Analysis of a Hypothetical Modelling Problem

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***Abstract*—In this paper, the behaviour of a hypothetical problem, concerning the change in density of strategies within a population over time, is analysed using several methods including deriving an analytic solution from a differential equation (DE) model and experiments using an agent-based simulation. It is shown that in almost all circumstances that as time tends to infinity, one strategy will dominate, and the other will become extinct. In an extension to the problem, this behaviour is shown to extend to systems with more than two strategies.**

I. Introduction

Differential equations can be used as a tool for modelling the behaviour of dynamic systems over time. In this report, such an equation, describing how the densities of a pair of strategies vary over time for a hypothetical problem involving a population of students, given the problem definition described in section II-A, is derived. The behaviour of such a system through the use of analytical, numerical and agent-based modelling techniques is also investigated.

Finally, in an extension to the problem, the behaviour of a system in which more than two strategies are possible is explored by evaluating the equilibrium solutions for an appropriate DE model, and also by simulating the system.

II. Modelling the Scenario

A. System Definition

Each semester, for an infinite number of semesters, an infinite number of students work on projects in randomly assigned groups of size n. Students follow either a hardworking strategy h or a lazy strategy l, and based on their strategies they put in E_h or E_l effort respectively, where we will assume $E_h > E_l$. At the end of each semester, each student compares their payoff with one other randomly selected student, a "reference partner", where the general equation for the payoff of a student following some strategy i is given by

$$\pi_i = M - aE_i \tag{1}$$

where a is a non-negative constant which describes the cost-effort ratio of strategy i, and M is their group's shared mark given by

$$M = \frac{n_h E_h + (n - n_h)E_l}{n}, \tag{2}$$

where n_h is the number of hard workers in the group. If the payoff π_i of a student's reference partner is greater than their own π_j, then they will adopt the reference partner's strategy with a probability

$$P(j \to i) = \begin{cases} \alpha(\pi_i - \pi_j) & \text{if } \pi_i > \pi_j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

in the next semester. Here, α is a strictly positive scaling factor which prevents $P(j \to i)$ from exceeding 1, and ensuring it is a valid probability.

B. Model Derivation

Given a, E_h, E_l, n and an initial density of hardworking students x_0 at time $t = 0$, we wish to understand how the density of hard workers $x(t)$ is expected to change with respect to time. To do this, we can develop a differential equation which captures the system definition described in section II-A. From the definition, we see the only way x can decrease is if a hardworking student selects a lazy student as a reference partner, and the reference partner's payoff was higher than their own. Hence we expect the number of hardworking students who will adopt a lazy strategy at each time step to be

$$x(1-x)\mathbf{E}\big[P(h \to l)\big] \tag{4}$$

where $\mathbf{E}\big[P(h \to l)\big]$ is the expected probability of switching strategies for all hardworking students. Since $P(h \to l)$ can be considered, in all cases, as a linear combination of π_h and π_l, we can use the linearity of expectation to rewrite equation 4 as

$$x(1-x)\sum_{\overline{\pi}_l > \overline{\pi}_h}(\overline{\pi}_l - \overline{\pi}_h) \tag{5}$$

where $(\overline{\pi}_l, \overline{\pi}_h)$ are the expected payoffs for hard workers and lazy workers respectively. Using a similar line of reasoning, we can find an expression for the expected number of hardworking students who will adopt a lazy strategy, and then combine the two (remembering to multiply equation 5 by a factor of -1 since it describes a **decrease** in x) to get the expression

$$\frac{dx}{dt} = \alpha\Big[-x(1-x)\sum_{\overline{\pi}_l > \overline{\pi}_h}(\overline{\pi}_l - \overline{\pi}_h) + x(1-x)\sum_{\overline{\pi}_h \geq \overline{\pi}_l}(\overline{\pi}_h - \overline{\pi}_l)\Big]. \tag{6}$$

By moving the factor of -1 from in front of the first x term into the summation, it becomes possible to combine the two summations since the ranges of their lower and upper bounds are mutually exclusive, and equation 6 can be rewritten as

$$\frac{dx}{dt} = \alpha x(1-x)(\overline{\pi}_h - \overline{\pi}_l). \tag{7}$$

We must now derive expressions for $\overline{\pi}_h$ and $\overline{\pi}_l$ using equations 1 and 2. When calculating $\overline{\pi}_i$ for some general strategy i, we only need consider groups with at least one student with strategy i in them, and then we would expect the remainder of the group to contain $(n-1)x$ hard workers and $(n-1)(1-x)$ lazy workers, hence

$$\overline{\pi}_i = E_i\left(\frac{1}{n} - a\right) + \frac{(n-1)xE_h + (n-1)(1-x)E_l}{n}. \tag{8}$$

