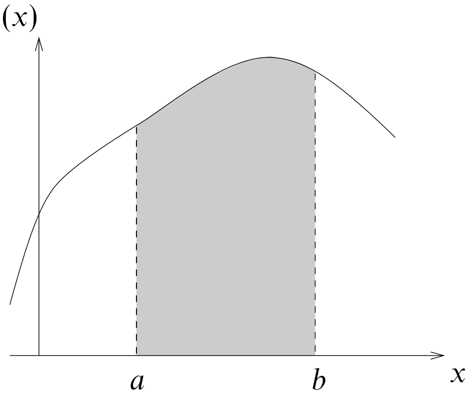

Figure 2.7 An integral as the area under a curve.

2.2 Integration

The notion of an integral as the area under a curve will be familiar to the reader. In figure 2.7, in which the solid line is a plot of a function $f(x)$, the shaded area represents the quantity denoted by

$$I = \int_a^b f(x)\,dx. \tag{2.21}$$

This expression is known as the *definite integral* of $f(x)$ between the *lower limit* $x = a$ and the *upper limit* $x = b$, and $f(x)$ is called the *integrand*.

2.2.1 Integration from first principles

The definition of an integral as the area under a curve is not a formal definition, but one that can be readily visualised. The formal definition of I involves subdividing the finite interval $a \leq x \leq b$ into a large number of subintervals, by defining intermediate points ξ_i such that $a = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = b$, and then forming the sum

$$S = \sum_{i=1}^{n} f(x_i)(\xi_i - \xi_{i-1}), \tag{2.22}$$

where x_i is an arbitrary point that lies in the range $\xi_{i-1} \leq x_i \leq \xi_i$ (see figure 2.8). If now n is allowed to tend to infinity in any way whatsoever, subject only to the restriction that the length of every subinterval ξ_{i-1} to ξ_i tends to zero, then S might, or might not, tend to a unique limit, I. If it does then the definite integral of $f(x)$ between a and b is defined as having the value I. If no unique limit exists the integral is undefined. For continuous functions and a finite interval $a \leq x \leq b$ the existence of a unique limit is assured and the integral is guaranteed to exist.

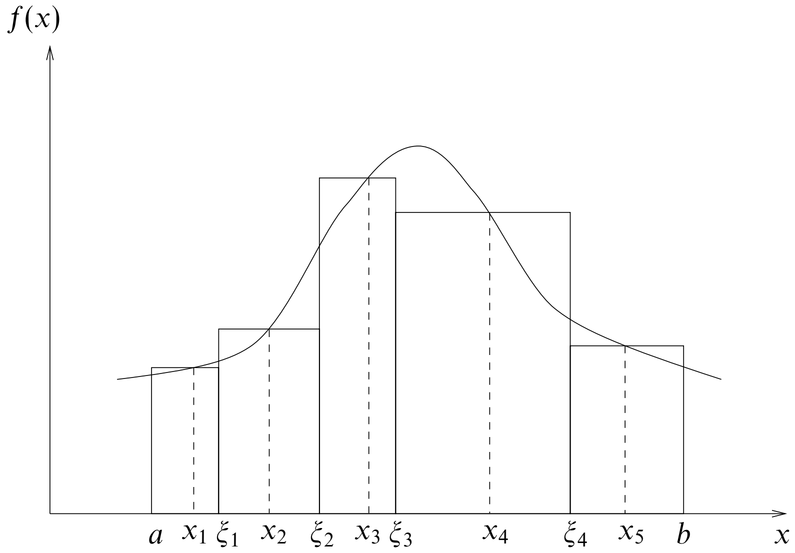

Figure 2.8 The evaluation of a definite integral by subdividing the interval $a \leq x \leq b$ into subintervals.

►*Evaluate from first principles the integral* $I = \int_0^b x^2\,dx$.

