_0, \Phi_0, 0)$ at p_0. Analogous to Theorem 1.4, V_0 consists of regular $\mathbb{Z}_2$-harmonic eigenvectors, and if the map (1.4) has empty kernel then V_0 is an open neighborhood of p_0 and $\Lambda : V_0 \to \mathbb{R}$ is transverse to 0. Once again, the conclusion holds replacing $\mathcal{P}$ by any tame Fréchet submanifold $\mathcal{P}' \subseteq \mathcal{P}$.

Remark 1.6. Assumption 3 can be shown to hold generically. Assumption 2 is known to be generic in analogous situations (see [25]), though we do not prove attempt to prove such a statement here. It is conjectured that Assumption 1 also holds generically. The genericity of embeddings is currently under investigation in ongoing work of Haydys-Mazzeo-Takahashi [23]. The genericity of smoothness and other questions on the regularity of the singular set $\mathcal{Z}_0$ involve significant detours into geometric measure theory (see [22, 71]) and are beyond the scope of the present article. Theorem 1.4 implies that smoothness is stable under smooth variations of the metric and perturbation.

1.2 Relations to Gauge Theory

As stated at the beginning of the article, the motivation for the study of $\mathbb{Z}_2$-harmonic spinors comes from gauge theory. $\mathbb{Z}_2$-harmonic spinors appear as limiting objects into two distinct settings in gauge theory: i) generalized Seiberg-Witten theory in 2,3, and 4 dimensions, and ii) Yang-Mills theory on manifolds with special holonomy in 6,7, and 8 dimensions.

1.2.1 Gauge Theory in Low-Dimensions. Most equations mathematical gauge theory fit into the framework of "generalized Seiberg-Witten equations" [4, 67]. Generalized Seiberg-Witten equations are systems of non-linear first-order PDEs on low-dimensional manifolds, whose variables typically include a connection A on a principal G-bundle for G a compact Lie group, and a spinor Ψ. Examples include the standard Seiberg-Witten equations [30, 38], the Vafa-Witten equations [50, 51, 63], the Kapustin-Witten equations [35, 36, 68, 69], the complex ASD equations [20, 54], and the ADHM-Seiberg-Witten equations [6, 21]. In each case, one wishes to understand the moduli space of all solutions modulo gauge transformations.

In the nicest cases, such as the standard Seiberg-Witten equations, the moduli space is compact. In general, however, there are sequences of solutions for which the L^2-norm of Ψ diverges. A variety of convergence theorems following pioneering work of Taubes [55] have shown that after renormalizing the spinor Ψ to have unit L^2-norm, such sequences converge to a type of $\mathbb{Z}_2$-harmonic spinor for many equations. In this sense, $\mathbb{Z}_2$-harmonic spinors are limiting objects appearing at the boundary of the moduli space. The simplest example of this behavior is the following:

Example 1.7. (Two spinor Seiberg-Witten Equations) Let (Y, g) be a Riemannian 3-manifold, $S \to Y$ a spin^c-structure, and E an auxiliary $SU(2)$-bundle with connection B. The two-spinor Seiberg-Witten equations are the following pair of equations for a $U(1)$-connection A on $\det(S)$ and a spinor $\Psi \in \Gamma(S \otimes_{\mathbb{C}} E)$:

$$\begin{cases} \not{D}_{AB}\Psi = 0 \\ \star F_A + \frac{1}{2}\mu(\Psi,\Psi) = 0, \end{cases} \tag{1.6}$$

where F_A is the curvature of A. By a theorem of Haydys-Walpuski [24], sequences of solutions to (1.6) subconverge (modulo gauge) to either another solution or, after renormalization of the spinor, to a $\mathbb{Z}_2$-harmonic spinor. See Section 2 of [41] for details. The perturbation B in Theorem 1.3 and Theorem 1.4 is the remanent of the $SU(2)$-connection denoted by the same symbol.

This type of convergence is quite general. As a second example, the *limiting configurations* at the boundary of Hitchin moduli space are the square roots of holomorphic quadratic differentials on Riemann surfaces, which are a dimensional reduction of $\mathbb{Z}_2$-harmonic spinors (see [55], Theorem 1.2). $\mathbb{Z}_2$-harmonic spinors are therefore a generalization of objects that have been well-studied in the context of Higgs bundles and the geometry of Hitchin moduli space near the boundary [15, 16, 33]. More generally, convergence theorems of a similar type have been proved by Taubes for **(i)** flat $SL(2,\mathbb{C})$ connections on a 3-manifold [55], **(ii)** the complex ASD equations on a 4-manifold [54], **(iii)** the Kapustin-Witten equations [59], **(iv)** the Vafa-Witten equations [58], and **(v)** the Seiberg-Witten equations with multiple spinors on a 4-manifold [57], and by Haydys-Walpuski and Walpuski-Zhang respectively for **(vi)** the Seiberg-Witten equations with multiple spinors on a 3-manifold [24] and **(vii)** the $\mathrm{ADHM}_{1,2}$ Seiberg-Witten equations on a 3-manifold [67]. In many of these cases the $\mathbb{Z}_2$-harmonic spinors that arise are $\mathbb{Z}_2$-harmonic 1-forms, i.e. "spinors" for the Dirac-type operator $(d + d^\star)$. Theorem 1.4 applies directly to the $\mathbb{Z}_2$-harmonic spinors in Example 1.7; while the deformation theory for many of these more general cases does not follow from Theorem 1.4, the differences from the present situation are expected to be primarily topological with the results following a similar analytic framework. A result similar to Theorem 1.4 for $\mathbb{Z}_2$-harmonic 1-forms follows from a result of S. Donaldson for multi-valued harmonic functions [9].

1.2.2 Fueter Sections.

The *Fueter equation* is a non-linear generalization of the Dirac equation on 3 and 4-manifolds for spinors taking values in a bundle of hyperkähler orbifolds rather than a Clifford module [43, 53]. Solutions of the Fueter equation are called **Fueter Sections**.

The Fueter equation arises naturally in the study of gauge theory on manifolds with special holonomy in dimensions 6, 7, or 8. On such manifolds, sequences of Yang-Mills instantons may converge with bubbling along a calibrated submanifold Y of codimension 4 [52, 62]. The bubbling behavior is expected to be captured by the data of a Fueter section of the bundle $\mathscr{M}_{ASD} \to Y$ whose fibers are the moduli spaces of framed anti-self-dual instantons on the fibers of the normal bundle to Y [65, 66]. Consequently, Fueter sections play key role in proposals for constructing gauge-theoretic invariants on these manifolds. In a closely related direction, Fueter sections also govern the deformation theory of calibrated submanifolds [26] and should therefore play a role in enumerative theories of these [6]. In particular, in both cases they are expected to contribute terms to wall-crossing formulas which relate these theories to generalized Seiberg-Witten theories on low-dimensional calibrated submanifolds and compensate for losses of compactness as parameters vary. For more in-depth expositions, see [4, 6, 10, 21, 24]. In other directions, there are putative applications of Fueter sections to symplectic geometry [5, 28, 47, 64], and to constructing generalized Floer theories on 3-manifolds [11, 12]. In all these cases, a well-developed theory of Fueter sections is lacking and many aspects remain speculative.

In at least the contexts of coming from gauge theory, it is expected that Fueter sections with singularities are unavoidable. Singularities arise when a Fueter section intersects the orbifold locus of the target hyperkähler orbifold. $\mathbb{Z}_2$-harmonic spinors are the simplest examples of Fueter sections with singularities, corresponding to the hyperkähler orbifold $X = \mathbb{H}/\mathbb{Z}_2$. The data of a $\mathbb{Z}_2$-harmonic spinors as defined in (1.1) is equivalent to that of Fueter section valued in a bundle with fiber $\mathbb{H}/\mathbb{Z}_2$ via choosing local lifts to a bundle with fiber $\mathbb{H}$. The line bundle ℓ captures the sign ambiguity in the choice of local lift, and the singular set $\mathcal{Z}$ arises from where the section intersects the singular stratum $\{0\} \in \mathbb{H}/\mathbb{Z}_2$ (see [41] Section 2 or [7] Section 4 for details).

For more general hyperkähler orbifolds X there is a stratification by stabilizer subgroups into subsets of codimension 4k, and a singular set arises where a Fueter section hits these strata. The reader is cautioned that even though these strata are codimension at least 4 and we consider a base manifold Y of dimension 3, solutions of the Fueter equation do not behave generically, and the existence of a codimension 2 singular set $\mathcal{Z}$ is stable under perturbation [8] in all known cases. Much of the work involving Fueter sections (e.g. [7, 19, 44, 65, 66]) has dealt only with the case that $\mathcal{Z} = \varnothing$. This article contributes a step toward understanding Fueter sections with singularities.

1.3 Outline

The paper is divided into three main parts: Sections 2–4 study the semi-Fredholm theory of the Dirac operator with a fixed singular set. Sections 5–6 study deformations of the singular set and prove Theorem 1.3. The non-linear deformation result, Theorem 1.4, is proved in Sections 7–8 using Nash-Moser theory. Throughout, we endeavor to give a largely self-contained exposition that does not assume previous familiarity with Nash-Moser theory or the microlocal analysis of singular elliptic operators.

We now outline these three parts in more detail and explain the strategy for proving Theorems 1.3 and 1.4. Section 2 begins with semi-Fredholm analogues of several standard results from elliptic theory for Dirac operator with a fixed singular set $\mathcal{Z}_0$. Although Fredholmness and elliptic bootstrapping in the usual sense fail, one still obtains various "semi"-elliptic estimates that display several properties analogous to the standard elliptic case. Many of the results in this section are particular cases of general results from microlocal analysis on elliptic edge operators [34, 45].

Section 3 studies the local expressions of solutions; while a $\mathbb{Z}_2$-harmonic spinor need not extend smoothly across $\mathcal{Z}_0$, the standard notion of regularity is replaced with the existence of an asymptotic expansion dictating the behavior along $\mathcal{Z}_0$. These asymptotic expansions and play a key role in all the local analysis in later sections. Section 4 then investigates the infinite-dimensional cokernel of the Dirac operator in more detail. As asserted in the introduction following (1.4), an isomorphism with a space of sections along $\mathcal{Z}_0$ is established. It is also shown that the cokernel concentrates along $\mathcal{Z}_0$ with exponential decay in the normal directions. Although the results of Section 4 are in some sense preliminary to the main purpose of the paper, the reader is cautioned that this section contains many of the more technical points of the article. Some readers may prefer to read only the statements in Section 4 on a first pass.

With the semi-Fredholm theory for fixed singular set established, Sections 5 - 6 proceed to study deformations of the singular set. The key point is that via pulling back by diffeomorphisms moving $\mathcal{Z}_0$ to nearby links, deformations of the singular set are equivalent to deformations of the metric along the family of pullback metrics while keeping the singular set fixed. Schematically,

$$\begin{pmatrix} \text{varying } \mathcal{Z} \\ \text{fixed } g_0 \end{pmatrix} \qquad \frac{\partial}{\partial \mathcal{Z}} \slashed{D}_{\mathcal{Z}} \quad \overset{\text{pullback}}{\Rightarrow} \quad \frac{\partial}{\partial g} \slashed{D}^g_{\mathcal{Z}_0} \qquad \begin{pmatrix} \text{varying } g \\ \text{fixed } \mathcal{Z}_0 \end{pmatrix}.$$

Considering now the right-hand side, the first-variation of the Dirac operator with respect to metrics is given by a well-known formula of Bourguignon-Gauduchon [3]. Calculating the family of pullbacks by diffeomorphisms leads to an explicit formula for the derivative $\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\slashed{\mathbb{D}}$ of the universal Dirac operator (given in Corollary 5.9). Theorem 1.3 is then proved by calculating the projection to the cokernel using the explicit description from Section 4. A key point in this is the notion of conormal regularity (see Section 6.1), which shows that the regularity of the projection depends on *both* the regularity of the spinor *and* its rate of vanishing along $\mathcal{Z}_0$; the loss of regularity is then an unavoidable consequence of the asymptotics of $\mathbb{Z}_2$-harmonic spinors.

Here, it is worth emphasizing that while there is a pleasing geometric reason for Theorem 1.3, the fact that the operator (1.4) is elliptic emerges quite miraculously from the formulas during the proof. Since differentiating the symbol does not preserve ellipticity, Bourguignon-Gauduchon's formula leads to a highly non-elliptic operator on Y; the content of Theorem 1.3 is to assert that under the isomorphisms from Section 4 associating this with an operator on sections of $\mathcal{Z}_0$, ellipticity somewhat surprisingly emerges! Theorem 6.1 provides a more technical version of Theorem 1.3, and an explicit formula for the elliptic operator (1.4) is given during the proof.

Sections 7–8 use Theorem 1.3 and a version of the Nash-Moser Implicit Function Theorem to prove Theorem 1.4. Section 7 gives a brief and practical introduction to Nash-Moser theory, and Section 8 shows that the universal Dirac operator satisfies the necessary hypotheses. The most challenging of these is to show that Theorem 1.3 persists on an open neighborhood of $(p_0, \mathcal{Z}_0, \Phi_0)$. In this, the difficulty is ensuring that some of the more subtle aspects of Sections 4 and 6 are stable.

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2 Semi-Fredholm Properties

Let (Y, g_0) denote a closed, oriented Riemannian 3-manifold, and fix a spin structure $\mathfrak{s}_0 \to Y$. Denote by S the associated spinor bundle, and Clifford multiplication by $\gamma_\circ : T^*Y \to \mathrm{End}(S)$. S carries its spin connection ∇^{spin} with respect to which $\gamma_\circ$ is parallel and a real inner product denoted $\langle -, - \rangle$. More generally, consider the set of perturbations to the spin connection $\nabla_B = \nabla^{\mathrm{spin}} + B$ where $B \in \Omega^1(\mathfrak{so}(S))$ is a real-linear endomorphism commuting with Clifford multiplication. The motivation for introducing this class of perturbations comes from the relation to gauge theory described in Example 1.7. Fix a choice B_0 of such a perturbation.

Now let $\mathcal{Z}_0 \subset Y$ be a smoothly embedded link, i.e. a union of disjoint embedded copies of S^1. Choose a real line bundle $\ell_0 \to Y - \mathcal{Z}_0$, and let A_0 denote the unique flat connection on ℓ with holonomy in $\mathbb{Z}_2$. The Clifford module (S_0, γ, ∇) defined using the fixed pair (g_0, B_0) as

$$S_0 := S \otimes_{\mathbb{R}} \ell_0 \qquad\qquad \gamma = \gamma_\circ \otimes 1 \qquad\qquad \nabla = \nabla_{B_0} \otimes \mathrm{Id} + 1 \otimes \nabla_{A_0} \tag{2.1}$$

carries a singular Dirac operator.

Definition 2.1. The **$\mathbb{Z}_2$-Dirac operator** associated to the Clifford module (S_0, γ, ∇) is defined on sections $\psi \in \Gamma(S_0)$ by

$$\not{D}_{\mathcal{Z}_0}\psi := \gamma(\nabla\psi).$$

In contexts where the singular set $\mathcal{Z}_0$ is fixed and no ambiguity will arise, we omit the subscript and write $\not{D}$.

In the case that $B_0 = 0$ and ℓ_0 extends over $\mathcal{Z}_0$ (and *a fortiori* if $\mathcal{Z}_0 = \emptyset$), this is the classical spin Dirac operator associated to the spin structure obtained from twisting $\mathfrak{s}_0$ by ℓ_0. The case of interest to us is that in which ℓ_0 does not extend over $\mathcal{Z}_0$ and instead restricts to the mobius line-bundle on the normal planes of $\mathcal{Z}_0$. Assumption 1 restricts to case.

As explained in the introduction, when Assumption 1 holds the Dirac operator $\not{D}$ is not an elliptic operator in the standard sense—it is singular along $\mathcal{Z}_0$, and its extension to spaces of sections is only semi-Fredholm. In this section we introduce appropriate Sobolev spaces of sections and describe the semi-Fredholm mapping properties of this Dirac operator. More general versions of these results for larger classes of singular operators can be found in [9, 17, 37, 45, 70]. Here, we give a self-contained exposition.

2.1 Function Spaces

To begin, we introduce "edge" Sobolev spaces starting with the case of lowest regularity. These are the natural function spaces for the analysis of certain classes of singular elliptic operators (see [45]). Let r denote a smooth weight function equal to $\mathrm{dist}(-, \mathcal{Z}_0)$ on a tubular neighborhood of $\mathcal{Z}_0$ and equal to 1 away from a slightly larger tubular neighborhood. For smooth sections compactly supported in $Y - \mathcal{Z}_0$, define the rH^1_e and L^2 norms respectively by

$$\|\varphi\|_{rH^1_e} := \left(\int_{Y\setminus\mathcal{Z}_0} |\nabla\varphi|^2 + \frac{|\varphi|^2}{r^2}\, dV\right)^{1/2} \qquad \text{and} \qquad \|\psi\|_{L^2} := \left(\int_{Y\setminus\mathcal{Z}_0} |\psi|^2\, dV\right)^{1/2},$$

where ∇ is the connection on S_0 defined above in (2.1), and dV denotes the volume form of the Riemannian metric g_0. In addition, we use $r^{-1}H^{-1}_e$ to denote dual norm of rH^1_e with respect to the L^2-pairing:

$$\|\xi\|_{r^{-1}H^{-1}_e} = \sup_{\|\varphi\|_{rH^1_e}=1} \langle \xi, \varphi\rangle_{L^2}.$$

Definition 2.2. The basic **edge Sobolev spaces** denoted rH^1_e, L^2, and $r^{-1}H^{-1}_e$ are defined by

$$\begin{aligned} rH^1_e(Y-\mathcal{Z}_0;S_0) &:= \{\ \varphi \mid \|\varphi\|_{rH^1_e} < \infty\} \\ L^2(Y-\mathcal{Z}_0;S_0) &:= \{\ \psi \mid \|\psi\|_{L^2} < \infty\} \\ r^{-1}H^{-1}_e(Y-\mathcal{Z}_0;S_0) &:= \{\ \xi \mid \|\xi\|_{r^{-1}H^{-1}_e} < \infty\} \end{aligned}$$

i.e. as the completions of compactly supported smooth sections with respect to the above norms respectively. By construction, $r^{-1}H^{-1}_e = (rH^1_e)^\star$ is the dual space with respect to the L^2-pairing.

When it is apparent from context, we will abbreviate these by rH^1_e, L^2, and $r^{-1}H^{-1}_e$ respectively. These spaces are equivalent for different choices of the weight function r. The former two rH^1_e and L^2 are Hilbert spaces with the inner products arising from the polarization of the above norms.

Although $Y - \mathcal{Z}_0$ is not compact, the weight ensures following version of Rellich's Lemma holds, proved by a standard diagonalization argument.

Lemma 2.3. The inclusion

$$rH^1_e(Y-\mathcal{Z}_0;S_0) \hookrightarrow L^2(Y-\mathcal{Z}_0;S_0)$$

is compact. □

2.2 Mapping Properties

The following proposition gives the fundamental mapping properties of the singular Dirac operator on the spaces defined in the previous subsection.

Proposition 2.4. The operator

$$\not{D} : rH^1_e(Y-\mathcal{Z}_0;S) \longrightarrow L^2(Y-\mathcal{Z}_0;S).$$

is (left) semi-Fredholm, i.e. it satisfies the below properties:

- $\ker(\not{D})$ is finite-dimensional, and
- $\mathrm{Range}(\not{D})$ is closed.

Proof. It is immediate from the definitions of rH^1_e, L^2 that $\not{D}$ is a bounded operator. Given $\varphi \in rH^1_e$, it suffices to show that there is a constant C such that the elliptic estimate

$$\|\varphi\|_{rH^1_e} \leqslant C\Big(\|\not{D}\varphi\|_{L^2} + \|\varphi\|_{L^2}\Big) \tag{2.2}$$

holds. Using the compactness of the embedding from Lemma 2.3, both conclusions of the lemma then follow from standard theory (see, e.g. [40] Section 10.4.1).

The estimate (2.2) follows from the Weitzenböck formula and integration by parts, as we now show, though some caution must be taken about the boundary term along $\mathcal{Z}_0$. Let $\varphi \in rH^1_e$ be a spinor, and for each $n \in \mathbb{N}$ let $N_{1/n}(\mathcal{Z}_0)$ denote a tubular neighborhood of $\mathcal{Z}_0$ of radius $1/n$. Additionally, let χ_n denote a cut-off function equal to 1 on $Y - N_{1/n}(\mathcal{Z}_0)$ and compactly supported in $N_{2/n}(\mathcal{Z}_0)$ satisfying

$$|d\chi_n| \leqslant \frac{C}{n} \leqslant \frac{C'}{r}.$$

Then, integrating by parts and using the $\not{D}$ is formally self-adjoint,

$$\begin{aligned}\int_{Y\setminus\mathcal{Z}_0} |\not{D}\varphi|^2 \, dV &= \lim_{n\to\infty}\int_{Y\setminus\mathcal{Z}_0} \langle \not{D}\varphi, \not{D}\varphi\rangle \chi_n \, dV \\ &= \lim_{n\to\infty}\int_{Y\setminus\mathcal{Z}_0} \langle \varphi, \not{D}\not{D}\varphi\rangle\chi_n \; + \langle\varphi, \gamma(d\chi_n)\not{D}\varphi\rangle \, dV.\end{aligned}$$

The Weitzenböck formula shows that

$$\not{D}\not{D} = \nabla^\star\nabla + F$$

wherein F is a zeroth order term arising from the scalar curvature and the derivatives of the perturbation B_0. Substituting this and integrating by parts again yields

$$\int_{Y\setminus\mathcal{Z}_0} |\not{D}\varphi|^2 \, dV = \int_{Y\setminus\mathcal{Z}_0} |\nabla\varphi|^2 + \langle\varphi, F\varphi\rangle + \lim_{n\to\infty}\int_{Y\setminus\mathcal{Z}_0} \langle \varphi, d\chi_n\cdot\nabla\varphi + \gamma(d\chi_n)\not{D}\varphi\rangle \, dV$$

where $\cdot$ denotes contraction of 1-form indices. Since F is smooth on Y hence uniformly bounded, rearranging and using Young's inequality yields

$$\begin{aligned}\int_{Y\setminus\mathcal{Z}_0} |\nabla\varphi|^2 \, dV &\leqslant C\left(\|\not{D}\varphi\|^2_{L^2} + \|\varphi\|^2_{L^2} + \lim_{n\to\infty}\int_{N_{2/n}(\mathcal{Z}_0)} |\nabla\varphi|^2 + d\chi_n^2|\varphi|^2 \, dV\right) &(2.3)\\ &\leqslant C\left(\|\not{D}\varphi\|^2_{L^2} + \|\varphi\|^2_{L^2} + \lim_{n\to\infty}\int_{N_{2/n}(\mathcal{Z}_0)} |\nabla\varphi|^2 + \frac{|\varphi|^2}{r^2} \, dV\right). &(2.4)\end{aligned}$$

Provided $\varphi \in rH^1_e$ the latter limit vanishes, hence

$$\|\nabla\varphi\|_{L^2} \leqslant C\Big(\|\not{D}\varphi\|_{L^2} + \|\varphi\|_{L^2}\Big). \tag{2.5}$$

To conclude, we show the left-hand side of (2.5) dominates the rH^1_e norm. For n sufficiently large, choose local coordinates $N_{1/n}(\mathcal{Z}_0) \simeq S^1 \times D_{1/n}$. Denote these by (t, r, θ) for t the coordinate in the S^1 factor and (r,θ) polar coordinates on $D_{1/n}$. For each fixed t_0, r_0, the fact that the holonomy around the loop (t_0, r_0, θ) for $\theta \in [0, 2\pi)$ is -1 implies that the operator ∇_θ has lowest eigenvalue $1/2$ on this loop (see the local expressions in Section 3.1). It follows that

$$\int_{N_{1/n}(\mathcal{Z}_0)} \frac{|\varphi|^2}{r^2}\, dV \leqslant \frac{1}{4}\int_{N_{1/n}(\mathcal{Z}_0)} \frac{1}{r^2}|\nabla_\theta \varphi|^2\, dV \leqslant \frac{1}{4}\|\nabla\varphi\|^2_{L^2},$$

and away from $N_{1/n}(\mathcal{Z}_0)$, the weight r is uniformly bounded. Combining this estimate with (2.5) (possibly increasing the constant on the $\|\varphi\|_{L^2}$ factor) yields (2.2), completing the lemma. □

Given the above lemma, there is finite-dimensional space of solutions.

Definition 2.5. A non-zero element Φ in the kernel of the operator

$$\not{D} : rH^1_e(Y - \mathcal{Z}_0; S_0) \longrightarrow L^2(Y - \mathcal{Z}_0; S_0) \tag{2.6}$$

is called a $\mathbb{Z}_2$**-harmonic spinor**.

Note that although the estimate (2.2) resembles the standard bootstrapping inequality, it does not imply that an L^2 solution of $\not{D}\psi = 0$ necessarily lies in rH^1_e. In order to establish (2.2) it was necessary to assume *a priori* that $\varphi \in rH^1_e$, else the boundary term along $\mathcal{Z}_0$ need not vanish and the proof fails. Since $\not{D}$ is uniformly elliptic on any compact subset $K \subset Y - \mathcal{Z}_0$, standard theory applies to show that $\varphi \in rH^1_{loc}$ (in fact C^∞_{loc}) but there is no guarantee it has finite rH^1_e-norm on $Y - \mathcal{Z}_0$. Indeed, as we will see in Section 4, the rH^1_e-kernel and the L^2-kernel are genuinely different spaces, with the latter infinite-dimensional. These L^2 kernel elements not in rH^1_e are not called $\mathbb{Z}_2$-harmonic spinors.

2.2.1 The Adjoint Operator.

Although the cokernel of (2.6) is not necessarily finite-dimensional as in standard elliptic theory, it can still be described as the solutions of the formal adjoint operator. As in the proof of Lemma 2.4, integration by parts shows that the relation

$$\langle \not{D}v, \varphi\rangle_{L^2} = \langle v, \not{D}\varphi\rangle_{L^2} \tag{2.7}$$

holds for $v, \varphi \in rH^1_e$. Here we have used that $\not{D}$ is formally self-adjoint since the unperturbed Dirac operator is and $B_0 \in \Omega^1(\mathfrak{so}(S))$. As a consequence of (2.7), the Dirac operator extends to a bounded map

$$\not{D} : L^2(Y - \mathcal{Z}_0; S_0) \longrightarrow r^{-1}H^{-1}_e(Y - \mathcal{Z}_0; S_0).$$

where for $v \in L^2$, the spinor $\not{D}v \in r^{-1}H^{-1}_e$ is the linear functional defined by the relation (2.7). To emphasize the domain of definition for various manifestations of the Dirac operator, we often write $\not{D}|_{rH^1_e}$ or $\not{D}|_{L^2}$.

We then have the following:

Lemma 2.6. The extension $\not{D}|_{L^2}$ defined by (2.7) is the (true) adjoint of $\not{D}|_{rH^1_e}$, and there is a closed decomposition

$$L^2(Y - \mathcal{Z}_0; S_0) = \ker(\not{D}|_{L^2}) \;\oplus\; \text{Range}(\not{D}|_{rH^1_e}).$$

Proof. Suppose that $\psi \in L^2$ is perpendicular to the range, i.e. $\langle \psi, \not{D}\varphi\rangle_{L^2} = 0$ for all $\varphi \in rH^1_e$. The definition of $\not{D}|_{L^2}$ via (2.7) shows that as a linear functional on rH^1_e, one has $\not{D}\psi = 0$. □

2.2.2 The Second Order Operator.

The (left) semi-Fredholmness of $\not{D}$ implies that the second order operator $\not{D}\not{D}$ is Fredholm for purely formal reasons.

Lemma 2.7. The second order operator $\not{D}\not{D} : rH^1_e(Y - \mathcal{Z}_0; S_0) \longrightarrow r^{-1}H^{-1}(Y - \mathcal{Z}_0; S_0)$ is Fredholm and $\ker(\not{D}\not{D}) = \ker(\not{D}|_{rH^1_e}) \simeq \text{coker}(\not{D}\not{D})$. In particular, there is an elliptic estimate

$$\|\varphi\|_{rH^1_e} \leqslant C(\|\not{D}\not{D}\varphi\|_{r^{-1}H^{-1}_e} + \|\pi_0(\varphi)\|_{L^2}). \tag{2.8}$$

where $\pi_0(\varphi)$ is the L^2-orthogonal projection onto $\ker(\not{D}|_{rH^1_e})$.

Proof. (Cf. [49] Proposition 4.4) By definition of $\slashed{D}|_{L^2}$ via 2.7, if $\varphi \in rH^1_e$ and $\varphi \in \ker(\slashed{D}\slashed{D})$, then

$$0 = \langle \slashed{D}\slashed{D}\varphi, \varphi\rangle_{L^2} = \|\slashed{D}\varphi\|_{L^2}$$

hence $\varphi \in \ker(\slashed{D}|_{rH^1_e})$, which is finite dimensional by Proposition 2.4.

To show that the range is closed and the cokernel finite-dimensional (and naturally isomorphic to $\ker(\slashed{D}|_{rH^1_e})$), let $f \in r^{-1}H^{-1}_e$ and consider the functional $E_f : rH^1_e \to \mathbb{R}$ given by

$$E_f(\varphi) := \int_{Y\setminus\mathcal{Z}_0} |\slashed{D}\varphi|^2 + \langle \varphi, f\rangle \, dV.$$

The Euler-Lagrange equations of E_f are

$$\slashed{D}\slashed{D}\varphi = f \qquad\qquad \langle f, \Phi\rangle = 0 \;\; \forall\; \Phi \in \ker(\slashed{D}|_{rH^1_e})$$

so it suffices to show that E_f admits a minimizer. By standard theory ([13] Chapter 8) this holds if E_f is (i) coercive, and (ii) weakly lower semi-continuous. The second of these is standard (see e.g. [13] Section 8.2.2). (i) means that

$$E_f(\varphi) \geqslant c_1\|\varphi\|^2_{rH^1_e} - c_2 \tag{2.9}$$

holds for some constants c_i, and φ in the L^2-orthogonal complement of $\ker(\slashed{D}|_{rH^1_e})$, which follows from the elliptic estimate (2.2) of Proposition 2.4 and Young's inequality. This establishes Fredholmness, and the estimate (2.8) is a routine consequence. □

As a consequence of the preceding lemma, we may let $P : r^{-1}H^{-1}_e \to rH^1_e$ denote the solution operator defined by

$$P(\xi) = \varphi \qquad \text{s.t.} \qquad \text{i) } \slashed{D}\slashed{D}\varphi = \xi \mod \ker(\slashed{D}|_{rH^1_e}) \tag{2.10}$$

$$\text{and} \qquad \text{ii) } \langle \varphi, \Phi\rangle_{L^2} = 0 \;\forall \Phi \in \ker(\slashed{D}|_{rH^1_e}). \tag{2.11}$$

To summarize, we have the following corollary:

Corollary 2.8. The following hold using the splitting $L^2 = \ker(\slashed{D}|_{L^2}) \oplus \text{Range}(\slashed{D}|_{rH^1_e})$ of Lemma 2.6.

(A) The second order operator $\slashed{D}\slashed{D}$ factors through the $\text{Range}(\slashed{D}|_{rH^1_e})$ summand of

$$\begin{tikzcd} rH^1_e \arrow[r, "\slashed{D}"] & \begin{matrix} \ker(\slashed{D}|_{L^2}) \\ \oplus \\ \text{Range}(\slashed{D}|_{rH^1_e}) \end{matrix} \arrow[r, "\slashed{D}"] & r^{-1}H^{-1}_e. \arrow[ll, bend left, "P"] \end{tikzcd}$$

In addition, we can further split $\ker(\slashed{D}|_{L^2}) = \ker(\slashed{D}|_{rH^1_e}) \oplus \ker(\slashed{D}|_{L^2})^{\perp}$ wherein the first summand is finite-dimensional.

(B) The projections to the two summands of L^2 may be written

$$\Pi^{\text{Range}} = \slashed{D}P\slashed{D} \qquad\qquad \Pi^{\ker} = 1 - \slashed{D}P\slashed{D}.$$

□

2.3 Higher Regularity

This subsection extends the results of the previous two to "edge" and "boundary" Sobolev spaces of higher regularity (see [45] again for a more general exposition). Beginning with the "boundary" spaces, define the space of "boundary" vector fields

$$\mathcal{V}_{\mathrm{b}} := \{V \in C^\infty(Y;TY) \;\;|\;\; V|_{\mathcal{Z}_0} \in C^\infty(\mathcal{Z}_0; T\mathcal{Z}_0)\}$$

as those tangent to $\mathcal{Z}_0$ at the boundary. Let ∇^{b} denote the covariant derivative with respect to such vector fields, so that in local coordinates (t,x,y) where t is a coordinate along $\mathcal{Z}_0$ and x, y coordinates in the normal directions it is given by

$$\nabla^{\mathrm{b}} = dx \otimes r\nabla_x \;+\; dy \otimes r\nabla_y \;+\; dt \otimes \nabla_t$$

and is equal to the standard covariant derivative ∇ away from $\mathcal{Z}_0$.

For $m \in \mathbb{N}$, define the H^m_{b}-norm on compactly supported smooth sections by

$$\|\psi\|_{H^m_{\mathrm{b}}} \;\;:=\;\; \left(\int_{Y\setminus\mathcal{Z}_0} |(\nabla^{\mathrm{b}})^m\psi|^2 \;+\ldots+ |\nabla^{\mathrm{b}}\psi|^2 \;+\; |\psi|^2 \, dV \right)^{1/2}. \tag{2.12}$$

Definition 2.9. The **mixed boundary and edge Sobolev spaces** are defined as (the closures of)

$$\begin{aligned}
rH^{m,1}_{\mathrm{b},e}(Y-\mathcal{Z}_0;S_0) &:= \Big\{\varphi \mid \|(\nabla^{\mathrm{b}})^m\varphi\|^2_{rH^1_e} + \ldots + \|\nabla^{\mathrm{b}}\varphi\|^2_{rH^1_e} \;+\; \|\varphi\|^2_{rH^1_e} < \infty\Big\}\\
H^m_{\mathrm{b}}(Y-\mathcal{Z}_0;S_0) &:= \Big\{\psi \mid \|(\nabla^{\mathrm{b}})^m\psi\|^2_{L^2} + \ldots + \|\nabla^{\mathrm{b}}\psi\|^2_{L^2} \;+\; \|\psi\|^2_{L^2} = \|\psi\|^2_{H^m_{\mathrm{b}}} < \infty\Big\}\\
r^{-1}H^{m,-1}_{\mathrm{b},e}(Y-\mathcal{Z}_0;S_0) &:= \Big\{\xi \mid \|(\nabla^{\mathrm{b}})^m\xi\|^2_{r^{-1}H^{-1}_e} + \ldots + \|\nabla^{\mathrm{b}}\xi\|^2_{r^{-1}H^{-1}_e} \;+\; \|\xi\|^2_{r^{-1}H^{-1}_e} < \infty\Big\}
\end{aligned}$$

equipped with the norms given by the positive square root of the quantities required to be finite. As for $m = 0$, changing the weight r result in equivalent norms. More generally, one can define the spaces for $m \in \mathbb{R}^{\geqslant 0}$ by interpolation.

We have the following version of the standard interpolation inequalities:

Lemma 2.10. The following interpolation inequalities hold for $m_1 < m < m_2$:

$$\|\psi\|_{H^m_{\mathrm{b}}} \leqslant C_{m_1,m,m_2}\|\psi\|^{\alpha}_{H^{m_1}_{\mathrm{b}}}\|\psi\|^{1-\alpha}_{H^{m_2}_{\mathrm{b}}} \qquad\qquad \|\varphi\|_{H^{m,1}_{\mathrm{b},e}} \leqslant C_{m_1,m,m_2}\|\varphi\|^{\alpha}_{H^{m_1,1}_{\mathrm{b},e}}\|\varphi\|^{1-\alpha}_{H^{m_2,1}_{\mathrm{b},e}}$$

where $\alpha = \frac{m_2-m}{m_2-m_1}$, and the constants may depend on the triple m_1, m, m_2.

Proof. Choose local cylindrical coordinates (t, r, θ) on a tubular neighborhood of $\mathcal{Z}_0$, where t a coordinate along $\mathcal{Z}_0$ and (r,θ) polar coordinates in the normal directions. The coordinate change $s = \log(r)$ is a diffeomorphism between $Y - \mathcal{Z}_0$ and the manifold Y° given by attaching a cylindrical end $T^2 \times (-\infty, r_0)$ near $\mathcal{Z}_0$. Under this coordinate change, H^m_{b} is taken to the standard Sobolev spaces $e^{-s}H^m$ with the an exponential weight. After multiplying by an exponential weight function, the inequalities for H^m_{b} follow from the standard ones on Y° (see, e.g. [14]).

For the mixed boundary and edge spaces, note that $\|[\nabla, \nabla^b]\varphi\|_{L^2} \leqslant \|\nabla\varphi\|_{L^2}$, and iterating these commutators shows that

$$\|\varphi\|^2_{H^{m,1}_{\mathrm{b},e}} \sim \|\nabla\varphi\|^2_{H^m_{\mathrm{b}}} + \|\tfrac{\varphi}{r}\|^2_{H^m_{\mathrm{b}}} \tag{2.13}$$

is an equivalent expression for the norm, after which the interpolation inequalities for $H^{m,1}_{\mathrm{b},e}$ follow from those for H^m_{b} applied to $\nabla\varphi$ and $\frac{\varphi}{r}$. □

Applying the elliptic estimate (2.2) to $(\nabla^{\mathrm{b}})^m$ and iterating commutators $[\nabla, \nabla^{\mathrm{b}}]$ also establishes the following higher-regularity elliptic estimates:

Corollary 2.11. There are constants C_m depending on up to $m+3$ derivatives of the pair (g_0, B_0) such that the following elliptic estimates hold for $\varphi \in rH^{m,1}_{\mathrm{b},e}$:

$$\begin{aligned} \|\varphi\|_{rH^{m,1}_{\mathrm{b},e}} &\leqslant C_m(\|\not{D}\varphi\|_{H^m_{\mathrm{b}}} + \|\varphi\|_{H^m_{\mathrm{b}}}) \\ \|\varphi\|_{rH^{m,1}_{\mathrm{b},e}} &\leqslant C_m(\|\not{D}\not{D}\varphi\|_{r^{-1}H^{m,-1}_{\mathrm{b},e}} + \|\varphi\|_{r^{-1}H^{m,-1}_{\mathrm{b},e}}) \end{aligned}$$

(note also that $\|\varphi\|_{H^m_{\mathrm{b}}} \leqslant C\|\varphi\|_{rH^{m-1,1}_{\mathrm{b},e}}$). □

Using this, we immediately deduce the higher-regularity version of the results of the previous subsection.

Corollary 2.12. For all $m > 0$, the following statements hold:

(A) There is an H^m_{b}-closed decomposition

$$H^m_{\mathrm{b}} = \ker(\not{D}|_{H^m_{\mathrm{b}}}) \oplus \mathrm{Range}(\not{D}|_{rH^{m,1}_{\mathrm{b},e}})$$

orthogonal with respect to the L^2-inner product. Moreover, the latter two spaces coincide with $\ker(\not{D}|_{H^m_{\mathrm{b}}}) = \ker(\not{D}|_{L^2}) \cap H^m_{\mathrm{b}}$ and $\mathrm{Range}(\not{D}|_{rH^{m,1}_{\mathrm{b},e}}) = \mathrm{Range}(\not{D}|_{rH^1_e}) \cap H^m_{\mathrm{b}}$.

(B) The second order operator $\not{D}\not{D}$ factors through the $\mathrm{Range}(\not{D}|_{rH^1_e}) \cap H^m_{\mathrm{b}}$ summand of

$$\begin{tikzcd} rH^{m,1}_{\mathrm{b},e} \arrow[r, "\not{D}"] & \begin{matrix} \ker(\not{D}|_{L^2}) \cap H^m_{\mathrm{b}} \\ \oplus \\ \mathrm{Range}(\not{D}|_{rH^1_e}) \cap H^m_{\mathrm{b}} \end{matrix} \arrow[r, "\not{D}"] & r^{-1}H^{m,-1}_{\mathrm{b},e}. \arrow[ll, bend left, "P"] \end{tikzcd}$$

(C) The projections to the two summands of L^2 respect regularity in the sense that

$$\Pi^{\mathrm{Range}} = \not{D}P\not{D} : H^m_{\mathrm{b}} \to H^m_{\mathrm{b}} \qquad\qquad \Pi^{\ker} = 1 - \not{D}P\not{D} : H^m_{\mathrm{b}} \to H^m_{\mathrm{b}}$$

are bounded operators. □

3 Local Expressions

This section studies the local expressions for the Dirac operator and its solutions in local coordinates on a tubular neighborhood of $\mathcal{Z}_0$. By Proposition 2.4 and Lemma 2.6, there is a dichotomy between two distinct types of solution:

$$\Phi \in \ker(\not{D}|_{rH^1_e}) \qquad\qquad \psi \in \ker(\not{D}|_{L^2}) \quad \text{s.t.} \quad \psi \notin rH^1_e \tag{3.1}$$

with the former being the $\mathbb{Z}_2$-harmonic spinors. These two types behave quite differently; in particular, there is a finite-dimensional space of $\mathbb{Z}_2$-harmonic spinors and these (if continuous) vanish along $\mathcal{Z}_0$ while general L^2-solutions need not.