Finally, we must derive an expression for α. To guarantee that $P(j \to i)$ has a lower limit of exactly zero and an upper limit of exactly one, α must be the reciprocal of the difference between the maximum possible payoff π_{max} and the minimum π_{min}. From equations 2 and 8, we see π_{max} occurs when a student has strategy i which makes the $E_i\left(1/n-a\right)$ term as large as possible, which depends on $sign\left(1/n-a\right)$, and when the remainder of the group is made up of hard workers. Similarly, π_{min} occurs when $E_i\left(1/n-a\right)$ is as small as possible and when the remainder of the group is made up of lazy workers.

$$\pi_{max} = \begin{cases} E_h\left(\frac{1}{n}-a\right)+\frac{(n-1)E_h}{n} & \text{if } \frac{1}{n}-a \geq 0 \\ E_l\left(\frac{1}{n}-a\right)+\frac{(n-1)E_h}{n} & \text{if } \frac{1}{n}-a < 0. \end{cases} \tag{9}$$

$$\pi_{min} = \begin{cases} E_l\left(\frac{1}{n}-a\right)+\frac{(n-1)E_l}{n} & \text{if } \frac{1}{n}-a \geq 0 \\ E_h\left(\frac{1}{n}-a\right)+\frac{(n-1)E_l}{n} & \text{if } \frac{1}{n}-a < 0. \end{cases} \tag{10}$$

Subtracting equation 10 from 9 to get the reciprocal of α, and noticing that only the first term in each pair of cases for both π_{max} and π_{min} depends on $\mathrm{sign}(1/n-a)$, we get

$$\frac{1}{\alpha} = (E_h - E_l)\left|\frac{1}{n}-a\right| + \frac{(n-1)(E_h-E_l)}{n}. \tag{11}$$

Substituting $\overline{\pi}_h$, $\overline{\pi}_l$, and α into equation 7, we get our final differential equation describing the problem

$$\frac{dx}{dt} = f(x) = \beta x(1-x)(\frac{1}{n}-a) \tag{12}$$

where $\beta = \alpha(E_h - E_l)$. Notice, by requiring that α be the reciprocal of the difference between the maximum possible payoff π_{max} and the minimum π_{min}, equation 12 becomes independent of E_h and E_l. This is an ordinary differential equation since x is purely a function of a single independent variable t, and it is first order since the highest derivative dx/dt is order one. Furthermore, it is non-linear since the right hand side is quadratic in x, and it is separable since all x terms, including the infinitesimal dx, can be arranged on one side of the equation, and all the t terms, which are dt and $(1/n-a)$, can be arranged on the opposite side. It is not homogeneous, since the equation is not invariant to dilation of x due to the $(1-x)$ term, and it is autonomous since the right hand side can be written purely as a function of x (in contrast to a function of x and t).

C. Equilibrium Solutions

From equation 12, we can read off five equilibrium solutions, in other words, situations in where $f(x)$ = 0. When $a = 1/n$, then there exists a non-isolated fixed point (FP) at all x, since for any strategy i, $\overline{\pi}_i$ is constant (from equation 8), and so $\overline{\pi}_h - \overline{\pi}_j = 0$, (see equation 7).

Two more FPs occur at $x^* \in \{0,1\}$, but the equilibrium types of these FPs depend on $\mathrm{sign}(1/n-a)$. When $1/n-a > 0$, then $x^* = 0$ is unstable since $f'(0) > 0$, and $x = 1$ is stable since $f'(1) < 0$. By the same logic, when $1/n-a < 0$ then $x = 0$ is stable and $x = 1$ is unstable. These four equilibria can be explained on the basis that if a population consists only of students following a single strategy, it is impossible for any of them to change strategy, since there are no students with alternative strategies to take as reference partners.

TABLE I: Intermediate Results, $n = 4$

t	x	K1	K2	K3	K4
0	0.5	NA	NA	NA	NA
1	0.438	-0.062	-0.062	-0.062	-0.062
2	0.378	-0.062	-0.060	-0.060	-0.059
3	0.321	-0.059	-0.057	-0.057	-0.054
4	0.269	-0.054	-0.052	-0.052	-0.049
5	0.223	-0.049	-0.046	-0.046	-0.043
6	0.182	-0.043	-0.040	-0.040	-0.037
7	0.148	-0.037	-0.034	-0.034	-0.032
8	0.119	-0.032	-0.029	-0.029	-0.026

