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ABSTRACT. In this paper, we prove and interpret the antitone Galois connection induced by satisfaction between sets of sentences and classes of structures in first-order logic.

Let $\mathcal{L}$ be a first-order language, and $\mathcal{S}$ be a set of $\mathcal{L}$-sentences, and $\mathcal{M}$ be a class of $\mathcal{L}$-structures.

Definition 1 (Subject of Set of Sentences)**.** Let $\Phi \subseteq \mathcal{S}$.

$$\mathrm{subj}(\Phi) = \{m \in \mathcal{M} \mid \forall \varphi \in \Phi (m \vDash \varphi)\}$$

Definition 2 (Theory of Class of Structures)**.** Let $M \subseteq \mathcal{M}$.

$$\mathrm{th}(M) = \{\varphi \in \mathcal{S} \mid \forall m \in M (m \vDash \varphi)\}$$

Theorem 1. Let $\Phi \subseteq \mathcal{S}$, and $M \subseteq \mathcal{M}$.

$$\Phi \subseteq \mathrm{th}(M) \iff M \subseteq \mathrm{subj}(\Phi)$$

Proof.

$$\Phi \subseteq \mathrm{th}(M) \iff \forall \varphi \in \Phi, \forall m \in M (m \vDash \varphi) \iff M \subseteq \mathrm{subj}(\Phi)$$

□

The syntax–semantics correspondence reverses inclusion: more axioms yield fewer models. This is a precise sense in which more constraints yield fewer possible worlds.

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