1) i) $(\neg p \Rightarrow q)$

$A \Rightarrow B \Leftrightarrow (\neg A \vee B) \Leftrightarrow \neg(A \wedge \neg B)$

Hence $(\neg p \Rightarrow q) \Leftrightarrow (\neg(\neg p) \vee q) \Leftrightarrow \underline{(p \vee q)}$

p	q	$(\neg p)$	$(\neg p \Rightarrow q)$	$(p \vee q)$
F	F	T	F	F
F	T	T	T	T
T	F	F	T	T
T	T	F	T	T
			*	*

$\therefore (\neg p \Rightarrow q) \Leftrightarrow \underline{(p \vee q)}$.

ii) $((p \vee q) \wedge r)$

$A \wedge B \Leftrightarrow \neg((\neg A) \vee (\neg B))$ Demorgan law.

(Distribution)

Hence $(p \vee q) \wedge r \Leftrightarrow p \wedge r \vee q \wedge r$

$\Leftrightarrow \neg((\neg p) \vee (\neg r)) \vee \neg((\neg q) \vee (\neg r))$

$(p \vee q)$	p	q	r	$\neg p$	$\neg r$	$\neg q$	$(\neg p) \vee (\neg r)$	$(\neg q) \vee (\neg r)$	$\neg(\ldots)$	$\neg(\ldots)$	$\ldots \vee \ldots$	$(p \vee q) \wedge r$
F	F	F	F	T	T	T	T	T	F	F	F	F
F	F	F	T	T	F	T	T	T	F	F	F	F
T	F	T	F	T	T	F	T	T	F	F	F	F
T	F	T	T	T	F	F	T	F	F	T	T	T
T	T	F	F	F	T	T	T	T	F	F	F	F
T	T	F	T	F	F	T	F	T	T	F	T	T
T	T	T	F	F	T	F	T	T	F	F	F	F
T	T	T	T	F	F	F	F	F	T	T	T	T
											*	*

iii) let $R = \neg((p \vee q) \Leftrightarrow r)$.

$(A \Leftrightarrow B) \Leftrightarrow (A \wedge B) \vee (\neg A \wedge \neg B) \Leftrightarrow (\neg(\neg A \vee \neg B) \vee \neg(\neg(\neg A) \vee \neg(\neg B)))$

$\Leftrightarrow \neg(\neg A \vee \neg B) \vee \neg(A \vee B)$

Hence $\underline{R \Leftrightarrow \left(\neg((\neg(p \vee q)) \vee (\neg r)) \vee \neg((p \vee q) \vee r)\right)}$.

$p\,q\,r$	$(p \vee q)$	$(p \vee q) \Leftrightarrow r$	$\neg r$	$\neg(p \vee q)$	$\neg(p \vee q) \vee \neg r$	$\neg(\dots)$	$(p \vee q) \vee r$	$\neg(\dots)$	$(\dots) \vee (\dots)$
FFF	F	T	T	T	T	F	F	T	T
FFT	F	F	F	T	T	F	T	F	F
FTF	T	F	T	F	T	F	T	F	F
FTT	T	T	F	F	F	T	T	F	T
TFF	T	F	T	F	T	F	T	F	F
TFT	T	T	F	F	F	T	T	F	T
TTF	T	F	T	F	T	F	T	F	F
TTT	T	T	F	F	F	T	T	F	T
		*							*

2) ~~a)~~

i) $\underbrace{\text{Arrakis}}_{a}$ has $\underbrace{\text{Sandworms}}_{s}$ $\rightsquigarrow$ $has(a, s)$

ii) $\underbrace{\text{Sandworms}}_{s}$ are the only $\underbrace{\text{animals}}_{a}$ that can produce $\underbrace{\text{spice}}_{S_p}$.

$$\forall x\left(\left(\overset{A(x)}{\cancel{\quad}} \wedge produce(x, S_p)\right) \rightarrow \overset{S(x)}{\cancel{\quad}}\right)$$

$A(x)$: x is an animal
$S(x)$: x is a sandworm.

iii) Every member of the $\underbrace{\text{guild navigator}}_{g}$ needs to have $\underbrace{\text{spice}}_{S_p}$ in order to navigate $\underbrace{\text{spaceships}}_{s}$.

i.e. $G(x)$: x is a guild navigator

$$\forall x\left(\left(navigate(x, s) \wedge \overset{G(x)}{\cancel{\quad}}\right) \rightarrow has(x, S_p)\right)$$

iv) The $\underbrace{\text{Kwisatz Haderach}}_{k}$ is the only one who can see the $\underbrace{\text{future}}_{\cancel{f}}$.