D. Full Solutions

To find a full solution for $x(t)$, equation 12 is first rearranged to get all x terms on one side and all t terms on the other, and then both sides are integrated with respect to their appropriate variable giving

$$\ln\frac{x}{x-1} = \beta(\frac{1}{n}-a)t + c_1 \tag{13}$$

where c_1 is the combination of constants of integration. Taking the exponential of both sides, and rearranging gives the general solution

$$x = \frac{e^{\beta(\frac{1}{n}-a)t}}{e^{\beta(\frac{1}{n}-a)t}+c_2}. \tag{14}$$

From this general solution, we can find particular solutions for the problem definition by calculating c_2 using initial conditions and substituting into equation 14. For $t = 0$, this gives

$$x_0 = \frac{e^0}{e^0 + c_2} \tag{15}$$

and therefore

$$c_2 = \frac{1}{x_0} - 1. \tag{16}$$

III. Numerical Integration

To numerically integrate the differential equation, a fourth order explicit Runge-Kutta method was used. The programming language Python was used to program the algorithm, the main class of which can be seen in listing A.

A. Evidence of Correct Operation

Table I shows the results, along with the K components of the algorithm, over several semesters for a scenario with the following parameterisation $x_0 = 0.5$, $a = 0.5$, $n = 4$, $E_h = 1$ and $E_l = 0$ as evidence of correctness.

B. Parameterisation

To select an appropriate value for the time-step δ, $x(t)$ was calculated for $t \in [0, 30]$ for $\delta \in \{0.1, 1, 5\}$. As seen from Figure 1, for all values of δ, the numerical approximation to $x(t)$ is stable. Furthermore, from Figure 2, we see the numerical solution $x(t)$ matches very well with the values calculated from the analytical solution. On this basis, $\delta = 1$ was selected. However, it was noted that $\delta = 1$ is not suitable in all cases, specifically when $|1/n-a|$ is large, since this can

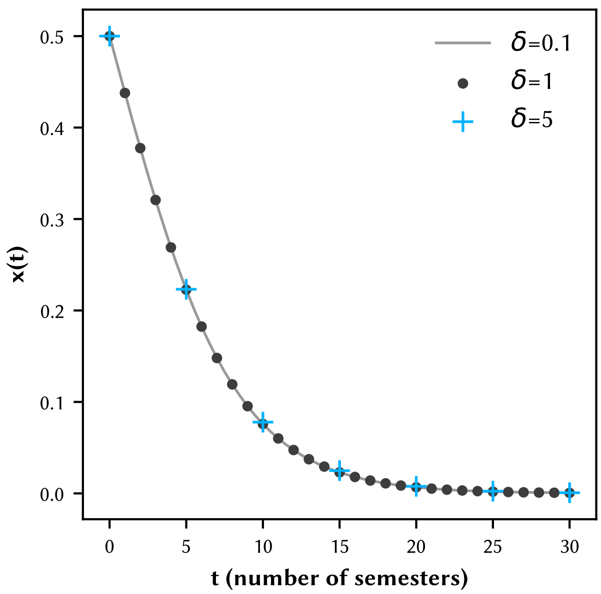

Fig. 1: Proportion of hard workers $x(t)$ for different sizes of time-step δ with $x_0 = 0.5$, $a = 0.5$, $n = 4$, $E_h = 1$, $E_l = 0$.

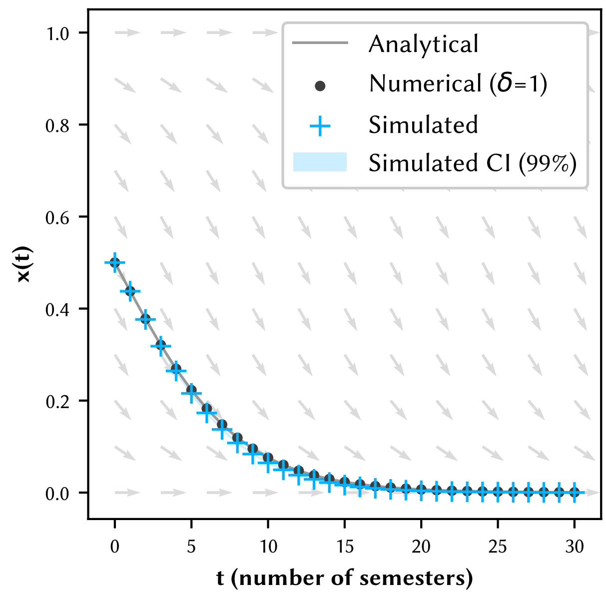

Fig. 2: Proportion of hard workers $x(t)$ as a function of time t for all methods, with $x_0 = 0.5$, $a = 0.5$, $n = 4$, $E_h = 1$ and $E_l = 0$. Trajectories of $x(t)$ are shown as arrows. Confidence interval (CI) for simulated model only just visible.

lead to numerical instability (such situations were avoided).