We begin in Section 3.1 with an explicit and illustrative discussion of the model case of $Y = S^1 \times \mathbb{R}^2$ equipped with the product metric. Sections 3.2 and 3.3 then deal with local expressions on a general 3-manifold.

3.1 The Model Operator

Let $Y_0 = S^1 \times \mathbb{R}^2$ denote the product equipped with coordinates (t, x, y) and the product metric $g_0 = dt^2 + dx^2 + dy^2$. Take $\mathcal{Z}_0 = S^1 \times \{0\}$ and $\ell_0 \to Y_0 - \mathcal{Z}_0$ the pullback of the mobius bundle on $\mathbb{R}^2 - \{0\}$.

The twisted spinor bundle of the product spin structure can be identified with $S = \underline{\mathbb{C}}^2 \otimes_{\mathbb{R}} \ell_0$. A section $\psi \in \Gamma(\underline{\mathbb{C}}^2 \otimes_{\mathbb{R}} \ell)$ may be written as

$$\psi = e^{i\theta/2} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \tag{3.2}$$

where $\psi^\pm$ are $\mathbb{C}$-valued function and (r, θ) are polar coordinates on $\mathbb{R}^2$. Indeed, on each normal plane $\mathbb{R}^2 - \{0\}$, the bundle $\underline{\mathbb{C}} \otimes_{\mathbb{R}} \ell$ can be constructed as the bundle with fiber $\mathbb{C}$ glued along two (thickened) rays by the transition functions $+1$ and -1. Consequently, $e^{i\theta/2}$, gives rise to a global nowhere-vanishing section of this bundle. Writing section in the form (3.2), the connection arising from the spin connection and ∇_{A_0} on ℓ_0 (with perturbation $B_0 = 0$) is simply $\nabla = \mathrm{d}$. The Dirac operator then takes the form

$$\not{D} = \begin{pmatrix} i\partial_t & -2\partial_z \\ 2\overline{\partial}_z & -i\partial_t \end{pmatrix} \tag{3.3}$$

where $z = x + iy$. That is to say, it is just the normal spin Dirac operator on Y_0, but the spinors have an additional $e^{i\theta/2}$ term which is differentiated as expected.

Remark 3.1. Although it is convenient for computation, the singular nature of the Dirac operator $\not{D}_{\mathcal{Z}}$ is hidden in the expression 3.3. It can be written in the following equivalent way which makes the singular nature manifest.

Multiplication $e^{-i\theta/2} : \underline{\mathbb{C}}^2 \otimes \ell_0 \simeq \underline{\mathbb{C}}^2$ provided an alternative trivialization, in which spinor are written $\psi = (\psi^+, \psi^-)$ where $\psi^\pm$ are still $\mathbb{C}$-valued functions. In this trivialization Dirac operator is instead given by

$$\not{D} = \begin{pmatrix} i\partial_t & -2\partial_z \\ 2\overline{\partial}_z & -i\partial_t \end{pmatrix} + \frac{1}{2}\gamma(d\theta) = \begin{pmatrix} i\partial_t & -2\partial_z \\ 2\overline{\partial}_z & -i\partial_t \end{pmatrix} + \frac{1}{4}\gamma\left(\frac{dz}{z} - \frac{d\overline{z}}{\overline{z}}\right)$$

where γ denotes Clifford multiplication. The singular nature of the operator is now apparent in the zeroth order term which is unbounded on L^2. Equivalently, $r\not{D}$ is an elliptic operator with bounded zeroth order term, but *the symbol degenerates along $\mathcal{Z}_0$*. This type of operator is called an elliptic operator with edge-type degeneracies or simply an **elliptic edge operator**. The theory of operators of this type has been studied extensively in microlocal analysis and many results in Section 2 hold in considerable generality (see [9, 17, 37, 45, 70] and the references therein).

Example 3.2. Let us now identify the L^2-kernel of $\not{D}$ on Y_0 (Cf. [49] Section 3). As in Lemma 2.6, this also identifies the cokernel of the operator on rH^1_e since $\mathrm{Coker}(\not{D}|_{rH^1_e}) \simeq \mathrm{Ker}(\not{D}|_{L^2})$ continues to hold. Here, the weight function is given by r globally on Y_0. Writing a general section in Fourier series as

$$\psi = \sum_{k,\ell} e^{i\ell t} e^{i(k+\frac{1}{2})\theta} \begin{pmatrix} \psi^+_{k,\ell} e^{-i\theta} \\ \psi^-_{k,\ell} \end{pmatrix}$$

and using the polar expressions

$$\overline{\partial}_z = \frac{1}{2}e^{i\theta}(\partial_r + \frac{i}{r}\partial_\theta) \qquad\qquad \overline{\partial}_z = \frac{1}{2}e^{-i\theta}(\partial_r - \frac{i}{r}\partial_\theta),$$

the Dirac equation 3.3 becomes the following system of ODEs for $\psi^\pm_{k,\ell}(r)$ which decouple for distinct pairs (k, ℓ):

$$\frac{d}{dr}\begin{pmatrix} \psi^+_{k,\ell} \\ \psi^-_{k,\ell} \end{pmatrix} = \begin{pmatrix} \frac{(k-\frac{1}{2})}{r} & -\ell \\ -\ell & -\frac{(k+\frac{1}{2})}{r} \end{pmatrix} \begin{pmatrix} \psi^+_{k,\ell} \\ \psi^-_{k,\ell} \end{pmatrix}. \tag{3.4}$$

This system of equations can be solved by substituting the second equation into the first, after which the general solution is given in terms of modified Bessel Functions. If $k \neq 0$, the pair (k, ℓ) admits no solutions in $L^2(S^1 \times \mathbb{R}^2)$; for $k = 0$ and the pairs $(0, \ell)$ with $\ell \neq 0$,

$$\Psi_\ell^{\text{Euc}} = \sqrt{|\ell|} e^{i\ell t} e^{-|\ell| r} \begin{pmatrix} \frac{1}{\sqrt{z}} \\ \frac{\text{sgn}(\ell)}{\sqrt{\bar{z}}} \end{pmatrix} \tag{3.5}$$

is an infinite-dimensional set of orthonormalized solutions in L^2, and $\ker(\not{D}|_{L^2})$ is their L^2-closure.

Disregarding the issues of the integrability of the $\ell = 0$ solutions as $r \to \infty$ (which is immaterial in the upcoming case of Y compact) and formally including this element leads to an isomorphism

$$L^2(S^1; \mathbb{C}) \simeq \ker(\not{D}|_{L^2}) \tag{3.6}$$

defined by the linear extension of $e^{i\ell t} \mapsto \Psi_\ell^{\text{Euc}}$. In this example there are no $\mathbb{Z}_2$-harmonic spinors.

There is a second choice of spin structure on $Y_0 = S^1 \times \mathbb{R}^2$ which also has monodromy -1 around the S^1 factor. In this case, spinor may be written with half integer Fourier modes $e^{i\ell t} e^{it/2}$, and the calculation is identical but the solutions are indexed by $\ell' \in \mathbb{Z} + \frac{1}{2}$. $\square$

Just as in this model example, the L^2-kernel on a general closed 3-manifold is infinite-dimensional, and the failure to prove Fredholmness was not simply a shortcoming of the techniques in Section 2. In the model case, the kernel spanned by Ψ_ℓ^{Euc} displays the following salient properties which generalize to the case of Y closed:

Expansion: Solutions Ψ_ℓ^{Euc} have asymptotic expansions with terms $r^{k-\frac{1}{2}}$ for $k \in \mathbb{Z}$.

Isomorphism: There is a natural isomorphism $\ker(\not{D}|_{L^2}) \simeq L^2(\mathcal{Z}_0; \mathbb{C})$ given by associating a kernel element to each eigenfunction of the Dirac operator $i\partial_t$ on $\mathcal{Z}_0$.

Rapid Decay: For eigenvalues $|\ell| >> 0$, solutions Ψ_ℓ^{Euc} decay exponentially away from $\mathcal{Z}_0$.

The first item follows from the power series expansion of $e^{-|\ell| r}$. The remainder of Section 3 deals with the first item, and the remaining two properties are the subject of Section 4.

Remark 3.3. Although there are no $\mathbb{Z}_2$-harmonic spinors in rH^1_e in Example 3.2, there are explicit solutions given in terms of modified Bessel functions for $(k, \ell) = (\pm 1, \ell)$ which have leading order $z^{1/2}$ and $\bar{z}^{1/2}$ and thus lie in rH^1_{loc} near $\mathcal{Z}_0$. All these solutions, however, grow exponentially as $r \to \infty$. Therefore, intuitively, the existence of a $\mathbb{Z}_2$-harmonic spinor on a closed manifold Y is a non-generic phenomenon and occurs only when one of these exponentially growing solutions can be patched together with a bounded solution on the complement of a neighborhood of $\mathcal{Z}_0$ in Y.

3.2 Local Expressions

From here on, we return to the case that (Y, g_0) is a closed, oriented Riemannian 3-manifold and $\mathcal{Z}_0$ a smoothly embedded link. In order to write local expressions, we endow a tubular neighborhood $N_{r_0}(\mathcal{Z}_0)$ of a component of $\mathcal{Z}_0$ diffeomorphic to a solid torus with a particular set of coordinates.

Let $\gamma : S^1 \to \mathcal{Z}_j$ denote an arclength parameterization of a chosen component $\mathcal{Z}_i$ of $\mathcal{Z}_0$ whose length is denoted by $|\mathcal{Z}_j|$, and fix a global orthonormal frame $\{n_1, n_2\}$ of the pullback $\gamma^* N\mathcal{Z}_0$ of the normal bundle to $\mathcal{Z}_0$. We are free to arrange that $\{\dot{\gamma}, n_1, n_2\}$ is an oriented frame of TY along $\mathcal{Z}_j$.

Definition 3.4. A system of **Fermi coordinates** for $r_0 < r_{\text{inj}}$ where r_{inj} is the injectivity radius of Y is the diffeomorphism $S^1 \times D_{r_0} \simeq N_{r_0}(\mathcal{Z}_0)$ for a chosen component of $\mathcal{Z}_0$ defined by

$$(t, x, y) \mapsto \text{Exp}_{\gamma(t)}(xn_1 + yn_2).$$

Here t is the normalized coordinate on the S^1 with $t \in \mathbb{R}/|\mathcal{Z}_i|\mathbb{Z}$. In these coordinates the Riemannian metric g_0 can be written

$$g_0 = dt^2 + dx^2 + dy^2 \ + \ O(r) \tag{3.7}$$

Given such a coordinate system, (t, r, θ) are used to denote the corresponding cylindrical coordinates, and $(t, z, \overline{z})$ the complex ones on the D_{r_0} factor.

Remark 3.5. There are different conventions on the usage of "Fermi coordinates", with some requiring that the curve is question is a geodesic. In this case, n_x and n_y can be chosen to locally solve an ODE so that $g = dt^2 + dx^2 + dy^2 + O(r^2)$. Here, we make no such assumption and the difference from the product metric is $O(r)$. Explicitly, the correction to the product metric is

$$[2x\mathfrak{m}_x(t) + 2y\mathfrak{m}_y(t)]dt^2 \ + \ [\mu(t)y]dtdx \ + \ [-\mu(t)x]dtdy \ + \ O(r^2)$$

where $\mathfrak{m}_\alpha(t) = \langle \nabla_{\dot\gamma}\dot\gamma, n_\alpha\rangle$ for $\alpha = x, y$ and $\mu(t) = \langle \nabla_{\dot\gamma} n_x, n_y\rangle = -\langle \nabla_{\dot\gamma} n_y, n_x\rangle$.

A choice of coordinates induces a trivialization of the frame bundle of Y on $N_{r_0}(\mathcal{Z}^i)$ by a global orthonormal frame $\{e_t, e_1, e_2\}$ which restricts to $\{\partial_t, \partial_x, \partial_y\}$ along $\mathcal{Z}_i$. We may now distinguish two cases:

Case 1: The spin structure restricts to as the product $\mathfrak{s}_0|_{N_{r_0}(\mathcal{Z}_i)} \simeq N_{r_0}(\mathcal{Z}_i) \times \text{Spin}(3)$, so that

$$S|_{N_{r_0}(\mathcal{Z}_i)} \simeq \underline{\mathbb{C}}^2 \otimes \ell_0 \tag{3.8}$$

Case 2: The spin structure restricts to $N_{r_0}(\mathcal{Z}_i)$ as the double cover of $Fr(Y)|_{N_{r_0}(\mathcal{Z}_i)} \simeq N_{r_0}(\mathcal{Z}_i) \times SO(3)$ non-trivial in the $\mathcal{Z}_i$ factor, so that

$$S|_{N_{r_0}(\mathcal{Z}_i)} \simeq \underline{\mathbb{C}}^2 \otimes \ell_t \otimes \ell_0 \tag{3.9}$$

where ℓ_t is the pullback of the mobius bundle on $\mathcal{Z}_j$.

We note that, in general, there are some rather subtle topological restrictions on which combinations of Case 1 and Case 2 can occur for the different components of $\mathcal{Z}_0$. For instance, if $Y = S^3$ and $\mathcal{Z}_0$ has a single component, then the unique spin structure on S^3 always restricts to Case 2 on a tubular neighborhood of $\mathcal{Z}_0$; if $\mathcal{Z}_0$ has multiple components then the number which fall in Case 1 must be even.

First consider Case 1. It may be assumed that the identification 3.8 is chosen so that the factors of $\mathbb{C}^2$ are given by the $\pm i$ eigenspaces of $\gamma(e_t)$, hence Clifford multiplication is given by

$$\gamma(e^t) = \sigma_t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \gamma(e^1) = \sigma_x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma(e^2) = \sigma_y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

As in the model case (Example 3.2), spinors can be written in this trivialization in the form (3.2) where $\psi^\pm$ are $\mathbb{C}$-valued function on $N_{r_0}(\mathcal{Z}_i)$. In Case 2, the same holds after changing the trivialization by $e^{i\ell t/2}$ which alters the Dirac operator by $\frac{i}{2}\gamma(dt)$. This leads to the following:

Lemma 3.6. In both Case (1) and Case (2), the $\mathbb{Z}_2$-Dirac operator in local coordinates around a component $\mathcal{Z}_i \subseteq \mathcal{Z}_0$ and the above trivialization takes the form

$$\not{D} = \not{D}_0 + \mathfrak{d}$$

where

- $\not{D}_0$ is the Dirac operator in the product metric on $N_{r_0}(\mathcal{Z}_i)$, given by (3.3)
- $\mathfrak{d}$ is a first order perturbation arising from the $O(r)$ terms of g_0, the perturbation B_0 and $\frac{i}{2}\gamma(dt)$ in Case 2, so that
$$|\mathfrak{d}\psi| \leqslant C(r|\nabla\psi| + |\psi|).$$

□

3.3 Asymptotic Expansions

In this subsection establishes that $\mathbb{Z}_2$-harmonic spinors have local power series expansions by half integer powers of r. This follows from invoking general regularity results for elliptic edge operators from [45], and should be seen as the appropriate form of elliptic regularity for the singular operator $\not{D}$.

Fix a choice of Fermi coordinates near $\mathcal{Z}_0$.

Definition 3.7. A spinor $\psi \in L^2(Y - \mathcal{Z}_0; S_0)$ is said to admit a **polyhomogenous expansion** with index set $\mathbb{Z}^+ + \frac{1}{2}$ if

$$\psi \sim \sum_{n\geqslant 0}\sum_{k,p\in\mathbb{Z}} \begin{pmatrix} c_{n,k,p}(t)e^{ik\theta} \\ d_{n,k,p}(t)e^{ik\theta} \end{pmatrix} r^{n+1/2}\log(r)^p e^{-i\theta/2}$$

where $c_{n,k,p}(t), d_{n,k,p}(t) \in C^\infty(S^1;\mathbb{C})$, and where $\sim$ denotes convergence in the following sense: for every $N \in \mathbb{N}$, the partial sums

$$\psi_N = \sum_{n\leqslant N}\sum_{k=-2n}^{2n+1}\sum_{p\leqslant n-1} \begin{pmatrix} c_{n,k,p}(t)e^{ik\theta} \\ d_{n,k,p}(t)e^{ik\theta} \end{pmatrix} r^{n+1/2}\log(r)^p e^{-i\theta/2}$$

satisfy the pointwise bounds

$$|\psi - \psi_N| \leqslant C_N r^{N+1+\frac{1}{4}} \qquad\qquad |\nabla_t^\alpha \nabla^\beta(\psi-\psi_N)| \leqslant C_{N,\alpha,\beta} r^{N+1+\frac{1}{4}-|\beta|} \tag{3.10}$$

for constants $C_{N,\alpha,\beta}$ determined by the background data and choice of local coordinates and trivialization. Here, β is a multi-index of derivatives in the directions normal to $\mathcal{Z}_0$.

The work of Mazzeo [45] implies the following regularity result about $\mathbb{Z}_2$-harmonic spinors (see also Appendix A of [26]).

Proposition 3.8. Suppose that $\Phi_0 \in rH^1_e(Y - \mathcal{Z}_0; S_0)$ is a $\mathbb{Z}_2$-harmonic spinor. Then Φ_0 admits a polyhomogenous expansion with index set $\mathbb{Z}^+ + \frac{1}{2}$. Moreover, $c_{n,k,p}$ and $d_{n,k,p}$ vanish unless $-2n \leqslant k \leqslant 2n+1$ and $p \leqslant n-1$. Thus Φ_0 has a local expression

$$\Phi_0 \sim \begin{pmatrix} c(t)\sqrt{z} \\ d(t)\sqrt{\overline{z}} \end{pmatrix} + \sum_{n\geqslant 1}\sum_{k=-2n}^{2n+1}\sum_{p=0}^{n-1} \begin{pmatrix} c_{n,k,p}(t)e^{ik\theta} \\ d_{n,k,p}(t)e^{ik\theta} \end{pmatrix} r^{n+1/2}\log(r)^p e^{-i\theta/2} \tag{3.11}$$

where $c(t), d(t), c_{k,m,n}(t), d_{k,m,n}(t) \in C^\infty(S^1;\mathbb{C})$. In this form, Assumption 2 is the requirement that $|c(t)|^2 + |d(t)|^2 > 0$ is nowhere-vanishing. The same result holds for an rH^1_e-solution of the operator $\not{D} - \lambda\mathrm{Id}$.

Proof. The existence of such an expansion is a consequence of the regularity theory in [45] (Section 7, Proposition 7.17) and the fact that the indicial roots are $j + \frac{1}{2}$ for $j \in \mathbb{Z}$ in this case. See also [23, 26]. The constraints on the expansion compared to Definition 3.7 then follow from writing the equation $\not{D}\Phi_0 - \lambda\Phi_0 = 0$ in Fermi coordinates as

$$\begin{pmatrix} 0 & -2\partial \\ 2\overline{\partial} & 0 \end{pmatrix}\Phi_0 = -\mathfrak{d}\Phi_0 - \begin{pmatrix} -i\partial_t & 0 \\ 0 & i\partial_t \end{pmatrix}\Phi_0 + \lambda\Phi_0$$

with $\mathfrak{d}$ as in Lemma 3.6, and formally solving term by term. □

The expression (3.10) depends on the choice of Fermi coordinates in the following way. Another choice of Fermi coordinates arises from an alternative choice of normal frame n_x, n_y differing by a transformation induced by a change of trivialization of the spin structure. Such a change of frame is given in complex coordinates on $N\mathcal{Z}_0$ by

$$n_1 + in_2 \mapsto e^{-2i\sigma(t)}(n_1 + in_2)$$

where $\sigma(t) : \mathcal{Z}_0 \to S^1$ (the minus sign in the exponent is due to the convention that Clifford multiplication is by cotangent vectors). The new complex coordinates (t, z', z') resulting from such a transformation are likewise related to the original coordinates by

$$(t, z', \overline{z}') = (t, e^{-2i\sigma(t)}z, e^{2i\sigma(t)}\overline{z}').$$

This shows the following:

Corollary 3.9. For a term of a polyhomogenous expansion

$$\psi(t, z, \overline{z}) = \begin{pmatrix} a(t)e^{ik\theta} \\ b(t)e^{ik\theta} \end{pmatrix} r^{n+1/2} \log(r)^p e^{-i\theta/2}$$

the coefficients are naturally sections $a(t) \in C^\infty(\mathcal{Z}_0; N\mathcal{Z}_0^{-k})$ and $b(t) \in C^\infty(\mathcal{Z}_0; N\mathcal{Z}_0^{-k+1})$. In particular, the leading coefficients $c(t), d(t)$ of (3.10) are sections of $N\mathcal{Z}_0^{-1}, N\mathcal{Z}_0$ respectively. $\square$

Remark 3.10. More generally, L^2-elements elements have similar asymptotic expansions, but it is no longer necessarily the case that the coefficients are smooth. In general, the coefficients only make sense as distributions (see Section 7 of [45] for a more general discussion). If $\psi \in \ker(\not{D}) \cap L^2(Y - \mathcal{Z}_0; S_0)$, then it admits a **weak asymptotic expansion** of the form

$$\psi \sim \begin{pmatrix} \frac{c_0(t)}{\sqrt{z}} \\ \frac{d_0(t)}{\sqrt{\overline{z}}} \end{pmatrix} + \sum_{n \geqslant 1} \sum_{k=-2n-1}^{2n+2} \sum_{p=0}^{n-1} \begin{pmatrix} c_{n,k,p}(t)e^{ik\theta} \\ d_{n,k,p}(t)e^{ik\theta} \end{pmatrix} r^{n-1/2} \log(r)^p e^{-i\theta/2}$$

where $c_{n,k,p}, d_{n,k,p} \in L^{-1/2-n}(S^1; \mathbb{C})$ are understood in a distributional sense and are sections of an appropriate power of $N\mathcal{Z}_0$ as in Corollary 3.9. *There is no nice sense in which these weak expansions converge.* In particular, if $\psi \in L^2$ has such an expansion, then the difference $|\psi - \psi_N|$ will not necessarily lie in L^2. Consequently, there is no robust notion in which the later terms are "smaller" than the earlier ones. If there were stronger notions of convergence for such expansions, it is possible that the use of Nash-Moser Theory could be eliminated in the proof of Theorem 1.4.

4 The Obstruction Space

This section studies the infinite-dimensional cokernel of the operator

$$\not{D} : rH^1_e(Y - \mathcal{Z}_0; S_0) \longrightarrow L^2(Y - \mathcal{Z}_0; S_0), \tag{4.1}$$

which coincides with $\ker(\not{D}|_{L^2})$ by Lemma 2.6. The main results are Propositions 4.2 and 4.3, which generalize the "isomorphism" and "rapid decay" properties from Example 3.2 respectively. The first of these, Proposition 4.2, explicitly identifies the cokernel of (4.1) with a space of spinors on $\mathcal{Z}_0$. The second, Proposition 4.3 gives an explicit description of the cokernel elements, showing their support concentrates exponentially along $\mathcal{Z}_0$. Both propositions are used heavily in the upcoming sections. An essential and frustrating point is that any weakening of Proposition 4.3 leads to several error terms being unbounded in Section 6 .

Definition 4.1. We define the **Obstruction Space** associated to the data $(\mathcal{Z}_0, g_0, B_0)$ by

$$\mathbf{Ob}(\mathcal{Z}_0) := \{\psi \in L^2 \mid \psi \in \ker(\not{D}|_{L^2})\}.$$

Although this definition appears to be a redundant renaming of $\ker(\not{D}|_{L^2})$, it is stated this way in preparation for the later generalization in Section 8. There, we consider varying the tuple $(\mathcal{Z}_0, g_0, B_0)$ and must deal with the fact that $\ker(\not{D}|_{L^2})$ may not form a vector bundle.

By Lemma 2.6 there is a closed orthogonal decomposition

$$L^2 = \mathbf{Ob}(\mathcal{Z}_0) \oplus \mathrm{Range}(\not{D}|_{rH^1_e}).$$

There is also a further L^2-orthogonal decomposition $\mathbf{Ob}(\mathcal{Z}_0) = \mathbf{Ob}(\mathcal{Z}_0)^\perp \oplus \ker(\not{D}|_{rH^1_e})$ into the space of $\mathbb{Z}_2$-harmonic spinors and its orthogonal complement within the obstruction space.

Let $S_0|_{\mathcal{Z}_0}$ denote the spinor bundle restricted to $\mathcal{Z}_0$. Clifford multiplication $\gamma(dt)$ by the unit tangent vector induces a splitting $S|_{\mathcal{Z}_0} \simeq \mathcal{S}_{\mathcal{Z}_0} \oplus \mathcal{S}_{\mathcal{Z}_0}$ where $\mathcal{S}_{\mathcal{Z}_0}$ is a rank 1 complex spinor bundle on $\mathcal{Z}_0$.

Proposition 4.2. There is an isomorphism

$$\mathrm{ob} \oplus \iota : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \ker(\not{D}|_{rH^1_e}) \longrightarrow \mathbf{Ob}(\mathcal{Z}_0), \tag{4.2}$$

where ι is the inclusion of span $\ker(\not{D}|_{rH^1_e})$.

The spinor bundle $\mathcal{S}_{\mathcal{Z}_0}$ carries a 1-dimensional Dirac operator denoted $\not{\partial}_{\mathcal{Z}_0}$ as follows. Using an arclength parameterization, identify each component $\mathcal{Z}_{(j)}$ of $\mathcal{Z}_0$ with a circle with coordinate t_j and length $|\mathcal{Z}_{(j)}|$. This induces a trivialization $\mathcal{S}_{\mathcal{Z}_0} \simeq \underline{\mathbb{C}}$ as the $+i$ eigenspace of $\gamma(dt_j)$, and we define $\not{\partial}_{\mathcal{Z}_0}$ to be $i\partial_t$. This Dirac operator $\not{\partial}_{\mathcal{Z}_0}$ may be diagonalized on $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})$ with

$$\mathrm{Spec}(\not{\partial}_{\mathcal{Z}_0}) = \bigsqcup_j \tfrac{2\pi}{|\mathcal{Z}_{(j)}|}\mathbb{Z}$$

and eigenvectors $\phi^{(j)}_\ell = e^{i\ell t_j}$ for $\ell \in 2\pi\mathbb{Z}/|\mathcal{Z}_{(j)}|$. These eigenvector provide a convenient basis for $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})$.

Proposition 4.3. (A) When $\not{D}$ is complex linear, there is a complex basis $\Psi^{(j)}_\ell$ of $\mathbf{Ob}(\mathcal{Z}_0)^\perp$ indexed by $(j,\ell) \in \mathrm{Spec}(\not{\partial}_{\mathcal{Z}_0})$ such that the $\mathbf{Ob}(\mathcal{Z}_0)$-component of a spinor $\psi \in L^2$ under (4.2) is given by

$$\mathrm{ob}^{-1}(\Pi^{\mathrm{Ob}}\psi) = \sum_{\mathrm{Spec}(\not{\partial}_{\mathcal{Z}_0})} \langle \psi, \Psi^{(j)}_\ell \rangle_{\mathbb{C}} \phi^{(j)}_\ell. \qquad \iota^{-1}(\psi) = \sum_k \langle \psi, \Phi_k \rangle \Phi_k. \tag{4.3}$$

where $\langle -,- \rangle_{\mathbb{C}}$ is the hermitian inner product, and Φ_k a (real) basis of $\ker(\not{D}|_{rH^1_e})$. Moreover,

$$\Psi^{(j)}_\ell = \chi_j \Psi^{\mathrm{Euc}}_\ell + \zeta^{(j)}_\ell + \xi^{(j)}_\ell$$

where

- Ψ^{Euc}_ℓ are the L^2-orthonormalized Euclidean obstruction elements from Example 3.2 (in the trivialization 3.6) and χ_j is a cutoff function supported on a tubular neighborhood of $\mathcal{Z}_{(j)}$.
- $\zeta^{(j)}_\ell$ is a perturbation with L^2-norm $O(\frac{1}{|\ell|})$ which decays exponentially away from $\mathcal{Z}_0$ in the following sense:

$$\|\zeta^{(j)}_\ell\|_{L^2(A^{(j)}_{n\ell})} \leqslant \frac{C}{|\ell|} \mathrm{Exp}\left(-\frac{n}{c_1}\right). \tag{4.4}$$

  where $A^{(j)}_{n\ell}$ denotes the collection of annuli

$$A^{(j)}_{n\ell} = \left\{ \tfrac{n}{|\ell|} R_0 \leqslant r^{(j)} \leqslant \tfrac{n+1}{|\ell|} R_0 \right\} \tag{4.5}$$

  for some constant R_0, and $r^{(j)}$ denotes the geodesic distance to $\mathcal{Z}_{(j)}$. Additionally, in Fermi coordinates on $N_{r_0}(\mathcal{Z}_{(j)})$ and in the trivialization of Lemma 3.6, $\zeta^{(j)}_\ell$ is a linear combination of only Fourier modes e^{ipt} in the range $\ell - |\ell|/2 \leqslant p \leqslant \ell + |\ell|/2$.

- $\xi_\ell^{(j)}$ is a perturbation of L^2-norm $O(\frac{1}{|\ell|^2})$ i.e. satisfying

$$\|\xi_\ell^{(j)}\|_{L^2} \leqslant \frac{C}{|\ell|^2}$$

for a universal constant C.

(B) In the case that $\not{D}$ is only $\mathbb{R}$-linear, then there is a real basis

$$\Psi_\ell^{\mathrm{Re},j} = \chi_j \Psi_\ell^{\mathrm{Euc}} + \zeta_\ell^{\mathrm{Re},j} + \xi_\ell^{\mathrm{Re},j} \qquad\qquad \Psi_\ell^{\mathrm{Im},j} = i(\chi_j \Psi_\ell^{\mathrm{Euc}}) + \zeta_\ell^{\mathrm{Im},j} + \xi_\ell^{\mathrm{Im},j}$$

satisfying identical bounds where the inner product in (4.3) is replaced by

$$\langle \psi, \Psi_\ell^{(j)} \rangle_{\mathbb{C}} = \langle \psi, \Psi_\ell^{\mathrm{Re},j} \rangle \ + \ i \langle \psi, \Psi_\ell^{\mathrm{Im},j} \rangle.$$

Propositions 4.2-4.3 are proved concurrently, with the proof occupy the remainder of the section. The proof has several steps, and goes roughly as follows. Because the Euclidean obstruction elements Ψ_ℓ^{Euc} decay exponentially away from $\mathcal{Z}_0$, and the Dirac operator differs from the Euclidean one by $O(r)$, these Euclidean obstruction elements are very good approximations to those on Y once $|\ell|$ is sufficiently large. Intuitively, pasting Ψ_ℓ^{Euc} onto Y using a cut-off function and correcting them by projection provides a Fredholm map

$$\mathbf{Ob}^{\mathrm{Euc}} \to \mathbf{Ob}(\mathcal{Z}_0) \tag{4.6}$$

whose index can be shown to be zero. Then, up to replacing a finite-dimensional space for small indices $|\ell|$, this yields an isomorphism with the model Euclidean case, which is identified with $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})$ by Example 3.2.

The reason for the caveat “intuitively” is that simply pasting the Euclidean elements Ψ_ℓ^{Euc} onto Y does not immediately yield good enough approximations to show (4.6) is Fredholm. To deal with this, we first construct better approximations Ψ_ℓ^N on the normal bundle $N\mathcal{Z}_0$ using a metric g_N equal to that on Y on a tubular neighborhood, after which above argument proceeds to show that (4.6) is an isomorphism. The map ob is obtained from (4.6) after a compact correction that ensures (4.3) holds (see Remark 4.4). Finally, it must be shown that all these construction have appropriate higher-regularity analogues on H^m_{b}. Thus the five steps of the proof are the following, each of which occupies a subsection:

Section 4.1: Construction of model obstruction elements Ψ_ℓ^N on the normal bundle $(N\mathcal{Z}_0, g_N)$.
Section 4.2: Patching argument to show pasting Ψ_ℓ^N onto (Y, g_0) is Fredholm onto $\mathbf{Ob}$.
Section 4.3: Index Calculation.
Section 4.4: Finite-dimensional correction for small $|\ell|$ and construction of ob.
Section 4.5: Higher-Regularity.

We prove the propositions in the case that $\not{D}$ is complex linear, and use $\langle -, - \rangle$ to denote the Hermitian inner product in this subsection. The real-linear case, which is needed later, differs only notationally. Additionally, since everything in the construction is local, we will tacitly assume

Assumption 4*. $\mathcal{Z}_0$ is smooth with a single component

and omit the superscript (j) from the notation. The general case is a trivial extension.

Remark 4.4. The reader is cautioned that the basis Ψ_ℓ is not in general orthonormal. The reason for this is that orthonormalizing disrupts the decay properties of ζ_ℓ. Additionally, the map ob is not given by the obvious linear extension of $\phi_\ell \mapsto \Psi_\ell$. Because the sequence of inner products does not calculate the expression for a spinor ψ in the given basis if it is not orthonormal, this naive map must be altered in order for the projection of ψ to still be calculated by its inner products (4.3). This is done in Section 4.4.

4.1 The Model Obstruction

This section begins the proof of Propositions 4.2-4.3 by constructing model basis for the obstruction on the normal bundle $N\mathcal{Z}_0$.

Let $r_0 > 0$ be small, and take χ_0 a cut-off function vanishing for $r > r_0$ and equal to 1 for $r < r_0/2$. Set

$$(N, g_N) := (N\mathcal{Z}_0, \chi_0 g_0 + (1-\chi_0)(dt^2 + dx^2 + dy^2))$$

where $N\mathcal{Z}_0 \simeq S^1 \times \mathbb{R}^2$ is the normal bundle of $\mathcal{Z}_0$ equipped with coordinates (t, x, y) as in Section 3.2. The spinor bundle $S_N \to N$ may be identified with $S_N = \underline{\mathbb{C}}^2 \otimes \ell_{\mathcal{Z}_0}$, after which the Dirac operator on (N, g_N) may be written

$$\not{D}_N := \not{D}_0 + \mathfrak{d}$$

with $\not{D}_0, \mathfrak{d}$ as in Lemma 3.6. If the spin structure falls in Case 2 as in (3.2), then we use $\chi_0\gamma(idt/2)$ in place of $\gamma(idt/2)$; additionally, we replace B_0 with $\chi_0 B_0$ so that this operator is equal to the Dirac operator $\not{D}$ on Y for $r \leqslant r_0/2$ and equal to $\not{D}_0$ for $r \geqslant r_0$.

The following lemma completes describes the obstruction on (N, g_N) for r_0 sufficiently small.

Lemma 4.5. For $r_0 << 1$, $\ker(\not{D}_N|_{rH^1_e}) = \varnothing$ and there is a basis

$$\Psi_\ell^N = \Psi_\ell^{\text{Euc}} + \zeta_\ell^N + \xi_\ell^N$$

of $\ker(\not{D}_N|_{L^2})$ satisfying the bounds of Proposition 4.3 such that

$$\begin{array}{rcl} L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})_0 & \to & \ker(\not{D}_N|_{L^2}) \\ \phi_\ell & \mapsto & \Psi_\ell^N \end{array}$$

is an isomorphism, where $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})_0$ means the orthogonal complement of the 0-eigenspace.

Remark 4.6. The $\ell = 0$ element is omitted simply because the $r^{-1/2}$ asymptotics fail to be L^2 on the non-compact space N. Fredholmness is insensitive to discarding the $\ell = 0$ mode, but it must be accounted for in the index calculation of Section 4.3.

Proof of the isomorphism in Lemma 4.5. As in Lemma 3.6, we have bounds

$$|\mathfrak{d}\varphi| \leqslant C(r|\nabla\varphi| + |\varphi|)$$

on $\text{supp}(\chi_0)$. It follows that $\|\mathfrak{d}\varphi\|_{L^2} \leqslant Cr_0\|\varphi\|_{rH^1_e}$. Thus for r_0 sufficiently small,

$$\|\varphi\|_{rH^1_e} \leqslant C\|\not{D}_0\varphi\|_{L^2} \leqslant C\|(\not{D}_0 + \mathfrak{d})\varphi\|_{L^2} + C\|\mathfrak{d}\|_{L^2} \qquad \Rightarrow \qquad \|\varphi\|_{rH^1} \leqslant C'\|\not{D}_N\varphi\|_{L^2}$$

which shows $\ker(\not{D}|_{rH^1_e}) = \varnothing$.

Since $\ker(\not{D}|_{rH^1_e}) = \varnothing$, it follows as in Lemma 2.7 that

$$\not{D}_N\not{D}_N : rH^1_e \to rH^{-1}_e$$

is an isomorphism. Denote by P_N its inverse. Then, as in Corollary 2.8, the projection to the range factor in $L^2 = \ker(\not{D}_N|_{L^2}) \oplus \text{Range}(\not{D}_N|_{rH^1_e})$ is given by $\Pi^{\text{Rg}} = \not{D}_N P_N \not{D}_N$. Define

$$\Psi_\ell^N := \Psi_\ell^{\text{Euc}} - \Pi^{\text{Rg}}(\Psi_\ell^{\text{Euc}}) = \Psi_\ell^{\text{Euc}} - \not{D}_N P_N \not{D}_N(\Psi_\ell^{\text{Euc}}) \tag{4.7}$$

so that $\Psi_\ell^N \in \ker(\not{D}_N|_{L^2})$.

We claim that the map defined by $\phi_\ell \mapsto \Psi_\ell^N$ is an isomorphism. Clearly $\phi_\ell \mapsto \Psi_\ell^{\text{Euc}}$ is an isomorphism to $\ker(\not{D}_0|_{L^2})$ by Example 3.2, so it suffices to show that $Id - \Pi^{\text{Rg}} : \ker(\not{D}_0|_{L^2}) \to \ker(\not{D}_N|_{L^2})$ is an isomorphism. For injectivity, note that for any $\Psi \in L^2$, integrating by parts show

$$\|\mathfrak{d}\Psi^{\text{Euc}}\|_{rH^{-1}_e} = \sup_{\|\varphi\|_{rH^1}=1} \langle \mathfrak{d}\Psi, \varphi\rangle \leqslant Cr_0\|\Psi\|_{L^2}. \tag{4.8}$$

Hence for any $\Psi^{\text{Euc}} \in \ker(\not{D}_0|_{L^2})$ one has

$$\|\Psi^{\text{Euc}}\|_{L^2} \leqslant \|(Id - \not{D}_N P_N \not{D}_N)\Psi^{\text{Euc}}\|_{L^2} + \|\not{D}_N P_N \not{D}_N \Psi^{\text{Euc}}\|_{L^2}$$

and $\not{D}_N P_N \not{D}_N \Psi^{\text{Euc}} = \not{D}_N P_N \mathfrak{d}\Psi^{\text{Euc}}$, hence (4.8) shows that $\|\not{D}_N P_N \not{D}_N \Psi^{\text{Euc}}\|_{L^2} \leqslant Cr_0\|\Psi^{\text{Euc}}\|_{L^2}$. It follows that for r_0 sufficiently small, $Id - \Pi^{\text{Rg}} : \ker(\not{D}_0|_{L^2}) \to \ker(\not{D}_N|_{L^2})$ is injective with closed range.

