Problem 1

a) Going from: $f(x+\delta) = f(x) + \delta f_T'(x) + \frac{\delta^2}{2!} f''(x) + \frac{\delta^3}{3!} f^{(3)}(x) + \cdots$

we get, by substituting δ by $-\delta$, 2δ, -2δ, and rearranging:

① (by δ): $f_T'(x) = \frac{1}{\delta}\left(f(x+\delta) - f(x) - \frac{\delta^2}{2!} f''(x) - \frac{\delta^3}{3!} f^{(3)}(x) - \frac{\delta^4}{4!} f^{(4)}(x) - \frac{\delta^5}{5!} f^{(5)}(x) + \cdots \right)$

② (by $-\delta$): $f_T'(x) = \frac{1}{\delta}\left(f(x) - f(x-\delta) + \frac{\delta^2}{2!} f''(x) - \frac{\delta^3}{3!} f^{(3)}(x) + \frac{\delta^4}{4!} f^{(4)}(x) + \frac{\delta^5}{5!} f^{(5)}(x) + \cdots \right)$

③ (by $+2\delta$): $f_T'(x) = \frac{1}{2\delta}\left(f(x+2\delta) - f(x) - \frac{4\delta^2}{2!} f''(x) - \frac{8\delta^3}{3!} f^{(3)}(x) - \frac{16\delta^4}{4!} f^{(4)}(x) + \cdots \right)$

④ (by -2δ): $f_T'(x) = \frac{1}{2\delta}\left(f(x) - f(x-2\delta) + \frac{4\delta^2}{2!} f''(x) - \frac{8\delta^3}{3!} f^{(3)}(x) + \frac{16\delta^4}{4!} f^{(4)}(x) + \cdots \right)$

$\frac{1}{2}$(① + ②) $\Rightarrow$ $f_T'(x) = \frac{1}{\delta}\left(\frac{f(x+\delta) - f(x-\delta)}{2} - \sum_{n=1}^{+\infty} \frac{\delta^{(2n+1)}}{(2n+1)} f^{(2n+1)}(x) \right)$ ⊛

$\frac{1}{2}$(③ + ④) $\Rightarrow$ $f_T'(x) = \frac{1}{\delta}\left(\frac{f(x+2\delta) - f(x-2\delta)}{4} - \sum_{n=1}^{+\infty} \frac{2^n \delta^{(2n+1)}}{(2n+1)} f^{(2n+1)}(x) \right)$ (#)

2(⊛) - (#) $\Rightarrow$ $f_T'(x) = \frac{1}{\delta}\left(f(x+\delta) - f(x-\delta) + \frac{(-f(x+2\delta) + f(x-2\delta))}{4} \right) + \underbrace{\left(\frac{2\delta^4}{5!} f^{(5)}(x) + \cdots \right)}_{\varepsilon \equiv \text{error term} \propto \delta^5}$

$$\Rightarrow \quad f_T'(x) = \frac{1}{\delta}\left(f(x+\delta) - f(x-\delta) + \frac{(-f(x+2\delta) + f(x-2\delta))}{4} \right) + \varepsilon$$

b) The smallest possible division on a computer is on the order of ε ($\approx 10^{-16}$). So, in practice:

$$\mathcal{E} = \frac{2\delta^4}{5!}\overset{\text{biggest}}{\swarrow}f^{(5)}(x) + \cdots + \overset{\text{smallest}}{\downarrow}\frac{G\xi f}{\delta}$$

If we ignore the terms in between and optimise to find the ~~smallest~~ value of δ that minimises E:

$$\frac{\partial \mathcal{E}}{\partial \delta} = \frac{8\delta^3}{5!} f^{(5)}(x) \cancel{+} - \frac{G\xi f}{\delta^2} = 0$$

$$\Rightarrow \delta^5 = \frac{5!\, G\xi f(x)}{8\, f^{(5)}(x)} \Rightarrow \delta = \sqrt[5]{\frac{5!\, G\xi f(x)}{8\, f^{(5)}(x)}}$$

Which gives me $\delta \simeq 10^{-3}$ for $\exp(x)$
$\delta \simeq 10^{-1}$ for $\exp(0{,}01x)$ } both are very different from the calculated value ☹