We first approximate the area under the curve $y = x^2$ between 0 and b by n rectangles of equal width h. If we take the value at the lower end of each subinterval (in the limit of an infinite number of subintervals we could equally well have chosen the value at the upper end) to give the height of the corresponding rectangle, then the area of the kth rectangle will be $(kh)^2h = k^2h^3$. The total area is thus

$$A = \sum_{k=0}^{n-1} k^2h^3 = (h^3)\tfrac{1}{6}n(n-1)(2n-1),$$

where we have used the expression for the sum of the squares of the natural numbers derived in subsection 1.7.1. Now $h = b/n$ and so

$$A = \left(\frac{b^3}{n^3}\right)\frac{n}{6}(n-1)(2n-1) = \frac{b^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right).$$

As $n \to \infty$, $A \to b^3/3$, which is thus the value I of the integral. ◄

Some straightforward properties of definite integrals that are almost self-evident are as follows:

$$\int_a^b 0\,dx = 0, \qquad \int_a^a f(x)\,dx = 0, \tag{2.23}$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx, \tag{2.24}$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \tag{2.25}$$

Combining (2.23) and (2.24) with c set equal to a shows that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \tag{2.26}$$

2.2.2 Integration as the inverse of differentiation

The definite integral has been defined as the area under a curve between two fixed limits. Let us now consider the integral

$$F(x) = \int_a^x f(u)\,du \tag{2.27}$$

in which the lower limit a remains fixed but the upper limit x is now variable. It will be noticed that this is essentially a restatement of (2.21), but that the variable x in the integrand has been replaced by a new variable u. It is conventional to rename the *dummy variable* in the integrand in this way in order that the same variable does not appear in both the integrand and the integration limits.

It is apparent from (2.27) that $F(x)$ is a continuous function of x, but at first glance the definition of an integral as the area under a curve does not connect with our assertion that integration is the inverse process to differentiation. However, by considering the integral (2.27) and using the elementary property (2.24), we obtain

$$\begin{aligned} F(x+\Delta x) &= \int_a^{x+\Delta x} f(u)\,du \\ &= \int_a^x f(u)\,du + \int_x^{x+\Delta x} f(u)\,du \\ &= F(x) + \int_x^{x+\Delta x} f(u)\,du. \end{aligned}$$

Rearranging and dividing through by Δx yields

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x}\int_x^{x+\Delta x} f(u)\,du.$$

Letting $\Delta x \to 0$ and using (2.1) we find that the LHS becomes dF/dx, whereas the RHS becomes $f(x)$. The latter conclusion follows because when Δx is small the value of the integral on the RHS is approximately $f(x)\Delta x$, and in the limit $\Delta x \to 0$ no approximation is involved. Thus

$$\frac{dF(x)}{dx} = f(x), \tag{2.28}$$

or, substituting for $F(x)$ from (2.27),

$$\frac{d}{dx}\left[\int_a^x f(u)\,du\right] = f(x).$$

From the last two equations it is clear that integration can be considered as the inverse of differentiation. However, we see from the above analysis that the lower limit a is arbitrary and so differentiation does not have a *unique* inverse. Any function $F(x)$ obeying (2.28) is called an *indefinite integral* of $f(x)$, though any two such functions can differ by at most an arbitrary additive constant. Since the lower limit is arbitrary, it is usual to write

$$F(x) = \int^{x} f(u)\,du \tag{2.29}$$

and explicitly include the arbitrary constant only when evaluating $F(x)$. The evaluation is conventionally written in the form

$$\int f(x)\,dx = F(x) + c \tag{2.30}$$

where c is called the *constant of integration*. It will be noticed that, in the absence of any integration limits, we use the same symbol for the arguments of both f and F. This can be confusing, but is sufficiently common practice that the reader needs to become familiar with it.