For surjectivity, we argue by contraction: suppose that $\eta \in \ker(\not{D}_N|_{L^2})$ were orthogonal to the image of $Id - \Pi^{\text{Rg}}$ and normalized so that $\|\eta\|_{L^2} = 1$. Since $\eta \in \ker(\not{D}_N|_{L^2}) = \text{Range}(\not{D}_N|_{rH^1_e})^{\perp}$ we have

$$0 = \langle \not{D}_N \varphi, \eta\rangle = \langle (\not{D}_0 + \mathfrak{d})\varphi, \eta\rangle$$

for all $\varphi \in rH^1_e$, hence $|\langle \not{D}_0\varphi, \eta\rangle| = |\langle \mathfrak{d}\varphi, \eta\rangle| \leqslant Cr_0\|\varphi\|_{rH^1_e}$, which is to say that component of η in $\text{Range}(\not{D}_0)$ is small. Consequently, there is a Ψ^{Euc}_η so that we may write $\eta = \Psi^{\text{Euc}}_\eta + w$ with $\Psi^{\text{Euc}}_\eta \in \ker(\not{D}_0|_{L^2})$ and

$$1 - Cr_0 \leqslant \|\Psi^{\text{Euc}}_\eta\|_{L^2} \leqslant 1 \qquad\qquad \|w\|_{L^2} \leqslant Cr_0$$

But this would imply

$$0 = \langle Id - \Pi^{\text{Rg}}(\Psi^{\text{Euc}}), \eta\rangle = \langle \Psi^{\text{Euc}}_\eta + \not{D}_N P_N \mathfrak{d}\Psi^{\text{Euc}}_\eta, \Psi^{\text{Euc}}_\eta + w\rangle \geqslant 1 - C' r_0,$$

a contradiction after possibly decreasing r_0. □

4.1.1 Exponential Decay:

We now turn to the issue of showing that there is a decomposition

$$\Psi^N_\ell = \Psi^{\text{Euc}}_\ell + \zeta^N_\ell + \xi^N_\ell$$

satisfying the desired exponential decay properties for ζ^N_ℓ.

To begin, we state a general lemma about solutions of the equation

$$\not{D}_0 \not{D}_0 u = f$$

is proved which is proved in Appendix A. Here, $\not{D}_0$ denotes the Dirac operator in the product metric. In the setting where f has restricted Fourier modes in the S^1 direction, then solutions with only high Fourier modes should enjoy good exponential decay properties away from $\mathcal{Z}_0$. To state the lemma, we denote by $A_{n\ell}$ the sequence of annuli 4.5 from Part (B) of Proposition 4.2, and define $B_{n\ell} = A_{(n-1)\ell} \cup A_{n\ell} \cup A_{(n+1)\ell}$.

Lemma 4.7. Let m be a non-negative integer, and assume that $|\ell| \geqslant 2m$. Suppose that $u_\ell \in rH^1_e(N)$ is the unique solution of

$$\not{D}_0 \not{D}_0 u_\ell = f_\ell \tag{4.9}$$

where $f_\ell \in r^{-1}H^{-1}_e$ satisfies the following two properties:

1. f_ℓ has only Fourier modes in t in the range

$$\ell - L_0 \leqslant p \leqslant \ell + L_0 \tag{4.10}$$

   where $|L_0| \leqslant |\ell|/2$.

2. For m as above, there are constants C_m, c_m independent of ℓ such that f_ℓ satisfies the bounds

$$\|f_\ell\|^2_{r^{-1}H^{-1}_e(B_{n\ell})} \leqslant \frac{C_m}{|\ell|^{2+2m}} \text{Exp}\left(-\frac{2n}{c_m}\right) \tag{4.11}$$

   on the sequence of annuli $B_{n\ell}$.

Then there are constants C'_m, c'_m independent of ℓ such that u_ℓ satisfies

$$\|u_\ell\|^2_{rH^1_e(A_{n\ell})} \leqslant \frac{C'_m}{|\ell|^{2+2m}} \mathrm{Exp}\left(-\frac{2n}{c'_m}\right). \tag{4.12}$$

Moreover, u_ℓ has only Fourier modes in the same range as f_ℓ. □

Given this lemma, we now complete the proof of Lemma 4.5 by showing that Ψ^N_ℓ admits a decomposition satisfying the desired bounds:

Proof of the form of Ψ^N_ℓ in Lemma 4.5. In (4.7), we defined

$$\Psi^N_\ell := \Psi^{\mathrm{Euc}}_\ell - \not{D}_N P_N \not{D}_N(\Psi^{\mathrm{Euc}}_\ell)$$

thus it suffices to show that there is a decomposition $-\not{D}_N P_N \not{D}_N(\Psi^{\mathrm{Euc}}_\ell) = \zeta^N_\ell + \xi^N_\ell$ with the latter two satisfying the desired estimates of Proposition 4.3.

We may write $\not{D}_N = \not{D}_0 + \mathfrak{d}$ where $\mathfrak{d}$ can be explicitly written in the form

$$\mathfrak{d} = \sum_{ij=1}^{3} a_{i,j}(t,x,y)\sigma_i\partial_j + \sum_{k=0}^{3}\Gamma_k(t,x,y)\sigma_k$$

where $|a_{ij}| \leqslant Cr$ and $|\Gamma| \leqslant C$ and $\sigma_0 = Id$ in the second sum. Decomposing $a_{ij}(t,x,y), \Gamma_k(t,x,y)$ into the Fourier modes in the t-direction on $N \simeq S^1 \times \mathbb{R}^2$, this operator can be written as

$$\mathfrak{d} = \mathfrak{d}^{\mathrm{low}} + \mathfrak{d}^{\mathrm{high}}$$

where $\mathfrak{d}^{\mathrm{low}}$ consists of the Fourier modes of a_{ij}, Γ_k with Fourier index $|p| \leqslant |\ell|/4$.

We now define ζ^N_ℓ, ξ^N_ℓ as follows. Since $\not{D}_0 \Psi^{\mathrm{Euc}}_\ell = 0$ by definition, one has

$$\not{D}_N P_N \not{D}_N(\Psi^N_\ell) = \not{D}_N P_N(\mathfrak{d}^{\mathrm{low}}\Psi^N_\ell + \mathfrak{d}^{\mathrm{high}}\Psi^N_\ell) = \not{D}_N P_N(f^{\mathrm{low}}_\ell + f^{\mathrm{high}}_\ell).$$

where $f^{\mathrm{low}}_\ell := \mathfrak{d}^{\mathrm{low}}\Psi^N_\ell$ and $f^{\mathrm{high}}_\ell := \mathfrak{d}^{\mathrm{high}}\Psi^N_\ell$. Recall that P_0, P_N denote the solution operators so that $u = P_0(f)$ denotes the unique solution of $\not{D}_0\not{D}_0 u = f$ and likewise for P_N with $\not{D}_N$. Set

$$\begin{array}{lclcl} \zeta^N_\ell & := & \not{D}_N u_\ell & \qquad & u_\ell := -P_0(f^{\mathrm{low}}_\ell) \\ \xi^N_\ell & := & \not{D}_N v_\ell & \qquad & v_\ell := -P_N(f^{\mathrm{high}}_\ell - (\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell) \end{array}$$

By construction, we have

$$\begin{aligned} \not{D}_N(\zeta^N_\ell + \xi^N_\ell) &= \not{D}_N\not{D}_N v_\ell + (\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell + \not{D}_0\not{D}_0 u_\ell \\ &= -f^{\mathrm{high}}_\ell + (\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell - (\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell - f^{\mathrm{low}}_\ell \\ &= -\not{D}_N(\Psi^N_\ell) \end{aligned}$$

And clearly $\zeta^N_\ell + \xi^N_\ell \in \mathrm{Range}(\not{D}_N)$, thus $\zeta^N_\ell + \xi^N_\ell$ is the unique $\ker(\not{D}_N|_{L^2})$-perpendicular solution of $\not{D}_N\psi = -\not{D}_N\Psi^{\mathrm{Euc}}_\ell$. Equivalently, $\zeta^N_\ell + \xi^N_\ell := -\not{D}_N P_N \not{D}_N(\Psi^N_\ell)$ is the desired correction.

To see that ζ_ℓ satisfies the desired decay properties, we apply Lemma 4.7 in the case that $m = 0$. The first hypothesis of that lemma is satisfied by construction, as $f^{\mathrm{low}}_\ell = \mathfrak{d}^{\mathrm{low}}\Psi^N_\ell$ has Fourier modes in the desired range by definition. To verify the second hypothesis, note that since $f^{\mathrm{low}}_\ell \in L^2$ we have

$$\|f^{\mathrm{low}}_\ell\|_{r^{-1}H^{-1}_e(B_{n,\ell})} \leqslant \sup_{\|u\|=1}\langle u, f^{\mathrm{low}}_\ell\rangle_{L^2} \leqslant \sup_{\|u\|=1}\|u\|_{rH^1_e}\|rf^{\mathrm{low}}_\ell\|_{L^2(B_{n\ell})} \leqslant \|rf^{\mathrm{low}}_\ell\|_{L^2(B_{n\ell})}$$

hence using the bounds $|a_{ij}| \leqslant Cr$ and $|\Gamma_k| \leqslant C$ for $\mathfrak{d}^{\mathrm{low}}$,

$$\int_{B_{n\ell}} r^2|f_\ell^{\text{low}}|^2 \, dV \;\leqslant\; C\frac{n^2}{|\ell|^2}R_0^2\int_{B_{n\ell}} |r\nabla_j\psi_\ell^{\text{Euc}}|^2+|r\nabla_t\psi_\ell^{\text{Euc}}|^2+|\psi_\ell^{\text{Euc}}|^2 \; rdrd\theta dt \tag{4.13}$$

$$\leqslant\; C\frac{n^2}{|\ell|^2}R_0^2\int_{B_{n\ell}} (1+r^2|\ell|^2+1)\frac{e^{-2|\ell|r}}{r}|\ell| \; rdrd\theta dt \tag{4.14}$$

$$\leqslant\; C\frac{n^5}{|\ell|^2}R_0^5e^{-nR_0} \leqslant \frac{C'}{|\ell|^2}e^{-2n/c_1}. \tag{4.15}$$

Thus we conclude from Lemma 4.7 that

$$\|u_\ell\|_{rH^1_e(A_{n\ell})} \leqslant \frac{C_0}{|\ell|}\text{Exp}\left(-\frac{n}{c_0}\right) \qquad \Rightarrow \qquad \|\zeta_\ell^N\|_{L^2(A_{n\ell})} \leqslant \frac{C_0}{|\ell|}\text{Exp}\left(-\frac{n}{c_0}\right)$$

as desired.

To finish, we show the desired bound on ξ_ℓ^N as well. Since $\not{D}_N : rH^1_e \to L^2$ and $P_N : rH^{-1}_e \to rH^1_e$ are bounded, it suffices to show that

$$\|f_\ell^{\text{high}} - (\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell\|_{r^{-1}H_e^{-1}} \leqslant \frac{C}{|\ell|^2}. \tag{4.16}$$

Addressing the two terms on the left separately, one has $\not{D}_N\not{D}_N - \not{D}_0\not{D}_0 = \mathfrak{d}\not{D}_0 + \not{D}_0\mathfrak{d} + \mathfrak{d}^2$ which shows

$$\|(\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell\|^2_{r^{-1}H_e^{-1}} \leqslant C\sum_n \sup_{A_{n\ell}}(r^2\|u_\ell\|^2_{rH^1_e(A_{n\ell})}) \leqslant \frac{C}{|\ell|^4}. \tag{4.17}$$

For the first term, we use the fact that the coefficients a_{ij}, Γ_k are smooth and $\mathfrak{d}^{\text{high}}$ have only Fourier modes p with $|p| \geqslant |\ell|/4$, in conjunction with the Sobolev embedding for each fixed $(x,y) \in D_{r_0}$. For example,

$$\|a^{\text{high}}\|_{C^0(Y)} \leqslant \sup_{x,y}\|a^{\text{high}}(t)\|_{C^0(S^1)} \leqslant C\sup_{x,y}\|a^{\text{high}}(t)\|_{L^{1,2}(S^1)} \leqslant \frac{C}{|\ell|^2}\sup_{x,y}\|a^{\text{high}}(t)\|_{L^{3,2}(S^1)} \leqslant \frac{Cr}{|\ell|^2} \tag{4.18}$$

and likewise for Γ^{high}. Combining the bounds (4.17) and (4.18) shows (4.16), completing the proof. Combining with the proof of the isomorphism via (4.7) completes the proof of Lemma 4.5.

□

A slight extension of the above shows the following stronger estimates, which will be used later to deliver estimates on the higher derivatives of ζ_ℓ, ξ_ℓ.

Corollary 4.8. For every m there is an alternative decomposition

$$\zeta_\ell^N + \xi_\ell^N = \zeta_\ell^{(m)} + \xi_\ell^{(m)}$$

where

- There are constants C_m and C'_m such that

$$\|\zeta_\ell^{(m)}\|_{L^2(A_{n\ell})} \leqslant \frac{C_m}{|\ell|}\text{Exp}\left(-\frac{n}{c_m}\right) \qquad \|(r\nabla_z)^\alpha(\nabla_t)^\beta\zeta_\ell^{(m)}\|_{L^2(A_{n\ell})} \leqslant \frac{C'_m|\ell|^\beta}{|\ell|}\text{Exp}\left(-\frac{n}{c'_m}\right). \tag{4.19}$$

  where $A_{n\ell}$ is as in Proposition 4.2 and where ∇_z is the covariant derivative in the normal directions.

- The latter perturbation satisfies

$$\|\xi_\ell^{(m)}\|_{L^2} \leqslant \frac{C_m}{|\ell|^{2+m}}. \qquad \|(r\nabla_z)^\alpha(\nabla_t)^\beta\xi_\ell^{(m)}\|_{L^2} \leqslant \frac{C'_m|\ell|^\beta}{|\ell|^{2+m}} \tag{4.20}$$

Moreover, ζ_ℓ contains only Fourier modes e^{ipt} with $\ell - \frac{|\ell|}{2} \leqslant p \leqslant \ell + \frac{|\ell|}{2}$. The constants C_m, c_m are independent of ℓ, and depend on up to the $L^{m+3,2}$-norm of the metric, and C'_m, c'_m on up to the $L^{m+|\alpha|+|\beta|+3,2}$-norm.

Proof. For $\alpha = \beta = 0$, this follows from applying Lemma 4.7 inductively. Instead of solving for ξ_ℓ with $f_\ell^{\text{high}} - (\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell$ on the right hand side, set $(f_\ell^{\text{low}})^1 = -(\not{D}_N\not{D}_N - \not{D}_0\not{D}_0)u_\ell$ and apply Lemma 4.7 again to the low Fourier modes to obtain a second correction ζ_ℓ^1 and set $\zeta_\ell^{(1)} = \zeta_\ell^0 + \zeta_\ell^1$. Proceeding in this fashion, each iteration yields an additional power of $|\ell|^{-1}$ in the new remainder. To control the range of Fourier modes, define the low modes instead by truncating at $L_0 = |\ell|/4m$. Using higher Sobolev norms in 4.18 allows the remainder after m iterations to be controlled by $C_m|\ell|^{-(m+2)}$, after which gives the bound on $\xi_\ell^{(m)}$.

The higher derivative estimates follow from repeating the argument applying estimates for nested sequences of commutators $[r\nabla_z, \not{D}_N]$ and $[\nabla_t, \not{D}_N]$. Each application of ∇_t adds a power of $|\ell|$, but the normal ∇^{b}-derivatives only constants. □

4.2 Fredholm Properties

This subsection continues the proof of Propositions 4.2 and 4.3 by showing that pasting the model basis Ψ_ℓ^N onto Y gives a Fredholm map onto $\mathbf{Ob}(\mathcal{Z}_0)$.

We begin by defining this patching map; this map is a preliminary version of the map ob from Proposition 4.2. Let χ_1 denote a cut-off function equal to 1 for $r \leqslant r_0/4$ such that $\text{supp}(\chi_1) \subseteq \{r \leqslant r_0/2\}$. Define

$$\begin{aligned} M : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}^k \quad &\to \quad \mathbf{Ob}(\mathcal{Z}_0) && (4.21)\\ (\phi_\ell, c_\alpha) \quad &\mapsto \quad \chi_1\Psi_\ell^N - \not{D}P\not{D}(\chi_1\Psi_\ell^N) + c_\alpha\Phi_\alpha && (4.22) \end{aligned}$$

where $\alpha = 1, ..., k = \dim(\ker \not{D}|_{rH^1})$. Here P is again the solution operator on Y defined in (2.10). Additionally, we set $\Psi_0^N := \Psi_0^{\text{Euc}}$ to span a real 2-dimensional subspace of the span of $(z^{-1/2}, 0)$ and $(0, \overline{z}^{-1/2})$, which is specified in the subsequent Section 4.3.

Lemma 4.9. The map M defined by (4.22) is Fredholm.

Proof. It suffices prove this ignoring the finite-dimensional span of ϕ_0 and Φ_k. With these modes discarded, Lemma 4.5 shows M may be viewed as a map

$$\begin{aligned} M : \ker(\not{D}_N|_{L^2}) \quad &\to \quad \mathbf{Ob}(\mathcal{Z}_0) \\ \Psi \quad &\mapsto \quad \chi_1\Psi - \not{D}v_\Psi \qquad \text{where} \qquad v_\Psi := P\not{D}(\chi_1\Psi). \end{aligned}$$

We define a pseudo-inverse

$$\begin{aligned} M^\dagger : \mathbf{Ob}(\mathcal{Z}_0) \quad &\to \quad \ker(\not{D}_N|_{L^2}) \\ \Phi \quad &\mapsto \quad \chi_1\Phi - \not{D}_N u_\Psi \qquad \text{where} \qquad u_\Phi := P_N\not{D}_N(\chi_1\Phi). \end{aligned}$$

To prove the lemma, we check that $M, M^\dagger$ are indeed pseudo-inverses so that $M^\dagger M = Id + A_1$ and $MM^\dagger = Id + A_2$ for compact operators A_1, A_2 from which Fredholmness follows. First, we note that standard elliptic theory implies the following: if $K \Subset Y - \mathcal{Z}_0$ is compactly contained in the complement of $\mathcal{Z}_0$, then the restriction

$$R : \mathbf{Ob}(\mathcal{Z}_0) \to rH_e^1(K) \tag{4.23}$$

is compact. Indeed, since $\not{D}$ is uniformly elliptic away from $\mathcal{Z}_0$, this follows from standard elliptic bootstrapping and Rellich's Lemma. The equivalent statement holds on $K_N \Subset N$, but compactness then also *a priori* requires that K_N be bounded in the non-compact N.

A quick computation shows

$$\begin{aligned}(MM^\dagger - I)\Phi &= (\chi_1^2-1)\Phi - \chi_1 \not{D}_N u_\Phi - \not{D} v_{M^\dagger\Phi}. && (4.24)\\ (M^\dagger M - I)\Psi &= (\chi_1^2-1)\Psi - \chi_1 \not{D} v_\Psi - \not{D}_N u_{M\Psi}. && (4.25)\end{aligned}$$

and we claim the terms on the right hand side are compact. For the first expression, $\text{supp}(\chi_1^2-1) \Subset Y - \mathcal{Z}_0$ hence compactness follows from what was said about the restriction map (4.23). Likewise, 4.23 implies that the map $\Phi \mapsto u_\Phi$ is compact since it may be written as the composition

$$u = P_N \circ d\chi_1 . \circ R|_{\text{supp}(d\chi_1)}.$$

Similarly, $\Psi \mapsto v_\Psi$ is compact. Since the remaining terms on the right hand side of factor through these, we conclude that $MM^\dagger - Id$ is compact. The only difference for $M^\dagger M - Id$ is that $(\chi_1^2 - 1)$ is not compactly supported on Y_N. Nevertheless, a standard diagonalization using the decay properties of $\Psi_\ell^N = \Psi_\ell^{\text{Euc}} + \zeta_\ell^N + \xi_\ell^N$ shows that it is compact on elements of $\mathbf{Ob}_N$ (choose subsequences on that simultaneously converge on $r \leqslant n$ and on the span of $|\ell| \leqslant n$). $\square$

4.3 The Index via Concentration

In this subsection we calculate the Fredholm index of the map $M : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}^n \to \mathbf{Ob}(\mathcal{Z}_0)$. This is done by introducing a family of perturbations depending on $\mu \in \mathbb{R}$ [1]

$$\not{D}_\mu := \not{D} + \mu J$$

where J is a complex anti-linear map with $J^2 = -Id$. As $\mu \to \infty$ the cokernel elements become increasingly concentrated near $\mathcal{Z}_0$, and for μ sufficiently large we may conclude that the μ-version of M_μ is an isomorphism. There are two subtleties in this. 1) one must be careful to ensure the family M_μ can be viewed on a fixed Banach space, as $\ker(\not{D}_\mu|_{rH^1_e})$ may jump in dimension as μ varies. 2) The role of the $\ell = 0$ modes for the index must be clarified. On $S^1 \times \mathbb{R}^2$, the ℓ^{th} Fourier mode ℓ has two solutions linearly independent over $\mathbb{C}$, one *decaying* exponentially like $e^{-|\ell|r}$ (this being Ψ_ℓ^{Euc}), and another *growing* exponentially like $e^{+|\ell|r}$ which is not L^2 hence not part of $\ker(\not{D}_0|_{L^2})$. For the $\ell = 0$ mode the situation is different: there are four (real) solutions given by the complex span of $(1/\sqrt{z}, 0)$ and $(0, 1/\sqrt{\overline{z}})$ *neither* of which are L^2. It is not at first clear which of these should contribute to the index, though we will show that a particular 2-dimensional subspace contributes. The point is that the perturbation $\mu \to \infty$ breaks a degeneracy between the exponentially growing and exponentially decaying modes for $\ell = 0$ and distinguishes a 2-dimensional subspace of exponentially decaying ones.

Lemma 4.10. The Fredholm map

$$M : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}^k \to \mathbf{Ob}(\mathcal{Z}_0)$$

from Lemma 4.9 has Index 0.

Proof. We can recast the Fredholm problem on fixed Banach spaces by considering the operator

$$\overline{\mathcal{Q}}_0 := \begin{pmatrix} M & 0 \\ 0 & \overline{\not{D}} \end{pmatrix} : \begin{matrix} L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}^k \\ \oplus \\ rH^1_e \end{matrix} \longrightarrow \begin{matrix} \mathbf{Ob}(\mathcal{Z}_0) \\ \oplus \\ \text{Range}(\not{D}) \oplus \mathbb{R}^k \end{matrix} = L^2(Y; S_0) \oplus \mathbb{R}^k$$

where $\overline{\not{D}} = (\not{D}, \sum_\alpha \langle \_, \Phi_\alpha \rangle \Phi_\alpha)$. By construction, $\overline{\not{D}}$ is Fredholm of Index 0, hence $\mathcal{Q}$ is Fredholm, and it suffices to show that $\mathcal{Q}$ has Index 0. Notice also that if we were to replace P in the definition of M with another parametrix P' for $\not{D}\not{D} : rH^1_e \to rH^{-1}_e$ then the resulting

$$\mathcal{Q}_0 := M' \oplus \overline{\not{D}} \tag{4.26}$$

[1] This approach was suggested to the author by Clifford Taubes.

differs by compact operators, hence is Fredholm of the same index as $\overline{\mathcal{Q}}_0$ (though it is no longer necessarily block diagonal).

Now set $\not{D}_\mu := \not{D} + \mu J$ for $\mu \geqslant 0$. Since the Weitzenböck formula becomes

$$\not{D}_\mu^\star \not{D}_\mu = (\not{D} - \mu J)(\not{D} + \mu J) = \not{D}^\star \not{D} + \mu^2, \tag{4.27}$$

the proofs of Proposition 2.4 and Lemma 2.7 apply to show that $\not{D}_\mu : rH^1_e \to L^2$ has finite-dimensional kernel and closed range, and $\not{D}_\mu^\star \not{D}_\mu : rH^1_e \to rH^{-1}_e$ is Fredholm. Define a pseudo-inverse by

$$\overline{P}_\mu(f) = u \qquad\qquad \text{where} \qquad\qquad \not{D}_\mu^\star \not{D}_\mu u = f \mod \ker(\not{D}_\mu|_{rH^1_e})$$

is the unique $\ker(\not{D}_\mu|_{rH^1_e})$-perpendicular solution. The proofs of Lemmas 4.5 and 4.9 apply equally well to show that

$$\overline{Q}_\mu = \begin{matrix} L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}^k \\ \oplus \\ rH^1_e \end{matrix} \longrightarrow L^2(Y; S) \oplus \mathbb{R}^k$$

is a Fredholm operator for each μ. It is *not*, however, necessarily continuous in μ. If there are jumps in the dimension of $\ker(\not{D}_\mu^\star \not{D}_\mu|_{rH^1_e})$, then $\overline{P}_\mu$ need not be a continuous family of parametrices. Instead, let P_μ be a continuous family of parametrices for $\not{D}_\mu^\star \not{D}_\mu$. Using this to form $\mathcal{Q}_\mu$ for each $\mu \geqslant 0$ we obtain a now continuous family of Fredholm operators. As noted in (4.26) the operators $\mathcal{Q}_\mu$ are connected by a continuous path of Fredholm operators to $\overline{\mathcal{Q}}_0$, hence have the same index. Q_μ is continuous family of Fredholm operators although the summand M_μ is not since $\ker(\not{D}_\mu|_{rH^1})$ may jump in dimension as μ varies.

Given the above, it suffices to calculate $\text{Ind}(\mathcal{Q}_\mu)$ for $\mu >> 0$. Since the Weitzenböck formula 4.27 implies that $\ker(\not{D}_\mu|_{rH^1_e}) = \varnothing$ for $\mu >> 0$, we can arrange by a further homotopy of parametrices that $\mathcal{Q}_\mu$ is formed using $\overline{P}^\mu = (\not{D}_\mu^\star \not{D}_\mu)^{-1}$ once μ is large. Removing the $\mathbb{R}^k$ summands form both the domain and range does not disrupt Fredholmness nor alter the index, so these may be ignored. Furthermore, there is new splitting $L^2 = \ker(\not{D}_\mu^\star|_{L^2}) \oplus \text{Range}(\not{D}_\mu|_{rH^1_e})$ in which one may now write

$$\mathcal{Q}^\mu = \begin{pmatrix} M_\mu & 0 \\ 0 & \not{D}_\mu \end{pmatrix} : \begin{matrix} L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \\ \oplus \\ rH^1_e \end{matrix} \longrightarrow \begin{matrix} \ker(\not{D}_\mu^\star|_{L^2}) \\ \oplus \\ \text{Range}(\not{D}_\mu) \end{matrix}$$

where $M_\mu(\Psi^N_\ell) = \chi_1 \Psi^N_\ell - \not{D}_\mu \overline{P}_\mu \not{D}_\mu(\chi_1 \Psi^N_\ell)$ as before. Since $\not{D}_\mu$ is injective, hence an isomorphism onto its range, it suffices now to show that M_μ is an isomorphism for $\mu >> 0$. Finally, since $\not{D}_\mu$ is injective once μ is sufficiently large independent of small variation in the metric, we may arrange by a further homotopy through Fredholm operators that the metric is a product for $r \leqslant r_0$. The proof is then completed by the subsequent two lemmas. $\square$

The next lemma shows that the perturbation μJ means the L^2-kernel enjoys an additional factor of $e^{-\mu r}$ in the exponential decay compared to the $\mu = 0$ case, thus it is concentrated more strongly near $\mathcal{Z}_0$. The proof is an elementary exercise is solving ODEs by diagonalizing matrices since the Fourier modes decouple.

Lemma 4.11. For (N, g_{prod}), the perturbed Dirac operator

$$\not{D}_{N,\mu} : rH^1_e \longrightarrow L^2$$

is injective, and its adjoint's extension to L^2 has a kernel $\ker(\not{D}_{N,\mu}^\star|_{L^2})$ characterized by the following.

- There is a real 2-dimensional subspace of $\ker(\not{D}_{N,\mu}^\star|_{L^2})$ in the $\ell = 0$ modes. It is given by the span over $\mathbb{R}$ of

$$\psi_0^+ = \begin{pmatrix} \frac{e^{-\mu r}}{\sqrt{z}} \\ 0 \end{pmatrix} \qquad\qquad \psi_0^- = \begin{pmatrix} 0 \\ \frac{e^{-\mu r}}{\sqrt{\overline{z}}} \end{pmatrix}$$

- There is a real 4-dimensional subspace of $\ker(\not{D}^\star_{N,\mu}|_{L^2})$ in the $\pm\ell$ modes spanned over $\mathbb{R}$ by spinors

$$\psi^{(j)}_{|\ell|} = \frac{e^{\pm i\theta/2}}{r^{1/2}} e^{-\sqrt{\ell^2+\mu^2}} e^{\pm i\ell t} v^{(j)}$$

where $v^{(j)} \in \mathbb{R}^4$ for $j = 1, \ldots, 4$. $\square$

Given the above we can now specify a particular definition of M_μ by specifying a real 2-dimensional subspace of the $\ell = 0$ modes: take the real span of $\phi_0 \mapsto (z^{-1/2}, 0)$ and $i\phi_0 \mapsto (0, \overline{z}^{-1/2})$. This is the two-dimensional subspace of the $\ell = 0$ modes which decays exponentially for $\mu > 0$ alluded to at the beginning of the subsection.

Lemma 4.12. For $\mu >> 0$,

$$M_\mu : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \to \ker(\not{D}^\star_\mu|_{L^2})$$

is an isomorphism.

Proof. By the previous Lemma 4.11, one has $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) = \ker(\not{D}^\star_{N,\mu}|_{L^2})$. As before, $M_\mu : \ker(\not{D}^\star_{N,\mu}|_{L^2}) \to \ker(\not{D}^\star_\mu|_{L^2})$ and $M_\mu^\dagger : \ker(\not{D}^\star_\mu|_{L^2}) \to \ker(\not{D}^\star_{N,\mu}|_{L^2})$ are given by

$$\begin{aligned} M_\mu(\Psi) &= \chi_1\Psi - \not{D}_\mu v_\Psi \qquad \text{where} \qquad v_\Psi := P_\mu \not{D}^\star_\mu(\chi_1\Psi). \\ M_\mu^\dagger(\Phi) &= \chi_1\Phi - \not{D}_{N,\mu} u_\Psi \qquad \text{where} \qquad u_\Phi := P_{N,\mu}\not{D}^\star_{N,\mu}(\chi_1\Phi). \end{aligned}$$

Here, $P_\mu, P_{N,\mu}$ are the true inverses.

By the explicit forms in Lemma 4.11, every $\Psi \in \ker(\not{D}^\star_\mu|_{L^2})$ on N satisfies

$$\|\Psi\|_{L^2(\mathrm{supp}(d\chi_1))} \leqslant Ce^{-\mu r_0/c_1}\|\Psi\|_{L^2} \tag{4.28}$$

on $\mathrm{supp}(d\chi_1)$. It then follows from the expression 4.25 that

$$\|(M_\mu^\dagger M_\mu - I)\Psi\|_{L^2} \leqslant Ce^{-\mu r_0/c_1}\|\Psi\|_{L^2},$$

hence for μ sufficiently large, $M_\mu^\dagger M_\mu$ is an isomorphism thus M_μ is injective.

Surjectivity follows by the same argument using (4.24) in place of (4.25) where (4.28) is replaced by the bound

$$\|\Phi\|_{L^2(\mathrm{supp}(d\chi_1))} \leqslant \frac{C}{\mu}\|\Phi\|_{L^2(Y)} \tag{4.29}$$

for $\Phi \in \ker(\not{D}^\star_\mu)$ on Y. To prove (4.29), let ρ denote a cut-off function supported equal to 1 on $Y - N_{r_0/8}(\mathcal{Z}_0)$ so that $\rho = 1$ on $\mathrm{supp}(\chi_1)$. Integrating by parts shows

$$\begin{aligned} \int_{Y\setminus\mathcal{Z}_0} \rho\langle J\Phi, \not{D}\Phi\rangle &= \int_{Y\setminus\mathcal{Z}_0} \rho\langle J\not{D}\Phi, \Phi\rangle + \langle d\rho.J\Phi, \Phi\rangle\, dV \\ &= -\int_{Y\setminus\mathcal{Z}_0} \rho\langle \not{D}\Phi, J\Phi\rangle + \int_{Y\setminus\mathcal{Z}_0} \langle d\rho.J\Phi, \Phi\rangle\, dV \end{aligned}$$

since $\not{D}J = J\not{D}$ and $J^\dagger = -J$. Consequently, since $d\rho$ is bounded by a universal constant,

$$2\mathrm{Re}\langle \rho J\Phi, \not{D}\Phi\rangle_{L^2} \leqslant C\|\Phi\|_{L^2}. \tag{4.30}$$

Then, if $\Phi \in \ker(\not{D}^\star_\mu)$,

$$0 = \langle \rho J\Phi, (\not{D} - \mu J)\Phi\rangle_{L^2} = -\mu\langle \rho\Phi, \Phi\rangle_{L^2} + \langle \rho J\Phi, \not{D}\Phi\rangle_{L^2} \overset{4.30}{\Rightarrow} \mu\|\Phi\|_{L^2(\rho=1)} \leqslant C\|\Phi\|_{L^2(Y)}.$$

The latter gives (4.29) which implies M_μ is surjective for μ sufficiently large. This completes the lemma and thus the proof of Lemma 4.10. $\square$

4.4 The Obstruction Map

In this subsection we complete the proofs of Propositions 4.2 and 4.3 by constructing a map ob satisfying the asserted properties.

The map ob is constructed from the map M defined by (4.21)-(4.22). To review, we have now shown via Lemma 4.10 that M is Fredholm of Index 0. It is convenient to amend M by making the images of the two summands orthogonal. Thus we revise M to be given by

$$\begin{aligned} M : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}^k &\to \mathbf{Ob}(\mathcal{Z}_0) && (4.31)\\ (\phi_\ell, c_\alpha) &\mapsto \chi_1\Psi_\ell^N - \not{D}P\not{D}(\chi_1\Psi_\ell^N) - \pi_1(\chi_1\Psi_\ell^N) + c_\alpha\Phi_\alpha. && (4.32) \end{aligned}$$

where π_1 is the L^2-orthogonal projection onto $\ker(\not{D}|_{rH^1})$. Since π_1 has finite rank thus is compact, this does not disrupt the Fredholmness or index.

Let $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})_{L_0}$ denote the subspace spanned by ϕ_ℓ for $|\ell| \geqslant L_0$.

Lemma 4.13. For L_0 sufficiently large, the restricted map

$$M|_{L_0} : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})_N \oplus \mathbb{R} \to \mathbf{Ob}(\mathcal{Z}_0)$$

is injective.

Given Lemma 4.13, $\operatorname{ind}(M) = 0$ means that the (complex) codimension of $\operatorname{Im}(M|_{L_0}) \subseteq \mathbf{Ob}$ is $2L_0+1$, and we can make the following definition:

Definition 4.14. The **Obstruction Basis** is defined as

$$\Psi_\ell := \begin{cases} \chi_1\Psi_\ell^N - \not{D}P\not{D}(\chi_1\Psi_\ell^N) - \pi_1(\chi_1\Psi_\ell^N) & |\ell| > L_0 \\ \Psi_\ell & |\ell| \leqslant L_0 \end{cases}$$

where Ψ_ℓ for $|\ell| \leqslant L_0$ is chosen to be an orthonormal basis of the orthogonal complement of $\operatorname{Im}(M|_{L_0}) \subseteq \mathbf{Ob}(\mathcal{Z}_0)$. It then follows that the amended map

$$\begin{aligned} M' : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}^k &\to \mathbf{Ob}(\mathcal{Z}_0)\\ (\phi_\ell, c_\alpha) &\mapsto \Psi_\ell + c_\alpha\Phi_\alpha \end{aligned}$$

is an isomorphism. Additionally, by in the upcoming proof of Lemma 4.13, we will see that Ψ_ℓ admits a decomposition

$$\Psi_\ell = \chi_1\Psi_\ell^{\mathrm{Euc}} + \zeta_\ell + \xi_\ell \tag{4.33}$$

satisfying the desired properties.

Proof of Lemma 4.13. Clearly, if the Φ_α-component is 0 then $c_\alpha = 0$ since the images of the summands are orthogonal. We show that if $\phi \in L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})_{L_0}$ then $M(\phi) = 0$ implies $\phi = 0$. Since $\phi_\ell \to \chi_1\Psi_\ell^N$ extends to a bounded linear isomorphism (with bounded inverse) $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \to \ker(\not{D}_N|_{L^2})$ by Lemma 4.5, it suffices to assume that $\phi = \sum c_\ell\phi_\ell$ is such that $\Psi^N := \sum c_\ell\Psi_\ell^N$ satisfies $\|\Psi^N\|_{L^2} = 1$.

Since Φ_α is polyhomogenous by Proposition 3.8, the Fourier mode restrictions of $\Psi_\ell^{\mathrm{Euc}} + \zeta_\ell^N$ and the bound on ξ_ℓ^N in Lemma 4.5 and Corollary 4.8

$$\langle \chi_1\Psi^N, \Phi_\alpha\rangle \leqslant CL_0^{-m} \tag{4.34}$$

for any $m > 0$. We show a similar bound on the term $\not{D}P\not{D}(\chi_1\Psi^N)$; in fact, since $\not{D}, P$ are bounded, it suffices to show it for $\not{D}(\chi_1\Psi^N)$. For this, one has

$$\begin{aligned} \|\not{D}(\chi_1\Psi^N)\|_{rH_e^{-1}} &= \|d\chi_1.\Psi^N|_{\mathrm{supp}(d\chi_1)}\|_{rH_e^{-1}}\\ &\leqslant \|d\chi_1.(\Psi^{\mathrm{Euc}} + \zeta^N)|_{\mathrm{supp}(d\chi_1)}\|_{L^2} + \|d\chi_1.\xi^N\|_{L^2} \leqslant C\mathrm{Exp}(-\tfrac{n}{c_1}) + CL_0^{-m-2}. \end{aligned}$$

Combining this with 4.34 we conclude

$$\begin{aligned}\|\chi_1\Psi^N - \not{D}P\not{D}(\chi_1\Psi^N) - \pi_1(\chi_1\Psi^N)\|_{L^2} &\geqslant \|\chi_1\Psi^N\|_{L^2} - CL_0^{-m} - CL_0^{-m+2}\\ &\geqslant \tfrac{1}{2}\Big\|\sum_\ell c_\ell\phi_\ell\Big\|_{L^2}\end{aligned}$$

for, say $m = 4$ and L_0 sufficiently large. Here we have used that $\|(1-\chi_1)\Psi^N\|_{L^2} = O(\text{Exp}(-L_0))$. It follows that $M|_{L_0}$ is injective. Moreover, the bounds above show that the correction terms satisfy $\|\not{D}P\not{D}(\chi_1\Psi^N_\ell) - \pi_1(\chi_1\Psi^N_\ell)\|_{L^2} \leqslant C_m|\ell|^{-m}$ for any m, hence they can be absorbed into ξ_ℓ to yield the decomposition (4.33) without disrupting the bounds of Lemma 4.5 and Corollary 4.8. □

To complete the proofs of Propositions 4.2 and 4.3 we construct the map ob from M'. This is necessary because M does not satisfy the property 4.3 that the projection to $\mathbf{Ob}(\mathcal{Z}_0)$ is easily calculated from the sequence of inner products. Indeed, since the basis Ψ_ℓ is not necessarily orthonormal, the coefficients of $\Psi = c_\ell\psi_\ell$ are not calculated by the L^2-inner product, i.e. in general

$$(M)^{-1}(\Pi^{\text{Ob}}\psi) \neq \left(\sum_{\ell\in\mathbb{Z}}\langle\psi,\Psi_\ell\rangle_{\mathbb{C}}\phi_\ell\ ,\ \sum_\alpha\langle\psi,\Phi_\alpha\rangle\Phi_\alpha\right).$$

Rather frustratingly, one cannot orthonormalize and retain the decay properties of Proposition 4.3 (disrupting these would lead to certain error terms being unbounded later, so the decay properties are essential). To amend this without orthonormalizing, we precompose M' with a change of basis $U : L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0}) \to L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$. Specifically, let U be defined by the linear extension of

$$U(c_k\phi_k) := \sum_{\ell\in\mathbb{Z}}\langle M'(c_k\phi_k),\Psi_\ell\rangle\ \phi_\ell = \sum_{\ell\in\mathbb{Z}}\langle c_k\Psi_k,\Psi_\ell\rangle\ \phi_\ell. \tag{4.35}$$

Lemma 4.15. For L_0 sufficiently large, $U : L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0}) \to L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$ is an isomorphism, and

$$\text{ob} := M' \circ U^{-1}$$

satisfies the properties (4.2) and (4.3).

Proof. (4.2) and (4.3) are immediate from the first statement. Indeed, ob is clearly an isomorphism if U is since M' is by construction. Additionally, using (4.35), one has that for a spinor $\psi \in L^2$

$$\begin{aligned}\text{ob}^{-1}(\Pi^{\text{Ob}}\psi) = UU^{-1}(\text{ob}^{-1}\Pi^{\text{Ob}}\psi) &= \sum_{\ell\in\mathbb{Z}}\langle M'U^{-1}(\text{ob}^{-1}\Pi^{\text{Ob}}(\psi)),\Psi_\ell\rangle\ \phi_\ell\\ &= \sum_{\ell\in\mathbb{Z}}\langle\text{ob}(\text{ob}^{-1}\Pi^{\text{Ob}}(\psi)),\Psi_\ell\rangle\phi_\ell = \sum_{\ell\in\mathbb{Z}}\langle\psi,\Psi_\ell\rangle\ \phi_\ell\end{aligned}$$

as desired.

