Hydrodynamic Instabilities in Active Smectic and Columnar Phases– Final Report

1703762

Department of Physics, University of Warwick, Coventry CV4 7AL, United Kingdom

(Dated: 5th March 2022)

The general hydrodynamic equations using non-equilibrium physics are derived for the smectic A and, through analogy, columnar chiral active systems. Stability analysis is then performed to compare with the Helfrich-Hurault effect and reaffirming what previous studies showed for the smectic system. Chiral stresses lead to a leading order correction for the stability of the columnar phase and box confinement leads to some resemblance of Helfric-Hurault effect in columnar phase.

1. Introduction

The Phases of Liquid Crystals

Isotropic liquids and crystalline solids are two states of matter that are well understood. There is in fact phases of matter known as liquid crystals that have liquid like behaviour in strictly less than, out of, three dimensions but also crystalline solid behaviour in the remainder. These systems can be distinguished by their optical properties[1]. To explore this, consider an isotropic liquid of rod like particles known as mesogens as illustrated in figure 1a. There is no order, or correlation, for the relative positions and orientations of the mesogens in the isotropic liquid phase. Formally, this means the density-density correlation function has no direction dependence between molecules (isotropic) and no relative distance dependence (a liquid) except fundamentally based on the density of the system.[1]

Liquid crystals on the contrary must have some direction dependence (anisotropy). The main mesomorphic phases (mesophases) are the nematic (figure 1b), smectic (figure 2) and columnar phases (figure 4). The distinction is based on the number of liquid-like dimensions. As with mesogens, liquid crystals can be composed of discotic (disc-like) particles as seen for the columnar phase (and some nematic) as in figure 4.[1]

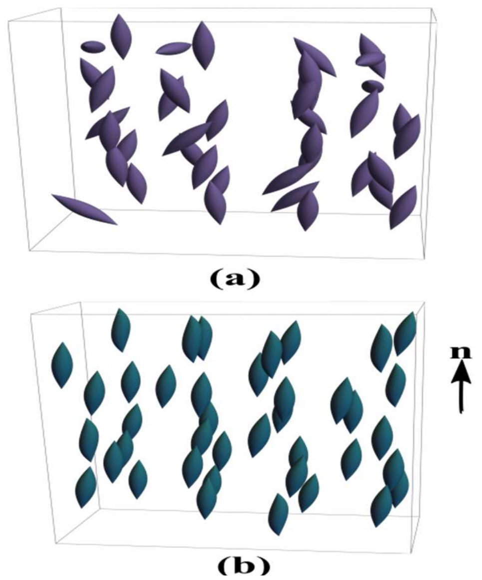

Figure 1 An isotropic liquid (a) of rod-like molecules with random positions and orientations vs a nematic liquid crystal (b) with again random positions but a preferred orientation, on average, n.

Liquid crystal systems are found in both nature and technological applications. Phase transitions between the liquid-crystal states can occur when varying the temperature, thermotropic, of the system for typically inorganic molecules or the concentration, lyotropic, which tend to occur for organic molecules[4]. Liquid crystals are most notably used in LCD displays and there are lyotropic biological systems such as fibrous proteins and amino acids and even arises in nanotechnology[X]. Liquid crystals are a recent and certainly not exhaustively studied field in physics with vast applications in and out of biology.

A conventional liquid can be described entirely by the velocity field. This still holds for liquid crystal systems but in addition, information needs to be captured on the orientation of the molecules[1]. This is achieved with the average unit orientation of the molecules, also known as the director $\boldsymbol{n}$. In addition, most liquid crystals exhibit a head-tail symmetry meaning a reflection of the system orthogonal to the director is identical ($\boldsymbol{n} \equiv -\boldsymbol{n}$). Consider again an isotropic liquid, as in figure 1a, but now impose that the mesogens have a uniformly preferred orientation and head-tail symmetry as in figure 1b. This describes the nematic phase liquid crystal.[1]

Suppose the system described in figure 1b now has positional order imposed in one dimension, this is modelled as a set of two-dimensional liquid layers stacked on each other with well-defined layer spacing and with a director normal to the layers (figure 2a). This is the smectic A mesophase. Formally, a density wave ρ can be used to characterise these layers. A phase field ϕ (away from a phase transition) such that $\rho = \rho_0 + \mathcal{R}e(\psi_0 e^{iq\phi})$ can be used as the order

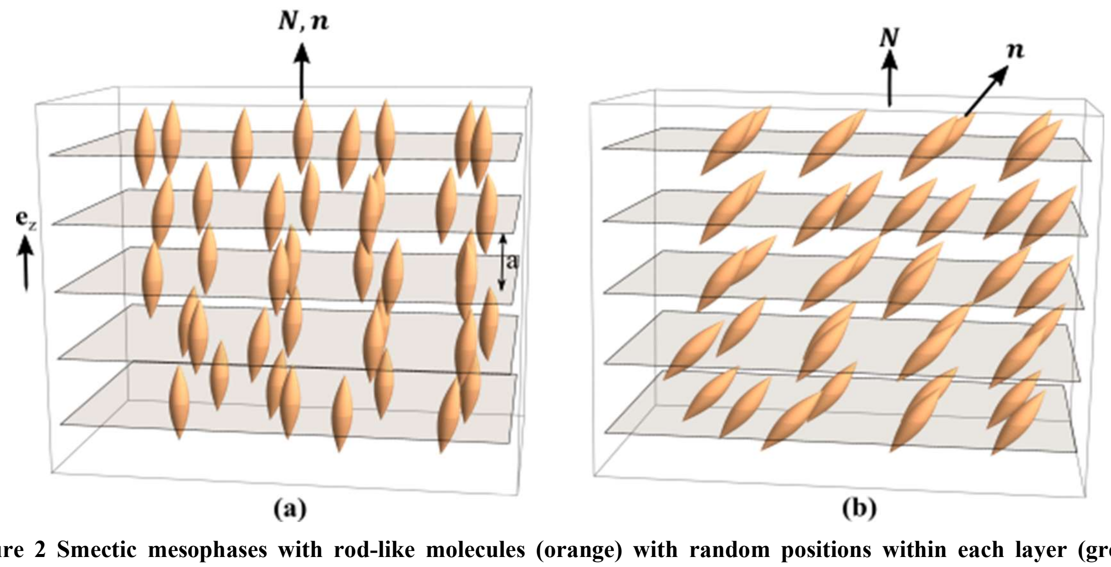

Figure 2 Smectic mesophases with rod-like molecules (orange) with random positions within each layer (grey). The director n and normal to the layers N are given for a smectic A (a) and smectic C (b) respectively with layer spacing a.

parameter for the system[5].

For ease, consider the Smectic-A ground state where $\phi = z$ and $\mathbf{n} = \nabla\phi = \mathbf{e}_z$. There are also smectic phases where the director makes a (non-zero) angle with the layer normal, which are called smectic C (figure 2b).

To study the properties of the smectic system, non-equilibrium physics is considered by perturbing the system about the ground state (figure 3) resulting in

$$\phi = z - u, \tag{1}$$

where u is the Eulerian displacement field. The elastic free energy for the deformations of the smectic can be written as

$$F = \int \frac{B}{8}(1 - |\nabla\phi|^2)^2 + \frac{K}{2}(\nabla.\boldsymbol{N})^2\, dV \tag{2}$$

where B and K are bulk, or compression, and bending moduli respectively, $\boldsymbol{N} = \frac{\nabla\phi}{|\nabla\phi|}$ is the layer normal[6]. The two terms correspond to a compression energy and bending energy. They penalise the deviations in the layer spacing and layer curvature respectively. There are different choices for the free energy, but all agree in the linear theory. [8]

The free energy can also be described by

$$\mathrm{F} = \int \frac{\mathrm{B}}{2}(\partial_z \mathrm{u})^2 + \frac{\mathrm{K}}{2}(\nabla_\perp^2 \mathrm{u})^2\, \mathrm{dV} \tag{3}$$

for small displacements up to quadratic order in u. Notice only gradients in u appear in the free energy. This is reasonable because the displacement of the layers does not directly contribute to the free energy. There are also no linear gradient terms in u since they correspond to pure rotations of the system and hence do not contribute to the free energy[1]. This free energy can be derived directly by substituting $\phi = z - u$ into (2). The free energy for the system is fundamental in describing the elastic forces due the solid crystal directions in the smectic as well as well as describing the evolution of the smectic system[9].