We also note that the definite integral of $f(x)$ between the fixed limits $x = a$ and $x = b$ can be written in terms of $F(x)$. From (2.27) we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_{x_0}^b f(x)\,dx - \int_{x_0}^a f(x)\,dx \\ &= F(b) - F(a),\end{aligned} \tag{2.31}$$

where x_0 is *any* third fixed point. Using the notation $F'(x) = dF/dx$, we may rewrite (2.28) as $F'(x) = f(x)$, and so express (2.31) as

$$\int_a^b F'(x)\,dx = F(b) - F(a) \equiv [F]_a^b.$$

In contrast to differentiation, where repeated applications of the product rule and/or the chain rule will always give the required derivative, it is not always possible to find the integral of an arbitrary function. Indeed, in most real physical problems exact integration cannot be performed and we have to revert to numerical approximations. Despite this cautionary note, it is in fact possible to integrate many simple functions and the following subsections introduce the most common types. Many of the techniques will be familiar to the reader and so are summarised by example.

2.2.3 Integration by inspection

The simplest method of integrating a function is by inspection. Some of the more elementary functions have well-known integrals that should be remembered. The reader will notice that these integrals are precisely the inverses of the derivatives found near the end of subsection 2.1.1. A few are presented below, using the form given in (2.30):

$$\int a\,dx = ax + c, \qquad \int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c,$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c, \qquad \int \frac{a}{x}\,dx = a\ln x + c,$$

$$\int a\cos bx\,dx = \frac{a\sin bx}{b} + c, \qquad \int a\sin bx\,dx = \frac{-a\cos bx}{b} + c,$$

$$\int a\tan bx\,dx = \frac{-a\ln(\cos bx)}{b} + c, \qquad \int a\cos bx\sin^n bx\,dx = \frac{a\sin^{n+1} bx}{b(n+1)} + c,$$

$$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c, \qquad \int a\sin bx\cos^n bx\,dx = \frac{-a\cos^{n+1} bx}{b(n+1)} + c,$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c, \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c,$$

where the integrals that depend on n are valid for all $n \neq -1$ and where a and b are constants. In the two final results $|x| \leq a$.

2.2.4 Integration of sinusoidal functions

Integrals of the type $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$ may be found by using trigonometric expansions. Two methods are applicable, one for odd n and the other for even n. They are best illustrated by example.

►*Evaluate the integral* $I = \int \sin^5 x\,dx$.

Rewriting the integral as a product of $\sin x$ and an even power of $\sin x$, and then using the relation $\sin^2 x = 1 - \cos^2 x$ yields

$$\begin{aligned}
I &= \int \sin^4 x \sin x\,dx \\
&= \int (1-\cos^2 x)^2 \sin x\,dx \\
&= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx \\
&= \int (\sin x - 2\sin x\cos^2 x + \sin x\cos^4 x)\,dx \\
&= -\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + c,
\end{aligned}$$

where the integration has been carried out using the results of subsection 2.2.3. ◄

►*Evaluate the integral* $I = \int \cos^4 x\, dx$.

Rewriting the integral as a power of $\cos^2 x$ and then using the double-angle formula $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ yields

$$\begin{aligned} I &= \int (\cos^2 x)^2\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \int \tfrac{1}{4}(1 + 2\cos 2x + \cos^2 2x)\, dx. \end{aligned}$$

Using the double-angle formula again we may write $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$, and hence

$$\begin{aligned} I &= \int \left[\tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{8}(1 + \cos 4x)\right] dx \\ &= \tfrac{1}{4}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x + c \\ &= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + c. \blacktriangleleft \end{aligned}$$

2.2.5 Logarithmic integration

Integrals for which the integrand may be written as a fraction in which the numerator is the derivative of the denominator may be evaluated using

$$\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + c. \qquad (2.32)$$

This follows directly from the differentiation of a logarithm as a function of a function (see subsection 2.1.3).

►*Evaluate the integral*

$$I = \int \frac{6x^2 + 2\cos x}{x^3 + \sin x}\, dx.$$

We note first that the numerator can be factorised to give $2(3x^2 + \cos x)$, and then that the quantity in brackets is the derivative of the denominator. Hence

$$I = 2\int \frac{3x^2 + \cos x}{x^3 + \sin x}\, dx = 2\ln(x^3 + \sin x) + c. \blacktriangleleft$$

2.2.6 Integration using partial fractions

The method of partial fractions was discussed at some length in section 1.4, but in essence consists of the manipulation of a fraction (here the integrand) in such a way that it can be written as the sum of two or more simpler fractions. Again we illustrate the method by an example.