To show U is an isomorphism we show that

$$U = Id + K$$

where $\|K\|_{L^2\to L^2} \leqslant CL_0^{-1/8}$. To see this, write $\Psi_\ell = \chi_1\Psi_\ell^{\text{Euc}} + \Xi_\ell$ where $\Xi_\ell = \zeta_\ell + \xi_\ell$. We claim the following four bounds hold where all inner products are the hermitian inner product on L^2:

(i) $\langle\Psi_k,\Psi_\ell\rangle = \delta_{k\ell}$ unless both $|k| > L_0$ and $|\ell| > L_0$.

(ii) $\langle\Xi_k,\Xi_\ell\rangle \leqslant \frac{C}{|k||\ell|}$.

(iii) $\langle\Xi_k,\chi_1\Psi_\ell^{\text{Euc}}\rangle \leqslant \frac{C}{|k||\ell|}$.

(iv) $\langle\chi_1\Psi_k^{\text{Euc}},\chi_1\Psi_\ell^{\text{Euc}}\rangle = \delta_{k\ell}+a_{k\ell}$ where $|a_{k\ell}| \leqslant \frac{C}{|k|^{1/2}|\ell|^{1/2}}$ and if $|k-\ell| \geqslant |k|^{1/4}|\ell|^{1/4}$ then $|a_{k\ell}| \leqslant \frac{C}{|k|^2|\ell|^2}$.

(i) holds by construction by Definition 4.14. And (ii) is immediate from the bounds on $\zeta_\ell+\xi_\ell$ and Cauchy-Schwartz in Lemma (4.13). For (iii), recall from Definition 4.14 that $\Xi_\ell = \Xi_\ell^\perp + \pi_k$ where $\Xi_\ell^\perp \perp \mathbf{Ob}(\mathcal{Z}_0)$ is the (negative of the) projection of $\chi_1\Psi_\ell^{\text{Euc}}$ to the orthogonal complement, and $\pi_k = \sum_\alpha \langle \chi_1 \Psi_k^{\text{Euc}}, \Phi_\alpha\rangle \Phi_\alpha$. Thus since $\langle \Xi_k^\perp, \Psi_\ell\rangle = 0$,

$$\langle \Xi_k, \chi_1\Psi_\ell^{\text{Euc}}\rangle = \langle \Xi_k^\perp, \chi_1\Psi_\ell^{\text{Euc}}\rangle - \langle \pi_k, \chi_1\Psi_\ell^{\text{Euc}}\rangle = \langle \Xi_k^\perp, \Xi_\ell\rangle + \langle \pi_k, \chi_1\Psi_\ell^{\text{Euc}}\rangle$$

so (iii) follows from (ii) and polyhomogeneity of Φ_α. Finally, for (iv) the integral may be written explicitly as

$$(1+\text{sgn}(k)\text{sgn}(\ell))\int_{N_{r_0}(\mathcal{Z}_0)} \chi_1^2 |k|^{\frac12}|\ell|^{\frac12}\frac{e^{-(|\ell|+|k|)r}}{r} e^{i(k-\ell)t}|g|^{1/2}d\text{vol}$$

were $|g|^{1/2} = 1 + O(r)$ the the volume form in normal coordinates. For $|g|^{1/2} = 1$ the integral is (exponentially close to) $\delta_{k\ell}$. Integrating the $O(r)$ term results in the first bound. Since the metric is smooth, the $e^{i(k-\ell)t}$ Fourier mode is bounded by $|k-\ell|^m$ for m large. Thus if $|k-\ell| \geqslant |k|^{1/4}|\ell|^{1/4}$ the stronger bound follows.

With (i)-(iv) established, we calculate the L^2-norm of K on $c(t) = \sum_k c_k\phi_k$,

$$\begin{aligned}\|(Kc(t))\|_{L^2}^2 &= \sum_{|\ell|\geqslant L_0}\Big|\sum_{|k|\geqslant L_0} c_k a_{k\ell} + c_k(\langle \chi_1\Psi_k^{\text{Euc}}, \Xi_\ell\rangle + \langle \Xi_k, \chi_1\Phi_\ell^{\text{Euc}}\rangle + \langle \Xi_k, \Xi_\ell\rangle)\Big|^2 &(4.36)\\ &\leqslant C\|c(t)\|_{L^2}\sum_{|\ell|\geqslant L_0}\Big(\sum_{|k|\geqslant L_0}|a_{k\ell}|^2 + \sum_{|k|\geqslant L_0}\frac{1}{|k|^2|\ell|^2}\Big) &(4.37)\end{aligned}$$

where we have used Cauchy-Schwartz and (i)-(iii). The second term is easily summable, with sum bounded by $\frac{1}{L_0}$. Next, we split the first sum into the two cases of (iv):

$$\sum_{|\ell|\geqslant L_0}\Big(\tfrac{1}{|\ell|}\sum_{|k-\ell|\leqslant|k\ell|^{1/4}}\tfrac{1}{|k|} + \sum_{|k-\ell|\geqslant|k\ell|^{1/4}}\tfrac{1}{|k|^4|\ell|^4}\Big) \tag{4.38}$$

Again the second sum is summable and bounded by $\frac{1}{L_0}$. For the first, observe that for each $|\ell|$ there are at most $|k|^{1/4}|\ell|^{1/4}$ non-zero $|k|$, and $|k-\ell|\leqslant |k|^{1/4}|\ell|^{1/4}$ implies $|k|\geqslant |\ell|/2$. Hence the first sum is bounded by

$$\sum_{|\ell|\geqslant L_0}\frac{1}{|\ell|}\sup_{|k-\ell|\leqslant|k\ell|^{1/4}}|k|^{1/4}|\ell|^{1/4}\frac{1}{|k|} \leqslant \sum_{|\ell|\geqslant L_0}\frac{1}{|\ell|^{3/2}} \leqslant \frac{C}{L_0^{1/4}}. \tag{4.39}$$

It follows that $\|K\|_{L^2\to L^2}\leqslant CL_0^{-1/8}$ hence is an isomorphism after possibly increasing L_0. This completes the proof of Lemma 4.15, thus the proofs of Propositions 4.2 and 4.3. □

To conclude this subsection, we briefly note the following higher-regularity extension of the previous lemma:

Lemma 4.16. The map U defined by 4.35 restricts to an isomorphism

$$U : L^{m,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0}) \to L^{m,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$$

for every $m > 0$.

Proof. As in the proof of the previous Lemma 4.15, write $U = Id + K$. It suffices to show that $K : L^{m,2} \to L^{m+1/8,2}$ is bounded, hence K is a smoothing operator of order $\frac18$. Knowing this, the lemma follows from the "elliptic estimate"

$$\|\phi\|_m \leqslant C_m\left(\|U\phi\|_m + \|\phi\|_{m-1/8}\right) \tag{4.40}$$

derived by writing $Id = U - K$ and using the triangle inequality and the fact that $U : L^2 \to L^2$ is an isomorphism.

To show that $K : L^{m,2} \to L^{m+1/8,2}$ is bounded, first observe that the $m = 0$ case follows from (4.39), where we instead use the summability of $\frac{1}{|\ell|^{5/4}}$ and add the weighting of $|\ell|^{1/4}$ to the overall sum. The $m > 0$ case follows repeating the proof using the bounds

$$\frac{|\ell|^{2m+2\delta}|a_{k\ell}|^2}{|k|^{2m}} \leqslant C_m|\ell|^{2\delta}|a_{k\ell}|^2 \qquad\qquad \frac{|\ell|^{2m+2\delta}|b_{k\ell}|^2}{|k|^{2m}} \leqslant C_m|\ell|^{2\delta}|b_{k\ell}|^2 \tag{4.41}$$

for $a_{k\ell}$ as before and $b_{k\ell}$ any of the latter inner products in (4.36), and using Cauchy-Schwartz with the grouping $(\frac{a_{k\ell}}{|k|^m})(c_k|k|^m)$. The bounds (4.41) follow easily from the Fourier mode restriction on ζ_ℓ and integration by parts using the higher-order bounds of Corollary 4.8. $\square$

4.5 The Higher Regularity Obstruction

This subsection refines Propositions 4.2 and 4.3 to cover the cases of higher regularity. The Dirac operator

$$\not{D} : H^{m,1}_{\mathrm{b},e}(Y - \mathcal{Z}_0; S) \to H^m_{\mathrm{b}}(Y - \mathcal{Z}_0; S)$$

has infinite-dimensional cokernel equal to $\mathbf{Ob} \cap H^m_{\mathrm{b}}$ by Corollary 2.12. It is not *a priori* clear that this cokernel coincides with the natural restriction $\mathbf{Ob}^m := \mathrm{Im}(\mathrm{ob}|_{L^{m,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})})$. The next lemma asserts that this is indeed the case.

Lemma 4.17. There is equality

$$\mathbf{Ob}^m = \mathbf{Ob} \cap H^m_{\mathrm{b}}$$

as subspaces $H^m_{\mathrm{b}}(Y - \mathcal{Z}_0; S_0)$. In particular, $\mathrm{ob}|_{L^{m,2}}$ restricts to an isomorphism making the following diagram commute.

$$\begin{array}{ccc} L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0}) & \xrightarrow{\ \mathrm{ob}\ } & \mathbf{Ob}\cap L^2(Y-\mathcal{Z}_0) \\ \iota\uparrow & & \iota\uparrow \\ L^{m,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0}) & \xrightarrow{\ \mathrm{ob}|_{L^{m,2}}\ } & \mathbf{Ob}\cap H^m_{\mathrm{b}}(Y-\mathcal{Z}_0) \end{array}$$

Proof. Lemma 4.16 shows that there are equivalences of norms

$$L^{m,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0}) \ \overset{U}{\sim}\ L^{m,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$$

It is therefore enough to show that

$$\sum_\ell c_\ell \Psi_\ell \in H^m_{\mathrm{b}} \qquad \Leftrightarrow \qquad \sum_\ell |c_\ell||\ell|^{2m} < \infty.$$

The right hand side equivalent to the H^m_{b}-norm of $\sum c_\ell \Psi_\ell^{\mathrm{Euc}}$, and the statement then follows from the fact that the projection operator $\not{D}P\not{D} : H^m_{\mathrm{b}} \to H^m_{\mathrm{b}}$ is bounded by Corollary 2.12. $\square$

5 The Universal Dirac Operator

This section begins the analysis of the Dirac operator allowing the singular set $\mathcal{Z}_0$ to vary. This is done by introducing a "universal" Dirac operator which is the infinite-dimensional family of Dirac operators parameterized by possible singular sets. The main result of this section, Proposition 5.5 in Subsection 5.2 calculates the derivative of this universal Dirac operator with respect to variations in the singular set. Throughout this section, care is taken to construct explicit trivializations of the relevant Banach vector bundles. The present situation is more subtle than the case of scalar-valued functions appearing in [9], and imprecision about certain isomorphisms can lead to incorrect formulas for the derivative with respect to variations in the singular set.

For the remainder of the article we assume $(\mathcal{Z}_0, \ell_0, \Phi_0)$ is regular in the sense that it satisfies Assumptions 1-3. Note that Assumption 3 requires that the Dirac operator is only $\mathbb{R}$-linear placing us in Case (B) of Proposition 4.3 (this requires $B_0 \neq 0$). We also continue to tacitly assume that $\mathcal{Z}_0$ is connected (Assumption 4*). From here on, $\langle -,- \rangle$ denotes the real inner product on spinors, and $\langle -,- \rangle_{\mathbb{C}}$ the Hermitian one.

5.1 Trivializations

We consider variations of the singular set $\mathcal{Z}_0$ as follows. Let

$$\mathcal{E}_0 \subseteq \mathrm{Emb}^{2,2}(\mathcal{Z}_0; Y)$$

denote an open neighborhood of $\mathcal{Z}_0$ in the space of embedded links of Sobolev regularity $(2,2)$. For each $\mathcal{Z} \in \mathcal{E}_0$, let $(S_{\mathcal{Z}}, \gamma, \nabla)$ denote the Clifford module defined analogously to S_0 in (2.1) so that $S_{\mathcal{Z}} := S_{\mathfrak{s}_0} \otimes \ell_{\mathcal{Z}}$. Here $\ell_{\mathcal{Z}} \to Y - \mathcal{Z}$ is the real line bundle whose holonomy representation agrees with that of ℓ_0 (up to homotopy) equipped with its unique flat connection with holonomy in $\mathbb{Z}_2$. The Dirac operator $\slashed{D}_{\mathcal{Z}}$ is defined as in Definition 2.1, and the Hilbert spaces $rH^1_e(Y - \mathcal{Z}, S_{\mathcal{Z}}), L^2(Y - \mathcal{Z}, S_{\mathcal{Z}})$ are defined for $\mathcal{Z} \in \mathcal{E}_0$ analogously to 2.2 but using a weight $r_{\mathcal{Z}} \approx \mathrm{dist}(-, \mathcal{Z})$.

Define families of Hilbert spaces

$$\begin{aligned} \mathbb{H}^1_e(\mathcal{E}_0) &:= \{(\mathcal{Z}, \varphi) \mid \mathcal{Z} \in \mathcal{E}_0 \ , \ \varphi \in rH^1_e(Y - \mathcal{Z}; S_{\mathcal{Z}})\} \\ \mathbb{L}^2(\mathcal{E}_0) &:= \{(\mathcal{Z}, \psi) \mid \mathcal{Z} \in \mathcal{E}_0 \ , \ \psi \in L^2(Y - \mathcal{Z}; S_{\mathcal{Z}})\} \end{aligned}$$

which come equipped with projections $p_1 : \mathbb{H}^1_e(\mathcal{E}_0) \to \mathcal{E}_0$ and $p_0 : \mathbb{L}^2(\mathcal{E}_0) \to \mathcal{E}_0$ respectively.

Lemma 5.1. There are trivializations

$$\begin{aligned} \Upsilon : \mathbb{H}^1_e(\mathcal{E}_0) &\simeq \mathcal{E}_0 \times rH^1_e(Y - \mathcal{Z}_0; S_0) \\ \Upsilon : \mathbb{L}^2(\mathcal{E}_0) &\simeq \mathcal{E}_0 \times \ L^2(Y - \mathcal{Z}_0; S_0) \end{aligned}$$

which endow the spaces on the left with the structure of locally trivial Hilbert vector bundles.

Assuming this lemma momentarily, we define

Definition 5.2. The **Universal Dirac Operator** is the section $\slashed{\mathbb{D}}$ defined by

$$\begin{tikzcd} p_1^*\mathbb{L}^2(\mathcal{E}_0) \arrow[d] \\ \mathbb{H}^1_e(\mathcal{E}_0) \arrow[u, bend right, swap, "\slashed{\mathbb{D}}(\mathcal{Z}, \varphi) := \slashed{D}_{\mathcal{Z}}\varphi"] \end{tikzcd}$$

Before proving Lemma 5.1, we first construct a chart around $\mathcal{Z}_0$ in $\mathrm{Emb}^{2,2}(\mathcal{Z}; Y)$. A choice of Fermi coordinates (t, x, y) on $N_{r_0}(\mathcal{Z}_0)$ gives an association $T_{\mathcal{Z}_0}\mathrm{Emb}^{2,2}(\mathcal{Z}; Y) \simeq L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0)$. Choosing a cut-off function $\chi(r) : N_{r_0} \to \mathbb{R}$ equal to 1 for $r \leqslant r_0/2$ and vanishing for $r \geqslant r_0$ we define an exponential map as follows: given $\eta \in L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0)$ with $\|\eta\|_{2,2} < \rho_0$ we set

$$F_\eta(t, z) = (t, z + \chi(r)\eta(t)). \tag{5.1}$$

Then define

$$\begin{aligned} \mathrm{Exp} : L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) &\to \mathrm{Emb}^{2,2}(\mathcal{Z}_0; Y) \\ \eta &\mapsto \mathcal{Z}_\eta := F_\eta[\mathcal{Z}_0], \end{aligned}$$

and take $\mathcal{E}_0 = B_{\rho_0}(\mathcal{Z}_0) \subset L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0)$ to be the open ball of radius ρ_0.

Lemma 5.3. For ρ_0 sufficiently small, F_η is a diffeomorphism and $\mathrm{Exp} : \mathcal{E}_0 \to \mathrm{Emb}^{2,2}(\mathcal{Z}_0; Y)$ is a homeomorphism onto its image.

Proof. Since $\|\eta\|_{C^1} \leqslant C\|\eta\|_{L^{2,2}} \leqslant C\rho_0$ by the Sobolev embedding theorem, it follows that

$$\mathrm{d}F_\eta = \begin{pmatrix} 1 & 0 & 0 \\ \chi\eta_x' & 1 + \partial_x\chi\eta_x & \partial_y\chi\eta_x \\ \chi\eta_y' & \partial_x\chi\eta_y & 1 + \partial_y\chi\eta_y \end{pmatrix}$$

is close to the identity for ρ_0 sufficiently small. It then follows from the Inverse Function Theorem that it is a local diffeomorphism, and since F_η preserves normal disks $\{t_0\} \times D_{\mathbf{r}_0}$ and is monotonically increasing in the η direction, it is injective hence a diffeomorphism.

For the second statement, note that $F_\eta(t, 0, 0) = (t, \eta(t))$ is distinct for distinct $\eta \in C^1$, hence Exp is injective. For surjectivity, since any $L^{2,2}$-embedding $\mathcal{Z}$ close to $\mathcal{Z}_0$ is also close in C^1, such an embedding must be a graph over $\mathcal{Z}_0$ in local coordinates. Thus $\mathcal{Z} = \mathcal{Z}_\eta$ for η the function defining this graph. Continuity of Exp and its inverse are obvious.

□

Remark 5.4. Note that F_η is not the flow of a time-independent vector field on Y. Although it is morally equivalent, this choice simplifies several formulas and avoids standard issues with thinking of $C^\infty(Y; TY)$ as the tangent space of the diffeomorphism group.

We now prove Lemma 5.1 by constructing the trivializations Υ. The only slight subtlety here is the association of spinor bundles for different metrics. To highlight the metric dependence, we denote by S_h the spinor bundle (without tensoring with ℓ_0) formed with the spin structure $\mathfrak{s}_0$ using the metric h.

Proof of Lemma 5.1. . Let $g_\eta := F_\eta^* g_0$ denote the pullback metric. We now define Υ as the map induced on sections by a fiberwise linear diffeomorphism (denoted by the same symbol) $\Upsilon_\eta : S_{g_0} \otimes \ell_{\mathcal{Z}_\eta} \simeq S_{g_0} \otimes \ell_{\mathcal{Z}_0}$ which is given as the composition of three maps $\Upsilon := \tau \circ \iota \circ F^*$.

$$S_{g_0} \otimes \ell_{\mathcal{Z}_\eta} \xrightarrow{F_\eta^*} F_\eta^*(S_{g_0} \otimes \ell_{\mathcal{Z}_0}) \xrightarrow{\iota} S_{g_\eta} \otimes \ell_{\mathcal{Z}_0} \xrightarrow{\tau_\eta} S_{g_0} \otimes \ell_{\mathcal{Z}_0}$$

where F_η^* is the pullback by the diffeomorphism F_η, ι is a canonical association of $S_{g_\eta} \simeq F_\eta^* S_{g_0}$ and of line bundles, and τ is an association of spinor bundles for different metrics. Note that Υ covers the diffeomorphism F_η on Y hence is not an isomorphism in the category of vector bundles on Y.

It is obvious that the first map F_η^* is a diffeomorphism linear on fibers. The remainder of the proof proceeds in four steps.

Step 1: First, note that it follows directly from the definitions that there are canonical isomorphisms

$$F_\eta^* S_{g_0} \simeq S_{g_\eta} \qquad\qquad F_\eta^* \ell_{\mathcal{Z}_0} \simeq \ell_{\mathcal{Z}_0}.$$

In addition, these isomorphisms associate the spin connections $F_\eta^* \nabla_{g_0}^{\mathrm{spin}}$ and $\nabla_{g_\eta}^{\mathrm{spin}}$ and the unique flat connections on the line bundles.

Step 2: Next, we define the map τ by parallel transport on cylinders. This construction follows Section 5 of [2]. Consider the natural family of interpolating metrics

$$g_{s\eta} = (F_{s\eta})^* g_0 \qquad\qquad \text{for } s \in [0, 1]$$

Note this family does not coincide with $(1 - s)g_0 + s g_\eta$. Now consider the (generalized) 4-dimensional cylinder

$$X_\eta = ([0, 1] \times Y, ds^2 + g_{s\eta}).$$

It is spin since $w_2(X_\eta) = w_2(Y) = 0$, and Spin structures on W are in 1-1 correspondence with those on Y. Let $S_X \to X_\eta$ be the spinor bundle on X_η arising from the spin structure corresponding to the fixed spin structure on Y. There is a natural isomorphism $S^+_X|_{Y\times\{s\}} \simeq S_{g_{s\eta}}$ (see [30] Section 4.3 or [27] top of page 4).

Let ∇_X denote the spin connection on S^+_X. Parallel transport along the curve $\gamma_y(s) = (s, y)$ in the s direction defines a linear isometry

$$\tau^{g_\eta}_{g_0}(y, s) : (S_{g_0})_y \to (S_{g_{s\eta}})_y.$$

Together, parallel transport for $s = 1$ along all such curves define an isomorphism $\tau^{g_\eta}_{g_0} : S_{g_0} \to S_{g_{s\eta}}$ which is a fiberwise isometry.

Step 3: Note that the previous two steps depend continuously on the parameter η. Denote by $\mathcal{Y} \to \mathcal{E}_0$ the universal 3-manifold bundle whose fiber above $\eta \in \mathcal{E}_0$ is $Y_\eta = (Y - \mathcal{Z}_0, g_\eta)$, and by $\mathfrak{S} \to \mathcal{Y}$ the vector bundle above it whose restriction to Y_η is the bundle $S_{g_\eta} \otimes \ell_{\mathcal{Z}_0}$.

There is then a trivialization of $\mathfrak{S}$ denoted $\tau : \mathcal{E}_0 \times (S_{g_0} \otimes \ell_{\mathcal{Z}_0}) \longrightarrow \mathfrak{S}$ defined by parallel transport along cylinders in the radial directions in $\mathcal{E}_0$. That is, we set

$$\tau(\eta, \psi) := (\tau^{g_\eta}_{g_0} \otimes 1)(\psi).$$

Likewise, we take $F^*(\eta, -) := F^*_\eta$ and $\iota(\eta, -) := \iota_\eta$ and define

$$\Upsilon := \tau \circ \iota \circ F^*.$$

Step 4: Given the fiberwise linear diffeomorphism Υ, what remains to be seen is that the induced map

$$\Upsilon : rH^1_e(Y - \mathcal{Z}_0, S_{g_0} \otimes \ell_{\mathcal{Z}_0}) \longrightarrow rH^1_e(Y - \mathcal{Z}_\eta, S_{g_0} \otimes \ell_{\mathcal{Z}_\eta})$$

gives an equivalence of norms. To see this, note that $\eta \in L^{2,2}(\mathcal{Z})$ and $L^{1,2}(\mathcal{Z}_0) \hookrightarrow C^0(\mathcal{Z}_0)$ by the Sobolev embedding, hence in Fermi coorinates the pullback metric has entries of the form $h(t)g_1(t, x, y)$ with $h(t) \in L^{1,2}(S^1)$ and $g_1(t, x, y)$ smooth (cf. Lemma 5.8). The Christoffel symbols of ∇_{B_0} are, in turn, has entries of the form $f(t)g_2(t, x, y)$ now with $f(t) \in L^2(S^1)$. The equivalence of norms is then a consequence of the following "mixed dimension" Sobolev multiplication for $f \in L^2(S^1)$ and $\varphi \in rH^1_e$:

$$\|f(t)\varphi\|_{L^2(S^1\times D^2)} \leqslant C\|f\|_{L^2(S^1)}\|\varphi\|_{rH^1_e},$$

which is proved by integrating over slices $\{t\} \times D^2$ and using the Sobolev restriction theorem.

□

5.2 Universal Linearization

Using the trivialization constructed in Lemma 5.1, we may now calculate the (vertical component of the) derivative of the universal Dirac operator considered as a map

$$\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\not{\mathbb{D}} : L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \times rH^1_e(Y - \mathcal{Z}_0; S_0) \longrightarrow L^2(Y - \mathcal{Z}_0; S_0). \tag{5.2}$$

After trivializing, differentiating with respect to a variation η in the singular set becomes differentiation of the Dirac operator with respect to the family of metrics $g_{s\eta}$ for $s \in [0, 1]$.

Proposition 5.5. In the local trivialization provided by Υ, the Linearization of the universal Dirac operator on the spaces 5.2 is given by

$$\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\not{\mathbb{D}}(\eta, \psi) = \mathcal{B}_{\Phi_0}(\eta) + \not{D}_{\mathcal{Z}_0}\psi \tag{5.3}$$

where

$$\mathcal{B}_{\Phi_0}(\eta) = \left(\frac{d}{ds}\Big|_{s=0} \tau^{g_{s\eta}}_{g_0} \circ \not{D}^{g_{s\eta}}_{\mathcal{Z}_0} \circ (\tau^{g_{s\eta}}_{g_0})^{-1}\right)\Phi_0$$

is the first variation of the Dirac operator with respect to the family of metrics $g_{s\eta}$ acting on the spinor Φ_0.

Remark 5.6. (Cf. Section 4.1 of [9]) Since the configuration $(\mathcal{Z}_0, \Phi_0)$ does not lie along the zero-section in $\mathbb{H}^1_e(\mathcal{E}_0)$, there is no canonical association

$$T_{(\mathcal{Z}_0,\Phi_0)}\mathbb{H}^1_e(\mathcal{E}_0) \simeq T_{\mathcal{Z}_0}\mathcal{E}_0 \oplus rH^1_e(Y - \mathcal{Z}_0).$$

Thus expression of the derivative (5.2) implicitly relies on a choice of connection — the pullback of the product connection by Υ^{-1}. Different choices of trivialization will result in different connections and different expressions for the derivative $\mathrm{d}\not{\mathbb{D}}$. Concretely, this choice manifests as the dependence of the family of metrics g_η on our choice of diffeomorphisms F_η. A different choice of family of diffeomorphisms differs from our choice of F_η by composing with (a family of) diffeomorphisms fixing $\mathcal{Z}_0$. Although there are many possible choices (see [42] and [61]) this choice simplifies many expressions. Of course, the salient properties of the linearization are independent of these choices.

Proof of Proposition 5.5. Take a path

$$\begin{array}{rcl}\gamma : (-\epsilon, \epsilon) & \to & \mathbb{H}^1_e(\mathcal{E}_0) \\ s & \mapsto & (\mathcal{Z}_{\eta(s)}, \Phi(s))\end{array}$$

such that $\gamma(0) = (\mathcal{Z}_0, \Phi_0)$. Using the chart $\mathrm{Exp} : L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \to \mathcal{E}_0$, we may assume that $\eta(s) = s\eta$. Let $\mathcal{H}$ be the section of $\mathbb{H}^1_e(\mathcal{E}_0)$ obtained from radial parallel transport of Φ_0 in the connection induced by the trivialization Υ. That is, set

$$\mathcal{H} = \Upsilon^{-1}(\mathcal{E}_0 \times \{\Phi_0\}).$$

We may write each $\Phi(s) \in rH^1_e(Y - \mathcal{Z}_{s\eta})$ as the point in $\mathcal{H}$ plus a vertical vector $\phi(s) = \Upsilon^{-1}(\psi(s))$, i.e.

$$\gamma(s) = (\mathcal{Z}_{s\eta}, \Upsilon^{-1}_{s\eta}(\Phi_0) + \phi(s)\ ) = (\mathcal{Z}_{s\eta}, \Upsilon^{-1}_{s\eta}(\Phi_0 + \psi(s))\ ).$$

The derivative in the trivialization given by Υ is

$$\frac{d}{ds}\Big|_{s=0} \Upsilon_{s\eta} \circ \not{\mathbb{D}}(\mathcal{Z}_{s\eta}, \Upsilon^{-1}_{s\eta}(\Phi_0 + \psi)) = \frac{d}{ds}\Big|_{s=0} \Upsilon_{s\eta} \circ \not{D}^{g_0}_{\mathcal{Z}_{s\eta}} \circ \Upsilon^{-1}_{s\eta}(\Phi_0 + \psi) \tag{5.4}$$

where Υ denotes the trivialization for both $\mathbb{H}^1_e(\mathcal{E}_0)$ and $\mathbb{L}^2(\mathcal{E}_0)$.

Recalling the definition of $\Upsilon = \tau \circ \iota \circ F^*$, the following diagram commutes, where the rightmost vertical arrow is the expression (5.4) which we wish to calculate.

$$\begin{tikzcd}
L^2(S_{g_0} \otimes \ell_{\mathcal{Z}_{s\eta}}) \arrow[r, "\iota_{s\eta} \circ F^*_{s\eta}"] & L^2(S_{g_{s\eta}} \otimes \ell_{\mathcal{Z}_0}) \arrow[r, "\tau^{g_{s\eta}}_{g_0}"] & L^2(S_{g_0} \otimes \ell_{\mathcal{Z}_0}) \\
rH^1_e(S_{g_0} \otimes \ell_{\mathcal{Z}_{s\eta}}) \arrow[u, "\not{D}^{g_0}_{\mathcal{Z}_{s\eta}}"] \arrow[r, "\iota_{s\eta} \circ F^*_{s\eta}"'] & rH^1_e(S_{g_{s\eta}} \otimes \ell_{\mathcal{Z}_0}) \arrow[u, "\not{D}^{g_{s\eta}}_{\mathcal{Z}_0}"] \arrow[r, "\tau^{g_{s\eta}}_{g_0}"] & rH^1_e(S_{g_0} \otimes \ell_{\mathcal{Z}_0}) \arrow[u, "\Upsilon_{s\eta}\not{D}^{g_0}_{\mathcal{Z}_{s\eta}}\Upsilon^{-1}_{s\eta}"'] \\
\begin{pmatrix}\text{varying } \mathcal{Z}_{s\eta} \\ \text{fixed } g_0\end{pmatrix} & \begin{pmatrix}\text{varying } g_{s\eta} \\ \text{fixed } \mathcal{Z}_0\end{pmatrix} & \begin{pmatrix}\text{fixed } g_0 \\ \text{fixed } \mathcal{Z}_0\end{pmatrix}
\end{tikzcd}$$

The middle vertical arrow denotes Dirac operator on the bundle $S_{g_\eta} \otimes \ell_{\mathcal{Z}_0}$ formed from with the pullback metric g_η and the unique flat connection on $\ell_{\mathcal{Z}_0}$.

By commutativity, the rightmost vertical arrow is equivalent to the conjugation of the middle arrow by $\tau^{g_\eta}_{g_0}$ and its inverse. Consequently, using the product rule (noting as well that $\psi(0) = 0$ and $\tau^{g_{\eta(0)}}_{g_0} = Id$),

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} \Upsilon_{s\eta} \circ \not{D}^{g_0}_{\mathcal{Z}_{s\eta}} \circ \Upsilon^{-1}_{s\eta}(\Phi_0 + \psi(s)) \quad &= \quad \frac{d}{ds}\Big|_{s=0} \left(\tau^{g_{s\eta}}_{g_0} \circ \not{D}^{g_{s\eta}}_{\mathcal{Z}_0} \circ (\tau^{g_{s\eta}}_{g_0})^{-1}\right)(\Phi_0 + \psi(s)) \\ &= \quad \left(\frac{d}{ds}\Big|_{s=0} (\tau^{g_{s\eta}}_{g_0} \circ \not{D}^{g_{s\eta}}_{\mathcal{Z}_0} \circ (\tau^{g_{s\eta}}_{g_0})^{-1}\right)\Phi_0 + \not{D}^{g_0}_{\mathcal{Z}_0}\dot{\psi}(0).\end{aligned}$$

as claimed. ∎

5.3 First Variation Formula

In order to analyze the derivative of the universal Dirac operator calculated in Proposition 5.5, a more explicit formula is needed for the variation of the Dirac operator with respect to metrics ($\mathcal{B}_{\Phi_0}(\eta)$ in 5.3). The formula for this variation is originally due to Bourguignon and Gauduchon [3]. A concise proof (in English) was later given in [2]. See also [39].

5.3.1 Metric Variations Suppose, forgetting any reference to the above situation momentarily, that g_s is a path of metrics on a Riemannian spin manifold X. Let $\dot{g}_s$ denote the derivative of this path at $s = 0$, and let

$$\tau_{g_0}^{g_s} : S_{g_s} \to S_{g_0}$$

be the association of spinor bundles via parallel transport on $[0,1]\times X$ as defined in the proof of Lemma 5.1. We obtain a 1-parameter family of operators

$$\tau_{g_0}^{g_s} \circ \not{D}_{g_s} \circ (\tau_{g_0}^{g_s})^{-1} : \Gamma(S_{g_0}) \to \Gamma(S_{g_0})$$

as the right arrow in the commutative diagram

$$\begin{tikzcd} \Gamma(S_{g_s}) \arrow[r, "\tau_{g_0}^{g_s}"] & \Gamma(S_{g_0}) \\ \Gamma(S_{g_s}) \arrow[u, "\not{D}_{g_s}"] \arrow[r, "\tau_{g_0}^{g_s}"] & \Gamma(S_{g_0}) \arrow[u] \end{tikzcd}$$

for every s. Letting $\{e_i\}$ be an orthonormal frame for the metric g_0 and $\{e^i\}$ its dual frame, Bourguignon and Gauduchon calculate:

Theorem 5.7. (Bourguignon–Gauduchon [3]) The first variation of the Dirac operator with respect to the family of metrics g_s is given by

$$\left(\frac{d}{ds}\Big|_{s=0} \tau_{g_0}^{g_s} \circ \not{D}_{g_s} \circ (\tau_{g_0}^{g_s})^{-1}\right)\Psi = -\frac{1}{2}\sum_{ij} \dot{g}_s(e_i, e_j)e^i.\nabla_j^{g_0}\Psi + \frac{1}{2}d\mathrm{Tr}_{g_0}(\dot{g}_s).\Psi + \frac{1}{2}\mathrm{div}_{g_0}(\dot{g}_s).\Psi \tag{5.5}$$

where . denotes Clifford multiplication in the g_0 metric.

Note that the first term is independent of the choice of frame for the same reason as the standard Dirac operator. Here, in an orthonormal frame, the $\mathrm{div}_{g_0}(k)$ is the 1-form $-(e_i \lrcorner \nabla_i k_{ij} e^i)e^j$. To give some quick intuition for this slightly unappetizing formula, the first term comes from differentiating the symbol of the Dirac operator (Clifford multiplication), and the second two terms arise from differentiating the Christoffel symbols.

5.3.2 Pullback Metric Formula We will apply Bourguignon-Gauduchon's formula (5.7) in the case that the family of metrics is the one given by the pullbacks

$$\dot{g}_\eta := \frac{d}{ds}\Big|_{s=0} g_{s\eta} = \frac{d}{ds}\Big|_{s=0} F_{s\eta}^* g_0. \tag{5.6}$$

As in Definition 3.4, the metric in Fermi coordinates (t, x, y) on the tubular neighborhood $N_{r_0}(\mathcal{Z}_0)$ has the form

$$g_0 = dt^2 + dx^2 + dy^2 + h \qquad \text{where} \qquad |h_{ij}| \leqslant Cr.$$

Lemma 5.8. The derivative of the family of pullback metrics 5.6 is given by

$$\dot{g}_\eta = \begin{pmatrix} 0 & \eta_x'\chi & \eta_y'\chi \\ \eta_x'\chi & 2\eta_x\partial_x\chi & \eta_x\partial_y\chi + \eta_y\partial_x\chi \\ \eta_y'\chi & \eta_x\partial_y\chi + \eta_y\partial_x\chi & 2\eta_y\partial_y\chi \end{pmatrix} + h_1 + h_2 \tag{5.7}$$

where

- $h_1$is a $O(1)$ term whose entries are formed from products of derivatives of h_{ij} and η.
- h_2 is a $O(r)$ term whose entries are formed from products of h_{ij} and products of η, η'.

Here, $\eta = \eta_x + i\eta_y$ and $\eta' = \frac{d}{dt}\eta$ and $\dot{g}_\eta$ denotes $\frac{d}{ds} g_{s\eta}$.

Proof. Since the diffeomorphism $F_\eta(s)$ is supported in the tubular neighborhood, it suffices to do the calculation in the local coordinates.

First, consider the case that $h = 0$. Recall

$$F_{s\eta}(t, x, y) = (t, x + s\chi(r)\eta_x(t), y + s\chi(r)\eta_y(t)),$$

hence

$$\mathrm{d}F_{s\eta} = \begin{pmatrix} 1 & 0 & 0 \\ s\chi\eta'_x & 1 + s\partial_x\chi\eta_x & s\partial_y\chi\eta_x \\ s\chi\eta'_y & s\partial_x\chi\eta_y & 1 + s\partial_y\chi\eta_y \end{pmatrix}.$$

A quick calculation shows in this case the pullback metric is

$$\dot{g}_\eta = \frac{d}{ds}\Big|_{s=0} (\mathrm{d}F_{s\eta})^T g_0 (\mathrm{d}F_{s\eta}) \tag{5.8}$$

$$\tag{5.9}$$

$$= \begin{pmatrix} 0 & \eta'_x\chi & \eta'_y\chi \\ \eta'_x\chi & 2\eta_x\partial_x\chi & \eta_x\partial_y\chi + \eta_y\partial_x\chi \\ \eta'_y\chi & \eta_x\partial_y\chi + \eta_y\partial_x\chi & 2\eta_y\partial_y\chi \end{pmatrix}. \tag{5.10}$$

Now assume $h \neq 0$, and let $\widetilde{h}_{ij} = h_{ij}(t, z + F_{s\eta})$. Then the term added to the above is

$$= \frac{d}{ds}\Big|_{s=0} (\mathrm{d}F_{s\eta})^T \cdot h(t, z + F_{s\eta}) \cdot (\mathrm{d}F_{s\eta})$$

$$= \frac{d}{ds}\Big|_{s=0} (\mathrm{d}F_{s\eta})^T \begin{pmatrix} \widetilde{h}_{11} + s\chi(\widetilde{h}_{12}\eta'_x + \widetilde{h}_{13}\eta'_y) & \widetilde{h}_{12} + s\partial_x\chi(\widetilde{h}_{12}\eta_x + \widetilde{h}_{13}\eta_y) & \widetilde{h}_{13} + s\partial_y\chi(\widetilde{h}_{12}\eta_x + \widetilde{h}_{13}\eta_y) \\ \widetilde{h}_{21} + s\chi(\widetilde{h}_{22}\eta'_x + \widetilde{h}_{23}\eta'_y) & \widetilde{h}_{22} + s\partial_x\chi(\widetilde{h}_{22}\eta_x + \widetilde{h}_{23}\eta_y) & \widetilde{h}_{23} + s\partial_y\chi(\widetilde{h}_{22}\eta_x + \widetilde{h}_{23}\eta_y) \\ \widetilde{h}_{31} + s\chi(\widetilde{h}_{32}\eta'_x + \widetilde{h}_{33}\eta'_y) & \widetilde{h}_{32} + s\partial_x\chi(\widetilde{h}_{32}\eta_x + \widetilde{h}_{33}\eta_y) & \widetilde{h}_{33} + s\partial_y\chi(\widetilde{h}_{32}\eta_x + \widetilde{h}_{33}\eta_y) \end{pmatrix}.$$

Write the matrix above as $\widetilde{h}_{ij} + sB_{ij}$, so that e.g. $B_{11} = \chi\widetilde{h}_{12}\eta'_x + \widetilde{h}_{13}\eta'_y$. Then since

$$\mathrm{d}F^T_{s\eta} = Id + s\begin{pmatrix} 0 & \chi\eta'_x & \chi\eta'_y \\ 0 & \partial_x\chi\eta_x & \partial_x\chi\eta_y \\ 0 & \partial_y\chi\eta_x & \partial_y\chi\eta_y \end{pmatrix}$$

and $\left(\widetilde{h}_{ij}\right)$ is symmetric, the above becomes

$$= \frac{d}{ds}\Big|_{s=0} \left[\left(\widetilde{h}_{ij}\right) + s\left(A_{ij} + A^T_{ij}\right) + O(s^2)\right] = \underbrace{\frac{d}{ds}\Big|_{s=0}\left(\widetilde{h}_{ij}\right)}_{:=h_1} + \underbrace{\left(A_{ij} + A^T_{ij}\right)}_{:=h_2}. \tag{5.11}$$

Call these terms h_1 and h_2 as indicated. Since

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0}\widetilde{h}_{ij} &= \frac{d}{ds}\Big|_{s=0} h_{ij}(t, x + s\chi\eta_x, y + s\chi\eta_y) = (\partial_x h_{ij})\chi\eta_x + (\partial_y h_{ij})\chi\eta_y \\ A_{ij}\Big|_{s=0} &= h_{k\ell}\chi\eta'_\alpha \quad \text{or} \quad h_{k\ell}\partial_\alpha\chi\eta_\beta \end{aligned}$$

where α, β range over x, y and (summation is implicit in the expression for A), these are of the forms claimed respectively for h_1 and h_2. $\square$

Combining the formula for the linearization of the universal Dirac operator of Proposition 5.5 with the formula of Bourguignon-Gauduchon (Theorem 5.7) and the calculation of the pullback metric in Lemma 5.8 allows us to immediately deduce the following more concrete expression for the linearization.