Listing 1: Code snippet for numerical integration of $f(x)$, ‘self’ keyword abbreviated to ‘s’.

```
class integrator:

  @staticmethod
  def get_f(n,a,beta):
    def f(x):
      return beta*x*(1-x)*((1/n)-a)
    return f

  def __init__(s,n,x0,a,beta,h):
    s.f = integrator.get_f(n,a,beta)
    s.x0 = x0
    s.x = x0
    s.delta = delta

  def advance(s,verbose=False):
    K1 = s.delta * s.f(s.x)
    K2 = s.delta * s.f(s.x + 0.5*K1)
    K3 = s.delta * s.f(s.x + 0.5*K2)
    K4 = s.delta * s.f(s.x + K3)
    s.x = s.x+(K1+2*(K2+K3)+K4)/ 6
    if verbose:
      return (s.x, K1, K2, K3, K4)
    else:
      return s.x
```

IV. Agent-Based Model

To simulate the behaviour of the hypothetical system described in section II-A, an agent-based model was built using the Python scripting language. The Numpy package was used to vectorise all operations, meaning that large numbers of students ($> 10^7$), bounded only by memory constraints, could be simulated. Since each individual simulation is stochastic, the results of a single run may not be consistent with the analytical or numerical solutions. Therefore, for each experiment, many simulations were run to get a more accurate estimate of the expected behaviour.

A. *Evidence of Correct Operation*

As evidence of the model's correct operation, outpit on a very small scale simulation can be found in Appendix A. In this simulation (with a population of eight, $x_0 = 0.5$, $a = 0.5$, $n = 4$, $E_h = 1$, and $E_l = 0$) four hardworking and four lazy students study for three semesters and therefore update their strategies three times. Since $1/n - a < 0$ we would expect this group to tend towards the lazy strategy, and this is indeed what we see. We see Student 3 (hardworking) begins their studies in a group with one other hardworking student and two lazy students (Group 2), which gives them a mark of 0.5. Since Student 3 is hardworking, their payoff is 0 (from equations 1 and 2). Student 3 then selects Student 5 as their reference partner, who started with a lazy strategy, and who therefore had a payoff of 0.5 since Group 1 also received a 0.5 mark. In this instance, Student 3 happens to change strategy to lazy, the probability of which was 50% (from equations 3 and 11). As proof of the stochastic nature of the simulation, notice that Student 1 does not change strategies between semesters $t = 2$ and $t = 3$, even though the payoff difference between them and their reference partner is positive.

V. Results

A. *Research Question 1*

From the coursework definition, we are asked to determine the composition of the population after four years (i.e. $t = 8$),

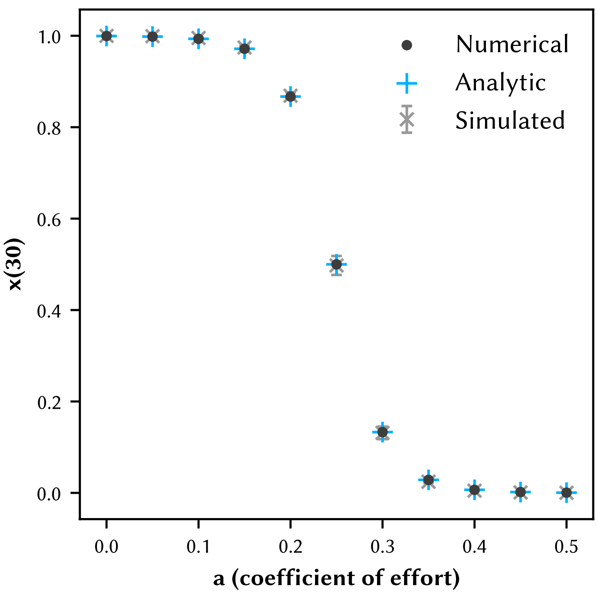

Fig. 3: Proportion of hard workers at time $t = 30$ for different values of a with $x_0 = 0.5$, $n = 4$, $E_h = 1$, $E_l = 0$. Error bars on simulation for 99% confidence interval with population of 10^4.