$$\forall x\left(seesFuture(x) \rightarrow \underset{x = k}{\cancel{k}}\right)$$

3)

i) $R_{x,y}$ is irreflexive iff

"for all x, x does not bear relation R to x".

iff $\forall x(\neg R_{x,x})$

ii) $R_{x,y}$ is intransitive iff

"if x has relation R to y and y has relation R to x then x doesn't have relation R to z"

iff $\forall x \forall y \forall z (R_{x,y} \wedge R_{y,z} \rightarrow \neg R_{x,z})$

iii) $R_{x,y}$ is not partially ordered iff

$$\exists x \exists y \exists z (\neg R_{x,x} \vee (R_{x,y} \wedge R_{y,z} \rightarrow \neg R_{x,z}) \vee (R_{x,y} \Rightarrow R_{y,x})).$$

4) A function is surjective if its range equals the codomain.

A function is injective if ~~each~~ each element of A maps to a unique element of B

i) $f : \mathbb{Z} \to \mathbb{N}$ where $\forall n \in \mathbb{Z} : f(n) = n^{2021} + 1$

f is injective since $\nexists\, a, b \in \mathbb{Z}$ s.t. $f(a) = f(b)$ & $a \neq b$

Proof: Suppose $\exists\, a, b \in \mathbb{Z}$ s.t. $f(a) = f(b)$ & $a \neq b$ then

$a^{2021} + 1 = b^{2021} + 1 \implies a^{2021} = b^{2021}$

$\implies a = b$.

Hence the statement is ~~the~~ false and f is injective.

f is not surjective since ~~no element maps to~~ f does not map to every natural number.

i.e. no integer $c \in \mathbb{Z}$ s.t. $f(c) = 3$

Let $f(c) = 3$ then $c^{2021} + 1 = 3 \implies c^{2021} = 2$

$\implies c = 2^{\frac{1}{2021}}$

but $2^{1/2021}$ is not ~~an~~ an integer.

~~Note~~ I believe f is ill-defined since $f : \mathbb{Z} \to \mathbb{N}$ so maps every element from $\mathbb{Z}$ to $\mathbb{N}$ but clearly $n^{2021} + 1 < 0$ $\forall n \leq \cancel{0}2, n \in \mathbb{Z}$, $f : \mathbb{Z} \to \mathbb{Z}$ I believe should be the proper ~~range~~ codomain.

ii) $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $\forall (n, k) \in \mathbb{N} \times \mathbb{N} : g(n, k) = 5^n 7^k 11^{n+k}$

g is not surjective since g doesn't map to every element of $\mathbb{N}$
→ i.e. let $g(a,b) = 2$, no product of 5^a, 7^b and 11^{a+b} can produce 2 to any powers a, b hence $2 \notin g(\mathbb{N} \times \mathbb{N})$

g is injective since each element of $g(\mathbb{N} \times \mathbb{N})$ is unique.

This is because 5, 7 and 11 are primes hence the ~~...~~ products of 5, 7 and 11 and their ~~...~~ powers are unique. Note $5^a 7^b 11^c$ is unique because so is $5^{a} 7^{b} 11^{c}$.

Actually easier to say cannot make any prime $\neq$ 5, 7 or 11 using powers of 5, 7, 11.

4) iv) $k : \mathbb{N} \to \mathbb{Z}$ where $\forall n \in \mathbb{N} : k(n) = (-1)^n$.

k is not surjective since $k(\mathbb{N}) = \{-1, 1\} \subset \mathbb{Z}$ is not the set of integers. ~~...~~

k is not injective since ~~...~~ $-1 = k(1) = k(3) = k(5) = \dots$ ~~and~~
& $1 = k(2) = k(4) = k(6) = \dots \in \mathbb{Z}$
$\Rightarrow$ mappings from the domain to codomain are not unique

16 elements

5) Let the sample space ~~$S = \{BBBB, BBBW, \ldots, BWWW\}$~~

15 elements $\longrightarrow S = \{BBBB, BBBW, \ldots, WWWW\} \setminus \{WWWW\}$

Let E be the event where there are 3W & 1B balls transferred

$E = \{WWWB, WWBW, WBWW, BWWW\}$.

Let F be the event of atleast one white

$\Rightarrow F = S \setminus \{BBBB\} = \text{Complement}(\{BBBB\})$.