Corollary 5.9. The linearization of the universal Dirac operator at $(\mathcal{Z}_0, \Phi_0)$ is given by

$$\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\mathbb{D}(\eta,\psi) = \left(-\frac{1}{2}\sum_{ij}\dot{g}_\eta(e_i,e_j)e^i.\nabla_j^{g_0} + \frac{1}{2}d\mathrm{Tr}_{g_0}(\dot{g}_\eta). + \frac{1}{2}\mathrm{div}_{g_0}(\dot{g}_\eta). + \mathcal{R}(B_0,\chi\eta).\right)\Phi_0 \quad (5.12)$$

$$+\not{D}\psi \quad (5.13)$$

where $\mathcal{R}(B_0,\eta)$ is a smooth term involving first derivatives of B_0 and linear in $\chi\eta$, and . denotes Clifford multiplication using the metric g_0. Explicitly, $\dot{g}_\eta$ is given in Fermi coordinates by

$$\begin{pmatrix} 0 & \eta_x'\chi & \eta_y'\chi \\ \eta_x'\chi & 2\eta_x\partial_x\chi & \eta_x\partial_y\chi+\eta_y\partial_x\chi \\ \eta_y'\chi & \eta_x\partial_y\chi+\eta_y\partial_x\chi & 2\eta_y\partial_y\chi \end{pmatrix} + h_1 + h_2$$

with h_1, h_2 as in the above Lemma 5.8.

Proof. In the case that $B_0 = 0$, this follows immediately from 5.7 and the above calculation of the pullback metric in Lemma 5.8. The line bundle is fixed after pulling back by F_η and plays no role. The perturbation B_0 pulls back to $F_{s\eta}^*B_0$, and differentiating this yields the term $\mathcal{R}(B_0,\chi\eta)$. □

A word of caution to the reader: the formula for this linearization is slightly deceptive in the following sense. The expression for $\mathcal{B}_{\Phi_0}(\eta)$, which is the first line in (5.12), appears to be a first order term plus a zeroeth order term. But these are the orders in the *spinor* Φ_0, and we are viewing it as an equation in the *deformation* η. The variation of the pullback metrics $\dot{g}_\eta$, as above, contains first derivatives of $\eta(t)$, and so the trace and divergence, which contain derivatives of $\dot{g}_\eta$ contain second derivatives of $\eta(t)$. Thus this equation is actually *second order* in η, with the second and third terms being leading order. This is part of the reason deformation η must be taken to be at least $L^{2,2}$.

Remark 5.10. For later use, we note that the proof of Lemma 5.8 shows that the complete formula for the pullback metric can be written

$$g_{s\eta} = g_0 + s\dot{g}_\eta + \mathfrak{q}(s\eta, s\eta)$$

where $\mathfrak{q}(s\eta,s\eta)$ is a matrix whose entries are $O(s^2)$ and are formed from finite sums of terms of the following form

- Products of at least two terms of the form $\chi\eta_\alpha'$, or $\partial_\beta\chi\eta_\alpha$, or $(\widetilde{h}-h) \leqslant C|\chi\eta|$.
- Higher order terms of the form $(\widetilde{h}-h-h_1) \leqslant C|\chi\eta|^2$.

where the bounds on the terms involving $\widetilde{h}$ follow from Taylor's theorem. □

6 Fredholmness of Deformations

In this section we prove Theorem 1.3 by explicitly calculating the obstruction component of the linearized universal Dirac operator.

Working in the trivialization of Lemma 5.1 and splitting the domain and codomain into their summands, the linearization has the following block lower-triangular matrix where $\Pi_0 : L^2 \to \mathbf{Ob}(\mathcal{Z}_0)$ denotes the orthogonal projection:

$$\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\overline{\mathbb{D}} = \begin{pmatrix} \Pi_0\mathcal{B}_{\Phi_0} & 0 \\ (1-\Pi_0)\mathcal{B}_{\Phi_0} & \not{D} \end{pmatrix} : \begin{matrix} L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \\ \oplus \\ rH^1_e \end{matrix} \longrightarrow \begin{matrix} \mathbf{Ob}(\mathcal{Z}_0) \\ \oplus \\ \mathrm{Range}(\not{D}|_{rH^1_e}). \end{matrix} \tag{6.1}$$

Composing with the isomorphism $\mathrm{ob}^{-1} : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \to \mathbf{Ob}(\mathcal{Z}_0)$ from Proposition 4.2, the upper left entry of (6.1) can be written as (T_{Φ_0}, π_1) where π is the L^2-orthogonal projection onto $\mathbb{R}\Phi_0$, and T_{Φ_0} is the composition:

$$\begin{tikzcd} L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \arrow[r, "\Pi_0\mathcal{B}_{\Phi_0}"] \arrow[rr, bend right, "T_{\Phi_0}"'] & \mathbf{Ob} \arrow[r, "\mathrm{ob}^{-1}"] & L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}). \end{tikzcd}$$

In particular, T_{Φ_0} is a map of Hilbert spaces of sections of vector bundles on $\mathcal{Z}_0$. The main result of the current section is the following theorem, which refines the statement of Theorem 1.3 in the introduction. Although T_{Φ_0} is *a priori* only bounded into L^2, the theorem shows it is Fredholm onto a dense subspace of this.

Theorem 6.1. The composition T_{Φ_0} is an elliptic pseudo-differential operator of order $1/2$. In particular, as a map

$$T_{\Phi_0} : L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \longrightarrow L^{3/2,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \tag{6.2}$$

it is Fredholm, and has index 0.

Given the block-diagonal decomposition (6.1), Theorem 6.1 and standard Fredholm theory and bootstrapping imply the following result on the full linearization (6.1). Here, recall that $\mathbf{Ob}^m = \mathbf{Ob}(\mathcal{Z}_0) \cap H^m_{\mathrm{b}}$.

Corollary 6.2. The following versions of the the linearized universal Dirac operator are Fredholm of Index 0 for all $m \geqslant 0$:

$$(m \geqslant 0) \quad \mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\overline{\mathbb{D}} : L^{m+2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \oplus rH^{m,1}_{\mathrm{b},e} \longrightarrow \mathbf{Ob}^{m+3/2} \oplus \Big((\mathrm{Range}(\not{D}) \cap H^m_{\mathrm{b}}\Big).$$

□

The proof of Theorem 6.1 occupies the remainder of the section. Subsection 6.1 discusses the relation between regularity on $Y - \mathcal{Z}_0$ and regularity in $L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})$ under ob, and Subsection 6.2 proves 6.1. Subsection 6.3 finishes the Index calculation by proving Lemma 6.9.

6.1 Conormal Regularity

The loss of regularity of $r = 3/2$ appearing in Theorem 6.1 is a consequence of the fact that **Ob** does not simply inherit the obvious notion of regularity from $Y - \mathcal{Z}_0$.

Key Observation: The regularity of $\Pi_0(\psi) \in \mathbf{Ob}(\mathcal{Z}_0)$ depends on both the regularity of ψ and its asymptotics along $\mathcal{Z}_0$. .

Using Proposition 4.3, the regularity of $\Pi_0(\psi)$ is a question about the rate of decay in $|\ell|$ of the sequence of inner products

$$\big\{ \langle \psi, \Psi_\ell \rangle_{\mathbb{C}} \big\}_{\ell \in \mathbb{Z}}. \tag{6.3}$$

Because the basis elements Ψ_ℓ concentrate exponentially around $\mathcal{Z}_0$ as $|\ell| \to \infty$, this rate of decay is intertwined with the growth of ψ along $\mathcal{Z}_0$. If, for example, ψ is compactly supported away from $\mathcal{Z}_0$, then Proposition 4.3 implies the sequence (6.3) decays faster than polynomially and $\mathrm{ob}^{-1}\Pi_0(\psi) \in$

$C^\infty(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$ is smooth regardless of the regularity of Ψ on Y. By Lemma 4.17, regularity of $\Pi_0\psi \in$ $\mathbf{Ob}(\mathcal{Z}_0)$ coincides with regularity of $\text{ob}^{-1}\Pi_0\psi \in L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$.

More generally, suppose that a spinor ψ can be written locally in Fermi coordinates and an accompanying trivialization as

$$\psi = \chi \begin{pmatrix} f^+(t)h^+(\theta) \\ f^-(t)h^-(\theta) \end{pmatrix} r^p \tag{6.4}$$

where $f^\pm \in L^k(S^1;\mathbb{C})$ and $h^\pm$ are smooth. Here, χ is a cutoff function supported in a neighborhood $N_{r_0}(\mathcal{Z}_0)$.

Definition 6.3. Suppose ψ has the form (6.4). The quantity

$$s = \boxed{1 + k + p}$$

is called the **conormal regularity** of ψ.

The following simple lemma governs many of the regularity considerations in the upcoming sections.

Lemma 6.4. Suppose that $\psi \in L^2$ has conormal regularity s. Then $\text{ob}^{-1}\Pi_0(\psi) \in L^{s,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$ and

$$\|\text{ob}^{-1}\Pi_0(\psi)\|_s \leqslant C_s(\|f^+\|_{L^{k,2}} + \|f^-\|_{L^{k,2}}).$$

Proof. Using Proposition 4.3, $\text{ob}^{-1}\Pi_0(\psi)$ is calculated by the sequence of inner products

$$\langle \psi, \Psi_\ell\rangle = \langle \psi, \Psi_\ell^{\text{Euc}} + \zeta_\ell + \xi_\ell\rangle \qquad \text{where} \qquad \Psi_\ell^{\text{Euc}} = \chi\sqrt{|\ell|}e^{i\ell t}e^{-|\ell| r}\begin{pmatrix} \frac{1}{\sqrt{z}} \\ \frac{\text{sgn}(\ell)}{\sqrt{\bar{z}}} \end{pmatrix}.$$

Assume first that $g_0 = dt^2 + dx^2 + dy^2$ on $N_{r_0}(\mathcal{Z}_0)$. Taking the inner product of ψ in 6.4 with Ψ_ℓ^{Euc} yields

$$\begin{aligned}
\langle \psi, \Psi_\ell^{\text{Euc}}\rangle &= \left\langle \chi\begin{pmatrix} f^+h^+ \\ f^-h^- \end{pmatrix} r^p \ , \ \sqrt{|\ell|}e^{i\ell t}\begin{pmatrix} \frac{e^{-|\ell| r}}{\sqrt{z}} \\ \text{sgn}(\ell)\frac{e^{-|\ell| r}}{\sqrt{\bar{z}}} \end{pmatrix}\right\rangle \\
&\leqslant \int_{S^1}\langle f^+ + Hf^-, e^{i\ell t}\rangle \int_{\mathbb{R}^2}\sqrt{|\ell|}e^{-|\ell| r}r^{p-1/2}\chi(r)\|h^\pm\|_{C^0} r dr d\theta dt \\
&\leqslant C\int_{S^1}\langle f^+ + Hf^-, e^{i\ell t}\rangle dt \int_0^\infty \sqrt{|\ell|}e^{-|\ell| r}r^{p+1/2}dr \\
&\leqslant C\left\langle \tfrac{1}{|\ell|^{p+1}}\left(f^+(t) + Hf^-(t)\right), e^{i\ell t}\right\rangle_{L^2(S^1;\mathbb{C})}
\end{aligned}$$

Since $f^\pm \in L^{k,2}(S^1;\mathbb{C})$, then $(f^+(t) + Hf^-(t)) \in L^{k,2}(S^1;\mathbb{C})$ as well, thus after applying the Fourier multiplier $1/|\ell|^{p+1}$ it lies in $L^{1+k+p,2}(S^1;\mathbb{C})$ as desired. For the case of a general metric, the integrals differs by a factor of $1 + O(r)$ and the latter only contributes a higher regularity term bounded by a constant times $|\ell|^{-(s+1)}$.

It is easy to show that the contributions arising from $\zeta_\ell + \xi_\ell$ satisfy the same bounds using Corollary 4.8 and integration by parts. Since these terms are dealt with explicitly in the proof of Theorem 6.1, the details are omitted here.

□

The following additional cases are a straightforward extension of the above and the example considered preceding (6.4).

Corollary 6.5. Let $\psi \in L^2$

(B) Suppose that $\text{supp}(\psi) \Subset Y - \mathcal{Z}_0$. Then $\text{ob}^{-1}\Pi_0(\psi) \in L^{s,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$ for all $s > 0$, and its $L^{s,2}$-norm is bounded by $C_s\|\psi\|_{L^2}$.

(C) Suppose ψ has the form

$$\psi = \begin{pmatrix} f^+(t)\varphi^+(t,r,\theta) \\ f^-(t)\varphi^-(t,r,\theta) \end{pmatrix} \tag{6.5}$$

where $f^\pm \in L^{k,2}(S^1;\mathbb{C})$ and $\varphi^\pm$ satisfy pointwise bounds $|\varphi^\pm| + |\nabla_t \varphi^\pm| + \ldots + |\nabla_t^k \varphi^\pm| < C(\varphi) r^p$. Then $\text{ob}^{-1}\Pi_0(\psi) \in L^{s,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})$ for $s = 1+k+p$, and its $L^{s,2}$-norm is bounded by $C_s C(\varphi)\|f^\pm\|_{L^{k,2}}$.

Remark 6.6. At this point, it is already apparent that there is a loss of regularity of 3/2 dictated by the $r^{1/2}$-asymptotics of $\mathbb{Z}_2$-harmonic spinors. Indeed, Corollary 5.9 shows that $\mathcal{B}_{\Phi_0}(\eta)$ schematically has the form $\eta'.\nabla\Phi_0 + \eta''.\Phi$. Since $\eta \in L^{2,2}$, and $\Phi_0 = O(r^{1/2})$ with $\nabla\Phi_0 = O(r^{-1/2})$, these terms have conormal regularity $s = 1+1-1/2$ and $s = 1+0+1/2$ respectively. Lemma 6.4 therefore already implies that (to leading order), $\Pi_0\mathcal{B}_{\Phi_0}(\eta) \subseteq \mathbf{Ob}^{3/2}$ hence lies is a proper dense subset of $\mathbf{Ob}(\mathcal{Z}_0)$. Decreasing the regularity of η below $L^{2,2}$, however, causes $\Pi^{\text{Rg}}(\mathcal{B}_{\Phi_0}(\eta))$ to be unbounded into L^2.

6.2 Obstruction Component of Deformations

This subsection carries out the main portion of the proof of Theorem 6.1 by proving an explicit formula for T_{Φ_0}.

This formula is expressed in terms of standard operators and the following zeroth order operator, for which we recall from Proposition 3.8 that $c(t) \in N\mathcal{Z}_0^{-1}$ and $d(t) \in N\mathcal{Z}_0$ denote the leading order (i.e. $r^{1/2}$) coefficients of Φ_0. Define an operator

$$\begin{aligned} \mathcal{L}_{\Phi_0} : L^2(\mathcal{Z}_0; N\mathcal{Z}_0) &\longrightarrow L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) && (6.6) \\ \xi(t) &\mapsto H(c(t)\xi(t)) - \overline{\xi}(t)d(t). && (6.7) \end{aligned}$$

where $H : L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \longrightarrow L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})$ is the Hilbert Transform given by the Fourier multiplier $\text{sgn}(\ell)$ where ℓ is the Fourier variable (and we take $\text{sgn}(0) = 1$). These formulas include an implicit association $\mathcal{S}_{\mathcal{Z}_0} \simeq \underline{\mathbb{C}}$ induced by the arclength parameterization.

Lemma 6.7. For $\eta(t) \in L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0)$, the operator T_{Φ_0} in Theorem 6.1 is given by

$$T_{\Phi_0}(\eta(t)) = -\tfrac{3|\mathcal{Z}_0|}{2}(\Delta+1)^{-\frac{3}{4}}\,\mathcal{L}_{\Phi_0}(\eta''(t)) \;+\; K(\eta) \tag{6.8}$$

where $|\mathcal{Z}_0|$ denotes the length, Δ denotes the positive-definite Laplacian on $\mathcal{S}_{\mathcal{Z}_0}$, $\mathcal{L}_{\Phi_0}$ is as in 6.7 above, and $\eta''(t)$ denotes the (covariant) second derivative on $N\mathcal{Z}_0$. K is a lower-order term.

The formula (6.8) is proved by calculating the sequence of inner products

$$\text{ob}^{-1}(\Pi_0\mathcal{B}_{\Phi_0}(\eta)) = \sum_\ell \langle \mathcal{B}_{\Phi_0}(\eta), \Psi_\ell\rangle_{\mathbb{C}}\ \phi_\ell \tag{6.9}$$

quite explicitly, where $\mathcal{B}_{\Phi_0}(\eta)$ is as in Corollary 5.9. The proof consists of five steps: Steps 1-2 calculate (6.9) in the case that g_0 is locally the product metric and Φ_0 is given by its leading order term, and Steps 3-5 show that the small parade of error terms arising from higher order terms contribute to the lower-order operator K.

Proof of Lemma 6.7. Suppose, to begin, that all the structure are given locally by the Euclidean ones. That is, assume

$$g_0 = dt^2 + dx^2 + dy^2 \qquad \Phi_0 = \begin{pmatrix} c(t)\sqrt{z} \\ d(t)\sqrt{\overline{z}} \end{pmatrix} \qquad \dot{g}_\eta = \begin{pmatrix} 0 & \eta_x'\chi & \eta_y'\chi \\ \eta_x'\chi & 2\eta_x\partial_x\chi & \eta_x\partial_y\chi + \eta_y\partial_x\chi \\ \eta_y'\chi & \eta_x\partial_y\chi + \eta_y\partial_x\chi & 2\eta_y\partial_y\chi \end{pmatrix},$$

and $B_0 = 0$; also assume that the obstruction elements of Proposition 4.2 have $\zeta_\ell + \xi_\ell = 0$ so that

$$\Psi_\ell = \chi\sqrt{|\ell|}e^{i\ell t}e^{-|\ell|r}\begin{pmatrix}\frac{1}{\sqrt{z}}\\ \frac{\text{sgn}(\ell)}{\sqrt{\overline{z}}}\end{pmatrix}$$

Step 1: product case, divergence term. Let e_i for $i = 1, 2, 3$ denote an orthonormal frame for g_0 with e^i the dual frame. Recall that for a symmetric 2-tensor k, $\text{div}_{g_0}k = (-\nabla_i k_{ij})e^j$.

$$\begin{aligned}
\tfrac{1}{2}\text{div}_{g_0}(\dot{g}_\eta).\Phi_0 &= -\frac{1}{2}\left[\sigma_2\chi\eta_x'' + \sigma_3\chi\eta_y''\right]\begin{pmatrix}c(t)\sqrt{z}\\ d(t)\sqrt{\overline{z}}\end{pmatrix} + (\mathbf{I})\\
&= -\frac{1}{2}\left[\chi\eta_x''\begin{pmatrix}-d(t)\sqrt{\overline{z}}\\ c(t)\sqrt{z}\end{pmatrix} + \chi\eta_y''\begin{pmatrix}id(t)/\sqrt{\overline{z}}\\ ic(t)/\sqrt{z}\end{pmatrix}\right] + (\mathbf{I})\\
&= -\frac{1}{2}\left[\begin{pmatrix}-\overline{\eta}''d(t)\chi\sqrt{\overline{z}}\\ \eta''c(t)\chi\sqrt{z}\end{pmatrix}\right] + (\mathbf{I})
\end{aligned}$$

where we have written $\eta(t) = \eta_x(t) + i\eta_y(t)$, and

$$(\mathbf{I}) = -\frac{1}{2}\Big[(\partial_x\chi\eta_x' + \partial_y\chi\eta_y')\sigma_t + (2\partial_{xx}\chi\eta_x + \partial_{xy}\chi\eta_y + \partial_{yy}\chi\eta_x)\sigma_x + (\partial_{xy}\chi\eta_y + \partial_{yy}\chi\eta_x + 2\partial_{yy}\chi\eta_y)\sigma_y\Big].\Phi_0.$$

Taking the inner product of the first term with Ψ_ℓ yields

$$\begin{aligned}
\langle\tfrac{1}{2}\text{div}_{g_0}(\dot{g}_\eta).\Phi_0, \Psi_\ell\rangle &= -\frac{1}{2}\Big\langle\chi\begin{pmatrix}-\overline{\eta}''d(t)\sqrt{\overline{z}}\\ \eta''c(t)\sqrt{z}\end{pmatrix}\ ,\ \sqrt{|\ell|}e^{i\ell t}\chi\begin{pmatrix}e^{-|\ell|r}/\sqrt{z}\\ \text{sgn}(\ell)e^{-|\ell|r}/\sqrt{\overline{z}}\end{pmatrix}\Big\rangle_{\mathbb{C}}\\
&= -\frac{1}{2}\int_{S^1}\Big\langle\begin{matrix}-\overline{\eta}''d(t)\\ \eta''c(t)\end{matrix}\ ,\ \begin{matrix}e^{i\ell t}\\ \text{sgn}(\ell)e^{i\ell t}\end{matrix}\Big\rangle_{\mathbb{C}}\,dt\ \int_{\mathbb{R}^2}\sqrt{|\ell|}\chi^2e^{-|\ell|r}rdrd\theta\\
&= -\frac{1}{2}\langle\text{sgn}(\ell)\eta''c - \overline{\eta}''d\ ,\ e^{i\ell t}\rangle_{L^2(\mathcal{Z}_0)}\int_{\mathbb{R}^2}\sqrt{|\ell|}\chi^2(r)e^{-|\ell|r}rdrd\theta\\
&= \langle-\frac{1}{2}\frac{|\mathcal{Z}_0|}{|\ell|^{3/2}}\mathcal{L}_{\Phi_0}(\eta''), e^{i\ell t}\rangle_{\mathbb{C}} + \langle K, e^{i\ell t}\rangle
\end{aligned}$$

since

$$\int_0^\infty\sqrt{|\ell|}e^{-|\ell|r}rdrd\theta = \frac{1}{|\ell|^{3/2}}$$

and the presence of $\chi^2(r)$ results in a difference from this of size $O(e^{-|\ell|r_0})$ which is denoted by K.

Then, since

$$\frac{1}{|\ell|^{3/2}} = \frac{1}{(|\ell|^2+1)^{3/4}} + O\left(\frac{1}{|\ell|^3}\right),$$

we can write

$$\text{ob}^{-1}(\tfrac{1}{2}\text{div}_{g_0}(\dot{g}_\eta).\Phi_0) = -\tfrac{|\mathcal{Z}_0|}{2}(\Delta+1)^{-\frac{3}{4}}\mathcal{L}_{\Phi_0}(\eta'') + K$$

where K is a psuedo-differential operator of lower order (the first term has order 1/2). For the term **(I)**, note that it is a sum of term compactly supported away from $\mathcal{Z}_0$, hence by Case (B) of Corollary 6.5, it contributes a smoothing operator which we may absorb into K.

Step 2: product case, symbol term. The "symbol" term from $\mathcal{B}_{\Phi_0}(\eta)$ is given by

$$\begin{aligned}
-\frac{1}{2}\dot{g}_\eta(e_i, e_j)e^i.\nabla_j\Phi_0 &= -\frac{1}{2}\left[\chi\eta_x'\sigma_t\nabla_x\Phi_0 + \chi\eta_y'\sigma_t\nabla_y\Phi_0\right] + (\mathbf{II})\\
&= -\frac{1}{4}\left[\chi\eta_x'\begin{pmatrix}ic(t)/\sqrt{z}\\ -id(t)/\sqrt{\overline{z}}\end{pmatrix} + \chi\eta_y'\begin{pmatrix}-c(t)\sqrt{z}\\ d(t)\sqrt{\overline{z}}\end{pmatrix}\right] + (\mathbf{II})\\
&= -\frac{1}{4}\left[\begin{pmatrix}i\eta'c(t)\chi/\sqrt{z}\\ -i\overline{\eta}'d(t)\chi/\sqrt{\overline{z}}\end{pmatrix}\right] + (\mathbf{II})
\end{aligned}$$

where

$$(\mathbf{II}) \quad = \quad -\frac{1}{2}\Big[(\chi\eta_x'\sigma_x + \chi\eta_y'\sigma_y)\nabla_t\Phi_0 + (2\partial_x\chi\eta_x\sigma_x + \partial_x\chi\eta_y\sigma_y + \partial_y\chi\eta_x\sigma_y)\nabla_x\Phi_0 + (2\partial_y\chi\eta_y\sigma_y + \partial_x\chi\eta_y\sigma_x + \partial_y\chi\eta_x\sigma_x)\nabla_y\Phi_0\Big].$$

Taking the inner product of the first term with Ψ_ℓ yields the following. This calculation is almost identical to the previous one, but with an additional integration by parts.

$$\begin{aligned}
\langle \tfrac{1}{2}\dot{g}_\eta(e_i,e_j)e^i.\nabla_j\Phi_0, \Psi_\ell\rangle &= -\frac{1}{4}\Big\langle \chi\begin{pmatrix} i\eta' c(t)/\sqrt{z} \\ -i\overline{\eta}' d(t)/\sqrt{\overline{z}}\end{pmatrix}, \sqrt{|\ell|}e^{i\ell t}\chi\begin{pmatrix} e^{-|\ell| r}/\sqrt{z} \\ \mathrm{sgn}(\ell)e^{-|\ell|r}/\sqrt{\overline{z}}\end{pmatrix}\Big\rangle_{\mathbb{C}} \\
&= -\frac{1}{4}\Big\langle \chi\begin{pmatrix} i\eta' c(t)/\sqrt{z} \\ -i\overline{\eta}' d(t)/\sqrt{\overline{z}}\end{pmatrix}, \tfrac{\sqrt{|\ell|}}{i|\ell|\mathrm{sgn}\ell}\partial_t e^{i\ell t}\chi\begin{pmatrix} e^{-|\ell| r}/\sqrt{z} \\ \mathrm{sgn}(\ell)e^{-|\ell|r}/\sqrt{\overline{z}}\end{pmatrix}\Big\rangle_{\mathbb{C}} \\
&= -\frac{1}{4}\Big\langle \chi\partial_t\begin{pmatrix} \eta' c(t)/\sqrt{z} \\ -\overline{\eta}' d(t)/\sqrt{\overline{z}}\end{pmatrix}, \tfrac{1}{\sqrt{|\ell|}}e^{i\ell t}\chi\begin{pmatrix} \mathrm{sgn}(\ell)e^{-|\ell| r}/\sqrt{z} \\ e^{-|\ell|r}/\sqrt{\overline{z}}\end{pmatrix}\Big\rangle_{\mathbb{C}} \\
&= -\frac{1}{4}\int_{S^1}\Big\langle \begin{matrix}\partial_t(\eta' c(t)) \\ -\partial_t(\overline{\eta}' d(t))\end{matrix}, \begin{matrix}\mathrm{sgn}(\ell)e^{i\ell t} \\ e^{i\ell t}\end{matrix}\Big\rangle_{\mathbb{C}}\, dt \quad \int_{\mathbb{R}^2}\frac{1}{\sqrt{|\ell|}}\chi^2 e^{-|\ell|r}drd\theta
\end{aligned}$$

In the second line we have multiplied the second argument by 1 in the form $1 = \frac{i\ell}{i|\ell|\mathrm{sgn}\ell}$ and noted $i\ell\psi_\ell = \partial_t\psi_\ell$, and then integrated by parts. Then,

$$\begin{aligned}
&= -\frac{1}{4}\langle \mathrm{sgn}(\ell)\eta'' c - \overline{\eta}'' d\ ,\ e^{i\ell t}\rangle_{L^2(S^1)}\int_{\mathbb{R}^2}\frac{1}{\sqrt{|\ell|}}\chi^2(r)e^{-|\ell|r}rdrd\theta \\
&\quad -\frac{1}{4}\langle \mathrm{sgn}(\ell)\eta' c' - \overline{\eta}' d'\ ,\ e^{i\ell t}\rangle_{L^2(S^1)}\int_{\mathbb{R}^2}\frac{1}{\sqrt{|\ell|}}\chi^2(r)e^{-|\ell|r}rdrd\theta \\
&= \langle -\frac{1}{4}\frac{|\mathcal{Z}_0|}{|\ell|^{3/2}}\mathcal{L}_{\Phi_0}(\eta''), e^{i\ell t}\rangle_{\mathbb{C}} + \langle -\frac{1}{4}\frac{|\mathcal{Z}_0|}{|\ell|^{3/2}}\mathcal{L}_{\nabla_t\Phi_0}(\eta'), e^{i\ell t}\rangle_{\mathbb{C}} + \langle K, e^{i\ell t}\rangle
\end{aligned}$$

Where K is again an error of size $O(e^{-|\ell|r_0})$ and $\mathcal{L}_{\nabla_t\Phi_0}$ is defined exactly as $\mathcal{L}_{\Phi_0}$ but with $c'(t), d'(t)$ in place of $c(t), d(t)$. Both $\mathcal{L}_{\nabla_t\Phi_0}$ and the term **(II)** are lower order by Lemma 6.4 and Case (B) of Corollary 6.5, so they may be absorbed into K. To see this, note these are comprised of terms of the form form $\eta'\nabla_t\Phi_0 = \eta' r^{1/2}$, hence of conormal regularity $s = 5/2$ or have a factor of $d\chi$ so are compactly supported away from $\mathcal{Z}_0$. The term same applies to the term $\frac{1}{2}d\mathrm{Tr}_{g_0}(\dot{g}_\eta).\Phi_0$.

Remark 6.8. A coincidence has occurred here. Lemma 6.4 implies that the two leading order terms from *Step 1* and *Step 2* are both order 1/2 as they have the same conormal regularity. The calculation shows they are actually *the same* up to a constant multiple and lower order terms. It is unclear if there is a more abstract reason for this (cf. Remark 5.6).

Now we return to the general case. In general, there are deviation from the product case for $\Phi_0, (g_0, B_0)$ and Ψ_ℓ. These are accounted for in *Step 3*-*Step 5* respectively.

Step 3: By Proposition 3.8 we can in general write

$$\Phi_0 = \begin{pmatrix} c(t)\sqrt{z} \\ d(t)\sqrt{\overline{z}}\end{pmatrix} + \Phi_1$$

where the higher order terms satisfy

$$|\Phi_1| + |\nabla_t^k\Phi_1| \leqslant C_k r^{3/2} \qquad\qquad |\nabla_z\Phi_1| + |\nabla_t^k(\nabla_z\Phi_1)| \leqslant C_k r^{1/2} \tag{6.10}$$

for any $k \in \mathbb{N}$ and identically for $\nabla_{\overline{z}}$. The resulting contribution to $\mathcal{B}_{\Phi_0}(\eta)$ is

$$-\frac{1}{2}\dot{g}_\eta(e_i,e_j)e^i.\nabla_j\Phi_1+\frac{1}{2}d\mathrm{Tr}_{g_0}(\dot{g}_\eta).\Phi_1+\frac{1}{2}\mathrm{div}_{g_0}(\dot{g}_\eta).\Phi_1 \tag{6.11}$$

and using 6.10 and Part (C) of Corollary 6.5 shows that each term has conormal regularity one higher than the corresponding term for the leading order of Φ_0. (6.11) therefore contributes an operator of order $-1/2$ which can be absorbed into K.

Step 4: As in Definition 3.4, the metric in Fermi coordinates is given by

$$g_0=dt^2+dx^2+dy^2+h$$

where $h=O(r)$. Compared to the case of the product metric, we now have $e_i=\partial_i+O(r)$ and

$$\nabla_i^{g_0}=\partial_i+\Gamma_i. \qquad\qquad \dot{g}_\eta=\dot{g}_\eta^{\mathrm{prod}}+h_1+h_2 \tag{6.12}$$

$$dV_{g_0}=(1+O(r))rdrd\theta dt \qquad\qquad \langle -,-\rangle_{g_0}=(1+O(r))\langle -,-\rangle_{\mathrm{Euc}}. \tag{6.13}$$

where h_1,h_2 are as in Corollary 5.9. As such, each additional term in $\mathcal{B}_{\Phi_0}(\eta)$ has *either* an additional power of r *or* one fewer derivative of η compared to the terms for the product case. Using Corollary 6.5 and the bounds

$$|\Phi_0|+|\nabla_t^k\Phi_0|\leqslant C_k r^{1/2} \qquad\qquad |\nabla_z\Phi_0|+|\nabla_t^k(\nabla_z\Phi_0)|\leqslant C_k r^{-1/2}$$

we see that all such terms have conormal regularity at least $s=5/2$. The term $\mathcal{R}(B_0,\chi\eta)=O(1)\eta$ arising from the perturbation similarly has conormal regularity $s>5/2$. They therefore factor through $L^{5/2,2}(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$ and can be added to the compact term K. In addition, changing $\frac{d}{dt}$ to the covariant derivative only contributes to the compact term.

Step 5: By Proposition 4.3 we may in general write

$$\Psi_\ell=\Psi_\ell^{\mathrm{Euc}}+\zeta_\ell^{(m)}+\xi_\ell^{(m)}$$

where the latter satisfy the bounds of Corollary 4.8. Set

$$K_1(\eta):=\sum_\ell\langle\mathcal{B}_{\Phi_0}(\eta),\zeta_\ell^{(m)}\rangle\phi_\ell \qquad\qquad K_2(\eta):=\sum_\ell\langle\mathcal{B}_{\Phi_0}(\eta),\xi_\ell^{(m)}\rangle\phi_\ell. \tag{6.14}$$

We claim that the second factors $K_2:L^{2,2}\to L^{5/2,2}\to L^{3/2}$ hence contributes a compact term. By Cauchy-Schwartz and the bound $\|\xi_\ell^{(m)}\|_{L^2}\leqslant C_m|\ell|^{-2-m}$,

$$\begin{aligned}
\|K_2(\eta)\|^2_{5/2,2} &= \sum_\ell|\langle\mathcal{B}_{\Phi_0}(\eta),\xi_\ell^{(m)}\rangle|^2|\ell|^5\\
&\leqslant \sum_\ell\|\mathcal{B}_{\Phi_0}(\eta)\|^2_{L^2}\ \|\xi_\ell^{(m)}\|^2_{L^2}\ |\ell|^5\\
&\leqslant C\|\mathcal{B}_{\Phi_0}(\eta)\|^2_{L^2}\sum_\ell\frac{|\ell|^5}{|\ell|^{4+2m}} \quad\leqslant\ C\|\eta\|^2_{L^{2,2}}\sum_\ell\frac{1}{|\ell|^{2m-1}}\leqslant C\|\eta\|^2_{L^{2,2}}
\end{aligned}$$

for, say, $m=2$. In the last line we have used that $|\mathcal{B}_{\Phi_0}(\eta)|\leqslant(|\eta|+|\eta'|+|\eta''|)r^{-1/2}$ and the latter is integrable on normal disks.

Likewise, we claim K_1 factors through $L^{3/2+\delta,2}$ for $\delta<1/2$. This time, we apply Cauchy-Schwartz on each annulus $A_{n\ell}$ (defined in 4.3). Write $K_1=K_1'+K_1''$ where

$$K_1'(\eta)=\langle\tfrac{1}{2}d\mathrm{Tr}_{g_0}(\dot{g}_\eta).\Phi_0+\tfrac{1}{2}\mathrm{div}_{g_0}(\dot{g}_\eta).\Phi_0,\zeta_\ell.\rangle \qquad\qquad K_1''(\eta)=\langle-\tfrac{1}{2}\dot{g}_\eta(e_i,e_j)e^i.\nabla_j\Phi_0,\zeta_\ell\rangle$$

and we keep the superscript (m) implicit. For the first of these,

$$\begin{aligned}\|K_1'(\eta)\|^2_{3/2+\delta,2} &\leqslant C\sum_\ell\sum_n \|\eta''|\Phi_0|\|^2_{L^2(A_{n\ell})}\ \|\zeta_\ell\|^2_{L^2(A_{n\ell})}\ |\ell|^{3+2\delta}\\ &\leqslant C\sum_\ell |\ell|^{3+2\delta}\sum_n \|\eta'' r^{1/2}\|^2_{L^2(A_{n\ell})}\ \frac{1}{|\ell|^2}\mathrm{Exp}\left(-\frac{n}{c_1}\right)\end{aligned}$$

Then, since $r\sim\frac{(n+1)R_0}{|\ell|}$ on $A_{n\ell}$, and each has area $O(|\ell|^{-2})$, the above is bounded by

$$\leqslant\ C\|\eta''\|^2_{L^2(S^1)}\sum_\ell|\ell|^{3+2\delta}\sum_n\frac{(n+1)^3}{|\ell|^5}\mathrm{Exp}\left(-\frac{n}{c_1}\right)\ \leqslant\ C\|\eta''\|^2_{L^2(S^1)}\sum_\ell\frac{1}{|\ell|^{2-2\delta}}\ \leqslant\ C\|\eta\|^2_{L^{2,2}}.$$

The K_1'' term is the same except we first use the Fourier mode restriction that ζ_ℓ has only Fourier modes p with $\ell-\frac{|\ell|}{2}\leqslant p\leqslant \ell+\frac{|\ell|}{2}$ to write $1\sim\frac{i\partial_t}{|\ell|}$ and then integrate by parts as in *Step 2*.

□

6.3 The Index of $\mathcal{L}_{\Phi_0}$

In this section we complete the proof of Theorem 6.1. This follows from the following about $\mathcal{L}^{\Phi_0}$. The key role and Fredholmness of a similar map was originally observed in [49]. Here, we present a simplified proof.

Lemma 6.9. When Assumption 2 holds,

$$\mathcal{L}_{\Phi_0}: L^2(\mathcal{Z}_0;N\mathcal{Z}_0)\to L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})$$

is an elliptic pseudo-differential operator of Index 0.

To begin, we have the following fact. Let $a(t)\in C^\infty(\mathcal{Z}_0;\mathbb{C})$ be a smooth function and let

$$[H,a]=H\circ a(t)-a(t)\circ H$$

denote the commutator.

Claim 6.9.1. The commutator

$$[H,a]: L^{m,2}(\mathcal{Z}_0;\mathbb{C})\to L^{m+1,2}(\mathcal{Z}_0;\mathbb{C})$$

is a smoothing operator of order 1.

Proof. Multiplication by $a(t)$ and H are both elliptic pseudodifferential operators of order 0, hence so is the commutator. Using the composition property of principal symbols, its principal symbol of order 0 is

$$\sigma_0([H,a])=\sigma_0(H)\sigma_0(a)-\sigma_0(a)\sigma_0(H)=0$$

hence it is a pseudodifferential operator of order -1. □

We now prove the lemma:

Proof of Lemma 6.9. Given $\xi\in L^2(\mathcal{Z}_0;\mathcal{S}_{\mathcal{Z}_0})\simeq L^2(\mathcal{Z}_0;\mathbb{C})$ we define a pseudo-inverse. Set

$$\mathcal{L}^\star_{\Phi_0}(\xi(t))=\overline{c}(t)H\xi(t)-d(t)\overline{\xi(t)}. \tag{6.15}$$

Using Claim 6.9.1 to move H past combinations of the smooth functions $c(t),d(t)$ and their conjugates (and noting $H^2=Id$), we compute

$$\begin{aligned}
\mathcal{L}_{\Phi_0} \circ \mathcal{L}^{\star}_{\Phi_0}(\xi(t)) &= ((Hc(t) - d(t) \circ \mathrm{conj}))(\bar{c}(t)H - d(t) \circ \mathrm{conj})(\xi(t))) \\
&= Hc\bar{c}H\xi - dc\overline{H\xi} - Hcd\bar{\xi} + d\bar{d}\xi \\
&= (|c|^2 + |d|^2)f + [H, |c|^2]H\xi - dc(\overline{H\xi} + H\bar{\xi}) - [H, cd]\bar{\xi} \\
&= ((|c|^2 + |d|^2)Id + K)\xi
\end{aligned}$$

for a smoothing operator K. In the last line we have used $\overline{Hf} + H\bar{\xi} = 2\xi_0$ where ξ_0 is the zeroeth Fourier mode, which is clearly a smoothing operator. It follows that

$$\frac{1}{|c|^2 + |d|^2}\mathcal{L}^{\star}_{\Phi_0}$$

provides a right pseudo-inverse for $\mathcal{L}_{\Phi_0}$ (commuting the denominator past H contributes to the compact term). An equivalent calculation for the reverse composition shows it is also a left pseudo-inverse, thus $\mathcal{L}_{\Phi_0}$ is Fredholm.