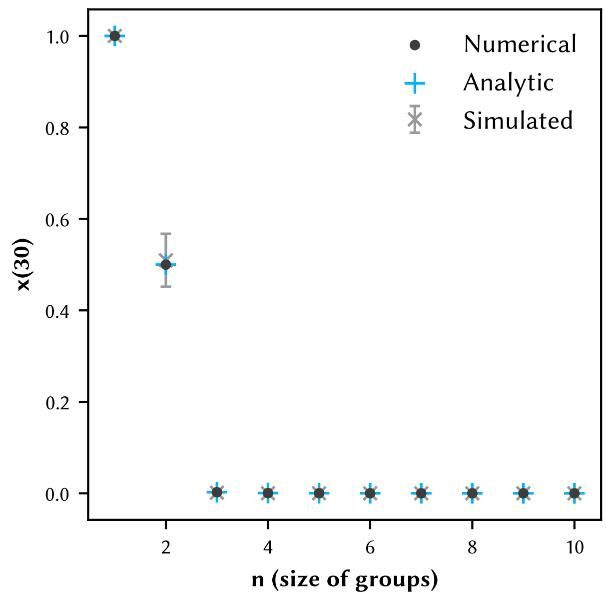

Fig. 4: Proportion of hard workers at time $t = 30$ for different values of n with $x_0 = 0.5$, $n = 4$, $E_h = 1$, $E_l = 0$. Error bars on simulation for 99% confidence interval with population of 10^4.

given the following parameterisation $x_0 = 0.5$, $a = 0.5$, $E_h = 1$ and $E_l = 0$. To complete the parameterisation, we choose $n = 4$. Using equations 14 and 16, it was found that approximately 11.9% of students continued to follow a hardworking strategy after four years (corresponding to $x(8) = 0.119$). The value found using the numerical integration method matched to five decimal places. The value from the simulation, calculated by averaging ten runs, each with 10^7 students, was found to be slightly lower at $0.107 \pm 4.1 \times 10^{-5}$ with 99% confidence. The difference in the values may be accounted for on the basis that the differential equation is an approximation of the system behaviour. Interestingly, it was noted that the difference between the simulation and the analytical/numerical solutions increases as $|1/n - a|$ increases, suggesting that the discrete system has a slightly different dynamic to that of the continuous system.

In the long run, all methods indicate that the proportion of hardworking students will drop to zero, and the proportion of lazy students will increase to one as seen in Figure 2. Using the analytical method, this state is only ever reached after an infinite number of semesters, but for practical purposes, less than 1% of students are hardworking after 25 semesters.

B. *Research Question 2*

We now look at how varying the values of $x_0 = 0.5$, $a = 0.5$, $n = 4$, $E_h = 1$, and $E_l = 0$ affects the behaviour of the system. Considering x_0 first, we see from the trajectories shown in Figure 2 by the grey arrows, that unless $x_0 = 1$, x will always tend to zero when $1/n < a$ since $sign(f) = sign(1/n - a)$ for $0 < x < 1$. This is consistent with the equilibrium analysis performed in section II-C.

In regard to a, in Figure 3, which shows the influence of a on $x(30)$ with $x_0 = 0.5$, we see all methods are in good agreement. When $a < 1/n$ we see that the population adopts the hardworking strategy, and when $a > 1/n$ the lazy strategy dominates. As $|1/n - a|$ gets smaller, the number of iterations required for one strategy to dominate increases. When $a = 1/n = 0.25$ we see that $x(30) = x_0$ as expected from equation 12, in other words, the proportions of hardworking and lazy workers remains constant.

The behaviour of the system as n varies is closely related to how it varies with a. From 4, we see when $n = 1$, then $1/n - a > 0$ and the hardworking strategy dominates, and when $n > 2$, then $1/n - a < 0$ and the lazy strategy dominates. This asymmetric behaviour in the system (unlike that seen when varying a) implies that if a professor wishes to get the best out of their students, they ought to ensure that groups sizes are set as small as possible.

VI. Extension

The system rules defined in the coursework description make many simplifications regarding the ways in which students behave and how they interact with one another. Perhaps one of the most significant is the assumption that there exist only two strategies, hardworking and lazy. In this section, we investigate the behaviour of the system when more than two strategies are possible. First, an analytical evaluation of the equilibrium solutions for a system with three strategies is presented (including a medium effort strategy) which lays the groundwork for consideration of situations in which the students' levels of effort are drawn from a continuous distribution i.e. an infinite number of strategies. The results of simulations which model the continuous strategy situation are then presented and discussed.

A. Analytical Approach To Multiple Strategies

Consider the situation in which another strategy, a medium effort strategy E_m where $E_h > E_m > E_l$, is introduced. A new variable, in addition to x, is then required to fully describe the system. Let this variable be y and let it describe the proportion of lazy workers. It is then the case that z, the proportion of students with a medium effort strategy, is given by $z = 1-x-y$. Using a similar approach to the way in which equation 8 was derived, via the conditional summations, it can be shown that for some strategy i

$$\frac{dw_i}{dt} = \alpha w_i(\overline{\pi}_i - \overline{\pi}), \tag{17}$$

where w_i is the proportion of students following strategy i and $\overline{\pi}$ is the expected payoff for the whole population. The main difference in the derivation for the two systems, is that when dealing with more than two strategies, the summation terms in equation 6 need to be summed over all strategies where the inequalities hold. Equation 17 is known as the replicator equation. In the case of three strategies, for some strategy i, then