►*Evaluate the integral*

$$I = \int \frac{1}{x^2 + x}\, dx.$$

We note that the denominator factorises to give $x(x+1)$. Hence

$$I = \int \frac{1}{x(x+1)}\, dx.$$

We now separate the fraction into two partial fractions and integrate directly:

$$I = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = \ln x - \ln(x+1) + c = \ln\left(\frac{x}{x+1} \right) + c. \blacktriangleleft$$

2.2.7 Integration by substitution

Sometimes it is possible to make a substitution of variables that turns a complicated integral into a simpler one, which can then be integrated by a standard method. There are many useful substitutions and knowing which to use is a matter of experience. We now present a few examples of particularly useful substitutions.

►*Evaluate the integral*

$$I = \int \frac{1}{\sqrt{1-x^2}}\, dx.$$

Making the substitution $x = \sin u$, we note that $dx = \cos u\, du$, and hence

$$I = \int \frac{1}{\sqrt{1-\sin^2 u}} \cos u\, du = \int \frac{1}{\sqrt{\cos^2 u}} \cos u\, du = \int du = u + c.$$

Now substituting back for u,

$$I = \sin^{-1} x + c.$$

This corresponds to one of the results given in subsection 2.2.3. ◄

Another particular example of integration by substitution is afforded by integrals of the form

$$I = \int \frac{1}{a + b\cos x}\, dx \qquad \text{or} \qquad I = \int \frac{1}{a + b\sin x}\, dx. \tag{2.33}$$

In these cases, making the substitution $t = \tan(x/2)$ yields integrals that can be solved more easily than the originals. Formulae expressing $\sin x$ and $\cos x$ in terms of t were derived in equations (1.32) and (1.33) (see p. 14), but before we can use them we must relate dx to dt as follows.

Since

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) = \frac{1+t^2}{2},$$

the required relationship is

$$dx = \frac{2}{1+t^2}\,dt. \qquad (2.34)$$

►*Evaluate the integral*

$$I = \int \frac{2}{1+3\cos x}\,dx.$$

Rewriting $\cos x$ in terms of t and using (2.34) yields

$$\begin{aligned}
I &= \int \frac{2}{1+3\left[(1-t^2)(1+t^2)^{-1}\right]}\left(\frac{2}{1+t^2}\right)dt \\
&= \int \frac{2(1+t^2)}{1+t^2+3(1-t^2)}\left(\frac{2}{1+t^2}\right)dt \\
&= \int \frac{2}{2-t^2}\,dt = \int \frac{2}{(\sqrt{2}-t)(\sqrt{2}+t)}\,dt \\
&= \int \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}-t}+\frac{1}{\sqrt{2}+t}\right)dt \\
&= -\frac{1}{\sqrt{2}}\ln(\sqrt{2}-t)+\frac{1}{\sqrt{2}}\ln(\sqrt{2}+t)+c \\
&= \frac{1}{\sqrt{2}}\ln\left[\frac{\sqrt{2}+\tan(x/2)}{\sqrt{2}-\tan(x/2)}\right]+c. \blacktriangleleft
\end{aligned}$$

Integrals of a similar form to (2.33), but involving $\sin 2x$, $\cos 2x$, $\tan 2x$, $\sin^2 x$, $\cos^2 x$ or $\tan^2 x$ instead of $\cos x$ and $\sin x$, should be evaluated by using the substitution $t = \tan x$. In this case

$$\sin x = \frac{t}{\sqrt{1+t^2}}, \qquad \cos x = \frac{1}{\sqrt{1+t^2}} \qquad \text{and} \qquad dx = \frac{dt}{1+t^2}. \qquad (2.35)$$

A final example of the evaluation of integrals using substitution is the method of completing the square (cf. subsection 1.7.3).