Since $\pi_1(\mathbb{C}^2 - \{0\}, *)$ is trivial, the pair $(c(t), d(t))$ is homotopic through pairs satisfying the condition $|c(t)|^2 + |d(t)|^2 > 0$ to the constant pair $(1, 0)$. The operator $\mathcal{L}_{\Phi_0}$ is therefore homotopic to the identity through Fredholm operators hence has index 0. $\square$

Theorem 6.1 is now immediate:

Proof of Theorem 6.1. Lemma 6.7 shows that the operator

$$\mathrm{ob}^{-1}(B_{\Phi_0}(\eta)) = -\frac{3|\mathcal{Z}_0|}{2}(\Delta + 1)^{-\frac{3}{4}}\mathcal{L}_{\Phi_0}(\eta''(t)) + K$$

is given as the sum of following compositions:

$$\begin{tikzcd}
L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \arrow[r, "-\frac{3|\mathcal{Z}_0|}{2}\frac{d}{dt^2}"] \arrow[d, leftrightarrow, "\simeq"'] & L^2(\mathcal{Z}_0; N\mathcal{Z}_0) \arrow[r, "\mathcal{L}_{\Phi_0}"] & L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \arrow[r, "(\Delta+1)^{-\frac{3}{4}}"] & L^{3/2,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \\
L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \arrow[r, "K"] & L^{3/2+\delta,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \arrow[urr, "\iota"] & &
\end{tikzcd}$$

where the diagonal arrow is the inclusion, hence compact. All the top arrows are Fredholm of Index 0 using Lemma 6.9, and Theorem 6.1 therefore follows from the composition law for pseudodifferential operators.

$\square$

Given Theorem 6.1, we now impose one more tacit assumption that this Fredholm operator of Index zero is actually invertible. This is expected to hold generically (see [25]), though we do not prove such a result here. At the end of Section 8, this assumption can be removed by the use of standard Kuranishi models.

Assumption 5*. The index zero map $T_{\Phi_0} : L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \to L^{3/2,2}(\mathcal{Z}_0, \mathcal{S}_{\mathcal{Z}_0})$ is an isomorphism.

7 Nash-Moser Theory

As explained in introduction, deducing the non-linear deformation result (Theorem 1.4) from the linear one (Theorem 1.3) requires the Nash-Moser Implicit Function Theorem because of the loss of regularity in the operator T_{Φ_0}. This section gives a brief and practical introduction to the framework of Nash-Moser Theory and states the relevant version of the implicit function theorem. The most complete reference for the full abstract theory is [18]. Here, we more closely follow the expositions in [1, 46, 48] which are more modest in scope but suffice for our purposes.

7.1 Tame Fréchet Spaces

Let $\mathcal{X}, \mathcal{Y}$ be Fréchet spaces given as the intersection of families of Banach spaces

$$\mathcal{X} := \bigcap_{m \geqslant 0} X_m \qquad\qquad \mathcal{Y} := \bigcap_{m \geqslant 0} Y_m \tag{7.1}$$

whose norms are monotonically increasing so that

$$\|x\|_0 \leqslant \|x\|_1 \leqslant \ldots \leqslant \|x\|_m,$$

and likewise for $\mathcal{Y}$. The topologies on $\mathcal{X}, \mathcal{Y}$ are the ones generated by the countable collection of norms, i.e. a set U is open if and only if for each point $x \in U$ there are $r > 0$ and $m \geqslant 0$ such that the ball $\{x \mid \|x\|_m < r\} \subset U$ measured in the m-norm is contained in U.

Definition 7.1. A Fréchet space $\mathcal{X}$ is said to be **tame** if it satisfies the two following criteria:

(I) For all $m_1 < m < m_2$ the interpolation inequalities

$$\|x\|_m \leqslant C_{m,m_1,m_2} \|x\|_{m_1}^{\alpha} \|x\|_{m+2}^{1-\alpha}$$

holds where $\alpha = \frac{m_2 - m}{m_2 - m_1}$.

(II) $\mathcal{X}$ is equipped with a family of smoothing operators

$$S_\varepsilon : \mathcal{X} \to \mathcal{X}$$

for all $\varepsilon \in (0, 1]$ satisfying the following conditions.

(i) $\|S_\varepsilon x\|_n \leqslant C_{mn} \varepsilon^{m-n} \|x\|_m$ for $n \geqslant m$ and $\|S_\varepsilon x\|_m \leqslant C_{mn} \|x\|_n$ for $n \leqslant m$.
(ii) $\|S_\varepsilon x - x\|_m \leqslant C_{mn} \varepsilon^{n-m} \|x\|_n$ for $n \geqslant m$.
(iii) $\|\frac{d}{d\varepsilon} S_\varepsilon x\|_n \leqslant C_{mn} \varepsilon^{m-n-1} \|x\|_m$ for all $m, n \geqslant 0$.

In practice, most reasonable choices of families of norms coming from Sobolev or Hölder norms are tame. Roughly speaking, smoothing operators S_θ are usually constructed by truncating local Fourier transforms at radius ε^{-1}. In particular, the Fréchet spaces introduced in Section 8.4 are tame and possess smoothing operators constructed effectively in this fashion.

Given two tame Fréchet spaces $\mathcal{X}$ and $\mathcal{Y}$,

Definition 7.2. A **tame Fréchet map** on an open subset $U \subseteq \mathcal{X}$

$$\mathcal{F} : U \to \mathcal{Y}$$

is a smooth map of vector spaces such that there is an r and the estimate

$$\|\mathcal{F}(x)\|_m \leqslant C_m \, (1 + \|x\|_{m+r}) \tag{7.2}$$

holds for all sufficiently large m.

The definitions of tame spaces and maps extend naturally to define a category of tame Fréchet manifolds with tame Fréchet maps between them (see [18] for details). The key point about tame estimates is that each norm depends only on a fixed finite number r of norms larger than it. Thus, for example, a map with an estimate of the form (7.2) where $r = 2m$ would not be tame.

7.2 The Implicit Function Theorem

Before stating a precise version of the Nash-Moser Implicit Function Theorem, let us briefly give some intuition. Here, our exposition follows [48].

Suppose that $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is a map with $\mathcal{F}(0) = 0$, and we wish to solve

$$\mathcal{F}(x) = f \tag{7.3}$$

for $f \in \mathcal{Y}$ small. When $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, the (standard) Implicit Function Theorem is proved using Newton iteration and the Banach Fixed Point Theorem. More specifically, one begins with an initial approximation $x_0 = 0$, and (provided that $\mathrm{d}_{x_0}\mathcal{F}$ is invertible) defines

$$x_{k+1} = x_k + (\mathrm{d}_{x_0}\mathcal{F})^{-1}(f - \mathcal{F}(x_k)). \tag{7.4}$$

The sequence $x_k \to x_\infty$ then converges to a unique fixed point solving equation (7.3) for $f \in \mathcal{Y}$ sufficiently small. Alternatively, one can modify the iteration step (7.4) by inverting $\mathrm{d}\mathcal{F}$ at x_k instead of at x_0, taking

$$x_{k+1} = x_k + (\mathrm{d}_{x_k}\mathcal{F})^{-1}(f - \mathcal{F}(x_k)). \tag{7.5}$$

This iteration scheme has a much faster rate of convergence: like $\sim 2^{-2^k}$.

Consider now the case of $\mathcal{X}, \mathcal{Y}$ tame Fréchet spaces when $\mathrm{d}\mathcal{F}$ displays a loss of regularity of r. Given an initial bound on $f \in Y_m$, then x_1 is bounded only in X_{m-r} thus $g - \mathcal{F}(x_1)$ in Y_{m-r} and x_2 in X_{m-2r}. In this way, the standard Newton iteration scheme will exhaust the prescribed regularity in a finite number of steps. To circumvent this loss of regularity, Nash introduced smoothing operators at each stage. More precisely, for some $\varepsilon_k \in (0, 1]$, we set

$$x_{k+1} = x_k + (\mathrm{d}_{S_{\varepsilon_k}(x_k)}\mathcal{F})^{-1} S_{\varepsilon_k}(f - \mathcal{F}(x_k)), \tag{7.6}$$

where the smoothing operators in the subscript are those on $\mathcal{X}$ and in the argument those on $\mathcal{Y}$. The key point is that the rate of convergence is rapid enough to overcome the disruption of the smoothing operators, but only if we use this smoothing to modify the improved iteration (7.5), rather than the original iteration (7.4). Thus, unlike to the Implicit Function Theorem on Banach spaces, the Nash-Moser Implicit Function Theorem requires the linearization be invertible on a neighborhood of the initial guess, and requires bounds on the second derivatives to control the linearization over this neighborhood. Specifically, the theorem requires the following hypotheses on a tame map $\mathcal{F} : U \to \mathcal{Y}$:

Hypothesis (I). There exists a $\delta_0 > 0$ and an $m_0 \geqslant 0$ such that for $x \in U_0 = B_{\delta_0}(0, m_0) \cap \mathcal{X}$, the open ball of radius δ_0 measured in the m_0 norm, then

$$\mathrm{d}_x\mathcal{F} : \mathcal{X} \to \mathcal{Y}$$

is invertible.

Hypothesis (II). With $x \in U_0$ as above, there are fixed $s, s' \in \mathbb{N}$ such that the unique solution u of

$$\mathrm{d}_x\mathcal{F}(u) = g$$

satisfies the tame estimate

$$\|u\|_m \leqslant C_m \left(\|g\|_{m+s} + \|g\|_{m_0} \cdot \|x\|_{m+s'}\right). \tag{7.7}$$

Hypothesis (III). With $x \in U_0$ as above, there are fixed $r, r' \in \mathbb{N}$ such that the second derivative satisfies the tame estimate

$$\|\mathrm{d}_x^2 \mathcal{F}(u, v)\|_m \leqslant C_m \, \Big(\|u\|_{m+r} \|v\|_{m_0} \; + \; \|u\|_{m_0} \|v\|_{m+r} \; + \; \|u\|_{m_0} \|v\|_{m_0} \cdot (1 + \|x\|_{m+r'}) \Big). \tag{7.8}$$

For our purposes, we require a slight extension of the standard Nash-Moser Implicit Function Theorem that keeps track of subspaces that have some specified additional property, denoted (P).

Definition 7.3. A property (P) that is satisfied on linear (not necessarily closed) subspaces $\mathbf{P}_{\mathcal{X}} \subseteq \mathcal{X}$ and $\mathbf{P}_{\mathcal{Y}} \subseteq \mathcal{Y}$ is said to be **propagated** by the iteration scheme if

$$\begin{aligned} u \in \mathbf{P}_{\mathcal{X}} \ , \ g \in \mathbf{P}_{\mathcal{Y}} \quad &\Rightarrow \quad S_\varepsilon(u) \in \mathbf{P}_{\mathcal{X}} \ \ , \ \ S_\varepsilon(g) \in \mathbf{P}_{\mathcal{Y}} \qquad \forall \varepsilon \in (0,1] \\ u \in \mathbf{P}_{\mathcal{X}} \quad &\Rightarrow \quad \mathcal{F}(u) \in \mathbf{P}_{\mathcal{Y}} \\ x \in \mathbf{P}_{\mathcal{X}} \ , \ g \in \mathbf{P}_{\mathcal{Y}} \quad &\Rightarrow \quad (\mathrm{d}_x \mathcal{F})^{-1} g \in \mathbf{P}_{\mathcal{X}}. \end{aligned}$$

In particular, in the iteration scheme (7.6), if f has property (P) then x_k has property (P) for all $k \geqslant 0$.

We will use the following version of the Nash-Moser Implicit Function Theorem. The proof is identical to that in [48], with the additional observation that Hypotheses **(I)**–**(III)** are only ever invoked at elements x_k occurring in the iteration, and at linear combinations of the x_k and their smoothings. The proof of smooth dependence on parameters is given in [18, III.1].

Theorem 7.4. (Nash-Moser Implicit Function Theorem) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are tame Fréchet spaces as in (7.1). Moreover, assume that a property (P) satisfied on linear subspaces $\mathbf{P}_{\mathcal{X}} \subseteq \mathcal{X}$ and $\mathbf{P}_{\mathcal{Y}} \subseteq \mathcal{Y}$ is propagated, and that Hypotheses **(I)**-**(III)** hold for $x \in U_0 \cap \mathbf{P}_{\mathcal{X}}$.

(A) There exists an $m_1 \geqslant m_0$ depending on s, s', r, r' and a $\delta_1 \geqslant 0$ such that if $f \in \mathcal{Y}$ with

$$f \in \mathbf{P}_{\mathcal{Y}} \qquad\qquad \text{and} \qquad\qquad \|f\|_{m_1} \leqslant \delta_1$$

then there exists a unique solution $x \in \mathcal{X}$ of

$$\mathcal{F}(x) = f.$$

(B) Suppose, in addition, that $\mathcal{F}$ and f are parameterized (via a smooth tame map) by another tame Fréchet space $\mathcal{P}$ with $f_{p_0} = 0$ at $p_0 \in \mathcal{P}$. If the Hypotheses **(I)**–**(III)** hold uniformly on an open neighborhood $V_0 \subset \mathcal{P}$ of p_0 and $\|f_p\|_{m_1} < \delta_1$ for all $p \in V_0$, then the unique solution x_p of

$$\mathcal{F}_p(x) = f_p$$

also depends smoothly on p locally near p_0.

□

In case (B), smooth tame dependence on p means that we replace $\|x\|_{m+s'}$ and $\|x\|_{m+r'}$ on the right-hand sides of Hypothesis **(II)** and **(III)** by $\|(p,x)\|_{m+s'}$ and $\|(p,x)\|_{m+r'}$. Case (B) asserts that

$$\mathcal{F}^{-1}(f_p) \subset \mathcal{P} \times \mathcal{X}$$

is locally a tame Fréchet submanifold that is a graph over $\mathcal{P}$.

8 Tame Estimates

In this final section we complete the proofs of Theorem 1.4 and Corollary 1.5 by verifying the hypotheses of the Nash-Moser Implicit Function Theorem 7.4 for the operator

$$\overline{\mathbb{D}}_p : \mathcal{P} \times \mathcal{X} \longrightarrow \mathcal{Y} \qquad\qquad \overline{\mathbb{D}}_p := (\mathbb{D}_p - \Lambda\,\mathrm{Id}\;,\; 1 - \|\Phi\|^2_{L^2})$$

on tame Fréchet spaces $\mathcal{X} = \{(\eta, \Lambda, \varphi)\}$ and $\mathcal{Y} = \{\psi, c\}$ introduced in Section 8.4. Here $\Lambda, c \in \mathbb{R}$ and $\mathcal{P} = \{(g, B)\}$ is the space of smooth metrics and perturbations (equipped with the standard Fréchet structure arising from the $L^{m,2}$ norms on Y).

In our case, the property (P) that is propagated by the iteration scheme is polyhomogeneity of the spinor. Set:

$$\begin{aligned}
\mathbf{P}_{\mathcal{X}} \;&:=\; \{(\eta, \Lambda, \varphi) \in \mathcal{X} \mid \varphi \text{ is polyhomogenous with index set } \mathbb{Z}^+ + \tfrac{1}{2}\} \\
\mathbf{P}_{\mathcal{Y}} \;&:=\; \{\quad (\psi, c) \in \mathcal{Y} \;\mid \psi \text{ is polyhomogenous with index set } \mathbb{Z}^+ - \tfrac{1}{2}\}
\end{aligned}$$

Here, we use a slightly weaker notion of polyhomogeneity than is given in Definition 3.8. More specifically, we do not constrain the θ modes, so that $\varphi \in \mathbf{P}_{\mathcal{X}}, \psi \in \mathbf{P}_{\mathcal{Y}}$ means that there are respectively asymptotic expansions

$$\varphi \sim \begin{pmatrix} c(t,\theta) \\ d(t,\theta) \end{pmatrix} r^{1/2} \;+\; \sum_{n\geqslant 1}\sum_{p=0}^{n} \begin{pmatrix} c_{n,p}(t,\theta) \\ d_{n,p}(t,\theta) \end{pmatrix} r^{n+1/2} (\log r)^p \tag{8.1}$$

$$\psi \sim \begin{pmatrix} c(t,\theta) \\ d(t,\theta) \end{pmatrix} r^{-1/2} + \sum_{n\geqslant 1}\sum_{p=0}^{n} \begin{pmatrix} c_{n,p}(t,\theta) \\ d_{n,p}(t,\theta) \end{pmatrix} r^{n-1/2} (\log r)^p \tag{8.2}$$

where $c_{n,p}, d_{n,p} \in C^\infty(S^1 \times S^1)$ and $\sim$ denotes convergence in the sense of Definition 3.7.

This section is divided into six subsections. Subsections 8.1–8.3 cover preliminary material used to verify the hypotheses of the Nash-Moser theorem. Specifically, subsections 8.1 and 8.2 are devoted to lemmas used in the verification of the Hypothesis **(I)**. Then in subsection 8.3 the precise form of the derivative and second derivative of $\mathbb{D}_p$ are derived using the non-linear version of Bourguignon-Gauduchon's Formula (5.7). Subsection 8.4 introduces the tame Fréchet spaces $\mathcal{X}, \mathcal{Y}$, and Subsection 8.5 derives tame estimates verifying Hypotheses **(I)-(III)**. The final subsection 8.6 invokes Theorem 7.4 to complete the proofs.

8.1 The Obstruction Bundle

This subsection covers preliminary lemmas used in the verification of Hypothesis **(I)** which asserts that the linearization of $\mathrm{d}\mathbb{D}$ is invertible on a neighborhood of $((g_0, B_0), \mathcal{Z}_0, \Phi_0)$. Although the invertibility of the linearization, at the end of the proof, comes down to the fact that there is an open neighborhood of invertible operators around the identity in a Banach space, the proper context in which to invoke this fact is somewhat subtle. The first step is to upgrade the obstruction space $\mathbf{Ob}(\mathcal{Z}_0)$ to a vector bundle. This is the content of the current subsection.

To motivate this construction briefly, observe that by Corollary 6.2 (when Assumption 5* holds) imply the linearization at $(\mathcal{Z}_0, \Phi_0)$ for $p = p_0$ is an invertible map after supplementing the domain and codomain with additional factors of $\mathbb{R}$:

$$\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\overline{\mathbb{D}}_{p_0} = \begin{pmatrix} \Pi_0\mathcal{B}_{\Phi_0} - \_\Phi_0 & 0 \\ & \\ (1-\Pi_0)\mathcal{B}_{\Phi_0} & \overline{\slashed{D}} \end{pmatrix} : \quad \begin{matrix} L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \oplus \mathbb{R} \\ \oplus \\ rH^1_e \end{matrix} \quad\longrightarrow\quad \begin{matrix} \mathbf{Ob}(\mathcal{Z}_0) \cap H^{3/2}_{\mathrm{b}} \\ \oplus \\ \mathrm{Range}(\slashed{D}|_{rH^1_e}) \oplus \mathbb{R} \end{matrix} \tag{8.3}$$

where $\overline{\slashed{D}} = (\slashed{D}\;,\;\langle \_, \Phi_0\rangle)$. For a nearby parameter $p \neq p_0$, however, the infinite-dimensional cokernel $\ker(\slashed{D}_p|_{L^2})$ is tilted slightly with respect to that for p_0. Because of the mixed regularity norm, the perturbation to the above linearization for $p \neq p_0$ is not bounded in the top component (and therefore does not behave in a tame fashion). To avoid this and show the linearization at $p \neq p_0$ is well-behaved, we must work in the analogous decomposition induced by p. Thus the first step is to show the family of obstruction spaces form a locally trivial vector bundle **Ob** over an open ball $V_0 \subset \mathcal{P}$.

Proving that **Ob** forms a locally trivial vector bundle involves some rather subtle issues. Besides the standard issue that the "dimension" of the cokernel may jump (which is why the obstruction space was defined as a thickening of cokernel in Section 4), several technical issues arise. The obvious approach to constructing a trivialization would be to argue that the construction of **Ob** in Section 4 is continuous in $p_0 \in U_0$. Although this appears intuitive, it is not clear such a statement has an easy proof, or is even necessarily true (see Remark 8.1). For this reason, *we do not use this natural approach to showing* **Ob** *forms a Banach vector bundle over* V_0; instead, we construct a trivialization by projection. For this second approach, the key point is that (while an arbitrary projection operator on L^2 has no reason to respect regularity) the projection to the cokernel is a pseudodifferential edge operator, so preserves the space H^m_{b} by Corollary 2.12 item (C).

Remark 8.1. The subtle part of the constructions in Section 4 that makes continuity not obvious is the seemingly innocuous choice of indexing the spectrum of the Dirac operator $\not{\partial}_{\mathcal{Z}_0}$ on $\mathcal{Z}_0$. As the metric g_0 changes, so does the induced metric on $\mathcal{Z}_0$. In general, comparing the spectra of two different Riemannian metrics is a quite messy endeavor and showing the bounds of Proposition 4.3 for the basis Ψ_ℓ are uniform and continuous is a subtle issue. While there may be ways to circumvent this since $\mathcal{Z}_0 = \sqcup S^1$, we avoid this approach keeping an eye towards generalizing these results to higher dimensions.

We begin by defining the bundle $\mathbf{Ob} \to V_0$, where V_0 is an open ball of radius δ_0 around p_0 measured in the m_0-norm. Here, $m_0 \in \mathbb{N}$ is an integer to be chosen later ($m_0 = 11$ works). Let $p \in V_0$. By parallel transport on cylinders as in Section 5.1, we may think of the Dirac operator for every p as an operator on the spinor bundle S_0, thus we tacitly write

$$\not{D}_p := \tau^h_{g_0} \circ \not{D}_{h,B} \circ (\tau^h_{g_0})^{-1} \tag{8.4}$$

for the Dirac operator with respect to a metric h and perturbation B (and the fixed singular locus $\mathcal{Z}_0$) on the spinor bundle S_0.

By the (standard) Implicit Function Theorem with the Fredholm operator $\not{D}^\star_p \not{D}_p : rH^1_e \to r^{-1}H^{-1}_e$, we observe

Lemma 8.2. For $0 < \delta_0$ sufficiently small, there is a unique eigenvector $(\Phi_p, \mu_p) \in rH^1_e$ such that

$$\not{D}^\star_p \not{D}_p \Phi_p = \mu_p \Phi_p$$

and equal to $(\Phi_0, 0)$ at $p_0 = (g_0, B_0)$. Moreover, these satisfy

$$\|\Phi_p - \Phi_0\|_{H^{m_0-2,1}_{\mathrm{b},e}} \ + \ |\mu_p| \ \ \leqslant \ \ C\|p - p_0\|_{m_0}.$$

and Φ_p is polyhomogeneous with index set $\mathbb{Z}^+ + \frac{1}{2}$. □

Next, let $rH^\perp_e$ denote the L^2-orthogonal complement of Φ_p in rH^1_e. A trivial extension of the arguments in Section 2 shows the following lemma. In the statement, $\not{D}^\star_p$ denotes the adjoint of the Dirac operator with respect to the L^2-inner product formed using g_0.

Lemma 8.3. For $0 < \delta_0$ sufficiently small, the following hold:

- $\not{D}_p : rH^\perp \to L^2$ is injective with closed range.
- $\not{D}^\star_p \not{D}_p : rH^\perp_e \to r^{-1}H^{-1}_e / \mathbb{R}\Phi_p$ is an isomorphism and if $\langle u, \Phi_p \rangle_{L^2} = 0$ then there is a uniform bound on the solution operator

$$P_p(f) = u \quad \text{s.t.} \quad \not{D}^\star_p \not{D}_p u = f \mod \Phi_p \qquad \Rightarrow \qquad \|P_p\|_{r^{-1}H^{-1}_e \to rH^1} \leqslant C.$$

$\square$

As a result of the first bullet point, $\mathfrak{R} := \text{Range}(\not{D}_p|_{rH_e^\perp}) \subseteq L^2$ is a smooth Banach subbundle, and we may define

Definition 8.4. The **Obstruction Bundle** denoted $\mathbf{Ob} \to V_0$ is defined by the L^2-orthogonal complement of $\mathfrak{R}$ so that there is an orthogonal splitting

$$L^2 = \mathbf{Ob} \oplus \mathfrak{R}$$

as smooth Banach vector bundles over V_0. For $m \leqslant m_0 - 3$, we denote the higher-regularity versions by $\mathbf{Ob}^m := \mathbf{Ob} \cap H^m_{\mathrm{b}}$ and $\mathfrak{R}^m := \mathfrak{R} \cap H^m_{\mathrm{b}}$.

Proposition 8.5. Provided δ_0 is sufficiently small, then for every $m \leqslant m_0$ and in particular for $m = 5/2$, the map

$$\begin{array}{rcl} \Xi_p : \mathbf{Ob}^m_p & \longrightarrow & L^{m,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R} \\ \Psi & \mapsto & \mathrm{ob}_0^{-1} \circ (\Psi - \not{D}_0 P_0 \not{D}_0^\star \Psi). \end{array}$$

provides a local trivialization of the smooth Banach vector bundles $\mathbf{Ob}^m$.

Proof. We begin with the $m = 0$ case. Since ob_0^{-1} is an bounded linear isomorphism with bounded inverse onto $\mathbf{Ob}_0$, it suffices to show that the projection

$$\mathbf{Ob}_p \to \mathbf{Ob}_0 \qquad\qquad \Psi \mapsto \Pi_0 \Psi = (\Psi - \not{D}_0 P_0 \not{D}_0^\star \Psi) \tag{8.5}$$

is an isomorphism (where $\mathbf{Ob}_0, \not{D}_0, P_0$ denote those for the original parameter p_0). Indeed, the reverse projection

$$\mathbf{Ob}_0 \to \mathbf{Ob}_p \qquad\qquad \Psi \mapsto \Pi_p \Psi = (\Psi - \not{D}_p P_p \not{D}_p^\star(\Psi))$$

is an inverse of (8.5) up to small error of size $O(\delta)$. To see this, write and element $\Psi = (\psi, c\Phi_0) \in \mathbf{Ob}_0$ and the composition on the first component is

$$\begin{array}{rcl} \psi \mapsto & & \Pi_0(\psi - \not{D}_p P_p \not{D}_p^\star(\psi) \\ & = & \Pi_0(\psi - \not{D}_p P_p \mathfrak{d}(\psi)) \\ & = & \Pi_0(\psi + O(\delta)) \end{array}$$

where $\mathfrak{d} = (\not{D}_p^\star - \not{D}_0^\star)$ is a 1st order operator with $\|\mathfrak{d}\|_{L^2 \to r^{-1} H_e^{-1}} \leqslant C\delta$. This holds since $\not{D}_p P_p \not{D}_0^\star \psi = 0$ by definition for $\psi \in \mathbf{Ob}_0$. Taking the reverse projection given by (8.5) shows

$$\Pi_0(\psi + O(\delta)) = \psi + O(\delta)$$

again. Similar arguments apply to the $c\Phi_0$ component, and we conclude that

$$\Pi_0 \circ \Pi_p = Id + O(\delta) : \mathbf{Ob}_0 \to \mathbf{Ob}_0$$

and so is an isomorphism for δ sufficiently small.

The proof for $0 < m \leqslant m_0 - 3$ is identical since

$$\not{D}_p P_p \not{D}_p^\star : H^m_{\mathrm{b}} \to H^m_{\mathrm{b}} \tag{8.6}$$

is an (edge) pseudodifferential operators of order 0 hence preserves regularity for $m < m_0 - 2$ and likewise for the reverse projection. To prove (8.6), first note that $\Phi_p \in H^m_{\mathrm{b}}$ since is polyhomogeneous by Proposition 3.8. The result then follows easily from the parameter p-version of Corollary 2.12. $\square$

8.2 Invertibility on a Neighborhood

In this subsection we verify a preliminary version of Hypothesis **(I)** for the linearization of the universal Dirac operator. We show, in particular, that the linearization at $(p_0, \mathcal{Z}_0, \Phi_0)$ is invertible as a bundle map, i.e. on $\mathbf{Ob}_p$ for $p \in V_0$. The complete verification of Hypothesis **(I)** is completed in Section 8.5 and follows easily from this after deriving the form of the linearization at a general $(p, \mathcal{Z}_0, \Phi_0)$ in the following subsection.

Extending the map T_{Φ_0} from Section 6 by adding the λ-component, define

$$\overline{T}_{\Phi_0} = \text{ob}_0^{-1} \circ \Pi_0 \Big[(\text{d}_{(\mathcal{Z}_0,\Phi_0)} \overline{\not{\mathbb{D}}}_0(\eta, 0, \lambda) \Big]$$

as the obstruction component of the linearization at the central fiber $\mathbf{Ob}_0$. Assumption 5* and elliptic bootstrapping means that this map is an isomorphism $\overline{T}_{\Phi_0} : L^{3,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \oplus \mathbb{R} \to L^{5/2,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}$.

Proposition 8.6. Provided that $m_0 \geqslant 10$ and $0 < \delta_0 << 1$ is sufficiently small, then for $p \in V_0$, the $\mathbf{Ob}_p$ components of the linearization at $(p_0, \mathcal{Z}_0, \Phi_0)$ in the trivialization provided by Ξ_p

$$\Xi_p \circ \Pi_p(\text{d}_{(\mathcal{Z}_0,\Phi_0)} \overline{\not{\mathbb{D}}}_0) : L^{3,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \oplus \mathbb{R} \longrightarrow L^{5/2,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R} \tag{8.7}$$

is an isomorphism, and the estimate

$$\|\eta\|_{L^{3,2}} + \|\lambda\| \leqslant C \|\text{d}_{(\mathcal{Z}_0,\Phi_0)} \overline{\not{\mathbb{D}}}_0(\eta, 0, \lambda)\|_{\mathbf{Ob}^{5/2} \oplus L^2} \tag{8.8}$$

hold uniformly on V_0.

Proof. At $p = p_0$, then $\Xi_0 \circ \Pi_0 = \text{ob}^{-1} \circ \Pi_0^2 = \text{ob}^{-1} \circ \Pi_0$ so the map (8.7) is simply $\overline{T}_{\Phi_0}$ hence and isomorphism.

It therefore suffices to show that for $p \in V_0$ with $m_0 \geqslant 10$ and $0 < \delta_0 << 1$ sufficiently small, then the following parameter p-version of the conormal regularity Lemma 6.4 from Section 6.1 holds: for $\overline{\eta} = (\eta, \lambda) \in L^{3,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \oplus \mathbb{R}$ one has

$$\Pi_p \overline{\mathcal{B}}_{\Phi_0}(\overline{\eta}) \in \mathbf{Ob}_p \cap H^{5/2}_{\text{b}} \qquad \text{and} \qquad \|\Pi_p(\overline{\mathcal{B}}_{\Phi_0}(\overline{\eta}))\|_{5/2} \leqslant C \|\overline{\eta}\|_{L^{3,2}}. \tag{8.9}$$

where the latter estimate holds uniformly over $p \in V_0$. Given this, Lemma 8.5 shows that 8.7 is a continuous family of bounded maps between fixed Banach spaces, hence is an isomorphism for δ sufficiently small.

To conclude the lemma, it is therefore enough to establish (8.9). This is proved by writing $\overline{\mathcal{B}}_{\Phi_0}$ in the Fermi coordinates of the new parameter $p = (g, B)$. The constructions of Section 4 apply equally well to the Dirac operator $\not{D}_p$ written in the Fermi coordinates and associated trivialization of the metric g rather than g_0. As a result, there is a basis $\Psi^p_\ell, \Pi_p(\Phi_p)$ of $\mathbf{Ob}_p$ satisfying the conditions of Propositions 4.2 – 4.3 with the bounds on ζ^p_ℓ, ξ^p_ℓ being uniform in p (the only minor caveat in this is that we use the L^2 inner product induced by g_0, so that $\not{D}_p$ is no longer formally self-adjoint). In a similar fashion to Lemma 4.17, we obtain a bounded linear isomorphism $\text{ob}_p \oplus \iota_p : L^{m,2}(\mathcal{Z}_0; \mathcal{S}^p_{\mathcal{Z}_0}) \oplus \mathbb{R} \to \mathbf{Ob}_p \cap H^m_{\text{b}}$ for $m \leqslant 3$ provided, say, $m_0 \leqslant 10$.

The projection to $\mathbf{Ob}_p$ is calculated by the sequence of inner-products

$$\text{ob}_p \Big(\sum_{\ell \in \mathbb{Z}} \langle \tau^p_0 \mathcal{B}_{\Phi_0}(\eta), \Psi^p_\ell \rangle_{\mathbb{C}} \; \phi^p_\ell \Big)$$

(excluding the $\Pi_p(\Phi_p)$ component, which is automatically bounded in H^m_{b} for $m + 3 < m_0$ by Lemma 8.2). Here, τ^p_0 is the parallel transport map $S_0 \to S_p$ as in Section 5.1, whose C^m-norm is bounded by the $(m + 4)$-norm of p by the smooth dependence of ODEs on parameters. Thus we may write $\mathcal{B}_{\Phi_0}$ as a collection of terms

$$v'(t)\Xi_0(t_p, x_p, y_p) \qquad \text{or} \qquad v''(t)\Xi_1(t_p, x_p, y_p) \tag{8.10}$$

where (t_p, x_p, y_p) are the Fermi coordinates constructed using the parameter p, and apply Case (C) of Corollary 6.5. Since $r_p \sim r$, the bounds $|\nabla^m_{t'} \Xi_i| \leqslant C r^{i-1/2}$ hold for $m \leqslant m_0$ equally well for r_p, and it remains to write $v'(t)$ (resp. $v''(t)$) in terms of the Fermi coordinates (t_p, x_p, y_p). Expanding in Taylor series along $\mathcal{Z}_0$ in the norm directions, $v'(t) = w(t_p) + F(t_p, x_p, y_p)$ where $w(t_p) \in L^{2,2}(S^1)$ and

$$\begin{aligned} |F(t_p, x_p, y_p)| &\leqslant Cr_p(|\partial_{x_p} v'(t)| + |\partial_{y_p} v'(t)| \\ &\leqslant Cr_p^2 |v''(t)| \end{aligned}$$

where we have written $x_p(t) = a_0(t)x + b_0(t)y + O(r)t + \ldots$ and likewise for y_p. The crucial point here is that, although expanding in Taylor series seems at first to only exchange orders of growth for tangential derivatives (thus preserve conormal regularity), an extra factor of r_p arises since normal planes in the metrics of p_0 and p differ to first order by a linear coordinate change of x, y. Expanding the integral as in the proof of Lemma 6.4 and using Cauchy-Schwartz (along with arguments akin to those in *Step 5* of the proof of Theorem 6.1 for ζ_ℓ^p, ξ_ℓ^p) shows that the type of term on the left in (8.10) has conormal regularity 5/2 thus satisfies (8.9). The term on the right is identical after multiplying by 1 in the form $(i\partial_t + 1)(i\partial_t + 1)^{-1}$ and integrating by parts. This establishes (8.9), completing the proof.

□

8.3 Quadratic and Error Terms

In this section we employ the non-linear version of Bourguignon-Gauduchon's formula [3] for the metric variation of the Dirac operator to calculate the linearization and second derivative at tuple $(p, \mathcal{Z}, \Phi)$ near $(p_0, \mathcal{Z}_0, \Phi_0)$ as well as the initial error term f_p.

To state Bourguignon-Gauduchon's formula, let $p = (h, B)$ be a parameter pair of a metric and perturbation on Y. Via the parallel transport map $\tau_{g_0}^h$ we can view the Dirac operator $\not{D}_p : \Gamma(S_{g_0}) \to \Gamma(S_{g_0})$ on the spinor bundle associated to the metric g_0 (here, we omit τ from the notation as in (8.4)). Let $a_{g_0}^h, \mathfrak{a}$ be defined respectively by

$$h(X, Y) = g_0(a_{g_0}^h X, Y) \qquad\qquad \mathfrak{a} = (a_{g_0}^h)^{-1/2}$$

where the latter is understood via the eigenvalues of $(a_{g_0}^h)^\star a_{g_0}^h$, which are non-zero for h sufficiently close to g_0.

Theorem 8.7. (Bourguignon-Gauduchon, [3]) The Dirac operator $\not{D}_p$ is given by

$$\not{D}_p \Psi = \left(\sum_i e^i . \nabla^B_{\mathfrak{a}(e_i)} \; + \; \frac{1}{4} \sum_{ij} e^i e^j . \left(\mathfrak{a}^{-1}(\nabla^{g_0}_{\mathfrak{a}(e_i)} \mathfrak{a}) e^j + \mathfrak{a}^{-1} (\nabla^h - \nabla^{g_0})_{\mathfrak{a}(e_i)} \mathfrak{a}(e^j) \right) . \right) \Psi \tag{8.11}$$

where e^i and . are an orthonormal basis and Clifford multiplication for g_0, and ∇^h denotes the unperturbed spin connection of the metric h and likewise for g_0.

8.3.1 Error Terms: We begin by applying Theorem 8.7 to calculate the initial error terms f_p for the application of the Nash-Moser Implicit Function Theorem (7.4). The initial error is given by

$$\boxed{f_p := \not{D}_p \Phi_0.} \tag{8.12}$$

Let $U_1 \subset \mathcal{P}$ denote the ball around p_0 of radius δ_1 measured in the $m_1 + 3$ norm. Here, m_1 (like m_0) is an integer to be chosen later. To simplify notation, we omit the reference to the spaces from the notation from the norms, so that e.g. $\| - \|_m$ means the $H^{m,1}_{\mathrm{b},e}$-norm for elements of the domain, the H^m_{b}-norm for elements of the codomain.

Lemma 8.8. The Dirac operator at parameter p can be written

$$\not{D}_p = \not{D}_0 + \mathfrak{D}_p \tag{8.13}$$

where the latter satisfies

$$\|\mathfrak{D}_p \varphi\|_{m_1} \leqslant C_{m_1} \|p\|_{m_1+3} \|\varphi\|_{m_1}. \tag{8.14}$$

It follows that $\|f_p\|_{m_1} \leqslant C\delta_1$.

Proof. Write $p = (g_0, B_0) + (k, b)$ for $\|(k,b)\|_{m_1+3} \leqslant \delta$. In an orthonormal frame for g_0 we have $a_{g_0}^{g_0+k} = \mathrm{Id} + k$ where we also use k to denote the corresponding matrix in this orthonormal frame. Then $\mathfrak{a} = (\mathrm{Id}+k)^{-1/2}$. Substituting this into Theorem 8.7 shows that

$$\not{D}_p\varphi = \not{D}_0\varphi + \mathfrak{d}_1\varphi + \mathfrak{d}_0\varphi$$

where $\mathfrak{d}_1, \mathfrak{d}_0$ are respectively a first order and zeroth order operator satisfying $\|\mathfrak{d}_1\varphi\|_{m_1} \leqslant C\|p\|_{m_1+3}\|\varphi\|_{m_1}$ and $\|\mathfrak{d}_0\varphi\|_{m_1} \leqslant C\|p\|_{m_1+3}\|\varphi\|_{m_1}$. To see this, note that the coefficients of of $\mathfrak{d}_0$ are formed from sums and products of entries of k (by expanding $(\mathrm{Id}+k)^{-1/2}$), and these all lie in $C^{m_1+1}(Y) \hookrightarrow L^{m_1+3,2}(Y)$ by the Sobolev embedding and the fact that C^{m_1+1} is an algebra. Likewise, coefficients of $\mathfrak{d}_0$ lie in C^{m_1} because they are formed from sums and products of up to first derivatives of k, b. Since every term is at least linear in p, and $\|(k,b)\|_{m_1+3} \leqslant \delta << 1$, the bound (8.14) follows.

Since $f_p = \not{D}_p\Phi_0 = \mathfrak{d}_0\Phi_0 + \mathfrak{d}_1\Phi_0$ and $\|\Phi_0\|_{m_1} \leqslant C_{m_1}$, the second statement is then immediate for $p \in U_1$. □

8.3.2 Quadratic Terms For the tame estimates on $\mathrm{d}\not{\mathbb{D}}_p$ and $\mathrm{d}^2\not{\mathbb{D}}_p$, we must first investigate the higher-order terms of $\not{\mathbb{D}}_p$. Expanding, we may write

$$\not{\mathbb{D}}_p((\mathcal{Z}_0,\Phi_0) + (\eta,\varphi)) = f_p \ + \ \mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\not{\mathbb{D}}_p(\eta,\varphi) \ + \ Q_p(\eta,\varphi)$$

where Q_p is comprised of second order and higher terms.