$$\overline{\pi}_i = E_i\left(\frac{1}{n} - a\right) + \frac{(n-1)\left(xE_h + yE_l + (1-x-y)E_m\right)}{n}, \tag{18}$$

and

$$\overline{\pi} = (1-a)\left(xE_h + yE_l + (1-x-y)E_m\right). \tag{19}$$

Using the replicator equation, the rate of change of hard workers is then given by

$$\frac{dx}{dt} = \gamma x\big((1-x)E_h - yE_l - (1-x-y)E_m\big) \tag{20}$$

where

$$\gamma = \alpha\left(\frac{1}{n} - a\right). \tag{21}$$

As with the case of two strategies, α is set the reciprocal of the difference between the maximum possible payoff π_{max} and the minimum π_{min}. Since $E_h > E_m > E_l$, then equations 9, 10 and therefore 11 still hold.

Inspecting equation 20, we see dx/dt it is zero when $x = 1$ (requiring $y = 0$), similarly dy/dt is zero when $x = 1$ and $y = 0$. Therefore there exists a FP at $x = 1$, $y = 0$. There exists another FP at $x = 0$, $y = 1$ by the same logic. Furthermore, when both x and y are zero, then both equations are zero again, hence we can infer that another FP exists when $z = 1$. There are, therefore, at least three equilibrium points, occurring when all students have the same strategy. To determine if there are any other equilibrium points, we can equate equation dx/dt to zero and divide through by x, perform the equivalent operation with dy/dt, and then solve the system of equations. This results in $E_h + E_l = 0$, a contradiction by our definitions of E_h and E_l, hence, there are no other FPs.

To determine the type of each FP, we must solve, at each FP, the characteristic equation for the 2×2 Jacobian $\mathbf{J}$ for this system of equations. Inspecting the derivative of dx/dt with respect to y and dy/dt with respect to x (the off-diagonal elements of $\mathbf{J}$), we see that at least one of these terms will always be zero for all FPs, and hence the solutions λ_1 and λ_2 of the characteristic equation of $\mathbf{J}$ are given by

$$\lambda_1 = \frac{d^2x}{dx^2} = \gamma\big((1-2x)E_h - yE_l - (1-2x-y)E_m\big) \tag{22}$$

and

$$\lambda_2 = \frac{d^2y}{dy^2} = \gamma\big((1-2y)E_l - xE_h - (1-x-2y)E_m\big). \tag{23}$$

When $x = y = 0$, that is when $z = 1$, then $\lambda_1 = \gamma(E_h - E_z)$ and $\lambda_2 = \gamma(E_y - E_z)$. Since $E_y < E_z < E_x$, then $\operatorname{sign}(\lambda_1) = -\operatorname{sign}(\lambda_2)$ and $x = y = 0$ is a saddle point. When $x = 1$ then $\lambda_1 = \gamma(E_z - E_h)$ and $\lambda_2 = \gamma(E_l - E_h)$ and so $\operatorname{sign}(\lambda_1) = \operatorname{sign}(\lambda_2)$, meaning $x = 1$ is a stable FP when $\gamma > 0$ (i.e. $1/n - a > 0$) and unstable when $\gamma < 0$. The same procedure can be applied to determine the stability of the FP $y = 0$, and it can be shown that when $x = 1$ is stable then $y = 0$ is always unstable and that the reverse is also true.

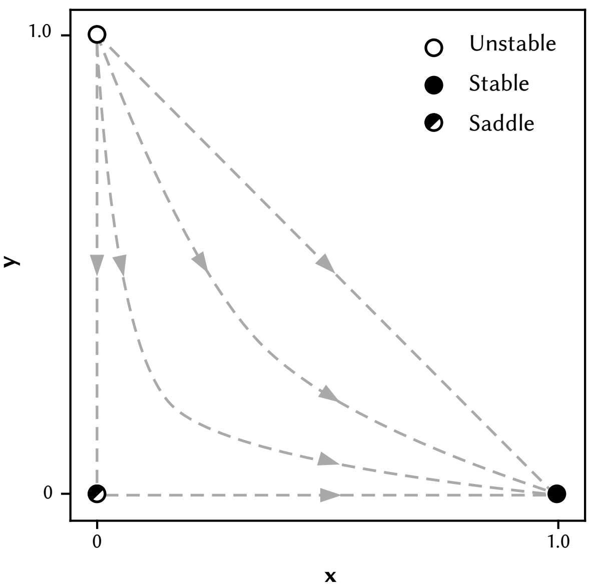

Fig. 5: Equilibrium points for a system with three strategies with $\gamma > 0$. Arrows indication direction of change with respect to time.