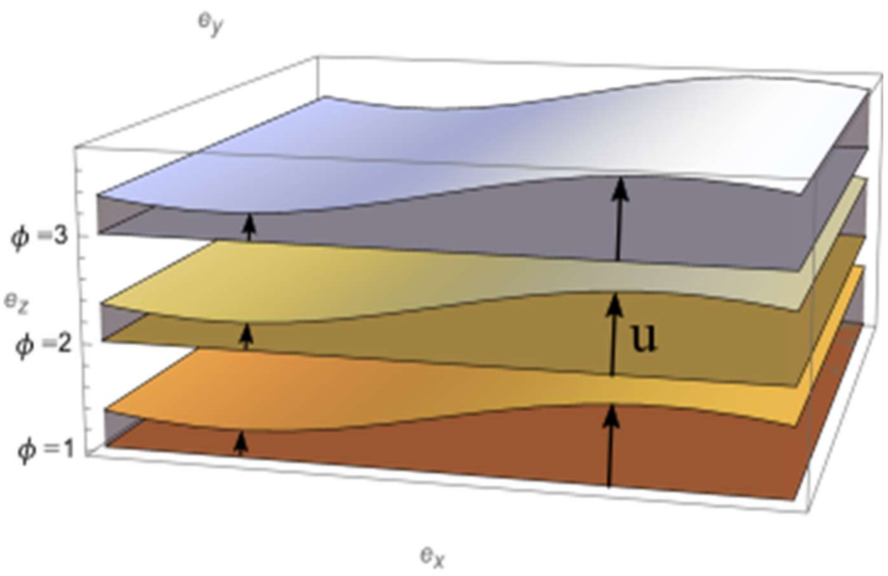

Figure 3 One dimensional displacement variable u for a Smectic A liquid crystal and phase field ϕ.

Consider a system of discotic, not rod-like, particles that are arranged into tubes as in figure 4. This is a system with fluid-like behaviour along the columns but crystal-like perpendicularly in two dimensions. This is a columnar phase liquid crystal with a director parallel to the column axes, as with the other phases there is once again head-tail symmetry. There are many examples of columnar phases that arise in biology and more recently being studied such as imogolite nanotubes[10].

In addition to conventional liquid crystals, we have chiral systems, containing chiral (differing from its mirror image) particles. The most studied in literature is the cholesteric.

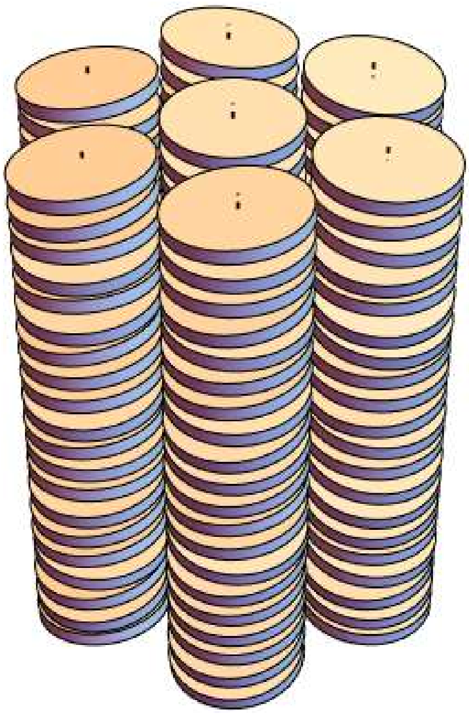

Figure 4 Hexagonal discotic columnar phase liquid crystal modelled as vertical tubes in a hexagonal two-dimensional crystalline structure.

A cholesteric is a chiral (nematic) liquid crystal phase, often made by doping ordinary nematics with chiral molecules[1]. This causes the structure to undergo helical distortion as shown in fig. 5. On a local scale the cholesteric is identical to the nematic phase with few physical implications[1]. The director for a cholesteric with helical axis parallel to $\hat{\boldsymbol{z}}$ is described by

$$\hat{\boldsymbol{n}} = (\cos(q_o z + \phi), \sin(q_o z + \phi), 0) \quad (1)$$

where q_o distinguishes between left and right handedness, arbitrary phase ϕ and a pitch as shown in figure 5. The pitch is usually called P and is equal to $\frac{2\pi}{q_0}$ [4]; due the head-tail symmetry, half the pitch describes the entire system. Notice the similarities between the smectic (figure 2a) and the cholesteric liquid crystal where the cholesteric is treated as 'layers' with the director parallel rather than orthogonal to them. The smectic and columnar phases can also be doped with chiral molecules, but these have been less covered in literature.

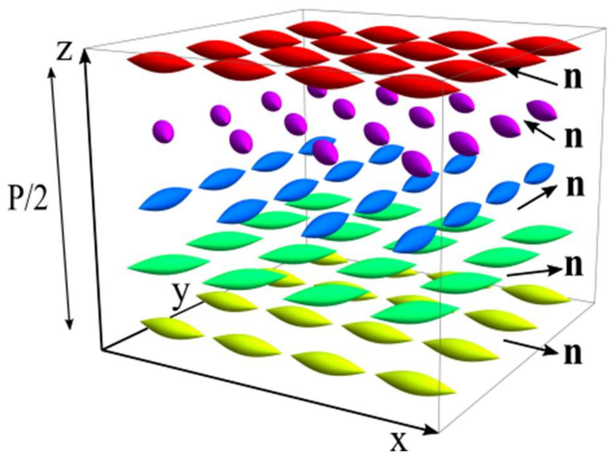

Figure 5 A nematic cholesteric with rod-like molecules and director n as described in (1) with Pitch P. The layers illustrate the alignment of the director but there is no positional order.

Active Materials

Active systems describe any system which are outside of thermal equilibrium due to energy being inputted from or to individual particles[11]. They can be artificial or natural. Behaviour of flocks of birds and schools of fish can be described this way[12]. Considerations are not only to the macroscopic but also microscopic length scales. From bacterial colonies, self-organising biopolymers to self-propelled particles[12] active matter physics plays an essential role. There has been a significant amount of research in active nematic systems in recent years but not considerably as much into active smectic and even less so into columnar systems and hence is the primary focus of this report.

Orientational order arises in many instances of active matter, for example in a flock of myxobacteria[4] as can be seen in fig 6. Nonequilibrium statistical mechanics provides a helpful theoretical framework for studying properties of active matter on the macroscopic length scale. More explicitly, generalised hydrodynamic equations are to be used to account for the active matter by perturbing the liquid crystal systems about well-defined states, for the smectic A and columnar phases. [13]

The Helfrich-Hurault effect[X] has been analysed in the smectic as well as columnar phase liquid crystals. It would be interesting to compare how the active system compares with the Helfrich-Hurault effect, primarily for the columnar phase. The effect in smectics has been studied previously as in [X] and the effect turns out to be analogous to the active system. The intent of this report is then to see how in comparison, how the Helfrich-Hurault differs in the columnar phase active system.

The chirality of the active system is a far more recent concept. Again, chiral nematics have been studied significantly and leading to the well-known helical distortion yet how the chirality affects the evolution of the system for less studied phases including the smectic and columnar are of more interest.

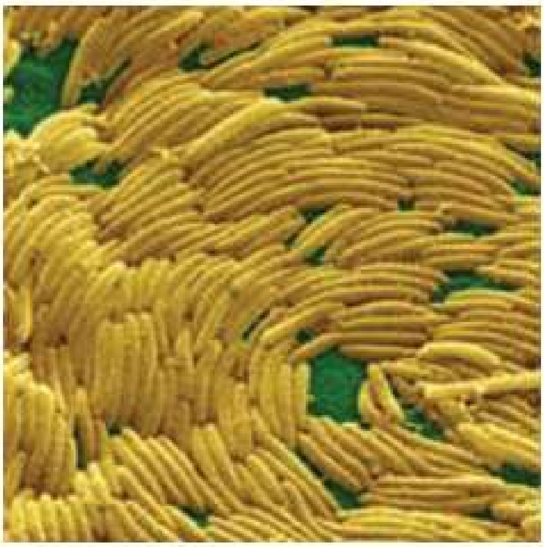

Figure 6 (colour). Liquid-Crystalline order in a myxobacterial flock. Reproduced from [4].

2. Hydrodynamics of Active Liquid Crystals

Conservation and Entropy Production

The use of generalised hydrodynamic theories has aided significantly in describing simple and complex fluid systems with equations expanded about well-understood states. On the contrary, many active physical systems do exist far away from an equilibrium state so that are some limitations[13]. To study the active systems, generalised hydrodynamic equations are derived, much like that for an isotropic fluid. The primary idea is to consider the forces and flux due to the entropy production rate $\dot{S}$ as performed by de Groot and Mazur, 1984[13]. Further to this, the chirality of the system then needs to be considered. The active system is treated as having a uniform temperature (contact with a heat bath with finite temperature T). The resulting entropy production rate is then given by

$$T\dot{S} = -\frac{dF}{dt} \tag{4}$$

where F is the free energy of the active system. Energy is locally injected into system constantly because of the active nature; hence it must be accounted for in the entropy production rate.[13]

In addition to the entropy production, the active system has two conserved quantities, the fluid mass and momentum. The fluid mass conservation reads

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{5}$$

where ρ and $\mathbf{v}$ are the density and Eulerian velocity of the fluid respectively. The mass conservation law (X) simplifies to the incompressibility condition

$$\nabla \cdot \mathbf{v} = 0 \tag{6}$$

by assuming a uniform density. Assuming (4), The pressure p enforces incompressibility and hence an additional thermodynamic equation of state isn't required, analogous to an incompressible fluid. For compressible systems such as when considering acoustic waves, an equation of state is necessary.