►*Evaluate the integral*

$$I = \int \frac{1}{x^2 + 4x + 7}\, dx.$$

We can write the integral in the form

$$I = \int \frac{1}{(x+2)^2 + 3}\, dx.$$

Substituting $y = x + 2$, we find $dy = dx$ and hence

$$I = \int \frac{1}{y^2 + 3}\, dy,$$

Hence, by comparison with the table of standard integrals (see subsection 2.2.3)

$$I = \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{y}{\sqrt{3}}\right) + c = \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + c. \blacktriangleleft$$

2.2.8 Integration by parts

Integration by parts is the integration analogy of product differentiation. The principle is to break down a complicated function into two functions, at least one of which can be integrated by inspection. The method in fact relies on the result for the differentiation of a product. Recalling from (2.6) that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v,$$

where u and v are functions of x, we now integrate to find

$$uv = \int u\frac{dv}{dx}\, dx + \int \frac{du}{dx}\, v\, dx.$$

Rearranging into the standard form for integration by parts gives

$$\int u\frac{dv}{dx}\, dx = uv - \int \frac{du}{dx}\, v\, dx. \tag{2.36}$$

Integration by parts is often remembered for practical purposes in the form *the integral of a product of two functions is equal to* {*the first times the integral of the second*} *minus the integral of* {*the derivative of the first times the integral of the second*}. Here, u is 'the first' and dv/dx is 'the second'; clearly the integral v of 'the second' must be determinable by inspection.

►*Evaluate the integral* $I = \int x \sin x\, dx$.

In the notation given above, we identify x with u and $\sin x$ with dv/dx. Hence $v = -\cos x$ and $du/dx = 1$ and so using (2.36)

$$I = x(-\cos x) - \int (1)(-\cos x)\, dx = -x\cos x + \sin x + c. \blacktriangleleft$$

The separation of the functions is not always so apparent, as is illustrated by the following example.

►*Evaluate the integral* $I = \int x^3 e^{-x^2}\,dx$.

Firstly we rewrite the integral as

$$I = \int x^2 \left(xe^{-x^2}\right)\,dx.$$

Now, using the notation given above, we identify x^2 with u and xe^{-x^2} with dv/dx. Hence $v = -\frac{1}{2}e^{-x^2}$ and $du/dx = 2x$, so that

$$I = -\tfrac{1}{2}x^2 e^{-x^2} - \int (-x)e^{-x^2}\,dx = -\tfrac{1}{2}x^2 e^{-x^2} - \tfrac{1}{2}e^{-x^2} + c. ◄$$

A trick that is sometimes useful is to take '1' as one factor of the product, as is illustrated by the following example.

►*Evaluate the integral* $I = \int \ln x\,dx$.

Firstly we rewrite the integral as

$$I = \int (\ln x)\,1\,dx.$$

Now, using the notation above, we identify $\ln x$ with u and 1 with dv/dx. Hence we have $v = x$ and $du/dx = 1/x$, and so

$$I = (\ln x)(x) - \int \left(\frac{1}{x}\right) x\,dx = x\ln x - x + c. ◄$$

It is sometimes necessary to integrate by parts more than once. In doing so, we may occasionally re-encounter the original integral I. In such cases we can obtain a linear algebraic equation for I that can be solved to obtain its value.

►*Evaluate the integral* $I = \int e^{ax}\cos bx\,dx$.