Figure 5 illustrates this system of three strategies. Since $\gamma > 0$ here, $y = 1$ is unstable and $x = 1$ is stable. The diagram illustrates that we would expect at least some lazy workers to transition through the intermediate medium effort strategy (assuming $z_0 \neq 0$) and from there medium effort workers will tend to adopt a hardworking strategy. After an infinite amount of time, we would expect all students to adopt a hardworking strategy. The reverse would be true for $\gamma < 0$. Interestingly, there is no situation whereby students tend to adopt, as t goes to infinity, the medium effort strategy.

By complicating the system to one containing three strategies, it raises the question of how the system might behave if a continuous range of efforts is possible, as opposed to a discrete number. The author hypothesises that one of the extreme strategies would still always dominate. In order to show this analytically, it would be necessary to show that in general, for any number of strategies, that FPs exist only when

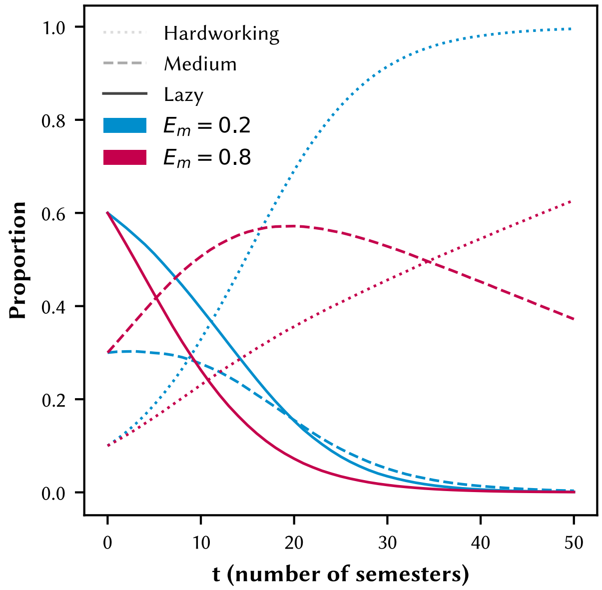

Fig. 6: Proportion of hardworking, lazy and medium effort strategies over time for two different values of E_m where $x_0 = 0.1$, $y_0 = 0.6$, $z = 0.3$, $a = 0.1$, $n = 4$, $E_h = 1$ and $E_l = 0$.

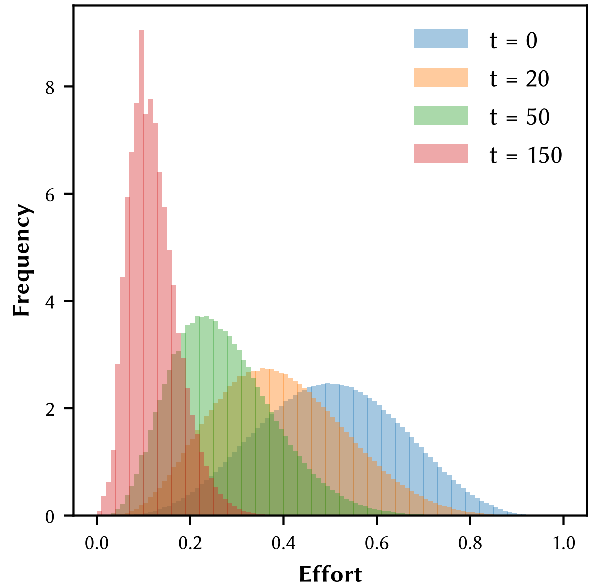

Fig. 7: Frequency density of efforts for different times, averaged over one hundred runs. Population efforts at $t = 0$ sampled from a beta distribution.

all students adopt a single strategy. It is clearly the case that FPs always exist when students adopt a single strategy (after all they will never have anyone to copy a different strategy from) but proving this covers the complete set of FPs is more difficult. Perhaps some proof by induction may be sufficient to show that the system of equations is limited in this way.

Secondly, it would be necessary to prove that the eigenvalues of the solutions are always real, and that there exists a single stable FP (corresponding to either the maximum or minimum effort), a single unstable FP (respectively corresponding to the minimum or maximum effort), and an infinite number of saddles points at the efforts in-between. It may be possible to prove this by extending the trick used to generate equations 22 and 23, by showing that the contribution from the off-diagonal components is always zero.