The middle term at p_0 is given by Corollary 5.9. For a general p, we can write the derivative of pullback metric as

$$\frac{d}{ds}\Big|_{s=0} F^*_{s\eta}(g_0 + k) = \dot{g}_\eta + \dot{k}_\eta, \tag{8.15}$$

where $\dot{g}_\eta$ is as calculated in (5.10) and analogously for k. Analogous to the formula for $\mathcal{B}_{\Phi_0}(\eta)$ in Corollary 5.9, we set

$$\mathfrak{B}_{\Phi_0,p}(\eta) := \left(-\frac{1}{2}\sum_{ij}\dot{k}_\eta(e_i,e_j)e^i.\nabla^{g_0}_j + \frac{1}{2}d\mathrm{Tr}_{g_0}(\dot{k}_\eta). + \frac{1}{2}\mathrm{div}_{g_0}(\dot{k}_\eta). + \mathcal{R}(b,\chi\eta).\right)\Phi_0 \tag{8.16}$$

to be the term arising from the perturbation (k,b) to p_0. Here $\mathcal{R}(b,\chi\eta)$ is a zeroth order term in η with coefficients depending on the perturbation b to B_0 and its derivatives.

Proposition 8.9. The universal Dirac operator at the parameter $p \in U_1$ for at a point $(\mathcal{Z}_0,\Phi_0) + (\eta,\varphi)$ with $\|(\eta,\varphi)\|_{m_0} \leqslant C\delta$ is given by

$$\not{\mathbb{D}}_p((\mathcal{Z}_0,\Phi_0) + (\eta,\varphi)) = f_p \ + \ \mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\not{\mathbb{D}}_p(\eta,\varphi) \ + \ Q_p(\eta,\varphi) \tag{8.17}$$

where

(1) $f_p = \not{D}_p\Phi_0$ as in Lemma 8.8.

(2) The derivative is given by

$$\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\not{\mathbb{D}}_p(\eta,\varphi) = \Big(\mathcal{B}_{\Phi_0}(\eta) + \not{D}_0\varphi\Big) \ + \ \Big(\mathfrak{B}_{\Phi_0,p}(\eta) + \mathfrak{D}_p(\varphi)\Big)$$

where $\mathcal{B}_{\Phi_0}(\eta)$ is as defined in 5.5 (cf. Corollary 5.9), and $\mathfrak{D}_p, \mathfrak{B}_{\Phi_0,p}$ are as in (8.13) and (8.16) respectively.

(3) The non-linear terms may be written

$$Q(\eta,\varphi) \quad = \quad (\mathcal{B}_\varphi + \mathfrak{B}_{\varphi,p})(\eta) \quad + \quad M^1_p(\eta',\eta')\nabla(\Phi_0+\varphi) \quad + \quad M^2_p(\eta',\eta'')(\Phi_0+\varphi) \quad + \quad F_p(\eta,\Phi_0+\varphi)$$

where

(i) $\mathcal{B}_\varphi, \mathfrak{B}_{\varphi,p}$ are defined identically to $\mathcal{B}_{\Phi_0}, \mathfrak{B}_{\Phi_0,p}$ but with φ replacing Φ_0.

(ii) M_1^p a finite sum of terms involving quadratic combinations of $\chi\eta', \eta d\chi, \chi\eta$, and linearly depending on $\nabla(\Phi_0+\varphi)$ and smooth endomorphisms m_i, e.g.

$$m_i(y)(\chi\eta')(\chi\eta')\nabla_j(\Phi_0+\varphi)$$

where $m_i(y)$ depend on g_0+k (and no derivatives).

(iii) M_2^p a finite sum of terms involving quadratic combinations of $\eta''\chi, \eta' d\chi, \eta d^2\chi, \eta'\chi, \eta d\chi, \eta\chi$, with at most one factor of η'', and linearly depending on $\Phi_0+\varphi$ and smooth endomorphisms m_i, e.g.

$$m_i(y)(\chi\eta'')(d\chi\eta').(\Phi_0+\varphi)$$

where $m_i(y)$ depend on up to first derivatives of g_0+k and B_0+b.

(iv) F_p is formed from a finite sum of similar terms but involving cubic and higher combinations of η, η', η'', with at most one factor of η''.

Proof. The constant (1) and linear (2) terms are immediate from, respectively, the definition (8.12) and the proof of Corollary 5.9 but using the pullback metric (8.15) in place of $\dot{g}_\eta$.

For $p=p_0$, quadratic terms (3) are calculated as follows. By Remark 5.10, the pullback metric can be written

$$g_\eta = g_0 + \dot{g}_\eta + \mathfrak{q}(\eta) \qquad \text{where} \qquad |\mathfrak{q}(\eta)| \leqslant C\big(\ |(\chi)\eta'| + |(d\chi)\eta| + |(\chi)\eta|\ \big)^2.$$

i.e. $\mathfrak{q}(\eta)$ vanishes to second order at $\eta=0$. Substituting g_η into the formula 8.11, one has that (in an orthonormal frame of g_0)

$$\mathfrak{a}_{g_0}^{g_\eta} = I - \tfrac{1}{2}\dot{g}_\eta + \mathfrak{q}'(\eta)$$

where $\mathfrak{q}'$ obeys the same bound as $\mathfrak{q}$. Some calculation (actually quite a lot) then yields the formula for $\not{\mathbb{D}}_0$ and subtracting off the known formulas for $\not{D}_0$ and $\mathrm{d}\not{\mathbb{D}}_0$ yields the result. The $\mathcal{B}_\varphi, M^1$, and M^2 terms come from the quadratic terms; and $F(\eta, \Phi_0+\varphi)$ from the cubic and higher order terms. The argument is identical for a general p using the pullback metric and perturbation $p_\eta := F_\eta^* p$. ☐

Straightforward differentiation now shows the following precise forms for the first and second derivatives. In these formulas, we use the notation that e.g. $F(p^3, q^2, s)$ to denote a term depending cubicly on p and its derivatives, quadratically on q and its derivatives, and linearly on s and its derivatives:

Corollary 8.10. The derivative at a point $(\mathcal{Z}_0, \Phi_0) + (\eta, \varphi)$ is given by

$$\begin{aligned}
\mathrm{d}_{(\eta,\varphi)}\not{\mathbb{D}}_p(v,\phi) \quad = \quad & \mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\not{\mathbb{D}}_p(v,\phi) \\
& + (\mathcal{B}_\varphi + \mathfrak{B}_\varphi)(v) \ + \ (\mathcal{B}_\phi + \mathfrak{B}_\phi)(\eta) \\
& + M^1(\eta', v')\nabla(\Phi_0+\varphi) \ + \ M^1(\eta',\eta')\nabla\phi \\
& + M^2(\eta', v'')(\Phi_0+\varphi) \ + \ M^2(v',\eta'')(\Phi_0+\varphi) \ + \ M^2(\eta',\eta'')\phi \\
& + F^1(\eta^3,\phi) \ + \ F^2(\eta^2, v, \Phi_0+\varphi)
\end{aligned}$$

where the subscript p is kept implicit on the right hand side. ☐

Alternatively, the terms linear in ϕ combine to form the Dirac operator

$$\not{D}_{p_\eta}\phi = \not{D}_p\phi + \ (\mathcal{B}_\phi + \mathfrak{B}_\phi)(\eta) + M^1(\eta',\eta')\nabla\phi + M^2(\eta',\eta'')\phi + F^1(\eta^3,\phi) \tag{8.18}$$

with respect to the pullback metric and perturbation $p_\eta := F_\eta^*(p)$.

Corollary 8.11. The second derivative at a point $(\mathcal{Z}_0, \Phi_0) + (\eta, \varphi)$ is given by

$$\begin{aligned}
\mathrm{d}^2_{(\eta,\varphi)}\mathbb{D}_p\Big((v,\phi),(w,\psi)\Big) \quad = \quad & (\mathcal{B}_\psi + \mathfrak{B}_\psi)(v) \ + \ (\mathcal{B}_\phi + \mathfrak{B}_\phi)(w) \\
& + M^1(w',v')\nabla(\Phi_0+\varphi) \ + \ M^1(\eta',v')\nabla\psi \ + \ M^1(\eta',w')\nabla\phi \\
& + M^2(w',v'')(\Phi_0+\varphi) \ + \ M^2(\eta',v'')\psi \ + \ M^2(v',w'')(\Phi_0+\varphi) \\
& + M^2(v',\eta'')\psi \ + \ M^2(\eta',w'')\phi \ + \ M^2(w',\eta'')\phi \\
& + F^3(\eta^2,w,\varphi) \ + \ F^4(\eta^2,v,\psi) \ + \ F^5(\eta,v,w,\Phi_0+\varphi)
\end{aligned}$$

where the subscript p is kept implicit on the right hand side. □

8.4 Tame Fréchet Spaces

This section introduces the tame Fréchet spaces used in the proof of Theorem 1.4 and Corollary 1.5.

While there is a natural Fréchet space of normal vector fields η (this being $C^\infty(\mathcal{Z}_0; N\mathcal{Z}_0)$ with the Fréchet structure arising from the $L^{m,2}$-norms), there are several possible choices of Fréchet spaces for the spinors, arising from different versions of the boundary and edge spaces. The relevant spinors are those lying in $\mathbf{P}_{\mathcal{X}}$ and $\mathbf{P}_{\mathcal{Y}}$ for the domain and codomain respectively, i.e. those spinors with polyhomogeneous expansions (8.1-8.2). While the spaces $\mathbf{P}_{\mathcal{X}}$ and $\mathbf{P}_{\mathcal{Y}}$ are themselves tame Fréchet spaces, these Fréchet structures are rather unwieldy and it is advantageous to enlarge the domain and codomain to spaces where it is easier to obtain estimates and then invoke Theorem 7.4 with the property (P) of polyhomogeneity which holds on $\mathbf{P}_{\mathcal{X}}$ and $\mathbf{P}_{\mathcal{Y}}$.

The mixed boundary and edge spaces $rH^{m,1}_{\mathrm{b},e}$ and H^m_{b} defined in Section 2 enlarge the domain and codomain and their norms facilitate much easier estimates using the material of Sections 2-4. Unfortunately, these spaces are slightly too large and it is impossible to control the higher order terms of the expansions (8.1-8.2) simply in terms of of these norms. To balance these conflicting advantages of $rH^{m,1}_{\mathrm{b},e}$ and $\mathbf{P}_{\mathcal{X}}$, we opt for intermediate spaces which supplement the $rH^{m,1}_{\mathrm{b},e}$ and H^m_{b}-norms with the norm of the higher order terms in (8.1-8.2) using a stronger weight.

Analogously to $rH^{m,1}_{\mathrm{b},e}$ and H^m_{b} denote $r^{1+\nu}H^{m,1}_{\mathrm{b},e}$ and $r^\nu H^m_{\mathrm{b}}$ the spaces formed by adding an overall weight of $r^{-2\nu}$ in the norm (2.12). Equivalently,

$$\varphi \in rH^{m,1}_{\mathrm{b},e} \quad \Leftrightarrow \quad r^\nu\varphi \in r^{1+\nu}H^{m,1}_{\mathrm{b},e}$$

so that the multiplication map r^ν is a bounded linear isomorphism, and similarly for $r^\nu H^m_{\mathrm{b}}$. Fix $\nu = 0.9$ and define Banach spaces

$$\begin{aligned}
r\mathcal{H}^{m,1} \quad &:= \quad \left\{\varphi \ \middle| \ \|\varphi\|_{r\mathcal{H}^{m,1}} := \Big(\|\varphi\|^2_{rH^{m,1}_{\mathrm{b},e}} \ + \ \|(r\partial_r - \tfrac{1}{2})\varphi\|^2_{r^{1+\nu}H^{m-1,1}_{e}}\ \Big)^{1/2} < \infty\right\} \\
\mathcal{H}^{m,0} \quad &:= \quad \left\{\psi \ \middle| \ \|\psi\|_{\mathcal{H}^{m,0}} := \Big(\ \|\psi\|^2_{H^m_{\mathrm{b}}} \ + \ \|(r\partial_r + \tfrac{1}{2})\psi\|^2_{r^\nu H^{m-1}_{\mathrm{b}}}\ \Big)^{1/2} < \infty\right\} \\
r^{-1}\mathcal{H}^{m,-1} \quad &:= \quad \left\{\psi \ \middle| \ \|\psi\|_{r^{-1}\mathcal{H}^{m,-1}} := \Big(\ \|\psi\|^2_{r^{-1}H^{m,-1}_{\mathrm{b},e}} \ + \ \|(r\partial_r + \tfrac{3}{2})\psi\|^2_{r^{-1+\nu}H^{m-1,-1}_{\mathrm{b},e}}\ \Big)^{1/2} < \infty\right\}.
\end{aligned}$$

with the indicated norms. These spaces are defined using the Fermi coordinates and norms of the base parameter p_0 and do not depend on $p \in \mathcal{P}$.

Using these, we now define the spaces used in the proofs of Theorem 1.4 and 1.5.

Lemma 8.12. The spaces

$$\mathcal{X} := \bigcap_{m\geqslant 0} X'_m \oplus \mathbb{R} \oplus X''_m \qquad\qquad \mathcal{Y} := \bigcap_{m\geqslant 0} Y'_m \oplus Y''_m \oplus \mathbb{R}$$

where

$$\begin{aligned}
X'_m \quad &:= \quad L^{m,2}(\mathcal{Z}_0; N\mathcal{Z}_0) & Y'_m &:= \mathbf{Ob} \cap H^m_{\mathrm{b}}(Y - \mathcal{Z}_0; S_0) \qquad (8.19)\\
X''_m \quad &:= \quad r\mathcal{H}^{m,1}(Y - \mathcal{Z}_0; S_0) & Y''_m &:= \mathfrak{R} \ \cap \ \mathcal{H}^{m,0}(Y - \mathcal{Z}_0; S_0) \qquad (8.20)
\end{aligned}$$

and $\mathbb{R}$ has the standard norm are tame Fréchet spaces as in Definition 7.1, and on an open neighborhood $U \subset \mathcal{X}$,

$$\overline{\mathbb{D}}_p : \mathcal{P} \times U \to \mathcal{Y}$$

is a tame Fréchet map.

Proof. The interpolation inequalities in item (I) of Definition 7.1 are immediate from those on the standard spaces $L^{m,2}(\mathcal{Z}_0; N\mathcal{Z}_0)$ and those from Lemma 2.10 which apply equally well for different weights. The smoothing operators whose existence is the content of item (II) of Definition 7.1 are constructed in Appendix B.

That $\overline{\mathbb{D}}_p$ is a tame Fréchet map is obvious for the H^m_{b}-norms, and for the $(r\partial_r - \frac{1}{2})$ terms follows from the commutation relations in the upcoming Lemma 8.15. ☐

Remark 8.13. Since **Ob** consists of solutions of the elliptic edge operator (this being $\not{D}$ or $\not{D} - \Lambda_p \mathrm{Id}$) which have expansions with index set $\mathbb{Z}^+ - \frac{1}{2}$, edge bootstrapping (see [45] Equation (7.7) and the accompanying discussion) implies that for $\psi \in$ **Ob** obeys

$$\|(r\partial_r + \tfrac{1}{2})\psi\|_{rH^{m-1}_{\mathrm{b}}} \leqslant C\|\psi\|_{H^m_{\mathrm{b}}}.$$

Since $\nu \leqslant 1$ it follows that the norm on Y'_m is equivalent to using the $\mathcal{H}^{m,0}$ norm.

As explained at the beginning of the subsection, the point is that the additional terms allows control of the higher order terms of expansions (8.1-8.2) in $\mathbf{P}_{\mathcal{X}} \cap r\mathcal{H}^{m,1}$. The following key lemma, proved in Appendix B makes this precise:

Lemma 8.14. Suppose that $\varphi \in r\mathcal{H}^{m,1}$ is a spinor. Then following bound holds pointwise on $Y - \mathcal{Z}_0$:

$$|(\nabla^{\mathrm{b}})^m \varphi| \leqslant C_m r^{1/2} \|\varphi\|_{r\mathcal{H}^{m+4,1}}.$$

☐

The final two lemmas needed before the verification of Hypotheses **(I)-(III)** are effectively book-keeping that show the Dirac operator $\not{D} : r\mathcal{H}^{1,1} \to \mathcal{H}^{1,0}$ behaves in similar fashion to on the spaces from Section 2. Fixing a parameter p, let $r\mathcal{H}^{\perp}$ be the L^2-orthogonal complement of Φ_p in $r\mathcal{H}^{1,1} \subset rH^1_e$. Denote the extended Dirac operator with the $\mathbb{R}$-factor included at a parameter p by

$$\overline{\not{D}}_p = \begin{pmatrix} \not{D}_p & 0 \\ \langle -, \Phi_0 + \varphi \rangle & \langle -, \Phi_0 + \varphi \rangle \end{pmatrix} \begin{matrix} r\mathcal{H}^{\perp} \\ \oplus \\ \mathbb{R}\Phi_p \end{matrix} \longrightarrow \begin{matrix} \mathfrak{R}_p \\ \oplus \\ \mathbb{R}. \end{matrix}$$

Lemma 8.15. Provided that $m_0 \geqslant 10$ and $0 < \delta_0 << 1$ is sufficiently small, then for $p \in V_0$ the extended Dirac operator

$$\overline{\not{D}}_p : r\mathcal{H}^{1,1} \oplus \mathbb{R} \to (\mathfrak{R}_p \cap \mathcal{H}^{1,0}) \oplus \mathbb{R}$$

is an isomorphism and the estimate

$$\|\varphi\|_{r\mathcal{H}^{1,1}} \leqslant C\|\overline{\not{D}}_p \varphi\|_{\mathcal{H}^{1,0} \oplus \mathbb{R}}$$

holds uniformly for $p \in V_0$.

Proof. We begin by showing that the (unextended) Dirac operator $\not{D}$ satisfies the following estimate: if $\not{D}\varphi = f$ then

$$\|\varphi\|_{rH^{1,1}_{\mathrm{b},e}} + \|(r\partial_r - \tfrac{1}{2})\varphi\|_{r^{1+\nu}H^1_e} \leqslant C\Big(\|f\|_{H^1_{\mathrm{b}}} + \|(r\partial_r + \tfrac{1}{2})f\|_{r^{\nu}L^2}) + \|\varphi\|_{r^{\nu}H^1_{\mathrm{b}}}\Big). \tag{8.21}$$

Since $\nu < 1$ by choice, the inclusion $rH^{1,1}_{\mathrm{b},e} \hookrightarrow r^{\nu}H^1_{\mathrm{b}}$ constituting the last term is compact.

We first prove (8.21) for $p = p_0$. That the first term is bounded by the right-hand side immediate from the estimate for $\not{D} : rH^{1,1}_{\mathrm{b},e} \to H^1_{\mathrm{b}}$ (Corollary 2.11). For the second term, we apply the elliptic estimate

$$\|\varphi\|_{r^{1+\nu}H^1_e} \leqslant C\Big(\|\not{D}\varphi\|_{r^\nu L^2} + \|\varphi\|_{r^\nu L^2}\Big) \tag{8.22}$$

for $\not{D} : r^{1+\nu}H^1_e \to r^\nu L^2$ to term $(r\partial_r - \frac{1}{2})\varphi$. This estimate cannot be derived by integration by parts as in Section 2 and instead follows from parametrix methods (see Theorem 6.1 of [45] or [70]). Then, since the commutation relations

$$\begin{aligned}(r\partial_r + \tfrac{1}{2})\partial_r &= \partial_r(r\partial_r - \tfrac{1}{2}) && (8.23)\\ (r\partial_r + \tfrac{1}{2})\tfrac{1}{r}\partial_\theta &= \tfrac{1}{r}\partial_\theta(r\partial_r - \tfrac{1}{2}). && (8.24)\end{aligned}$$

hold writing $\not{D} = \not{D}_0 + \mathfrak{d}$ as in Lemma 3.6 shows that

$$\not{D}(r\partial_r - \tfrac{1}{2})\varphi = (r\partial_r + \tfrac{1}{2})\not{D}\varphi + B\varphi$$

where B is a lower order term such that $B : r^\nu H^1_b \to r^\nu L^2$ is bounded. Applying (8.22) and substituting this expression yields (8.21) for $p = p_0$.

The fact that $\overline{\not{D}}_{p_0}$ is an isomorphism then follows from a standard proof by contradiction (e.g. [40] Lemma 10.4.9). For $p \neq p_0$ it is straightforward to show that writing $\not{D}_p = \not{D}_{p_0} + \mathfrak{d}_p$ and using the commutations (8.23-8.24) yields

$$\|\mathfrak{d}_p\varphi\|_{\mathcal{H}^{1,0}} \leqslant C\delta_0\|\varphi\|_{r\mathcal{H}^{1,1}}$$

completing the lemma.

□

Finally, the projection operators to **Ob** and $\mathfrak{R}$ are well-behaved on the new spaces analogously to Corollary 2.12 item (C).

Lemma 8.16. The projection operators

$$\Pi^{\mathrm{Range}} = \not{D}_p P_p \not{D}_p^\star : \mathcal{H}^m \to \mathfrak{R}_p \cap \mathcal{H}^m \qquad\qquad \Pi^{\ker} = 1 - \not{D}_p P_p \not{D}_p^\star : \mathcal{H}^m \to \mathbf{Ob}_p \cap \mathcal{H}^m$$

are bounded.

Proof. For the H^m_{b}-term of the $\mathcal{H}^m$-norm this follows directly from Corollary 2.12. For the second term, notice that by (8.22) and the analogous estimate for $\not{D}_p^\star\not{D}_p : r\mathcal{H}^{m,1} \to r^{-1}\mathcal{H}^{m,-1}$, one has

$$\not{D}_p P_p \not{D}_p^\star : r^\nu H^m_{\mathrm{b}} \to r^\nu H^m_{\mathrm{b}} \tag{8.25}$$

is bounded. Writing

$$(r\partial_r + \tfrac{1}{2})D_p P_p \not{D}_p^\star = D_p P_p \not{D}_p^\star(r\partial_r + \tfrac{1}{2}) + [(r\partial_r), D_p P_p \not{D}_p^\star]$$

and applying (8.25) to the first term, then using that $[(r\partial_r + \frac{1}{2}), D_p P_p \not{D}_p^\star] : H^{m-1}_{\mathrm{b}} \to rH^m_{\mathrm{b}}$ is bounded for the second term yields the result.

To prove that the commutator $[(r\partial_r + \frac{1}{2}), D_p P_p \not{D}_p^\star] : H^{m-1}_{\mathrm{b}} \to rH^m_{\mathrm{b}}$ is bounded, it suffices to consider the case of the product metric on the model space $S^1 \times \mathbb{R}^2$ as in Example 3.2. The result for $p = p_0$ follows easily from the same argument after writing $\not{D}_{p_0} = \not{D}_0 + \mathfrak{d}$ where $|\mathfrak{d}\psi| \leqslant C(r|\nabla\psi| + |\psi|)$ as in Lemma 3.6. For a general p, the same argument applies in the Fermi coordinates formed using p for the boundary Sobolev spaces defined using p, whose norms are uniformly (tamely) equivalent.

In the product case, the commutation relations (8.23-8.24) imply

$$\begin{aligned}\not{D}_0(r\partial_r - \tfrac{1}{2})\varphi &= (r\partial_r + \tfrac{1}{2})\not{D}_0\varphi - \gamma_t\nabla_t\varphi\\ P_0(r\partial_r + \tfrac{3}{2})f &= (r\partial_r - \tfrac{1}{2})P_0 f + P_0(\gamma_t\nabla_t\not{D}_0 + \not{D}_0\gamma_t\nabla_t)P_0 f\end{aligned}$$

where $P_0 f = u \Leftrightarrow \not{D}_0\not{D}_0 u = f$, and $\gamma_t = \gamma(dt)$. The latter expression follows from applying the first twice with $\not{D}_0\not{D}_0$ and then applying P_0. Using these, one has

$$\begin{aligned}
(r\partial_r + \tfrac{1}{2})\not{D}_0 P_0 \not{D}_0 &= \not{D}_0(r\partial_r - \tfrac{1}{2})P_0\not{D}_0 + \gamma_t\nabla_t P_0\not{D}_0 \\
&= \not{D}_0 P_0(r\partial_r + \tfrac{3}{2})\not{D}_0 - \not{D}_0 P_0(\gamma_t\nabla_t\not{D}_0 + \not{D}_0\gamma_t\nabla_t)P_0\not{D}_0 + \gamma_t\nabla_t P_0\not{D}_0 \\
&= \not{D}_0 P_0\not{D}_0(r\partial_r + \tfrac{1}{2}) + \not{D}_0 P_0\gamma_t\nabla_t - \not{D}_0(P_0(\gamma_t\nabla_t\not{D}_0 + \not{D}_0\gamma_t\nabla_t)P_0)\not{D}_0 - \gamma_t\nabla_t P_0\not{D}_0
\end{aligned}$$

so that

$$[(r\partial_r + \tfrac{1}{2}), \not{D}_0 P_0\not{D}_0] = \not{D}_0 P_0\gamma_t\nabla_t - \not{D}_0(P_0(\gamma_t\nabla_t\not{D}_0 + \not{D}_0\gamma_t\nabla_t)P_0)\not{D}_0 + \gamma_t\nabla_t P_0\not{D}_0.$$

Then, the last term is the composition of the bounded operators

$$H^m_{\mathrm{b}} \xrightarrow{\not{D}_0} r^{-1}H^{m,-1}_{\mathrm{b}} \xrightarrow{P_0} rH^{m,1}_{\mathrm{b},e} \xrightarrow{\nabla_t} rH^{m-1,1}_{\mathrm{b},e} \hookrightarrow rH^{m-1}_{\mathrm{b}}$$

and similarly for the first and middle terms.

□

8.5 Tame Estimates for the Linearization

In this subsection, we use the explicit formulas for $\mathrm{d}\not{\mathbb{D}}_p$ and $\mathrm{d}^2\not{\mathbb{D}}_p$ from Corollaries 8.10 and 8.11 to verify Hypotheses **(I)-(III)** of the Nash-Moser Implicit Function Theorem from Section 7.2.

Recall that $V_0 \subset \mathcal{P}$ denotes the open ball of radius δ_0 around p_0 measured in the m_0-norm, and let $U_0 \subset \mathcal{X}$ denote the ball of the same radius around $(\mathcal{Z}_0, \Phi_0)$.

Lemma 8.17. Hypothesis **(I)** of Theorem 7.4 holds for $\overline{\not{\mathbb{D}}}_p$, i.e. there is an $m_0 \geqslant 0$ such that for δ_0 sufficiently small, $p \in V_0$ that implies the linearization

$$\mathrm{d}_{(\varphi,\eta)}\overline{\not{\mathbb{D}}}_p : \mathcal{X} \to \mathcal{Y}$$

is invertible for $(\eta, \varphi) \in U_0 \cap \mathbf{P}_{\mathcal{X}}$.

Proof. Take $m_0 = 11$.

We first investigate the obstruction component of the linearization. Up to decreasing δ_0 by a constant factor, $p \in V_0, (\eta, \varphi) \in U_0$ implies that the pullback parameter $p_\eta \in V_0'$ where V_0' is the δ-ball in the $m = 10$-norm, hence Proposition 8.6 applies. For $(v, \lambda, 0) \in X_0' \oplus \mathbb{R} \oplus X_0''$, we can write the $\mathbf{Ob}_{p_\eta}$-component of the linearization as in the trivialization of Lemma 8.5 as a map $L^{3,2}(\mathcal{Z}_0, N\mathcal{Z}_0) \oplus \mathbb{R} \to L^2(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0}) \oplus \mathbb{R}$ given by

$$\begin{aligned}
\Xi_{p_\eta} \circ \Pi_{p_\eta}(\mathrm{d}_{(\eta,\varphi)}\overline{\not{\mathbb{D}}}_p(v, 0, \lambda)) &= \Xi_{p_\eta} \circ \Pi_{p_\eta}(\mathrm{d}_{(\mathcal{Z}_0,\Phi_0)}\overline{\not{\mathbb{D}}}_0(v, 0, \lambda) + \ldots + F^2(\eta^2, v, \Phi_0 + \varphi)) && (8.26)\\
&= T_\circ(v) - \lambda\Phi_0 + \mathfrak{t}_{p_\eta}(v, \lambda) && (8.27)\\
&:= \overline{T}_{p_\eta} && (8.28)
\end{aligned}$$

where $\overline{T}_\circ(v) = \Xi_{p_\eta} \circ \Pi_{p_\eta}\overline{\mathcal{B}}_{\Phi_0}(v)$ and $\mathfrak{t}_{p_\eta}(v, \lambda)$ is given as in Corollary 8.10) by

$$\begin{aligned}
\mathfrak{t}_{p_\eta}(v, \lambda) \;:=\; \Xi_{p_\eta} \circ \Pi_{p_\eta}\Big[&\mathfrak{B}_{\Phi_0,p}(v) + (\mathcal{B}_\varphi + \mathfrak{B}_\varphi)(v) + M^1(\eta', v')\nabla(\Phi_0 + \varphi) && (8.29)\\
&+ M^2(\eta', v'')(\Phi_0 + \varphi) + M^2(v', \eta'')(\Phi_0 + \varphi) + F^2(\eta^2, v, \Phi_0 + \varphi) - \lambda\varphi\Big] && (8.30)
\end{aligned}$$

Proposition 8.6 applies to show that $\overline{T}_\circ : L^{3,2} \oplus \mathbb{R} \to L^{5/2,2} \oplus \mathbb{R} \simeq \mathbf{Ob}_{p_\eta} \cap H^{5/2}_{\mathrm{b}}$ is an isomorphism, so in order to show that (8.28) is also an isomorphism, it is enough to prove that $\mathfrak{t}_{p_\eta}(v, \lambda) \in L^{5/2,2}(\mathcal{Z}_0; \mathcal{S}_{\mathcal{Z}_0})$ and there is a bound

$$\|\mathfrak{t}_{p_\eta}(v)\|_{5/2} \leqslant C\delta_0\|v\|_{L^{3,2}}. \quad (8.31)$$

This is effectively identical to the proof of (8.9) in Proposition 8.6, and follows by considering the conormal regularity and Case (C) or Corollary 6.5. Indeed, by Remark 6.6 and Proposition 8.9, it is easy to see that each term of $\mathfrak{t}_{p_\eta}$ is of the form either $v'(t)\Xi_0(t_p, x_p, y_p)$ or $v''(t)\Xi_1(t_p, x_p, y_p)$ just as (8.9). Here, both Ξ_0, Ξ_1 can be written respectively as the sum of terms $m_{p_\eta}(y)\nabla(\Phi_0 + \varphi)$ and $m_0(y)\nabla\varphi$ or $m_{p_\eta}(y)(\Phi_0 + \varphi)$ and $m_0(y)\varphi$, where m_{p_η}, m_0 are smooth endomorphisms bounded in terms of the norms of p_η, p_0 respectively. In particular, by Lemma 8.14 each such term has bounds $|\nabla_t^m \Xi_i| \leqslant Cr^{i-1/2}\|(p, \eta, \varphi)\|_{m+m_0}$ for $m \leqslant 3$. (8.31) then follows exactly as (8.9) by writing $v(t)$ in the Fermi coordinates of p_η. We conclude, after possibly reducing δ_0, that $\overline{T}_{p_\eta} : L^{3,2} \oplus \mathbb{R} \to L^{5/2,2} \oplus \mathbb{R}$ is an isomorphism.

The full linearization acting on (v, λ, ϕ) now has the block-diagonal form

$$\mathrm{d}_{(\eta,\varphi)}\overline{\overline{\not{D}}}_p = \begin{pmatrix} \overline{T}_{p_\eta} & * \\ \mathcal{B}^{\mathrm{Rg}} & \overline{\not{D}}_{p_\eta} \end{pmatrix} : \begin{matrix} L^{3,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \oplus \mathbb{R} \\ \oplus \\ r\mathcal{H}^{1,1} \end{matrix} \longrightarrow \begin{matrix} \mathbf{Ob}^{5/2}_{p_\eta} \\ \oplus \\ (\mathfrak{R}_{p_\eta} \cap \mathcal{H}^{1,0}) \oplus \mathbb{R}. \end{matrix} \tag{8.32}$$

where the top left entry is as above and $\mathcal{B}^{\mathrm{Rg}}$ is the $\mathfrak{R}_{p_\eta}$-component of the terms (8.30). By what was said above in conjunction with Lemma 8.15, the diagonal entries are both isomorphism.

We claim that the bottom left entry $\mathcal{B}^{\mathrm{Rg}}$ is bounded. Since $\mathcal{B}^{\mathrm{Rg}} = (\not{D}_p P_p \not{D}_p^\star)\mathcal{B}$ by definition, Lemma 8.16 shows that it suffices to prove that

$$\mathcal{B} : L^{3,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \oplus \mathbb{R} \to r\mathcal{H}^{1,1}$$

is bounded. The boundedness into H^1_{b} is obvious using the bounds on Φ_0, and $\|(\eta, \varphi)\|_{m_0=11} \leqslant C\delta_0$. For the boundedness into $r^\nu L^2$, note that since $\varphi \in r\mathcal{H}^{1,1} \cap \mathbf{P}_{\mathcal{X}}$ is polyhomogeneous with index set $\mathbb{Z}^+ + \frac{1}{2}$, the operator $(r\partial_r + \frac{1}{2})$ annihilates the order $r^{-1/2}$ term of $\nabla(\Phi_0 + \varphi)$ and all other terms are $O(r^{1/2})$ so are integrable with the stronger weight. This shows that 8.32 is invertible in the case that $* = 0$.

In general if $* \neq 0$, the alternative decomposition (8.18) implies that the top right entry $*$ has image in the the 1-dimensional subspace spanned by $\not{D}_{p_\eta}\Phi_{p_\eta} \neq 0$. By the polyhomogeneity of Φ_{p_η} from Lemma 8.2, this automatically lands in $\mathbf{Ob}^{5/2}_{p_\eta}$ where it has size $O(\delta_0)$ by Lemma 8.2. It therefore follows that for δ_0 sufficiently small, (8.32) is an isomorphism, and the estimate

$$\|(v, \lambda, \phi)\|_{L^{3,2}\oplus\mathbb{R}\oplus r\mathcal{H}^{1,1}} \leqslant C\|\mathrm{d}_{(\eta,\varphi)}\overline{\overline{\not{D}}}_p(v, \lambda, \phi)\|_{\mathbf{Ob}^{5/2}\oplus\mathcal{H}^{1,0}\oplus\mathbb{R}}.$$

holds.

Consequently, the linearization is injective, and and surjective onto $\mathbf{Ob}^{5/2}_{p_\eta} \oplus \mathfrak{R}_{p_\eta} \oplus \mathbb{R}$. It remains to show that if the right-hand side is smooth, then $(v, \lambda, \phi) \in \mathcal{X}$ is also smooth, which follows from bootstrapping using the tame estimate of the next lemma. ∎

Remark 8.18. The above lemma is, in some sense, the crux of the entire proof of Theorem 1.4. A key point in the proof is that one *must* use the splitting of L^2 determined by p_η, which justifies in hindsight the mildly laborious preparation of Sections 8.1-8.2. Trying to prove these estimates using a different splitting leads the top right entry in the analogue of (8.32) to have infinite-dimensional image and be unbounded in the $H^{5/2}$-norm (in contrast to $*$). In such a splitting, the proof of Lemma 8.17 fails, and the inverse of $\mathrm{d}_{(\eta,\varphi)}\overline{\overline{\not{D}}}_p$ appears non-tame.

Lemma 8.19. Hypothesis **(II)** of Theorem 7.4 holds for $\overline{\overline{\not{D}}}_p$, i.e. there are $s, s' \in \mathbb{N}$ such that the following holds provided δ_0 is sufficiently small: for $p \in V_0$ and $(\varphi, \eta) \in U_0$ the unique solution $u = (v, \phi, \dot{\lambda})$ of

$$\mathrm{d}_{(\varphi,\eta)}\overline{\overline{\not{D}}}_p u = g$$

obeys the tame estimate

$$\|u\|_m \leqslant C_m \Big(\|g\|_{m+s} + \|(p, \eta, \varphi)\|_{m+s'}\|g\|_{m_0}\Big). \tag{8.33}$$

uniformly over $V_0 \times (U_0 \cap \mathbf{P}_{\mathcal{X}})$ for all $m \geqslant m_0$.

Proof. This follows essentially from differentiating the previous proof. We will show that there are tame elliptic estimates of the following form for $\overline{T}_{p_\eta}$ and $\overline{\not{D}}_{p_\eta}$ individually: if $\overline{T}_{p_\eta}v = g_0$ and $\overline{\not{D}}_{p_\eta}\phi = g_1$ then

$$\begin{aligned} \|(v,\lambda)\|_m &\leqslant C_m\Big(\|g_0\|_{m+3/2} \;+\; \|(p,\varphi,\eta)\|_{m+s'}\|g_0\|_{m_0}\Big) & (8.34)\\ \|\phi\|_m &\leqslant C_m\Big(\|g_1\|_m \;+\; \|(p,\varphi,\eta)\|_{m+s'}\|g_1\|_{m_0}\Big) & (8.35)\end{aligned}$$

for $m_0 = 11$. Indeed, given these, one concludes the lemma as follows: write $g = (g_0, g_1)$, so that by the decomposition (8.32) one has $\overline{T}_{p_\eta}(v) = g_0 + *$ and $\overline{\not{D}}_{p_\eta} = g_1 - \mathcal{B}^{\text{Rg}}$ where $* = c\not{D}_{p_\eta}\Phi_{p_\eta}$ with c the Φ_{p_η} component of $u \in X$. Applying (8.34) shows

$$\begin{aligned} \|(v,\lambda)\|_m &\leqslant C_m\Big(\|g_0 + *\|_{m+3/2} \;+\; \|(p,\varphi,\eta)\|_{m+s'}\|g_0 + *\|_{m_0}\Big) & (8.36)\\ &\leqslant C_m\Big(\|g_0\|_{m+3/2} \;+\; \|(p,\varphi,\eta)\|_{m+s'}\|g_0\|_{m_0} \;+\; |c|\cdot\|(p,\eta,\varphi)\|_{m+4} & (8.37)\\ &\qquad + |c|\cdot\|(p,\varphi,\eta)\|_{m+s'}\|(p,\eta,\varphi)\|_{m_0}\Big) & (8.38)\\ &\leqslant C_m\Big(\|g_0\|_{m+3/2} \;+\; \|(p,\varphi,\eta)\|_{m+s'}\|g\|_{m_0}\Big) & (8.39)\end{aligned}$$

Where we have used Lemma 8.2 to bound $\|\not{D}_{p_\eta}\Phi_{p_\eta}\|_{m+3/2}$ by the L^2-norm of $c\Phi_{p_\eta}$ and the $m+4$ norm of p_η. By Lemma 8.17, the L^2-norm is bounded by $|c| \leqslant \|u\|_0 \leqslant \|g\|_{m_0}$and $\|(p,\eta,\varphi)\|_{m_0} \leqslant C$.

Similarly, for the second component, (8.35) shows

$$\begin{aligned} \|\phi\|_m &= C_m\Big(\|g_1 - \mathcal{B}^{\text{Rg}}(v,\lambda)\|_m \;+\; \|(p,\varphi,\eta)\|_{m+s'}\|g_1 - \mathcal{B}^{\text{Rg}}(v,\lambda)\|_0\Big)\\ &= C_m\Big(\|g_1\|_m \;+\; \|(v,\lambda)\|_{m+s} + \|(p,\varphi,\eta)\|_{m+s'}\|g\|_{m_0} + \|(p,\varphi,\eta)\|_{m+s'}\|(v,\lambda)\|_4\Big).\end{aligned}$$

In this, we have used that there is a (tame) boundedness estimate

$$\|\mathcal{B}^{\text{Rg}}(v,\lambda)\|_m \leqslant C_m\Big(\|(v,\lambda)\|_{m+s} + \|(p,\eta,\varphi)\|_{m+4}\|(v,\lambda)\|_{m_0}\Big).$$

Such an estimate from first applying Lemma 8.16, after which it suffices to show the estimate for $\mathcal{B}(v,\lambda)$ rather than just the range components (in this, we implicitly use that fact that the $\mathcal{H}^{m,0}$ norm is tamely equivalent for different metrics from Lemma 8.16). For $\mathcal{B}(v,\lambda)$, the estimate follows from interpolation and Young's inequality (see the subsequent Lemma 8.20). Substituting the previous estimate (8.39) on $\|(v,\lambda)\|_m$ and using that $\|(v,\lambda)\|_4 \leqslant \|g\|_{m_0}$ by Lemma 8.17 then shows that

$$\|(v,\phi,\lambda)\|_m \;\leqslant\; C_m\Big(\|g_1\|_{m+3/2} \;+\; \|(p,\varphi,\eta)\|_{m+s'}\|g\|_{m_0}\Big)$$

as desired.

