Problem set 4

1) If $f(t)$ is the function to be shifted by α on the x axis, we can do it with a convolution with $\delta(t-\alpha)$.

Proof: $h(t) \equiv f(t) * \delta(t-\alpha)$

$$h(t) = \int f(-t'+t)\,\delta(t'-\alpha)\,dt'$$

$$h(t) = \sum_{t'} \delta(t'-\alpha)\,f(t-t')$$

As seen in class, $\delta(t'-\alpha) = \text{IFT}(F'(k))$ with $F'(k) = \exp\left(\frac{-i2\pi\alpha k}{N}\right)$

$$\Rightarrow \delta(t'-\alpha) = \frac{1}{N}\sum_k \exp\left(i\frac{2\pi(t'-\alpha)k}{N}\right)$$

As seen in class, $f(t-t') = \frac{1}{N}\sum_{k'} \overline{F(k')}\exp\left(+i\frac{2\pi k't'}{N}\right)\exp\left(-i\frac{2\pi k't}{N}\right)$

with ~~$F(k') = FFT(f(t-t'))$~~

$F(k') = FFT(f(t))$

$$\Rightarrow h(t) = \frac{1}{N^2}\sum_{t'}\sum_{k'}\sum_k \overline{F(k')}\exp\left(+i\frac{2\pi k(t'-\alpha)}{N}\right)\exp\left(-i\frac{2\pi k't}{N}\right)\exp\left(i\frac{2\pi k't'}{N}\right)$$

$$= \frac{1}{N^2}\sum_{k'}\sum_k \overline{F(k')}\exp\left(-i\frac{2\pi k\alpha}{N}\right)\exp\left(-i\frac{2\pi k't}{N}\right)\underbrace{\sum_{t'}\exp\left(i\frac{2\pi(k'+k)t'}{N}\right)}_{N\delta(k'+k)}$$

$$\Rightarrow h(t) = \frac{1}{N^2}\cdot N\sum_k \overline{F(-k)}\exp\left(+i2\pi\frac{k(t-\alpha)}{N}\right)$$

t-shift

$$\Rightarrow h(t) = \frac{1}{N}\sum_k F(k)\exp\left(i\frac{2\pi k(t-\alpha)}{N}\right) = \frac{1}{N}\sum_k F(k)F'(k)\exp\left(i\frac{2\pi kt}{N}\right)$$

$$\Rightarrow h(t) = \text{IFT}(F(k)F'(k)) = f(t) * \delta(t-\alpha) = f(t-\alpha)$$

5) a) $$\sum_{n=0}^{N-1}\exp\left(-\frac{2\pi i k n}{N}\right) = \sum_{n=0}^{N-1}\left[\exp\left(-2\pi\frac{ik}{N}\right)\right]^n \equiv S$$

$$S\exp\left(-\frac{i2\pi k}{N}\right) = \sum_{n=0}^{N-1}\left[\exp\left(-\frac{2\pi ik}{N}\right)\right]^{n+1} \equiv R$$

$$S - R = \sum_{n=0}^{N-1} \left[\exp\left(-i2\pi \frac{k}{N}\right)\right]^n - \exp\left(-i2\pi \frac{k}{N}\right)^{n+1}$$

$$S - R = \exp\left(-\frac{i2\pi k}{N}\right)^0 - \exp\left(\frac{-i2\pi k}{N}\right)^{(N-1)+1}$$

$$\Rightarrow S\left(1 - \exp-\left(\frac{i2\pi k}{N}\right)\right) = 1 - \exp\left(-i2\pi \frac{kN}{N}\right)$$

$$\Rightarrow \boxed{S = \frac{1 - \exp(-i2\pi k)}{1 - \exp\left(\frac{-i2\pi k}{N}\right)}}$$

b) * If $k \in \mathbb{Z}^*$; $\exp(-i2\pi k) = 0$ ①

If $k \in \mathbb{Z}^*$ and $k \neq mN$; with $m \in \mathbb{Z}$; $\exp\left(\frac{-i2\pi k}{N}\right) \neq 1$ ②

① & ② $\Rightarrow \boxed{S = 0}$ for $k \in \mathbb{Z}^*$ and $k \neq mN$.