Integrating by parts, taking e^{ax} as the first function, we find

$$I = e^{ax}\left(\frac{\sin bx}{b}\right) - \int ae^{ax}\left(\frac{\sin bx}{b}\right)dx,$$

where, for convenience, we have omitted the constant of integration. Integrating by parts a second time,

$$I = e^{ax}\left(\frac{\sin bx}{b}\right) - ae^{ax}\left(\frac{-\cos bx}{b^2}\right) + \int a^2 e^{ax}\left(\frac{-\cos bx}{b^2}\right)dx.$$

Notice that the integral on the RHS is just $-a^2/b^2$ times the original integral I. Thus

$$I = e^{ax}\left(\frac{1}{b}\sin bx + \frac{a}{b^2}\cos bx\right) - \frac{a^2}{b^2}I.$$

Rearranging this expression to obtain I explicitly and including the constant of integration we find

$$I = \frac{e^{ax}}{a^2 + b^2}(b \sin bx + a \cos bx) + c. \tag{2.37}$$

Another method of evaluating this integral, using the exponential of a complex number, is given in section 3.6. ◄

2.2.9 Reduction formulae

Integration using reduction formulae is a process that involves first evaluating a simple integral and then, in stages, using it to find a more complicated integral.

►*Using integration by parts, find a relationship between I_n and I_{n-1} where*

$$I_n = \int_0^1 (1 - x^3)^n \, dx$$

and n is any positive integer. Hence evaluate $I_2 = \int_0^1 (1 - x^3)^2 \, dx$.

Writing the integrand as a product and separating the integral into two we find

$$\begin{aligned} I_n &= \int_0^1 (1 - x^3)(1 - x^3)^{n-1} \, dx \\ &= \int_0^1 (1 - x^3)^{n-1} \, dx - \int_0^1 x^3 (1 - x^3)^{n-1} \, dx. \end{aligned}$$

The first term on the RHS is clearly I_{n-1} and so, writing the integrand in the second term on the RHS as a product,

$$I_n = I_{n-1} - \int_0^1 (x) x^2 (1 - x^3)^{n-1} \, dx.$$

Integrating by parts we find

$$\begin{aligned} I_n &= I_{n-1} + \left[\frac{x}{3n}(1 - x^3)^n \right]_0^1 - \int_0^1 \frac{1}{3n}(1 - x^3)^n \, dx \\ &= I_{n-1} + 0 - \frac{1}{3n} I_n, \end{aligned}$$

which on rearranging gives

$$I_n = \frac{3n}{3n + 1} I_{n-1}.$$

We now have a relation connecting successive integrals. Hence, if we can evaluate I_0, we can find I_1, I_2 etc. Evaluating I_0 is trivial:

$$I_0 = \int_0^1 (1 - x^3)^0 \, dx = \int_0^1 dx = [x]_0^1 = 1.$$

Hence

$$I_1 = \frac{(3 \times 1)}{(3 \times 1) + 1} \times 1 = \frac{3}{4}, \qquad I_2 = \frac{(3 \times 2)}{(3 \times 2) + 1} \times \frac{3}{4} = \frac{9}{14}.$$

Although the first few I_n could be evaluated by direct multiplication, this becomes tedious for integrals containing higher values of n; these are therefore best evaluated using the reduction formula. ◄

2.2.10 Infinite and improper integrals

The definition of an integral given previously does not allow for cases in which either of the limits of integration is infinite (an *infinite integral*) or for cases in which $f(x)$ is infinite in some part of the range (an *improper integral*), e.g. $f(x) = (2-x)^{-1/4}$ near the point $x = 2$. Nevertheless, modification of the definition of an integral gives infinite and improper integrals each a meaning.

In the case of an integral $I = \int_a^b f(x)\,dx$, the infinite integral, in which b tends to ∞, is defined by

$$I = \int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty} F(b) - F(a).$$

As previously, $F(x)$ is the indefinite integral of $f(x)$ and $\lim_{b\to\infty} F(b)$ means the limit (or value) that $F(b)$ approaches as $b \to \infty$; it is evaluated *after* calculating the integral. The formal concept of a limit will be introduced in chapter 4.