B. *Simulating Multiple Strategies*

To facilitate the evaluation of problems with three strategies, modifications to the simulation program were created. Figure 6 shows the results of two different simulations, averaged over many runs, for different values of E_m. Given that we know that $y = x = 0$ is a saddle point from Figure 5, we might expect under certain conditions that the proportion of medium workers may initially grow, as lazy workers convert to medium effort strategies and then subsequently decline as medium effort students adopt hardworking approaches.

This is confirmed experimentally for both $E_m = 0.8$ and $E_m = 0.2$ as shown in figure 6. In both situations, the lazy strategy drops off quickly, and the hardworking strategy dominates in the long term. We see from that when $E_h = 0.8$ there is a significant amount of growth in the proportion of students with the medium effort strategy before it begins to decline again. This is also the case when $E_m = 0.2$, however, the growth is barely visible in the figure.

C. *Simulating a Continuous Range of Efforts*

Finally, the behaviour of a simulated system, in which a continuous range of efforts is possible, was investigated. An initial population with efforts sampled from a beta distribution was first generated. The value of $1/n - a$ was chosen to be negative, so as to encourage students to adopt lazier strategies. Figure 7 illustrates the change in the frequency density distributions of efforts over time for this population. The first thing to note is that even when a large number (in comparison to previous simulations) of semesters have passed, there is still quite a broad spread of efforts. It was observed in simulations with longer running times that the rate at which students adopt lower effort strategies decreases exponentially with time. The general trend however supports the earlier hypothesis that all students will tend to adopt one of the extreme strategies as time goes to infinity.

VII. Conclusion

For a system of two or three strategies, it has been shown using multiple methods that when $1/n - a \neq 0$ then the strategy with the highest expected payoff is expected to dominate in the long run. In the case of three or more strategies, it has been shown that whilst intermediate effort strategies may to grow in popularity initially, in the long term, they inevitably give way to the strategy with the highest expected payoff.

As a consequence, even when allowing for a continuous range of efforts, the model remains unrealistic, since in the case where $1/n - a < 0$, the low effort strategies will dominate in the long term, resulting in very low marks for everyone, which is not what is observed in reality. In a further extension to this topic, it would be interesting to investigate if including a term relating to a minimum passing grade required for continuation to the next semester might generate more realistic behaviour.

Appendix

A. Example Output from Simulation

t: 1
Simulator: 0.500

Group	StudentID	Strategy	PartnerID	PairingIndex	Mark	Cost	Payoff	PayoffDiff	NextStrat
0	5	Lazy	1	6	0.5	0.0	0.5	−0.5	Lazy
0	0	Hard	6	2	0.5	0.5	0.0	0.5	Lazy
0	6	Lazy	1	6	0.5	0.0	0.5	−0.5	Lazy
0	2	Hard	0	1	0.5	0.5	0.0	0.0	Hard
1	4	Lazy	3	5	0.5	0.0	0.5	−0.5	Lazy
1	3	Hard	5	0	0.5	0.5	0.0	0.5	Lazy
1	1	Hard	2	3	0.5	0.5	0.0	0.0	Hard
1	7	Lazy	7	7	0.5	0.0	0.5	0.0	Lazy

t: 2
Simulator: 0.250

Group	StudentID	Strategy	PartnerID	PairingIndex	Mark	Cost	Payoff	PayoffDiff	NextStrat
0	2	Hard	4	7	0.25	0.5	−0.25	0.5	Lazy
0	3	Lazy	0	4	0.25	0.0	0.25	0.0	Lazy
0	5	Lazy	3	1	0.25	0.0	0.25	0.0	Lazy
0	7	Lazy	2	0	0.25	0.0	0.25	−0.5	Lazy
1	0	Lazy	4	7	0.25	0.0	0.25	0.0	Lazy
1	6	Lazy	4	7	0.25	0.0	0.25	0.0	Lazy
1	1	Hard	6	5	0.25	0.5	−0.25	0.5	Hard
1	4	Lazy	2	0	0.25	0.0	0.25	−0.5	Lazy

t: 3
Simulator: 0.125

Group	StudentID	Strategy	PartnerID	PairingIndex	Mark	Cost	Payoff	PayoffDiff	NextStrat
0	2	Lazy	7	1	0.25	0.0	0.25	0.00	Lazy
0	7	Lazy	5	6	0.25	0.0	0.25	−0.25	Lazy
0	1	Hard	3	4	0.25	0.5	−0.25	0.25	Hard
0	0	Lazy	6	7	0.25	0.0	0.25	−0.25	Lazy
1	3	Lazy	5	6	0.00	0.0	0.00	0.00	Lazy
1	4	Lazy	5	6	0.00	0.0	0.00	0.00	Lazy
1	5	Lazy	7	1	0.00	0.0	0.00	0.25	Lazy
1	6	Lazy	6	7	0.00	0.0	0.00	0.00	Lazy