Newton's second law applied to a fluid element (momentum conservation) gives

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p + \nabla \cdot \sigma \tag{7}$$

where p is the pressure of the system and σ is the stress tensor[13]. $\frac{D}{Dt}$ is the total derivative.

Formally σ can be obtained by considering the entropy production rate (4) with the conservation laws. For brevity, this step is left out but resulting in

$$\nabla \cdot \sigma = \mu \nabla^2 \mathbf{v} + f_{el} + f_{act} + f_{chir} \tag{8}$$

where f_{act} and f_{chir} are the active and chiral active force terms due to the chirality and activity of the system. The term f_{el} corresponds to an elastic restoration force which comes from the Hookean nature of the mesogens in the solid directions within the liquid-crystal[9].

The resulting equations from (7) and (8) are analogous to the incompressible Navier-Stokes equation for isotropic fluids[14] but with additional elastic and (chiral) active terms.

Active Stresses

To describe the active force, we must use a vector. The director $\boldsymbol{n}$ isn't suitable due to the head-tail symmetry ($\mathbf{n} = -\mathbf{n}$). This active force represents the stress caused by the local force dipoles. The most appropriate form the active term is the divergence of $\boldsymbol{nn}$, also known as the Q-tensor[15].

$$-\xi \nabla \cdot (\boldsymbol{nn}) \tag{9}$$

where ξ is some phenomenological quantity. To account for the chiral active force, they require the same properties as the active force but also account for the handedness. The ideal choice for the chiral active force term is then

$$-\xi_c \nabla \times \nabla \cdot (\boldsymbol{nn}) \tag{10}$$

with a constant ξ_c representing the degree of chirality. Hence, the governing equations describing the evolution of the chiral active liquid crystal system is

$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + f_{el} - \xi \nabla \cdot (\mathbf{nn}) - \xi_c \nabla \times \nabla \cdot (\mathbf{nn}), \tag{11a}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{11b}$$

[15] and an equation that describes the 'solid structure'. This equation is heavily dependent on the liquid crystal in consideration.

3. Active Cholesterics

Derivation of Active Hydrodynamics

Considering explicitly the case of a smectic A liquid crystal with phase field ϕ and director $\boldsymbol{n} = \frac{\nabla\phi}{|\nabla\phi|}$ as described in figure 3, the equations governing the structure of the Smectic liquid crystal as well as the elastic force term in the equation of motion (11a) can be derived[14]. The phase field evolves according to advection with the fluid flow and the relaxation to equilibrium. This reads as

$$D_t\phi \equiv \partial_t\phi + (\boldsymbol{v}\cdot\nabla)\phi = -\frac{1}{\gamma}\frac{\delta F}{\delta\phi} \quad (12)$$

where $\frac{\delta F}{\delta\phi}$ is the functional derivative of the free energy of the Smectic and γ is a relaxation coefficient. The elastic force is given by

$$f_{el} = \nabla\phi\frac{\delta F}{\delta\phi}, \quad (13)$$

resulting in the smectic hydrodynamic equations

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{v} + \nabla\phi\frac{\delta F}{\delta\phi} - \xi\nabla\cdot(\mathbf{nn}) - \xi_c\nabla\times\nabla\cdot(\mathbf{nn}), \quad (14a)$$

$$\boldsymbol{n} = \frac{\nabla\phi}{|\nabla\phi|}, \quad (14b)$$

$$D_t\phi \equiv \partial_t\phi + (\boldsymbol{v}\cdot\nabla)\phi = -\frac{1}{\gamma}\frac{\delta F}{\delta\phi}. \quad (14c)$$

Linear Instability

The fundamental hydrodynamic instability can be studied for the active system by considering small perturbations in the displacement field u. This equates to reducing the system of equations (14) by ignoring non-linear terms in u and $\boldsymbol{v}$. First consider the equation of the structure

$$D_t\phi \equiv \partial_t\phi + (\boldsymbol{v}\cdot\nabla)\phi = -\frac{1}{\gamma}\frac{\delta F}{\delta\phi} \quad (15)$$

and recall $\phi = z - u$. The convective term $(v.\nabla)\phi$ simply reduces to v_z. To keep linear order terms in the functional derivative $\frac{\delta F}{\delta\phi}$, the free energy F is reduced to an equation up to quadratic order in u. The functional derivative is then derived using standard functional techniques and ignoring the boundary terms due to strong anchoring conditions[1].

$$-\frac{\delta F}{\delta\phi} = \frac{\delta F}{\delta u} = K\nabla_\perp^4 u - B\,\partial_z^2 u \quad (16)$$

Analogous simplifications can be made for the force equation (14a). First note that the director becomes

$$\boldsymbol{n} = \boldsymbol{e}_z - \nabla u, \quad (17)$$

hence, the active and chiral active forces,

$$\nabla\cdot(\mathbf{nn}) = -(\nabla^2 u)\mathbf{e_z} - \nabla(\partial_z u) \quad (18)$$

$$\nabla\times\nabla\cdot(\mathbf{nn}) = -\nabla\times(\nabla^2 u\cdot\mathbf{e_z}) \quad (19)$$

since $\nabla\times(\nabla s) = 0\ \forall s \in \mathbb{R}$.

Note that the chiral force is always perpendicular to the perturbation $u\boldsymbol{e}_z$. Furthermore, the elastic force $\nabla\phi \frac{\delta F}{\delta\phi}$ becomes $-\left(\frac{\delta F}{\delta u}\right)\mathbf{e}_z$. This leads to the linear hydrodynamic system of equations

$$\partial_t u = v_z - \frac{1}{\gamma}(K\nabla_\perp^4 u - B\,\partial_z^2 u) \qquad (20a)$$

$$\begin{aligned}\mu\nabla^2\mathbf{v} = \nabla p + (K\nabla_\perp^4 u - B\,\partial_z^2 u)\mathbf{e}_z \\ - \xi[(\nabla^2 u)\mathbf{e_z} + \nabla(\partial_z u)] \\ - \xi_c[\nabla \times (\nabla^2 u \cdot \mathbf{e_z})]\end{aligned} \qquad (20b)$$

$$\nabla \cdot \mathbf{v} = 0. \qquad (20c)$$

At this stage, the elastic force term in (14a) is ignored to simplify the analysis as it doesn't result in any characteristic features [14]. Again, with incompressibility (20 c), an equation of state isn't necessary to obtain a relation for the pressure p. Using the familiar method from incompressible fluid dynamics, taking the divergence of both sides of the equation of motion (20 b), results in a relation for the pressure in terms of the displacement field u. Many terms simplify using vector identities and (20 c), explicitly, $\nabla \cdot (\nabla p) = \nabla^2 p$, $\nabla \cdot \nabla \times w = 0\ \forall w \in \mathbb{R}^3$ and $\nabla \cdot (\nabla^2 \mathbf{v}) = \nabla^2(\nabla \cdot \mathbf{v}) = 0$ resulting in

$$\nabla^2 p = 2\,\xi\,\partial_z(\nabla^2 u), \qquad (21)$$

$$p = 2\,\xi\,\partial_z u. \qquad (22)$$

Substituting the pressure into the equation (20b) obtains the relation between the displacement and flow fields,

$$\begin{aligned}\nabla^2\mathbf{v} = \frac{\xi}{\mu}[\nabla(\partial_z u) - (\nabla^2 u)\mathbf{e_z}] \\ - \frac{\xi_c}{\mu}\nabla \times (\nabla^2 u)\,\mathbf{e_z}.\end{aligned} \qquad (23)$$

This system can then be solved using Fourier techniques. The spatial Fourier transform with corresponding wavevector $\boldsymbol{k} = (k_x, k_y, k_z)$ in cartesian coordinates and defining $k_\perp^2 = k_x^2 + k_y^2$ gives

$$\begin{aligned}-k^2\tilde{\boldsymbol{v}} = \frac{\xi}{\mu}(-\boldsymbol{k}k_z + k^2\boldsymbol{e}_z)\tilde{u} \\ - \frac{\xi_c}{\mu}(i\boldsymbol{k}) \\ \times (-k^2\boldsymbol{e}_z)\tilde{u},\end{aligned} \qquad (24a)$$

$$\partial_t\tilde{u} = \tilde{v}_z - \frac{1}{\gamma}(Kk_\perp^4 + Bk_z^2)\tilde{u}, \qquad (24b)$$

noting the second term in (24a) is perpendicular to $\boldsymbol{e}_z$. Substituting $\tilde{v}_z$ from (24a) into (24b) gives

$$\partial_t\tilde{u} = \left[\frac{\xi}{\mu}\left(1 - \left(\frac{k_z}{k}\right)^2\right) - \frac{1}{\gamma}(Kk_\perp^4 + Bk_z^2)\right]\tilde{u}. \qquad (25)$$

The differential equation then implies that $\tilde{u}$ and hence u [16] is stable when the coefficient is negative and unstable when positive. Since the choice of wavevector $\boldsymbol{k}$ can be made arbitrarily small(since no restrictions on the size of the system were made), the requirement is that u is stable when the phenomenological constant $\xi < 0$ (contractile) and unstable when $\xi > 0$ (extensile). This is reminiscent of the basic instability in active nematic systems[17].

