Problem 1

We have: $\dfrac{f(t+dt,x)-f(t-dt,x)}{2dt} = -v\dfrac{f(t,x+dx)-f(t,x-dx)}{2dx}$

For $\alpha \equiv \dfrac{v}{dx}dt$, we have:

$$f(t+dt,x) = f(t-dt,x) - \alpha\left(f(t,x+dx) - f(t,x-dx)\right)$$

Since $f(t,x) = \mathcal{E}^t \exp(ikx)$, we have:

$$\mathcal{E}^{dt}\mathcal{E}^t \exp(ikx) = \mathcal{E}^t \exp(ikx)\left(\mathcal{E}^{-dt} - 2i\alpha\sin(dx\,k)\right)$$

$$\Rightarrow \quad \left(\mathcal{E}^{dt}\right)^2 + 2i\alpha\sin(k\,dx) - 1 = 0$$

Solving for $\mathcal{E}^{dt}$, we get: $\mathcal{E}^{dt} = -i\alpha\sin(k\,dx) \pm \sqrt{1-\alpha^2\sin^2(k\,dx)}$

We have stability only if $|\mathcal{E}^{dt}| \leq 1$.

For $|\alpha| < 1$ (CFL condition), we get

$$|\mathcal{E}^{dt}| = \left(\left(\mathcal{E}^{dt}\right)\cdot\left(\mathcal{E}^{dt}\right)^*\right)^{1/2} = \left(1-\alpha^2\sin^2(k\,dx) + \alpha^2\sin^2(k\,dx)\right)^{1/2}$$
$$= 1$$

When the CFL condition is satisfied, the leapfrog method is stable