Thus to prove the lemma, we verify (8.34) and (8.35). The latter follows from differentiating elliptic estimates in the standard way. To elaborate briefly, we begin with the estimate for $\not{D}_{p_\eta}$ and the H^m_{b} term in the norms. One shows by iterating commutators that there is an elliptic estimate of the form

$$\|\phi\|_{rH^{m,1}_{\text{b},e}} \leqslant C_m\Big(\|\overline{\not{D}}_{p_\eta}\phi\|_{H^m_{\text{b}}} \;+\; \|(p,\eta,\varphi)\|_{s'}\|\phi\|_{rH^{m-1,1}_{\text{b},e}} \;+\; \dots \;+\; \|(p,\eta,\varphi)\|_{m+s'}\|\phi\|_{rH^1_e}\Big) \qquad (8.40)$$

for each m and $s' < m_0$. Given such an estimate, each middle term $\|(p,\eta,\varphi)\|_{k+s'}\|\phi\|_{rH^{m-k-1,1}_{\text{b},e}}$ can be absorbed into the $k = 0, m$ ones by Young's inequality and interpolation with $m_2 = m + s'$ and $m_1 = s'$ on the first factor and $m_2 = m - 1$ and $m_2 = 0$ on the second factor. The tame estimates are then a consequence of induction by substituting the tame estimate on $\|\phi\|_{rH^{m-1,1}_{\text{b},e}}$ beginning with the base case provided by Lemma 8.17, and using that $\|(p,\eta,\varphi)\|_{s'} \leqslant 1$. The same exact argument applies for the spaces $r\mathcal{H}^{m,1}$ and $\mathcal{H}^{m,0}$ using the elliptic estimate and commutation relations from Lemma 8.15, and the tame estimate (8.35) follows.

Similarly, for (8.34) it suffices to show

$$\|v\|_{L^{m+2,2}} \leqslant C_m\Big(\|\overline{T}_{p_\eta} v\|_{L^{m+3/2,2}} \ + \ \|(p,\eta,\varphi)\|_{s'}\|v\|_{L^{m-1+3/2,2}} \ + \ \ldots \ + \ \|(p,\eta,\varphi)\|_{m+s'}\|v\|_{3/2}\Big), \quad (8.41)$$

which is again proved by iterating commutators, but taking care to ensure that the conormal regularity is preserved. First apply the $m=0$ estimate to $\nabla_t^m v$. To use the term $\mathcal{B}_\varphi(v)$ as an example, one has

$$\mathcal{B}_\varphi(\nabla_t^m v) = \nabla_t^m \mathcal{B}_\varphi(v) + \nabla_t \mathcal{B}_\varphi(\nabla_t^{m-1} v) + \mathcal{B}_{\nabla_t\varphi}(\nabla_t^{m-1}v) + \ldots + \mathcal{B}_{\nabla_t^m\varphi}(v).$$

By the same argument as in Proposition 8.6 and Lemma 8.17, all except the first terms have conormal regularity 3/2, since φ is polyhomogenous. This leads to

$$\begin{aligned}
\|v\|_{L^{m+2,2}} \ &\leqslant \ C_m\Big(\|\mathrm{ob}^{-1}\Pi_{p_\eta}(\nabla_t^m\mathcal{B}_\varphi(v) + \ldots + \nabla_t^m F^2(\eta^2,v,\varphi))\|_{L^{m+3/2,2}} \\
&\qquad + \|(p,\eta,\varphi)\|_{s'}\|v\|_{L^{m-1+3/2,2}} \ + \ \ldots \ + \ \|(p,\eta,\varphi)\|_{m+s'}\|v\|_{3/2}\Big) \\
&\leqslant \ C_m\Big(\|\nabla_t^m\Pi_{p_\eta}(\mathcal{B}_\varphi(v) + ... + F^2(\eta^2,v,\varphi))\|_{H^{3/2}_{\mathrm{b}}} \\
&\qquad + \|(\nabla \not{D} P \not{D}^\star)\nabla^{m-1}\mathcal{B}_\varphi(v)\|_{H^{3/2}_{\mathrm{b}}} + \ldots + \|(\nabla^m \not{D} P\not{D}^\star)\mathcal{B}_\varphi(v)\|_{H^{3/2}_{\mathrm{b}}} + ... \\
&\qquad + \|(p,\eta,\varphi)\|_{s'}\|v\|_{L^{m-1+3/2,2}} \ + \ \ldots \ + \ \|(p,\eta,\varphi)\|_{m+s'}\|v\|_{3/2}\Big)
\end{aligned}$$

where we have used Corollary 4.17 on the first term, and then expanded the commutators with the projection operator. Since $\not{D}_{p_\eta}P_{p_\eta}\not{D}^\star_{p_\eta}$ is zeroth order with coefficients depending on (p,η,φ), it is routine to check that the second line can be absorbed into the terms of the last line. Another application of Corollary 4.17 to the first term (noting the equivalence of norms arising from U is also tame) leads to 8.41, completing the lemma.

$\square$

Lemma 8.20. Hypothesis **(III)** of Theorem 7.4 holds for $\overline{\not{\mathbb{D}}}_p$, i.e. there are $r,r' \in \mathbb{N}$ such that the following holds provided δ_0 is sufficiently small: for $p \in V_0$ and $(\varphi,\eta) \in U_0$, the second derivative obeys the tame estimate

$$\|\mathrm{d}^2_{(\eta,\varphi)}\overline{\not{\mathbb{D}}}_p(u,v)\|_m \leqslant C_m \ \Big(\|u\|_{m+r}\|v\|_{m_0} \ + \ \|u\|_{m_0}\|v\|_{m+r} \ + \ \|u\|_{m_0}\|v\|_{m_0}\cdot(1+\|(p,\eta,\varphi)\|_{m+r'})\Big). \quad (8.42)$$

for $u,v \in X$ uniformly over $V_0 \times (U_0 \cap \mathbf{P}_{\mathcal{X}})$ for all $m \geqslant m_0$.

Proof. This tame estimate follows directly from using the boundedness of the terms comprising $\mathrm{d}^2_{(\eta,\varphi)}\overline{\not{\mathbb{D}}}_p$ in conjunction with the interpolation inequalities.

As in Corollary 8.11, the second derivative is given by

$$\begin{aligned}
\mathrm{d}^2_{(\eta,\varphi)}\not{\mathbb{D}}_p\Big((v,\phi),(w,\psi)\Big) \quad = \quad & (\mathcal{B}_\psi + \mathfrak{B}_\psi)(v) \ + \ (\mathcal{B}_\phi + \mathfrak{B}_\phi)(w) \\
& + M^1(w',v')\nabla(\Phi_0+\varphi) \ + \ M^1(\eta',v')\nabla\psi \ + \ M^1(\eta',w')\nabla\phi \\
& + M^2(w',v'')(\Phi_0+\varphi) \ + \ M^2(\eta',v'')\psi \ + \ M^2(v',w'')(\Phi_0+\varphi) \\
& + M^2(v',\eta'')\psi \ + \ M^2(\eta',w'')\phi \ + \ M^2(w',\eta'')\phi \\
& + F^3(\eta^2,w,\varphi) \ + \ F^4(\eta^2,v,\psi) \ + \ F^5(\eta,v,w,\Phi_0+\varphi)
\end{aligned}$$

For the sake of the proverbial deceased horse, we will prove the lemma for the term $M^2(w',v'')(\Phi_0+\varphi)$; it is straightforward to verify that the same argument applies equally well to the remaining terms.

To begin, we bound that H^m_{b}-term in the norm. By Proposition 8.9 Item (3) part (iii), this term is itself a sum of terms of the form $m_p(y)w'v''(\Phi_0+\varphi)$. Differentiating the part involving φ of such a term,

$$\begin{aligned}
\|\nabla^m_{\mathrm{b}}(m_p(y)w'v''\varphi)\|_{L^2} &\leqslant C_m \sum_{0\leqslant k\leqslant m} \|\nabla^k_{\mathrm{b}}(v'w'')\nabla^{m-k}_{\mathrm{b}}(m_p\varphi))\|_{L^2} \\
&\leqslant C_m \sum_{0\leqslant k\leqslant m} \|\nabla^k_{\mathrm{b}}(v'w'')\|_{L^{2,2}(S^1)}\|\nabla^{m-k}_{\mathrm{b}}(m_p\varphi)\|_{H^2_{\mathrm{b}}} \\
&\leqslant C_m \sum_{0\leqslant k\leqslant m} \|v'w''\|^{1-\frac{k}{m}}_{L^{2,2}(S^1)}\,\|v'w''\|^{\frac{k}{m}}_{L^{m+2,2}(S^1)}\,\|m_p\varphi\|^{1-\frac{k}{m}}_{H^2_{\mathrm{b}}}\,\|m_p\varphi\|^{\frac{k}{m}}_{H^{m+2}_{\mathrm{b}}}
\end{aligned}$$

where we have used the Sobolev embedding $C^0 \hookrightarrow H^2(S^1)$ and then the interpolation inequalities with $m_2 = m+2, m_1 = 2$. By Young's inequality with exponents $p = \frac{m}{k}$ and $q = \frac{m}{m-k}$, one finds the above is bounded by

$$\begin{aligned}
&\leqslant C_m\Big(\|v'w''\|_{L^{m+2,2}(S^1)}\|m_p\varphi\|_{L^{2,2}(S^1)} + \|v'w''\|_{L^{2,2}(S^1)}\|m_p\varphi\|_{H^{m+2}_{\mathrm{b}}}\Big) \\
&\leqslant C_m\Big(\|v'\|_{L^{m+4,2}(S^1)}\|w''\|_{L^{4,2}(S^1)} + \|v'\|_{L^{4,2}(S^1)}\|w''\|_{L^{m+4,2}(S^1)} \\
&\quad + \|v'\|_{L^{4,2}(S^1)}\|w''\|_{L^{4,2}(S^1)}\Big(\|m_p\|_{H^{m+4}_{\mathrm{b}}}\|\varphi\|_{H^4_{\mathrm{b}}} + \|m_p\|_{H^4_{\mathrm{b}}}\|\varphi\|_{H^{m+4}_{\mathrm{b}}}\Big)\Big) \\
&\leqslant C_m\Big(\|v\|_{L^{m+5,2}}\|w\|_{L^{6,2}} + \|v\|_{L^{5,2}}\|w\|_{L^{m+6,2}} + \|v\|_{L^{5,2}}\|w''\|_{L^{6,2}} \cdot \|(p,\eta,\varphi)\|_{m+6}\Big)
\end{aligned}$$

where we have repeated the interpolation and Young's steps from above with $m_2 = m+4$ and $m_1 = 4$ on both products, and then used the fact that $6 \leqslant m_0$ so that $\|m_p\|_{H^4_{\mathrm{b}}} + \|\varphi\|_{H^4_{\mathrm{b}}} \leqslant C$. This shows the desired estimate for $r, r' = 6$. The same steps apply to the $r^\nu H^{m-1}_{\mathrm{b}}$ term in the norm using the commutation relations from Lemma 8.15. The other terms are similar, with the constant term in $(1 + \|(p,\eta,\varphi)\|_{m+r'})$ on the right hand side arising from the terms not involving (p,φ,η) such as $\mathcal{B}_\psi(v)$. □

8.6 Proofs of Theorem 1.4 and Corollary 1.5

In this subsection, we invoke the Nash-Moser Implicit Function Theorem 7.4 to conclude the proofs of Theorem 1.4 and Corollary 1.5, beginning with the latter.

Proof of Corollary 1.5. Lemmas 8.17, 8.19, and 8.20 verify hypotheses **(I)**, **(II)**, and **(III)** from Section 7.2 respectively on $V_0 \times (U_0 \cap \mathbf{P}_{\mathcal{X}})$. The formula (8.7) and Lemma 8.8 (which extends easily to the spaces $\mathcal{H}^{m,0}$) show that $f_p \in \mathbf{P}_{\mathcal{Y}}$ and $\|f_p\|_m \leqslant C\|p\|_{m+s}$.

It remains to show that the property (P) of being polyhomogeneous is propagated by the iteration in the sense of Definition 7.3. Lemma 8.12 and its proof in Appendix B show that the smoothing operators $S_\varepsilon, S^{\mathrm{b}}_\varepsilon$ preserve polyhomogeneity. It is evident from the definition of the Dirac operator $\not{D}_p$ that for any metric (including pullback metrics p_η)

$$\varphi \in \mathbf{P}_{\mathcal{X}} \quad \Rightarrow \quad \overline{\not{\mathbb{D}}}_p(\eta,\varphi) \in \mathbf{P}_{\mathcal{Y}}$$

preserves polyhomogeneity. To show polyhomogeneity (P) is propagated, we therefore verify that

$$g \in \mathbf{P}_{\mathcal{Y}} \quad \Rightarrow \quad (\mathrm{d}_{(\eta,\varphi)}\overline{\not{\mathbb{D}}}_p)^{-1} g \in \mathbf{P}_{\mathcal{X}}.$$

Let $u = (\mathrm{d}_{(\eta,\varphi)}\overline{\not{\mathbb{D}}}_p)^{-1} g$ be the solution, and suppose that g is polyhomogeneous with index set $\mathbb{Z}^+ - \frac{1}{2}$. By the block diagonal decomposition (8.32) from Lemma 8.17, one has $\overline{T}_{p_\eta}(v) = \Pi_{p_\eta}(g)$ in the case that the upper right entry $*$ vanishes. In this case, since $g \in Y$ is smooth, then $\Pi_0(g) \in \mathbf{Ob} \cap H^m_{\mathrm{b}}$ for every

$m \geqslant 0$ and admits a weak expansion as in Remark 3.10. Lemma 8.14 applies identically in the case of index set $\mathbb{Z}^+ - \frac{1}{2}$ to show that the coefficients are smooth, hence $\Pi_0(g)$ is polyhomogeneous with index set $\mathbb{Z}^+ - \frac{1}{2}$. Since the latter is a vector space, the right-hand side of the second equation

$$\not{D}_{p_\eta}\phi = (1 - \Pi_0)(g)$$

is polyhomogeneous with index set $\mathbb{Z}^+ - \frac{1}{2}$. Since the coefficients of $\not{D}_{p_\eta}$ are smooth, this left-hand side must also be polyhomogeneous with index set $\mathbb{Z}^+ + \frac{1}{2}$ (see [45], Proposition 7.17) with the caveat of possibly having logarithm terms appear with the $r^{1/2}$ coefficient. The case where $*$ is non-zero is the same as Φ_{p_η} is polyhomogeneous.

The final point is to rule out the appearance of logarithm terms $r^{1/2}\log(r)$. This is a consequence of the restrictions on the θ_p-Fourier modes in Fermi coordinates that appear with the $r^{1/2}$ coefficient and follows from formally solving the first term at each stage. Logarithm terms $e^{ik\theta}r^{1/2}\log(r)$ would arise from the right-hand side having terms $r^{-1/2}e^{\pm 3i\theta/2}$, but since the right-hand side is always a combination of $\Pi_0(g)$ which has only modes $r^{-1/2}e^{\pm i\theta/2}$ and the derivatives $\nabla_z, \nabla_{\overline{z}}$ of a polyhomogeneous spinor from the previous stage such terms never arise. This property is preserved by the smoothing operators $S_\varepsilon^{\mathrm{b}}$ by construction (see Appendix B). We conclude that the property (P) of being polyhomogeneous in the sense of having expansions of the form (8.1)-(8.2) is propagated.

By the Nash-Moser Implicit Function Theorem 7.4, there is an open neighborhood $U \subset \mathcal{P}$ of smooth parameters such that for $p \in U$ there exists a unique solution $(\mathcal{Z}_p, \Phi_p, \Lambda_p)$ to the equation

$$\not{\mathbb{D}}_p(\mathcal{Z}_p, \Phi_p) = \Lambda_p\Phi_p \tag{8.43}$$

and the triples $(\mathcal{Z}_p, \Phi_p, \Lambda_p)$ define a smooth tame graph of $U \subset U \times \mathcal{X}$. This completes the proof of Corollary 1.5 in the presence of Assumptions 4* and 5*. To eliminate the first is trivial as the deformations of $\mathcal{Z}_0$ are local along each component. In the absence of Assumption 5*, the standard Kuranishi picture (see, e.g. Section 3.3 of [7]) applies to show that the set of parameters for which (8.43) holds is described by the zero set of a smooth tame map

$$\kappa_p : U \times \mathbb{R}^n \to \mathbb{R}^n$$

where $n = \dim(\ker(T_{\Phi_0}))$ is the dimension of the kernel of the index 0 map from Section 6.3. $\square$

Proof of Theorem 1.4. The projection $\pi(\mathscr{M}_{\mathbb{Z}_2}) \subseteq U \cap \mathcal{P}$ of the universal moduli space of $\mathbb{Z}_2$-harmonic spinors to the parameter space is defined by the zero-set

$$\pi(\mathscr{M}_{\mathbb{Z}_2}) = \Lambda^{-1}(0) \cap U$$

of the eigenvalue $\Lambda : U \to \mathbb{R}$ of Corollary 1.5, and there is locally a unique $\mathbb{Z}_2$-harmonic spinor $(\mathcal{Z}_p, \Phi_p)$ up to normalization and sign for each $p \in \Lambda^{-1}(0)$, hence the projection π is a local homeomorphism.

In the presence of Assumption 5*, the map $\Lambda : U \to \mathbb{R}$ is transverse to 0. To see this, let $p(s)$ be a path of parameters which we will choose momentarily. By Corollary 1.5, such a path implicitly defines triples $(\mathcal{Z}_s, \Phi_s, \Lambda_s)$ for s sufficiently small. Differentiating (8.43) at $s = 0$ yields the relation that

$$\not{D}_{\dot{\mathcal{Z}}}\Phi_0 + \not{D}_{\dot{p}}\Phi_0 + \not{D}_0\dot{\Phi} = \dot{\Lambda}\Phi_0 \tag{8.44}$$

where $\cdot$ denotes the s-derivative. We now choose $p(s) = (g(s), B(s))$ so that the derivative $(\dot{g}, \dot{B})$ has the following properties. Let $\dot{B}$ be a smooth perturbation supported on a neighborhood disjoint from $N_{r_0}(\mathcal{Z}_0)$ such that $\langle \gamma(\dot{B})\Phi_0, \Phi_0\rangle \neq 0$. Given this, we define $\dot{g}$ in terms of $\dot{B}$ as follows. By Assumption 4*, we know that $T_{\Phi_0} : L^{2,2}(\mathcal{Z}_0; N\mathcal{Z}_0) \to \ker(\not{D}_0|_{L^2})$ is injective with closed range and 1-dimensional cokernel. Let $\Phi_1 \subseteq \ker(\not{D}|_{L^2})$ denote the orthogonal complement of its range. Since $\overline{T}_{\Phi_0}$ is an isomorphism, we must have have $\langle \Phi_1, \Phi_0\rangle \neq 0$. Decompose $\Pi_0(\gamma(\dot{B})\Phi_0) = (c\Phi_1, \xi)$, and set $\dot{g} = \dot{g}_\eta$ where $T_{\Phi_0}(\eta) = -\xi$. Taking the inner product of (8.45) with Φ_1 then yields

$$\cancel{\langle \slashed{D}_{\dot{\mathcal{Z}}}\Phi_0, \Phi_1\rangle} + \langle -c\Phi_1, \Phi_1\rangle + \cancel{\langle \slashed{D}_0\dot{\Phi}, \Phi_1\rangle} = \dot{\Lambda}\langle \Phi_0, \Phi_1\rangle \tag{8.45}$$

wherein the first term vanishes by definition of Φ_1 and the third via integration by parts. We conclude that $\dot{\Lambda} \neq 0$ and that $\Lambda : V_0 \to \mathbb{R}$ is transverse to 0.

When Assumption 5* fails, the Kuranishi picture applies again to show that the $\pi(\mathscr{M}_{\mathbb{Z}_2})$ is given locally by the zero-set of a smooth tame map

$$(\kappa_p, \Lambda) : U \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$$

where $n = \dim(\ker(T_{\Phi_0}))$ is the dimension of the kernel of the index 0 map from Section 6.3. This concludes the proof of Theorem 1.4. $\square$

A Appendix I: Exponential Decay

The goal of this appendix is to prove Lemma 4.7. To recall the notation, $\slashed{D}_0$ here denotes the Dirac operator on $(S^1 \times \mathbb{R}^2, dt^2 + dx^2 + dy^2)$. Given a solution

$$\slashed{D}_0\slashed{D}_0 u_\ell = f_\ell \tag{A.1}$$

of the second order equation where f_ℓ satisfies the assumptions of 4.7, we prove that the solution u_ℓ satisfies a discretized exponential decay as in the conclusion of the lemma. The proof relies on a discretized version of the maximum principle. Similar arguments may be found in [31] (see Appendix A.2.1).

Proof. The arguments of Proposition 2.4 and 2.7 are equally valid in the case of the non-compact Y_0 (provided the scalar curvature s and perturbation B are compactly supported) since the boundary term at infinity vanishes when integrating by parts. In fact, in this case $s = B = 0$ hence the Weitzenböck formula implies that rH^1_e-kernel is empty. It follows that

$$\slashed{D}_0\slashed{D}_0 : rH^1_e \longrightarrow rH^{-1}_e$$

is an isomorphism and that (A.1) admits a unique solution. Since $\slashed{D}_0$ preserves Fourier modes, u_ℓ has Fourier modes in the range 4.10 as well.

Denote by χ_n a cutoff-function equal to 1 on A_{n_ℓ} such that

$$\text{supp}(d\chi_n) \subseteq A_{(n-1)\ell} \cup A_{(n+1)\ell} \qquad\qquad |d\chi_n| \leqslant \frac{c|\ell|}{R_0}.$$

Taking the inner product of A.1 with $\chi_n^2 u_\ell$, integrating by parts, and using Young's inequality yields the following where $B_{n\ell} = A_{n-1,\ell} \cup A_{n,\ell} \cup A_{n+1,\ell}$ as before:

$$\begin{aligned}
\int_{B_{n\ell}} \chi_n^2 |\slashed{D}_0 u_\ell|^2 \, dV_0 &= -\int_{B_{n\ell}} \langle 2\chi_n d\chi_n . u_\ell, \slashed{D}_0 u_\ell\rangle \, dV_0 + \int_{B_{n\ell}} \langle \chi_n^2 u_\ell, -g_\ell\rangle \, dV_0 && \text{(A.2)}\\
&\leqslant 2c^2 \frac{|\ell|^2}{R_0^2} \|u_\ell\|_{L^2(B_{n\ell})} + \frac{1}{2}\|\chi_n \slashed{D}_0 u_\ell\|^2_{L^2(B_{n\ell})} && \text{(A.3)}\\
&\quad + \frac{1}{2c_1}\|g_\ell\|^2_{rH_e^{-1}(B_{n\ell})} + \frac{c_1}{2}\|\chi_n^2 u_\ell\|^2_{rH^1_e(B_{n\ell})} && \text{(A.4)}
\end{aligned}$$

where the inner product is that defined using the product metric g^{Euc} and likewise for the volume form dV_0.

The $\|\slashed{D}_0 u_\ell\|^2$ and $\|\chi_n^2 u_\ell\|^2$ terms on the right side can be absorbed into the left once c_1 is chosen sufficiently small. The first is clear from the factor of 1/2, and for the second apply the elliptic estimate $\slashed{D}_0 : rH^1_e \to \text{Range}(\slashed{D}_0)$ to $\chi_n^2 u_\ell$ to obtain

$$\|\chi_n^2 u_\ell\|^2_{rH^1_e(B_{n\ell})} \leqslant C\|\not{D}_0(\chi_n^2 u_\ell)\|^2_{L^2(Y_0)} \leqslant C\left(\|\chi_n^2 \not{D}_0 u_\ell\|^2_{L^2(B_{n\ell})} + \|d\chi_n . u_\ell\|^2_{L^2(B_{n\ell})}\right). \tag{A.5}$$

By choosing c_1 sufficiently small (and using that $|\chi_n^2| \leqslant |\chi_n|$), the $\|\chi_n^2 \not{D}_0 u_\ell\|^2$ term can be absorbed on the left hand side of A.2, and the $\|d\chi_n . u_\ell\|^2$ term can be absorbed into A.3 by increasing c^2.

Substituting A.5 again and increasing the constants yields

$$\|u_\ell\|^2_{rH^1_e(A_{n\ell})} \leqslant \|\chi_n^2 u_\ell\|^2_{rH^1_e(A_{n\ell})} \leqslant C_1 \frac{|\ell|^2}{R_0^2}\|u_\ell\|_{L^2(B_{n\ell})} + \frac{1}{2c_1}\|g_\ell\|_{rH^{-1}_e(B_{n\ell})}$$

which shows, invoking the assumption on g_ℓ, that

$$\|u_\ell\|^2_{rH^1_e(A_{n\ell})} \leqslant C_1 \frac{|\ell|^2}{R_0^2}\|u_\ell\|_{rH^1_e(B_{n\ell})} + \frac{C'_m}{|\ell|^{2+2m}}e^{-2n/c_m}. \tag{A.6}$$

Then, because of the restriction of Fourier modes on η_ℓ,

$$C_1 \frac{|\ell|^2}{R_0^2}\int_{B_{n\ell}} |u_\ell|^2 \, dV \leqslant \frac{4C_1}{R_0^2}\left(\int_{A_{(n-1)\ell}} |\nabla u_\ell|^2 \, dV + \int_{A_{n\ell}} |\nabla u_\ell|^2 \, dV + \int_{A_{(n+1)\ell}} |\nabla u_\ell|^2 \, dV\right).$$

Now set $\mathfrak{a}_n = \|u_\ell\|^2_{L^{1,2}(A_{n\ell})}$, and choose R_0 so that $\frac{4C_1}{R_0}^2 < \frac{1}{200}$. Equation A.6 implies the discrete differential inequality

$$\mathfrak{a}_n - \tfrac{1}{100}(\mathfrak{a}_{n-1} + \mathfrak{a}_{n+1}) \leqslant \frac{C'_m}{|\ell|^{2+2m}}e^{-2n/c_m}.$$

To conclude, we apply a discrete version of the maximum principle: let $\mathfrak{s}_n = \frac{2C_m}{|\ell|^{2+2m}}e^{-2n/c_m}$. Possibly by increasing c_m to c'_m, this rather trivially satisfies

$$\mathfrak{s}_n - \tfrac{1}{100}(\mathfrak{s}_{n-1} + \mathfrak{s}_{n+1}) \leqslant \frac{C'_m}{|\ell|^{2+2m}}e^{-2n/c'_m}.$$

Hence the difference $\mathfrak{r}_n = \mathfrak{a}_n - \mathfrak{s}_n$ satisfies

$$\mathfrak{r}_n - \tfrac{1}{100}(\mathfrak{r}_{n-1} + \mathfrak{r}_{n+1}) \leqslant 0 \tag{A.7}$$

and, possibly increasing C'_m, $\mathfrak{r}_0 \leqslant 0$ and $r_n \to 0$ as $n \to \infty$ (the latter requirement is simply from integrability). The "maximum principle" then implies $\mathfrak{r}_n \leqslant 0$ for all n, since an interior maximum with $\mathfrak{r}_n \geqslant \mathfrak{r}_{n-1}, \mathfrak{r}_{n+1}$ would violate A.7. We conclude that u_ℓ satisfies

$$\|u_\ell\|^2_{L^{1,2}(A_{n\ell})} \leqslant \frac{C'_m}{|\ell|^{2+2m}}\mathrm{Exp}\left(-\frac{2n}{c'_m}\right) \tag{A.8}$$

which completes the lemma. □

B Appendix II: Boundary and Edge Regularity

This appendix gives proofs of two facts about regularity in the boundary and edge Sobolev spaces, namely Lemma 8.12 and Lemma 8.14.

Recall the Fréchet spaces defined above in Lemma 8.12. To restate the assertion of Lemma 8.12 succinctly:

Lemma B.1. For $0 < \varepsilon \leqslant 1$ there exist smoothing operators

$$S_\varepsilon : C^\infty(\mathcal{Z}_0; N\mathcal{Z}_0) \to C^\infty(\mathcal{Z}_0; N\mathcal{Z}_0) \qquad S^{\rm b}_\varepsilon : \bigcap_{m\geqslant 0} X''_m \to \bigcap_{m\geqslant 0} X''_m \qquad S^{\rm b}_\varepsilon : \mathcal{Y} \to \mathcal{Y}$$

satisfying properties (i)-(iii) of Definition 7.1 and preserving the property (P) of polyhomogeneity as in defined by (8.1)–(8.2). Additionally, in Fermi coordinates around $\mathcal{Z}_0$, $S^{\rm b}_\varepsilon$ does not introduce new Fourier modes in θ.

Proof. On $\mathcal{X}_0 = C^\infty(\mathcal{Z}_0; N\mathcal{Z}_0)$ the operator S_ε may be defined straightforwardly as the truncation of the Fourier series at $|\ell| \sim \frac{1}{\varepsilon}$ in a global trivialization. It is instructive for what follows, however, to construct S_ε using Schwartz kernels.

Alternatively, S_ε may be defined as a convolution operator using a Schwartz kernel that smoothly approximates the δ-distribution along the diagonal in $(\mathcal{Z}_0)^2 = \mathcal{Z}_0 \times \mathcal{Z}_0$. More precisely, let $\chi(r)$ be a cut-off function equal to 1 near $r = 0$ and vanishing for $r > 1$. Fix a collection $U_j \times \mathbb{C}$ for $j = 1, .., n$ of trivializations of $N\mathcal{Z}_0$ on contractible open sets, and for each j, choose nested cut-off functions ξ_j, β_j such that $\text{supp}(\beta_j) \Subset \{\xi_j = 1\}$. Then define

$$S_\varepsilon(\eta)(t) := \frac{1}{\varepsilon}\sum_{j=1}^n \xi_j(t) \int_{\mathcal{Z}_0} \chi\left(\tfrac{|t-t'|}{\varepsilon}\right) \beta_j(t')\eta(t')dt'. \tag{B.1}$$

where the constant $\frac{1}{\varepsilon}$ serves to normalize χ in L^2. Properties (i)-(iii) now follow easily.

The construction of $S^{\rm b}_\varepsilon$ is analogous, but now we de-singularize the δ-distribution on the diagonal in the blown-up product defined as follows. Let $B = \mathcal{Z}_0 \times \mathcal{Z}_0 \subset Y \times Y$, and let $D(B)$ denote a disk bundle of finite radius in the normal bundle. Define

$$Y^2_{\rm b} := (Y - (N_{r_0}\mathcal{Z}_0))^2 \cup D(B).$$

This blow-up is a compact 6-manifold with corners, having three boundary strata of codimension 1 consisting of the interiors of $\partial(N_{r_0}\mathcal{Z}_0 \times Y), \partial(Y \times N_{r_0}\mathcal{Z}_0), \partial(D(B))$ which intersect along codimension 2 corners. This space can be given local coordinates $(s, \rho, \theta, \theta', t, t')$ in a neighborhood of the diagonal, where $s = [r, r']$ is a projective coordinate along the blow-up boundary, and $\rho = r'$.

Away from these strata, $S^{\rm b}_\varepsilon$ can be defined analogously to (B.1); near the boundary strata it is defined as a product

$$S^{\rm b}_\varepsilon := S^\theta_\varepsilon \circ S'_\varepsilon$$

where S^θ_ε is defined by truncation of the θ-Fourier modes in a local trivialization, and S'_ε is given in Fermi coordinates (s, ρ, t, t') by

$$S'_\varepsilon(\psi)(r, t, \theta) := \frac{1}{\varepsilon^2}\chi \int_{Y-\mathcal{Z}_0} \chi\left(\tfrac{|s-1|}{\varepsilon}\right)\chi\left(\tfrac{|t-t'|}{\varepsilon}\right)\frac{1}{r'}(\beta\psi)dt'dr' \tag{B.2}$$

where the factor of $1/r'$ appears because $|r - r'| \sim r's$ and the δ-distribution is homogeneous of order -1.

The properties (i)-(iii) for the spaces $H^m_{\rm b}$ follow analogously to the compact case. That $S^{\rm b}_\varepsilon$ introduces no new Fourier modes in θ is manifest from the definition, and the fact that polyhomogeneity is preserved is a consequence of the pushforward theorem or of direct inspection of the integral (B.2) (see [17] Section 3.1). Since the ratio r/r' is uniformly bounded where $\chi \neq 0$, the commutators $[\nabla^e, S^{\rm b}_\varepsilon]$ and $[r^\alpha, S^{\rm b}_\varepsilon]$ are uniformly bounded, properties (i)-(iii) for the space $\bigcap_{m\geqslant 0} H^{m,1}_{{\rm b},e}$ follows from the equivalent description of the norm (2.13). The same applies for the terms $(r\partial_r \pm \frac{1}{2})\psi$ and therefore for the spaces $r\mathcal{H}^{m,1}$ and $\mathcal{H}^{m,0}$.

$\square$

What remains is to prove Lemma 8.14, which requires several steps.

Lemma B.2. If $\varphi \in r^\alpha H_\mathrm{b}^3$ for $\alpha > 1$ then the φ satisfies the pointwise bound

$$|\varphi(x)| \leqslant C\|\varphi\|_{rH_\mathrm{b}^3}. \tag{B.3}$$

Proof. We first prove the lemma in the 1-dimensional case.

Consider $\mathbb{R}^+ = (0,\infty)$ with the measure rdr and suppose that $\varphi \in rH_b^1(rdr)$ with $\mathrm{supp}(\varphi) \subseteq (0,1]$. Then we claim that there is a constant C so that

$$|\varphi(x)| \leqslant C\|\varphi\|_{rH_\mathrm{b}^1(rdr)} = C\left(\int_{\mathbb{R}^+} \frac{|\varphi|^2}{r^2} + |\nabla\varphi|^2 \ rdr\right)^{1/2}. \tag{B.4}$$

This follows from a dyadic decomposition. Since r is uniformly bounded on $[1/2,2]$ and $L^{1,2}[1/2,2] \hookrightarrow C^0[1/2,2]$ by the standard Sobolev embedding, we have $|\varphi(1)|^2 \leqslant c\int |\varphi|^2 + |\nabla\varphi|^2 dr \leqslant c\|\varphi\|^2_{rH_\mathrm{b}^1}$. Then, by the Fundamental Theorem of Calculus,

$$\begin{aligned} |\varphi(1/2)| &\leqslant |\varphi(1)| + \int_{1/2}^1 |\varphi'(\rho)|d\rho \leqslant |\varphi(1)| + \left(\int_{1/2}^1 \rho|\varphi'(\rho)|^2 d\rho\right)^{1/2}\left(\int_{1/2}^2 \frac{1}{\rho}d\rho\right)^{1/2} \\ &\leqslant |\varphi(1)| + (\log 2)^{1/2}\|\varphi\|_{rH_\mathrm{b}^1([1/2,1],rdr)} \end{aligned}$$

Similarly, $|\varphi(1/4)| \leqslant |\varphi(1/2)| + (\log 2)^{1/2}\|\varphi\|_{rH_\mathrm{b}^1([1/4,1/2],rdr)} \leqslant |\varphi(1)| + (\log 2)^{1/2}\|\varphi\|_{rH_\mathrm{b}^1([1/4,1],rdr)}$ where the second inequality follows from substituting the above. In general, using the estimate on $|\varphi(1)|$ we conclude that

$$|\varphi(2^{-k})| \leqslant C\|\varphi\|_{rH_\mathrm{b}^1(rdr)}.$$

(B.4) then follows from applying the Fundamental Theorem of calculus again for $x \in [2^{-k}, 2^{-k+1}]$.

In general, for $\varphi \in H_\mathrm{b}^m(Y - \mathcal{Z}_0; S_0)$, the lemma follows from the above by applying (B.4) to rays of constant (t,θ) and after using the Sobolev restriction theorem (which increases the number of derivatives needed).

□

Next, we have the following fundamental fact about ODEs. For it, we use the 1-dimensional b-spaces $r^\alpha L_\mathrm{b}^{1,2}([0,1], rdr)$ and $r^\alpha L^2([0,1], rdr)$ defined by the norms

$$\|u\|_{r^\alpha L_\mathrm{b}^{1,2}} = \left(\int_0^1 (|r\partial_r u|^2 + |u|^2)r^{-2\alpha} rdr\right)^{1/2} \qquad \|u\|_{r^\alpha L_\mathrm{b}^2} = \left(\int_0^1 |u|^2 r^{-2\alpha} rdr\right)^{1/2}.$$

Lemma B.3. Provided $\alpha > 3/2$ then

$$(r\partial_r - \tfrac{1}{2}) : r^\alpha L_\mathrm{b}^{1,2}(0,1] \to r^\alpha L_\mathrm{b}^2(0,1]$$

is an isomorphism and there holds

$$\|u\|_{r^\alpha L_\mathrm{b}^{1,2}} \leqslant \|(r\partial_r - \tfrac{1}{2})u\|_{r^\alpha L_\mathrm{b}^2} \tag{B.5}$$

Proof. Setting $r = e^s$ for $s \in (-\infty, 0]$ the problem is equivalent to the analogous statement for

$$\partial_s - \tfrac{1}{2} : e^{(1-\alpha)s}L^{1,2}((-\infty,0],ds) \longrightarrow e^{(1-\alpha)s}L^2((-\infty,0],ds)$$

which is conjugate to

$$\tfrac{1}{e^{(\alpha-1)s}}(\partial_s - \tfrac{1}{2})e^{(\alpha-1)s} = (\partial_s + \alpha - \tfrac{3}{2}) : L^{1,2}((-\infty,0],ds) \to L^2((-\infty,0],ds).$$

The claim then follows directly from integrating by parts since the boundary term $(\alpha - \frac{3}{2})|u(0)|^2 > 0$ is strictly positive. □

We now conclude the proof of Lemma 8.14.

Proof of Lemma 8.14. If φ is compactly supported in the region $Y - N_{r_0/2}(\mathcal{Z}_0)$ where r is uniformly bounded below, the lemma is immediate from the standard Sobolev Embedding Theorem. We may therefore assume that φ is supported in a tubular neighborhood of $\mathcal{Z}_0$. Since $\varphi \in r\mathcal{H}^{m,1} \cap \mathbf{P}_{\mathcal{X}}$ by assumption, we may write

$$\varphi = A(t,\theta)r^{1/2} + B(t,\theta,r)$$

in local coordinates, after which it suffices to show the bound for each term individually.

Applying (B.5) to derivatives and integrating over the t,θ variables leads to

$$\|r\partial_r u\|_{r^\alpha H^m_{\mathrm{b}}} + \|u\|_{r^\alpha H^m_{\mathrm{b}}} \leqslant C\|(r\partial_r - \tfrac{1}{2})u\|_{r^\alpha H^m_{\mathrm{b}}}$$

for $\alpha > 3/2$ and in particular for $\alpha = 1+\nu$. Applying this to $B(t,\theta,r)$ and discarding the first term shows that

$$\|B(r,t,\theta)\|_{r^{1+\nu}H^{\mathrm{m}}_{\mathrm{b}}} \leqslant C\|(r\partial_r - \tfrac{1}{2})B\|_{r^{1+\nu}H^m_{\mathrm{b}}} = C\|(r\partial_r - \tfrac{1}{2})\varphi\|_{r^{1+\nu}H^m_{\mathrm{b}}} \leqslant C\|\varphi\|_{r\mathcal{H}^{m+1,1}} \tag{B.6}$$

since $(r\partial_r - \frac{1}{2})$ annihilates $A(t,\theta)r^{1/2}$. Then, applying Lemma B.2 to $B(t,\theta,r)r^{-1/2}$ shows that

$$|B(t,\theta,r)r^{-1/2}| \leqslant \|B(r,t,\theta)\|_{r^{3/2}H^m_{\mathrm{b}}} \leqslant \|B(r,t,\theta)\|_{r^{1+\nu}H^{\mathrm{m}}_{\mathrm{b}}} \leqslant C\|\varphi\|_{r\mathcal{H}^{m+1,1}}$$

and the result for $B(t,\theta,r)$ follows after multiplying by $r^{1/2}$.

For the first term, the triangle inequality and (B.6) shows that

$$\|A(t,\theta)r^{1/2}\|_{rH^m_{\mathrm{b}}} = \|A(t,\theta)r^{1/2} + B\|_{rH^m_{\mathrm{b}}} - \|B(r,t,\theta)\|_{rH^{\mathrm{m}}_{\mathrm{b}}} \leqslant C\|\varphi\|_{r\mathcal{H}^{m+2,1}}.$$

Finally, since $\|A(t,\theta)r^{1/2}\|_{rH^m_{\mathrm{b}}} \sim \|A(t,\theta)\|_{L^{m,2}(T^2)}$, the bound for the first term follows from the Sobolev embedding on T^2 after increasing $m+2$ to $m+4$. □

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