It differs from the nematic active system by being unstable for only extensile activity. Explicitly, the threshold ξ_{Th} for the instability is zero for the smectic system. The nematic is unstable for both contractile and extensile stresses. The stability due to the activity are analogous to the Helfrich-Hurault instability to be discussed in the following section.

Notice that the chiral active force makes no impact on the stability of the solution regardless of the degree of chirality (ξ_c) nor does it even impact the undulations in the smectic layers u. This result is identical to the active nematic system[17]. The chiral term emerges in the velocity flow field as can be seen in (24a). The contributions to the flow are then always perpendicular to the displacement $u\boldsymbol{e}_z$ and the wavevector $\boldsymbol{k}$ as revealed by (24b).

Helfrich-Hurault Instability

Applying mechanical strain to a smectic system introduces a fundamental instability in the liquid-crystal leading to layer undulations as described in figure 7. This is known as the Helfrich-Hurault effect which was originally studied by applying a magnetic or electric field normally to the layers of a smectic A liquid crystal[18]. This same phenomenon can be studied by undulating two plates containing a Smectic-A liquid crystal as shown in figure 7.

Suppose the liquid crystal has a uniform layer spacing, a, and the two plates are separated by a distance d, there are then $\frac{d}{a}$ layers between the two plates. Undulating the boundary by a small distance δ ($|\delta| \ll d$) along the z-direction, the layer spacing becomes $l = a(1 + \frac{\delta}{d})$ along the z direction but the true spacing is given by $d = \frac{l}{\cos\theta}$ where $\theta \approx |\nabla_{\perp} u|$.

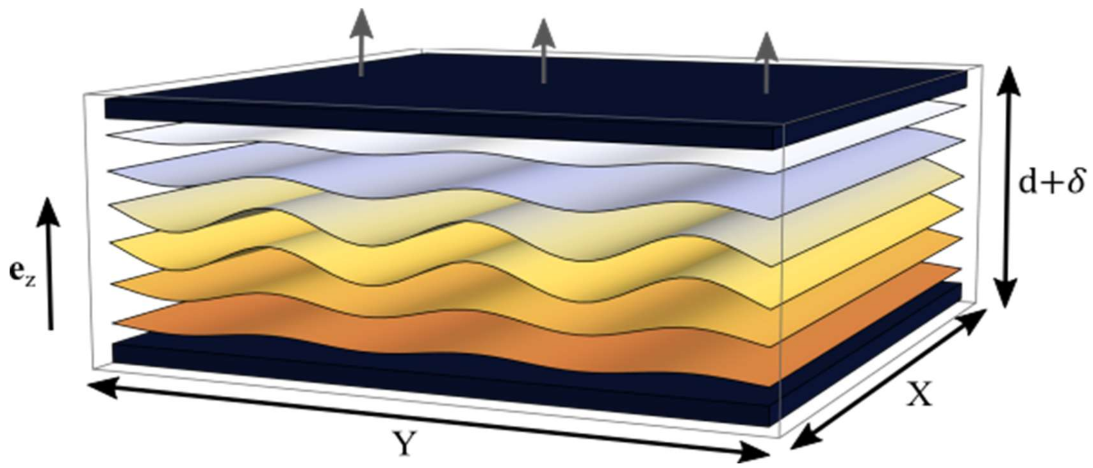

Figure 7 Two plates containing a smectic A liquid crystal, initially separated by a distance d. The Helfrich-Hurault instability caused by perturbing a plate by a distance δ along the z-direction with larger undulations induced towards the centre (in z) of the system.

The resulting phase field ϕ and layer normal $\boldsymbol{N}$ must then be given by

$$\phi = \frac{d}{d+\delta} z - u, \tag{26a}$$

$$\nabla\phi = \left(1 - \frac{\delta}{d} - \partial_z \mathrm{u}\right) \boldsymbol{e}_z - \nabla_\perp u, \tag{26b}$$

$$N = \frac{\nabla\phi}{|\nabla\phi|} = \boldsymbol{e}_z \left(1 - \frac{\delta}{d}\right) - \nabla_\perp u + \cdots. \tag{26c}$$

The free energy F for the smectic system is given by

$$F = \int \frac{B}{8}(1 - |\nabla\phi|^2)^2 + \frac{K}{2}(\nabla.\boldsymbol{N})^2 \, dV. \tag{27}$$

Substituting the expressions (26a), (26b), (26c) into the free energy keeping terms up to quadratic order in u obtains an expression for F in terms of the displacement u and the plate displacement α,

$$F = \int \frac{B}{2}\alpha^2 + \frac{B}{2}(\partial_z u)^2 - B\alpha|\nabla_\perp u|^2 + \frac{K}{2}(\nabla_\perp^2 u)^2 \, dV \tag{28}$$

where $\alpha = \frac{\delta}{d}$.

The smectic system is confined between the two plates, this means that the displacement field u can be expressed in the form

$$u = u_0 \sin\frac{\pi z}{d} \sin qx \tag{29}$$

where u_0 is a constant and q is the wavevector along x. This form is only an approximation since, for convenience, it ignores any y-component, and assumes the wavevector along z to take its the smallest value of $\frac{\pi}{d}$. To do this properly, one could take a Fourier series for u but this is excessive for this level of analysis.

With this choice of u, the free energy integral can then be calculated to give

$$F = \int \frac{B}{2}\alpha^2 + \frac{B}{2}(\partial_z u)^2 - B\alpha|\nabla_\perp u|^2 + \frac{K}{2}(\nabla_\perp^2 u)^2 \, dV \tag{30}$$

with the area of plate A, the free energy is then

$$F = \int \frac{B}{2}\alpha^2 + \frac{B}{2}(\partial_z u)^2 - B\alpha|\nabla_\perp u|^2 + \frac{K}{2}(\nabla_\perp^2 u)^2 \, dV \tag{31}$$

Minimising the free energy with respect to the wave vector q gives a minimum value F_{min} with corresponding wavevector q_{min}.

$$F_{min} = 0 \tag{32}$$

$${q_{min}}^2 = \frac{B\alpha}{2K} \tag{33}$$

By then setting $F_{min} = 0$, the threshold wavevector q_{Th} and undulation α_{Th} are found,

$$\alpha_{Th} = \frac{2K\pi^2}{Ba^3} + \frac{a}{8K^2}, q_{th}^2 = \frac{B\alpha_{th}}{2K} \tag{34}$$

This shows that the mechanical tension behaves as active smectics. There is a fundamental instability in the smectic when applying a mechanical stress to the liquid-crystal by undulating the boundary by a displacement δ. This is analogous to active smectics where the activity ξ is positive, or extensile, causes the system to become unstable. Unlike for the smectics analysis, there is a finite threshold in the wavevector q and boundary undulation δ for which the system becomes unstable. [18]

This is due to the box confinement of the material. Further analysis for the smectic system would show that this threshold activity ξ_{Th} and corresponding wavevector q_{Th} exists by considering the box confinement of u and imposed boundary conditions[18]. For the smectic analysis, the wavevector q was treated as small and taken to zero but, the confinement forces the wavevector to take a minimum value $\frac{\pi}{L}$ where L is the system size in the direction of the wavevector.