$E \subseteq F$.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)}{P(F)} \quad \text{since } E \subseteq F$$

$$P(F) = 1 - P(\{BBBB\}) = 1 - \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} = \underline{\frac{13}{14}}.$$

$$P(E) = \underbrace{\left(\frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} \times \frac{5}{5}\right)}_{1/56} + \left(\frac{3}{8} \times \frac{2}{7} \times \frac{5}{6} \times \frac{1}{5}\right) + \ldots$$

$$= 4 \times 1/56 = \underline{1/14}.$$

$$\Rightarrow P(E|F) = \frac{(1/14)}{(13/14)} = \underline{\underline{1/13}}.$$

Hence if 4 balls transferred, Prob. 3 are white and 1 is black given I know 1 is white, it's 1/13.

6) $P(+ \mid t) = 1-b$ $+, -$ for whether diagnosis detects tuberculosis respectively

$P(+ \mid \bar{t}) = a$ $t, \bar{t}$ for has tuberculosis and healthy respectively.

$P(t) = c$

$P(\bar{t} \mid +) = ?$

Baye's theorem states for two events A, B

$$P(A \mid B) = P(B \mid A) \cdot \frac{P(A)}{P(B)}, \quad P(B) \neq 0.$$

Hence $P(\bar{t} \mid +) = P(+ \mid \bar{t}) \cdot \dfrac{P(\bar{t})}{P(+)}$.

$P(+) = P(+ \mid t) + P(+ \mid \bar{t})$, and $P(\bar{t}) = 1 - P(t)$

$$\Rightarrow P(\bar{t} \mid +) = a \cdot \frac{(1-c)}{(1-b)+a}.$$

7) Event E = ~~Sample space~~ S = {BB, BG, GB}.

note each elementary event is equiprobable (= 1/3)

all outcomes with atleast one boy, ⊂ S := Sample space = {BB, BG, GG, ...}

$P(BB) = 1/3$ hence

There is a one in three chance that a family has two boys, given they have atleast one boy.

~~I am still not convinced whether you mean~~

This is contrary to intuition of 1/2 which you could get if the question was worded slightly differently.

8)
a) Confidence level 95% (1−0.05).

~~Since the~~
We use an unpaired t-test with ~~a~~ significance level 5% i.e.

$$t_\alpha = \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}.$$

Let ~~the~~ 1 correspond to beef and 2 correspond to poultry data.

We find $\mu_1 = 162.416$, $\mu_2 = 113.4$, $n_1 = 12$
$\sigma_1 = 23.700...$, $\sigma_2 = 24.48...$, $n_2 = 10$

Null hypothesis (H_0): The calorie content of is the same for both poultry and beef hotdogs.

Other hypotheses: ~~There is a difference~~ The calorie content for poultry is lower than beef hotdogs.
" " " " " higher than beef hotdogs.

$$t_\alpha = \frac{162.416... - 113.4}{\left(\frac{23.7...^2}{\sqrt{12}} + \frac{24.4...^2}{\sqrt{10}}\right)^{1/2}} = 2.61448... \simeq 2.61$$ (3sf)

~~2~~(10+12−2) degrees of freedom.

$t_{95\%,20} = 1.72$. Since $2.61 \notin (-1.72, 1.72)$ we reject H_0. Since $\mu_1 > \mu_2$ then we can say with 95% confidence level, the calorie content for poultry is lower than beef hotdogs.

beef

brand	before	after	diff
A	186	181	-5
B	181	191	10
C	176	186	10
D	149	129	-20
E	184	178	-6
F	190	194	4
G	158	139	-19
H	139	122	-17
I	175	195	20
J	148	158	10
K	152	158	6
L	111	104	-7
μ	162.4166667	161.25	-1.16667
σ	23.70062651	31.26754	13.12758

brand	poultry
1	129
2	132
3	102
4	106
5	94
6	102
7	87
8	99
9	170
10	113
μ	113.4
σ	24.46857

8) b) Since the hotdogs are different brands, we aim to use a paired t-test.

Null hypothesis (H_0): The new diet had no effect on the calories of the beef hotdogs.

Alternative hypothesis: The new diet had an effect.

To perform the t-test, the beef hotdog differences in calories are calculated, assuming the order the hotdogs are listed correspond to their brand.

The mean (μ) and std. deviation (σ) of the differences are calculated.

$\mu = -1.16\dot{6}$, $\sigma = 13.12758...$ without rounding.

There are $12 - 1 = 11$ degrees of freedom (k).

~~Hence~~ $t_{N\%,k} = \dfrac{\mu}{\sigma/\sqrt{n}}$ for the paired t-test.

Hence $t_{N\%,11} = \dfrac{-1.16\dot{6}}{(13.12...)/\sqrt{12}} = -0.30784... \simeq -0.308$ (3dp)

$t_{95\%,11} = 2.2010$.

Since clearly $-0.308 \in (-2.20, 2.20)$, we accept the null hypothesis.

The new feed has no effect on calories of beef hotdogs.