* $\lim\limits_{k \to 0} S = \frac{0}{0}$. We use the Hôpital rule:

$$\lim_{k \to 0} \frac{[1 - \exp(-i2\pi k)]'}{\left[1 - \exp\left(\frac{-i2\pi k}{N}\right)\right]'} = \lim_{k \to 0} \frac{i2\pi \exp(-i2\pi k)}{i\frac{2\pi}{N} \exp\left(\frac{-i2\pi k}{N}\right)}$$

$$\lim_{k \to 0} S = \lim_{k \to 0} N \frac{\exp(-i2\pi k)}{\exp\left(-i2\pi \frac{k}{N}\right)} = \boxed{N = \lim_{k \to 0} S}$$

c) If we have a sine ~~wave~~ of the form $\sin(2\pi \omega x) = f(x)$

Then:

$$DFT(f(x)) = \sum_{x=0}^{N-1} \sin(2\pi\omega x) \exp\left(-i2\pi \frac{kx}{N}\right)$$

$$= \frac{1}{2i} \sum_{x=0}^{N-1} \left[\exp(i2\pi\omega x) - \exp(-i2\pi\omega x)\right] \exp\left(-i2\pi \frac{kx}{N}\right)$$

$$DFT(f(x)) = \frac{1}{2i}\left(\sum_{x=0}^{N-1} \exp\left(\frac{-i2\pi x(k - \omega N)}{N}\right) - \sum_{x=0}^{N-1} \exp\left(\frac{-i2\pi x(k + N\omega)}{N}\right)\right)$$

Using 5b) we have: $DFT(f(x)) = \frac{1}{2i}\left(N\delta(k - \omega N) - N\delta(k + N\omega)\right)$

e) $FFT\left(0,5-0,5\cos\left(\frac{2\pi x}{N}\right)\right) = \sum_{x=0}^{N-1}\frac{1}{2}\exp\left(-i\frac{2\pi kx}{N}\right) - \sum_{x=0}^{N-1}\frac{1}{2}\cos\left(\frac{2\pi x}{N}\right)\exp\left(-i\frac{2\pi kx}{N}\right)$

$$= \frac{1}{2}\sum\exp\left(-i\frac{2\pi kx}{N}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\sum\left[\exp\left(i\frac{2\pi x}{N}\right)+\exp\left(-i\frac{2\pi x}{N}\right)\right]\exp\left(-i\frac{2\pi kx}{N}\right)$$

$$= \frac{1}{2}\sum\exp\left(-i\frac{2\pi kx}{N}\right) - \frac{1}{4}\sum\exp\left(\frac{-i2\pi x(k-1)}{N}\right) - \frac{1}{4}\sum\exp\left(\frac{-i2\pi x(k+1)}{N}\right)$$

$$= \left[\frac{1}{2}\delta(k) - \frac{1}{4}\delta(k-1) - \frac{1}{4}\delta(k+1)\right]N$$

$$FFT(\text{window}) = \begin{array}{cccccc} \Big[\frac{N}{2} & -\frac{1}{4}N & 0\ 0 & \dots & 0\ 0 & -\frac{1}{4}N\Big] \\ k=0 & k=1 & & & & k=-1 \end{array}$$

A product in real space is a convolution in Fourier space. So we have:

$$\int FFT\left(0,5-0,5\cos\left(\frac{2\pi x}{N}\right)\right) FFT\left(\sin(2\pi\omega x)\right) d\lambda$$

$$= N^2\int\left(\frac{1}{2}\delta(k-\lambda) - \frac{1}{4}\delta(k-\lambda-1) - \frac{1}{4}\delta(k-\lambda+1)\right)\left(\frac{1}{2i}\left(\delta(k-N\omega)-\delta(k+N\omega)\right)\right)d\lambda$$

$$= \frac{N^2}{4i}\int\Big[\delta(k-\lambda)\delta(k-N\omega) - \delta(k-\lambda)\delta(k+N\omega) - \frac{1}{2}\delta(k-\lambda-1)\delta(k-N\omega)$$

$$+\frac{1}{2}\delta(k-\lambda-1)\delta(k+N\omega) - \frac{1}{2}\delta(k-\lambda+1)\delta(k-N\omega) + \frac{1}{2}\delta(k-\lambda+1)\delta(k-N\omega)\Big]d\lambda$$

$$= \frac{N^2}{4i}\Big[\delta(k-N\omega) - \delta(k+N\omega) - \frac{1}{2}\left(\delta(k-1-N\omega)\right) + \frac{1}{2}\delta(k-1+N\omega)$$

$$- \frac{1}{2}\delta(k+1-N\omega) + \frac{1}{2}\delta(k+1+N\omega)\Big] = \text{windowed F.T}$$

As we can see that each point and his immediate neighbours contribute, as expected. (~~The coefficient also change, even at the central points, indicating that a window~~)

3) The correlation goes from 100% when the shift is 0 to infinitesimal

values when the shift is big enough that there's no more overlap. It grows back to 100%, indicating that the DFT has "wrapped around" for a shift as wide as my interval of integration.