Compression Waves

The fluid has so far been treated as incompressible. Treating the system as compressible we can study the properties of sound waves. In essence there are changes to the system of hydrodynamic equations that need to be accounted for.

$$\begin{aligned}\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla \mathrm{p} + \mu\nabla^2\mathbf{v} \\ + \frac{1}{3}\mu\nabla(\nabla\cdot\mathbf{v}) - \frac{\delta \mathrm{F}}{\delta \mathrm{u}} \\ - \xi\nabla\cdot(\mathbf{nn}) \\ - \xi_c\nabla\times\nabla\cdot(\mathbf{nn}),\end{aligned} \tag{35a}$$

$$\partial_t\rho + \nabla.(\rho\mathbf{v}) = 0, \tag{35b}$$

$$D_t\phi \equiv \partial_t\phi + (v.\nabla)\phi = -\frac{1}{\gamma}\frac{\delta \mathrm{F}}{\delta \phi}, \tag{35c}$$

$$p = C.\rho^{\gamma_1}. \tag{35d}$$

The fluid mass conservation equation (35b) is no longer simplified to the incompressibility condition. There is an additional term $\frac{1}{3}\mu\nabla(\nabla\cdot\mathbf{v})$ in (35 a) that must be considered since compressibility means $\nabla.\mathbf{v} \neq 0$. The term $\rho\frac{D\boldsymbol{v}}{Dt}$ can be ignored for low Reynolds number as before. Unlike the previous calculations, one requires an additional equation of state to solve the system. By taking the thermal conductivity to be sufficiently low, then the compressions will be adiabatic [1] then the resulting pressure is of the form (35 d) where γ_1 is the ratio of the specific heat capacities at constant volume and pressure [1]. C is a constant. The structure equation (35 c) and remaining terms in the force equation (35 a) are as before.

As before, to make progress analysing the system of equations, linearise in u and $\boldsymbol{v}$ but now also consider small perturbations in the density ρ to linear order $\rho = \rho_0 + \varrho$. This results in the linear system of equations

$$\rho_0\,\partial_t \mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \frac{1}{3}\mu\nabla(\nabla\cdot\mathbf{v}) - \frac{\delta F}{\delta u} - \xi\nabla\cdot(\mathbf{nn}) - \xi_c\nabla\times\nabla\cdot(\mathbf{nn}), \tag{36a}$$

$$\nabla.\mathbf{v} = -\frac{\partial_t \varrho}{\rho_0}, \tag{36b}$$

$$\partial_t u = v_z - \frac{1}{\gamma}(K\nabla_\perp^4 u - B\,\partial_z^2 u), \tag{36c}$$

$$p = C(\rho_0 + \gamma\varrho), \nabla p = D\nabla\varrho. \tag{36d}$$

Using Fourier techniques as before, taking the Spatial transform of the system of equations, after substituting the pressure (36 d) into equation (36 a).

$$\rho_0\,\partial_t\tilde{\boldsymbol{v}} - \mu k^2\tilde{\boldsymbol{v}} - \frac{1}{3}\mu\boldsymbol{k}(\boldsymbol{k}\cdot\tilde{\boldsymbol{v}}) = -D(i\boldsymbol{k})\tilde{\varrho} + \xi(-\boldsymbol{k}k_z + k^2\boldsymbol{e}_z)\tilde{u} - \xi_c(i\boldsymbol{k})\times(-k^2\boldsymbol{e}_z)\tilde{u}, \tag{37a}$$

$$\partial_t\tilde{u} = \tilde{v}_z - \frac{1}{\gamma}(Kk_\perp^4 + Bk_z^2)\tilde{u}. \tag{37b}$$

$$\partial_t\tilde{\varrho} = -\rho_0(i\boldsymbol{k})\cdot\tilde{\boldsymbol{v}} \tag{37c}$$

The system can then be expressed as a matrix equation of the form $\partial_t\tilde{\boldsymbol{x}} = A\tilde{\mathbf{x}}$ where A is a 5x5 complex matrix.

To consider the acoustic waves, the dispersive terms can be ignored since they do not oscillate and hence cannot contribute to the wave solutions. The components that are perpendicular and parallel to the wavevector are then to be considered separately and note that acoustic sound waves are longitudinal.

Due to time restraints, no further progress was made with finding the compression waves for the smectic phase.

4. Chiral Active Columnars

Derivation of Hydrodynamics through Analogy to Smectics

There are a limited number of studies considering the hydrodynamic equations of columnar phase active systems. To make a reasonable attempt at describing the system, the expressions for the free energy and the linearised equations of motion are obtained by analogy to the linear expressions used for the smectic mesophase. The displacement field, as described in figure 8, is now

$$\mathbf{u} = \left(u_x, u_y, 0\right). \tag{38}$$

The quadratic free energy density f for the smectic phase is given by

$$f = \frac{B}{2}(\partial_z u)^2 + \frac{K}{2}(\nabla_\perp^2 u)^2 \tag{39}$$

as described in (3). The term prior with $(\partial_z u)^2$ corresponds to a compressive stress in the direction of the displacement. By direct comparison, this corresponds to the gradients in the displacement plane or $(\nabla_\perp \mathbf{u})^2$ in the columnar phase. The latter term containing $(\nabla_\perp^2 u)^2$ again by direct comparison becomes $|\partial_z^2 \mathbf{u}|^2$. Hence the free energy to quadratic order in $\boldsymbol{u}$ for the columnar phase is given by

$$F = \int \frac{B}{2}(\nabla_\perp \mathbf{u})^2 + \frac{K}{2}|\partial_z^2 \mathbf{u}|^2 \, dV \tag{40}$$

where B and K are the bulk (or compression) and bending moduli respectively.

Notice the free energy is a tensor expression unlike for the smectic phase and the notation $(\nabla_\perp \mathbf{u})^2$ is equivalent to $\sum_{i,j=x,y} \partial_i u_j$ in index notation. The free energy uses a single elastic modulus B. It can be shown that the 2nd rank tensor $\nabla_\perp \mathbf{u}$ can be expressed as

$$\begin{aligned}\nabla_\perp \mathbf{u} = \frac{\partial_x u_x + \partial_y u_y}{2} I_2 &\\ + \frac{\partial_x u_y - \partial_y u_x}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &\\ + \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & -\Delta_1 \end{bmatrix} &\end{aligned} \tag{41}$$

where $\Delta_1 = \frac{1}{2}\left(\partial_x u_x - \partial_y u_y\right)$, $\Delta_2 = \frac{1}{2}\left(\partial_x u_y + \partial_y u_x\right)$ and I_2 is the 2x2 identity matrix. The 2nd term corresponds to pure rotations of the system and hence do not contribute to the free energy. This leads to the free energy expression with two elastic moduli B and G

$$\begin{aligned}F = \int \frac{B}{2}(\nabla_\perp \cdot \mathbf{u})^2 + \frac{G}{2}\Big(\left(\partial_x u_x - \partial_y u_y\right)^2 &\\ + \left(\partial_x u_y + \partial_y u_x\right)^2\Big)) &\\ + \frac{K}{2}|\partial_z^2 \mathbf{u}|^2 \, dV. &\end{aligned} \tag{42}$$

Although this form for the free energy is more accurate, it is of interest whether the two moduli would introduce any characteristics when performing stability analysis. Functional techniques and strong anchoring conditions[1] again gives a functional derivative linear in $\boldsymbol{u}$,

$$\frac{\delta F}{\delta \mathbf{u}} = -B\nabla_\perp^2 \mathbf{u} + K\,\partial_z^4 \mathbf{u}, \tag{43}$$

$$\begin{aligned}\frac{\delta F}{\delta \mathbf{u}} = -B\nabla_\perp(\nabla_\perp \cdot \mathbf{u}) - G\nabla_\perp^2 \mathbf{u} &\\ + K\,\partial_z^4 \mathbf{u}. &\end{aligned} \tag{44}$$

The system of hydrodynamic equations takes the same form as that described in (14). The active and chiral active forces keep the same form $-\xi\nabla\cdot(\mathbf{nn})$ and $-\xi_c\nabla\times\nabla\cdot(\mathbf{nn})$ although the expression for the director has differed. The director is defined along the direction of the column axis ($\boldsymbol{e}_z$ in the inactive system). By displacing the columns in the x-y plane, the director changes by the gradients in z. Linearly in $\boldsymbol{u}$ the director is then

$$\boldsymbol{n} = \mathbf{e_z} + \partial_z\mathbf{u}. \tag{45}$$

This gives the expressions

$$\nabla\cdot(\mathbf{nn}) = \mathbf{e_z}\,\partial_z(\nabla\cdot\mathbf{u}) + \partial_z^2\mathbf{u} \tag{46}$$

$$\nabla\times\nabla\cdot(\mathbf{nn}) = \nabla\times\mathbf{e_z}\,\partial_z(\nabla\cdot\mathbf{u}) + \nabla\times\partial_z^2\mathbf{u} \tag{47}$$

for the active force terms. The system still has a pressure gradient $-\nabla\mathrm{p}$ and elastic force contribution $-\frac{\delta F}{\delta\mathbf{u}}$ but the functional derivative is now a vector expression as described in (44). We still assume incompressibility (11b). The equation for the structure has two terms on the right-hand side corresponding to the advection and relaxation to equilibrium of the displacement. For the columnar phase, there is a still advection and this relaxation to equilibrium but is now a vector expression in the plane parallel to the displacement.

This gives the linear system of hydrodynamic equations

$$\nabla.\mathbf{v} = 0 \tag{48a}$$

$$0 = -\nabla\mathrm{p} + \mu\nabla^2\mathbf{v} - \frac{\delta F}{\delta\mathbf{u}} - \xi\nabla\cdot(\mathbf{nn}) - \xi_c\nabla\times\nabla\cdot(\mathbf{nn}) \tag{48b}$$

$$\partial_t\mathbf{u} = \mathbf{v}_\perp - \Gamma\frac{\delta F}{\delta\mathbf{u}} \tag{48c}$$

again, assuming a low Reynolds number to neglect the inertial term in (Y).

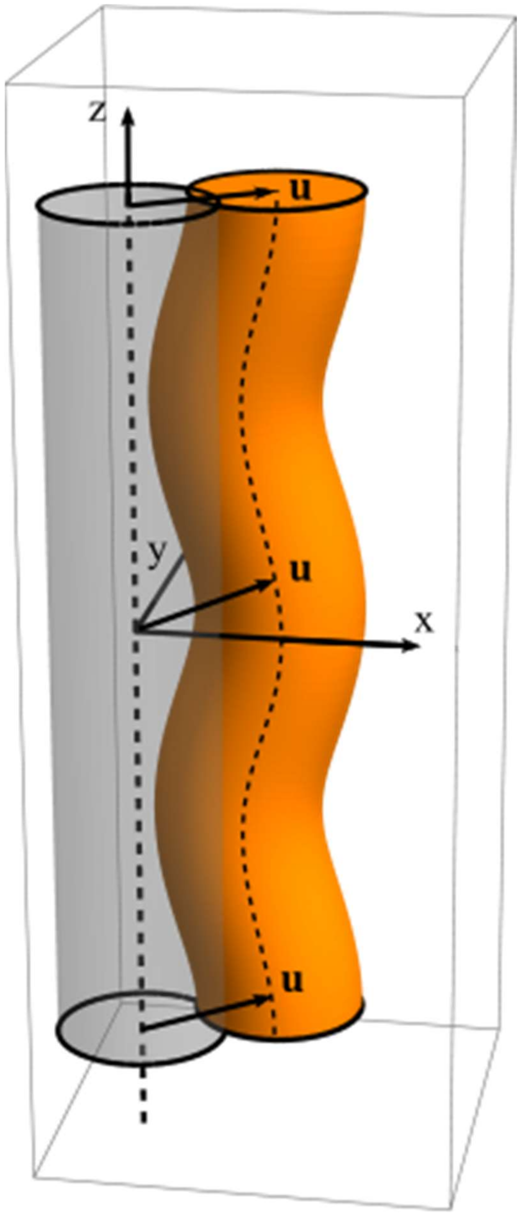

Figure 8 Two dimensional displacement variable u for a columnar phase liquid crystal.

Basic Instability:

To make progress analysing the system, first consider the simplest form for the displacement field $\boldsymbol{u}$ with only a z dependence,

$$\mathbf{u} = \mathbf{u}(\mathrm{z}) = \left(\mathrm{u_x}(\mathrm{z}), \mathrm{u_y}(\mathrm{z}), 0\right). \qquad (49)$$

This forces the pressure and velocity field to also depend only on z,

$$\mathrm{p} = \mathrm{p}(\mathrm{z}), \mathbf{v} = \mathbf{v}(\mathrm{z}) \qquad (50)$$

This significantly simplifies the terms in the hydrodynamic equations (48) with vanishing gradients in x and y. Notice the incompressibility (48a) with these assumptions means that the fluid flows only in the x-y plane,

$$\nabla \cdot \mathbf{v} = \partial_{\mathrm{z}} \mathrm{v_z} = 0 \;\Rightarrow \mathrm{v_z} = 0. \qquad (51)$$

Also notice that for the basic instability, using the two elastic moduli (40) instead of the single elastic modulus (39) free energy makes no impact on the results since the elastic terms vanish in the functional derivative,

$$\frac{\delta \mathrm{F}}{\delta \mathbf{u}} = \mathrm{K}\, \partial_{\mathrm{z}}^4 \mathbf{u}. \qquad (52)$$

The active and chiral terms also simplify,

$$\nabla \cdot (\mathbf{nn}) = \partial_{\mathrm{z}}^2 \mathbf{u}, \qquad (53)$$
$$\nabla \times \nabla \cdot (\mathbf{nn}) = \partial_{\mathrm{z}}^3 (\mathbf{e}_{\mathrm{z}} \times \mathbf{u}), \qquad (54)$$

resulting in the force balance equation (48b) becoming,

$$\begin{aligned} 0 = -\mathbf{e_z}\, \partial_{\mathrm{z}} \mathrm{p} + \mu\, \partial_{\mathrm{z}}^2 \mathbf{v} - K\, \partial_{\mathrm{z}}^4 \mathbf{u} \\ - \xi\, \partial_{\mathrm{z}}^2 \mathbf{u} \\ - \xi_c\, \partial_{\mathrm{z}}^3 (\mathbf{e_z} \times \mathbf{u}). \end{aligned} \qquad (55)$$

$\mathbf{u}$ and $\mathbf{v}$ are both in the x-y plane, this must mean that the pressure is uniform, $p = p_0$ for a constant pressure p_0, since the pressure gradient is the only term in $\mathbf{e_z}$. This means the pressure gradient vanishes in (55) and so then it is straight forward to show that

$$\mathbf{v} = \frac{\xi}{\mu} \mathbf{u} + \frac{\mathrm{K}}{\mu} \partial_{\mathrm{z}}^2 \mathbf{u} + \frac{\xi_{\mathrm{c}}}{\mu} \partial_{\mathrm{z}} (\mathbf{e_z} \times \mathbf{u}). \qquad (56)$$

By substituting this relation between $\boldsymbol{v}$ and u into the relaxation of order (48c) then

$$\partial_{\mathrm{t}} \mathbf{u} = \left(\frac{\xi}{\mu} + \frac{K}{\mu} \partial_{\mathrm{z}}^2 - \Gamma \mathrm{K}\, \partial_{\mathrm{z}}^4\right) \boldsymbol{u} + \frac{\xi_c}{\mu} \partial_{\mathrm{z}} (\mathbf{e_z} \times \mathbf{u}). \qquad (57)$$

This differential equation can then be solved by taking a Fourier transform in z with corresponding wavevector q to obtain

$$\begin{aligned} \partial_{\mathrm{t}} \tilde{\mathbf{u}}(\mathrm{q}) = \left(\frac{\xi}{\mu} - \frac{K}{\mu} q^2 - \Gamma \mathrm{K} \mathrm{q}^4\right) \tilde{\mathbf{u}} \\ + \frac{\xi_c}{\mu} (iq)(\mathbf{e_z} \times \tilde{\mathbf{u}}). \end{aligned} \qquad (58a)$$

The expression can then be rewritten as a matrix equation which can be solved by reading off the eigenvectors and eigenvalues [X].

$$\partial_{\mathrm{t}} \begin{pmatrix} \tilde{u}_x \\ \tilde{u}_y \end{pmatrix} = \left(\frac{\xi}{\mu} - \frac{K}{\mu} q^2 - \Gamma \mathrm{K} \mathrm{q}^4\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{u}_x \\ \tilde{u}_y \end{pmatrix} + \frac{\xi_c}{\mu} q \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \tilde{u}_x \\ \tilde{u}_y \end{pmatrix}, \qquad (58b)$$

The matrix equation has eigenvalues $\left(\frac{\xi}{\mu}-\frac{K}{\mu}q^2-\Gamma Kq^4\right)\pm\frac{\xi_c}{\mu}q$ and corresponding eigenvectors $\binom{1}{\pm i}$ hence we obtain solutions

$$\tilde{\boldsymbol{u}}_{\pm}\propto\binom{1}{\pm i}\exp\left[\left(\frac{\xi}{\mu}\pm\frac{\xi_c}{\mu}q-\frac{K}{\mu}q^2-\Gamma Kq^4\right)t\right] \quad (59)$$

and taking the corresponding inverse transform results in

$$\boldsymbol{u}_{\pm}=\mathcal{R}e\left[\binom{1}{\pm i}\exp\left[\left(\frac{\xi}{\mu}\pm\frac{\xi_c}{\mu}q-\frac{K}{\mu}q^2-\Gamma Kq^4\right)t\right]\exp(iqz)\right]= \quad (60)$$

$$\binom{\cos qz}{\mp\sin qz}\exp\left[\left(\frac{\xi}{\mu}\pm\frac{\xi_c}{\mu}q-\frac{K}{\mu}q^2-\Gamma Kq^4\right)t\right] \quad (61)$$

where we assume a single wave vector solution. The sign of the exponential growth rate dictates the stability of the system.

The wavevector q can be treated as a small non-zero value, $q\geq\frac{\pi}{d}$ (d, the system size along the column axis). As with the smectic analysis, the activity ξ mostly dictates whether the system will be stable or unstable. Again, an extensile $(\xi>0)$ activity leads to instability. For a contractile $(\xi<0)$ system, the columns return to the inactive state. Unlike the smectic case, the chirality does affect the stability. The degree of chirality ξ_c is the leading correction, $O(q)$, to the exponential growth rate. This means that although it will not usually affect whether the system is stable or unstable, it will increase or decrease how quickly it extends or contracts. The displacement causes the columns to form a helix around the z-axis with two modes, either right or left helixes.

The chiral term changes depending on the direction of the helix and reduces or increases the growth rate of the displacements about the helical axis.

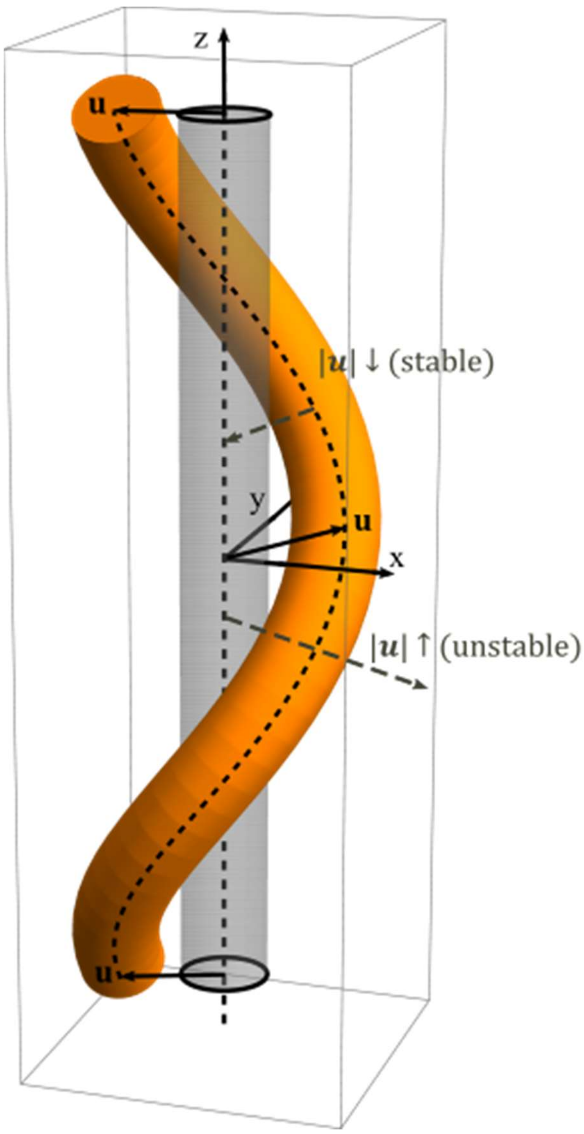

Figure 9 A left Helical Distortion induced within a chiral active columnar system.

Achiral Box Confinement:

Although the simple instability does show interesting properties for chiral active columnar phase, using a more realistic form for $\boldsymbol{u}$ could uncover other intriguing features, primarily in making a comparison with the Helfrich-Hurault effect in columnar phase liquid crystals[19]. The activity for a smectic system is already shown to be analogous to the smectic Helfrich-Hurault effect. Determining whether this is also true for the columnar phase is of interest. Now consider confining the system to an open-ended box with side lengths L as described in figure (10). The displacement then vanishes at the boundaries. Take

$$\boldsymbol{u} = u_0(z)\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\mathbf{e_x} \quad (62)$$

This is the simplest Fourier series that satisfies the boundary conditions. We set the $\mathbf{e_y}$ component to zero for simplicity as well being able to make a direct comparison with Helfric-Hurault instability analysis by Kleman-Oswald [19]. The terms in the equation of motion (48b) and structure (48c) equations again are considered. The divergence of the displacement becomes

$$\nabla_\perp \cdot \mathbf{u} = \frac{\pi}{L}u_0(z)\cos\frac{\pi x}{L}\sin\frac{\pi y}{L}. \quad (63)$$

Also note $\nabla_\perp^2 \equiv -\frac{2\pi^2}{L^2}$.

The divergence of the equation of motion (48b) then gives

$$\nabla^2 \mathrm{p} = \left(2(\mathrm{B}+\mathrm{G})\left(\frac{\pi}{L}\right)^2 - K\,\partial_\mathrm{z}^4 - 2\xi\,\partial_\mathrm{z}^2\right)(\nabla_\perp \cdot \mathbf{u}) \quad (64)$$

The form for the pressure is then chosen to match the divergence of the displacement

$$p = p_0(z)\cos\frac{\pi x}{L}\sin\frac{\pi y}{L} \quad (65)$$

to obtain the relation

$$\left[\partial_z^2 - 2\left(\frac{\pi}{L}\right)^2\right]p_0 = -\frac{\pi}{L}\left[(B+G)\frac{2\pi^2}{L^2} + K\partial_z^4 + 2\xi\partial_z^2\right]u_0. \quad (66)$$

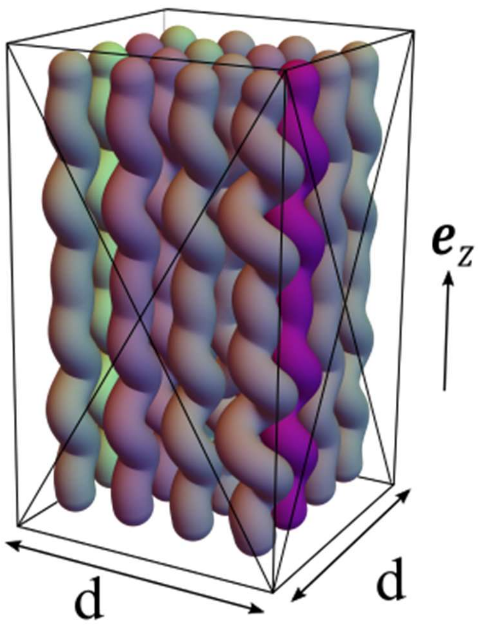

Figure 10 Columnar liquid crystal system confined to an open-ended box of sidelengths d with 'fluid direction' e_z. The 'tubes' aligned along e_z are displaced in $e_\perp$.

And taking the Fourier transform in z with corresponding wavevector q obtains

$$\left(\frac{2\pi^2}{L^2}+q^2\right)\tilde{p}_0=\frac{\pi}{L}\left(2(\mathrm{B}+\mathrm{G})\left(\frac{\pi}{L}\right)^2+K\mathrm{q}^4 - 2\xi\,\mathrm{q}^2\right)\tilde{\mathrm{u}}_0\,. \tag{67}$$

The Fourier transform along z of the equation of motion (48b) equation obtains

$$\begin{aligned}\mu(\nabla_\perp^2-q^2)\tilde{\boldsymbol{v}} &= (\nabla_\perp+\boldsymbol{e}_z.iq)\tilde{p}-B\nabla_\perp(\nabla_\perp.\tilde{\boldsymbol{u}})\\ &\quad -G\nabla_\perp^2\tilde{\boldsymbol{u}}+Kq^4\tilde{\boldsymbol{u}}\\ &\quad +\xi(\boldsymbol{e}_z.iq(\nabla_\perp.\tilde{\boldsymbol{u}})-q^2\tilde{\boldsymbol{u}})\end{aligned} \tag{68}$$

Making a similar choice in the confinement of $\boldsymbol{v}$, and then for the equation of motion (68) the perpendicular gradients are replaced with $\frac{\pi}{L}$ appropriately to obtain the vector equations (69) and (70) respectively,

$$\left(\frac{2\pi^2}{L^2}+q^2\right)\mu\tilde{\boldsymbol{v}}_0=\begin{pmatrix}\frac{\pi}{L}\\ -\frac{\pi}{L}\\ -iq\end{pmatrix}\tilde{p}_0+\begin{pmatrix}\xi q^2-Kq^4-(B+2G)\frac{\pi^2}{L^2}\\ \frac{B\pi^2}{L^2}\\ -iq\xi\frac{\pi}{L}\end{pmatrix}\tilde{\mathrm{u}}_0,$$

$$\boldsymbol{v}=\boldsymbol{v_0}(z)\left(\cos\frac{\pi x}{L}\sin\frac{\pi y}{L}\quad \cos\frac{\pi x}{L}\cos\frac{\pi y}{L}\quad \sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\right)^T.$$

The chosen form for v doesn't completely satisfy the boundary conditions. There is no flux across the boundary, but the no-slip condition isn't satisfied. A more suitable form for flow field could be chosen but it is acceptable for performing the stability analysis. Directly substituting the pressure from (67) results in a relation between $\tilde{v}$ and $\tilde{u}$ (71),

$$\begin{aligned}\partial_t\tilde{u}_0=\Bigg[&\frac{\mathrm{k}_\perp^2}{\mu k^4}(2\mathrm{k}_\perp^2(B+G)+Kk_z^4-2\xi k_z^2)\\ &+\frac{\xi k_z^2-Kk_z^4-(B+2G)\mathrm{k}_\perp^2}{\mu k^2}\\ &-\Gamma\left(Kk_z^4+\mathrm{k}_\perp^2(B+2G)\right)\Bigg]\tilde{u}_0\end{aligned}$$

The expression can then be further manipulated by substituting $\frac{\mathrm{k}_\perp^2}{k^2}$ with $\frac{1}{2}(1-\frac{\mathrm{k}_z^2}{k^2})$. Also note that $(k_z^2k^2-2k_z^2\mathrm{k}_\perp^2)$ is equivalent to k_z^4 when factoring the active (ξ) terms.

$$\begin{aligned}\partial_t\tilde{u}_0=\frac{1}{\mu k^4}\Big[&\xi(k_z^4)+\mathrm{k}_\perp^2(2\mathrm{k}_\perp^2(B+G)+Kk_z^4)\\ &-k^2(Kk_z^4+(B+2G)\mathrm{k}_\perp^2)\\ &-\mu\Gamma k^4\left(Kk_z^4+\mathrm{k}_\perp^2(B+2G)\right)\Big]\tilde{u}_0\end{aligned}$$

The above is (72).There is once again a term arising from the phenomenological constant ξ that for extensile ($\xi>0$) activity, can lead to instability. Unlike with the basic analysis, there is now a threshold for the instability (ξ_{Th}) for the system to become unstable. The system is always stable when it is contractile ($\xi<0$) as with the smectic system but the size of the active term for the growth rate now depends upon the wavevectors k_z and $k_\perp$. To give (73)

$$\partial_t \tilde{u}_0 = \frac{1}{\mu k^4}[\xi(k_z^4) - K(k_z^6 + k_z^4 k_\perp^2 + \mu\Gamma k_z^4 k^4) - 2G\mathrm{k}_\perp^4 - k_z^2 k_\perp^2(B+2G) - \mu\Gamma k^4(B + 2G)\,\mathrm{k}_\perp^2]\tilde{u}_0$$

Due to this one cannot assume $k_z \ll k_\perp$ but is of a similar order of magnitude since the threshold may have an optimal wavevector, similar in magnitude to $k_\perp$ caused by the confinement to the box. This, after all, is what is observed for the smectic system with the Helfrich-Hurault instability. As with the basic stability, the system tends to be unstable for positive ξ. This is because $\left(\frac{k_z}{k}\right)^4$ is approximately constant. The threshold for the instability is when the temporal change in $\boldsymbol{u}$ is identically zero which results in a threshold value for the activity, ξ_{Th}, in terms of the wavevector k_z. (74)

$$\xi_{Th}(k_z) = \frac{K(k_z^6 + k_z^4 k_\perp^2 + \mu\Gamma k_z^4 k^4) + k_z^2 k_\perp^2(B+2G) + \mu\Gamma k^4(B+2G)\mathrm{k}_\perp^2 + 2G\mathrm{k}_\perp^4}{k_z^4}$$

The critical wavevector q_c for the threshold is then be determined by differentiating the expression with respect to k_z^2 and, hence finding the minimum. Note that k^2 differentiates to unity. (75)

$$\frac{\partial\xi_{Th}(k_z)}{\partial k_z^2} = \frac{K(k_z^6 + 2\mu\Gamma k_z^6 k^2) - k_z^2 k_\perp^2(B+2G) + (-2k^4 + 2k_z^2 k^2)\mu\Gamma(B+2G)\mathrm{k}_\perp^2 - 4G\mathrm{k}_\perp^4}{k_z^6}$$

The critical wavevector ($k_z = q_c$) for the threshold can then be found by setting the numerator to zero. The k_z^6 and $k_\perp$,k_z product terms of the same order are ignored since the wavevectors are considered small here.

The critical wavevector is then given by

$$q_c^2 = -\frac{4\mathrm{Gk}_\perp^2}{(B+2G)} \tag{76}$$

which resembles the result obtained for the Helfric-Hurault analysis performed by Kleman and Oswald[19]. The threshold, again keeping terms only up to 4th order, is

$$\xi_{Th} = \frac{k_z^2 k_\perp^2(B+2G) + 4G\mathrm{k}_\perp^4}{k_z^4}, \tag{77}$$

and substituting the critical wavevector obtains

$$\xi_{Th} = 0. \tag{78}$$

The values for the critical threshold and wavevector suggest some further analysis before making full conclusions. Due to time restrictions and other issues, this hasn't been considered yet.

Choosing to use the one or two elastic moduli approximation for the free energy does make an impact on the result. Notice that in (43,44) the functional derivative is restricted to the x-axis by our choice of u for the one elastic modulus but is not the case for two. There is a y-component based on the gradients in $\boldsymbol{u}$, $-\mathrm{B}\nabla_\perp(\nabla_\perp \cdot \mathbf{u})$. This is interesting because in the equation of the structure (68), $v_y = 0$ if the functional derivative has no $\boldsymbol{e}_y$ component.

5. Conclusion

The general hydrodynamic equations using non-equilibrium physics are derived for the smectic A chiral active system. The stability analysis for the smectic system proved to give the results, as expected, analogous to the Helfrich-Hurault instability. This being that the phenomenological quantity ξ that represents the active contribution to the system, was stable for contractile and unstable for extensile activity. Furthermore, the chiral term in the smectic phase did not a make an effect on whether solution would be stable or unstable despite the degree of chirality. The chiral active stresses in the columnar phase liquid crystal led to a leading order correction for the stability of the system, for the basic instability. Furthermore, the corresponding displacement field suggests a helical distortion with similarities to that which arrives in the nematic chiral system.

There are similarities between the Helfric-Hurault effect in columnar phase columnar phase and the achiral box confinement active system. Less progress was made than was intended with regards to analysing the system for the box confinement due to time restraints.

The natural extension of the analysis would be to consider the chiral active system with box confinement for the chiral active columnar phase. Also, further considerations with acoustic waves to consider the anisotropy in the sound speed.

6. References

[1] P. G. de Gennes and J. Prost, The Physics of Liquid Crystals (Second Edition), (Oxford University Press, Oxford, 1993).

[2] Mason, T.O. and Shimanovich, U., 2018. Fibrous protein self-assembly in biomimetic materials. *Advanced Materials*, *30*(41), p.1706462.

[4] Marchetti, M. Cristina, et al. "Hydrodynamics of soft active matter." *Reviews of modern physics* 85.3 (2013): 1143.

[5] McMillan, W.L., 1971. Simple molecular model for the smectic A phase of liquid crystals. *Physical Review A*, *4*(3), p.1238.

[6] P. G. de Gennes and J. Prost, The Physics of Liquid Crystals (Second Edition), (Oxford University Press, Oxford, 1993).

[8] Joshi, A.A., Whitmer, J.K., Guzmán, O., Abbott, N.L. and de Pablo, J.J., 2014. Measuring liquid crystal elastic constants with free energy perturbations. *Soft matter*, *10*(6), pp.882-893.

[9] Chaikin, P.M., Lubensky, T.C. and Witten, T.A., 1995. *Principles of condensed matter physics* (Vol. 10). Cambridge: Cambridge university press.

[10] Yang, H., Wang, C. and Su, Z., 2008. Growth mechanism of synthetic imogolite nanotubes. *Chemistry of Materials*, *20*(13), pp.4484-4488.

[11] R. A. Simha and S. Ramaswamy, Phys. Rev. Lett. 89, 058101 (2002). [12]

[13] Marchetti, M. Cristina, et al. "Hydrodynamics of soft active matter." *Reviews of modern physics* 85.3 (2013): 1143.

[14] Foias, C., Manley, O., Rosa, R. and Temam, R., 2001. *Navier-Stokes equations and turbulence* (Vol. 83). Cambridge University Press.

[15] Kole, S.J., Alexander, G.P., Ramaswamy, S. and Maitra, A., 2021. Layered chiral active matter: Beyond odd elasticity. *Physical Review Letters*, *126*(24), p.248001.

[16] Bahouri, H., Chemin, J.Y. and Danchin, R., 2011. *Fourier analysis and nonlinear partial differential equations* (Vol. 343, pp. 523-pages). Berlin: Springer.

[17] Doostmohammadi, A., Ignés-Mullol, J., Yeomans, J.M. et al. Active nematics. Nat Commun 9, 3246 (2018)

[18] Bevilacqua, G. and Napoli, G., 2005. Reexamination of the Helfrich-Hurault effect in smectic-A liquid crystals. *Physical Review E*, *72*(4), p.041708.

[19] M. Kléman, P. Oswald. Columnar discotic mesophases : elasticity, dislocations, instabilities. Journal de Physique, 1982, 43 (4), pp.655-662