►*Evaluate the integral*

$$I = \int_0^\infty \frac{x}{(x^2+a^2)^2}\,dx.$$

Integrating, we find $F(x) = -\frac{1}{2}(x^2+a^2)^{-1} + c$ and so

$$I = \lim_{b\to\infty}\left[\frac{-1}{2(b^2+a^2)}\right] - \left(\frac{-1}{2a^2}\right) = \frac{1}{2a^2}. ◄$$

For the case of improper integrals, we adopt the approach of excluding the unbounded range from the integral. For example, if the integrand $f(x)$ is infinite at $x = c$ (say), $a \le c \le b$ then

$$\int_a^b f(x)\,dx = \lim_{\delta\to 0} \int_a^{c-\delta} f(x)\,dx + \lim_{\epsilon\to 0} \int_{c+\epsilon}^b f(x)\,dx.$$

►*Evaluate the integral* $I = \int_0^2 (2-x)^{-1/4}\,dx$.

Integrating directly,

$$I = \lim_{\epsilon\to 0}\left[-\tfrac{4}{3}(2-x)^{3/4}\right]_0^{2-\epsilon} = \lim_{\epsilon\to 0}\left[-\tfrac{4}{3}\epsilon^{3/4}\right] + \tfrac{4}{3}2^{3/4} = \left(\tfrac{4}{3}\right)2^{3/4}. ◄$$

2.2.11 Integration in plane polar coordinates

In plane polar coordinates ρ, ϕ, a curve is defined by its distance ρ from the origin as a function of the angle ϕ between the line joining a point on the curve to the origin and the x-axis, i.e. $\rho = \rho(\phi)$. The area of an element is given by

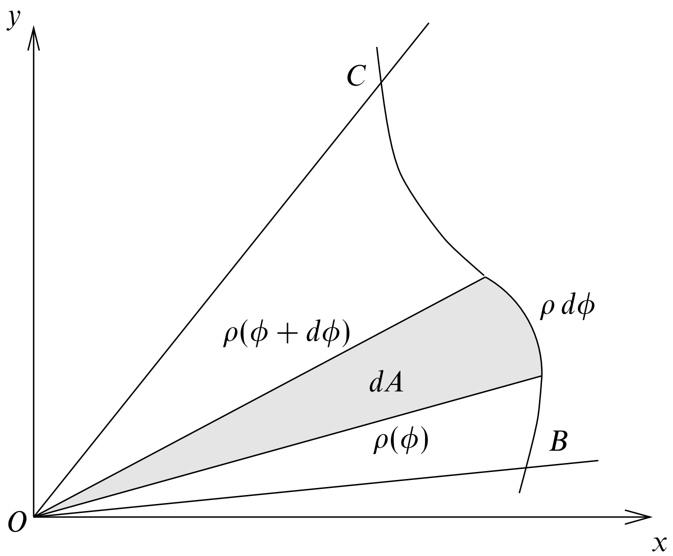

Figure 2.9 Finding the area of a sector OBC defined by the curve $\rho(\phi)$ and the radii OB, OC, at angles to the x-axis ϕ_1, ϕ_2 respectively.

$dA = \frac{1}{2}\rho^2\, d\phi$, as illustrated in figure 2.9, and hence the total area between two angles ϕ_1 and ϕ_2 is given by

$$A = \int_{\phi_1}^{\phi_2} \tfrac{1}{2}\rho^2\, d\phi. \tag{2.38}$$

An immediate observation is that the area of a circle of radius a is given by

$$A = \int_0^{2\pi} \tfrac{1}{2}a^2\, d\phi = \left[\tfrac{1}{2}a^2\phi\right]_0^{2\pi} = \pi a^2.$$

►*The equation in polar coordinates of an ellipse with semi-axes a and b is*

$$\frac{1}{\rho^2} = \frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}.$$

Find the area A of the ellipse.

Using (2.38) and symmetry, we have

$$A = \frac{1}{2}\int_0^{2\pi} \frac{a^2b^2}{b^2\cos^2\phi + a^2\sin^2\phi}\, d\phi = 2a^2b^2\int_0^{\pi/2} \frac{1}{b^2\cos^2\phi + a^2\sin^2\phi}\, d\phi.$$

To evaluate this integral we write $t = \tan\phi$ and use (2.35):

$$A = 2a^2b^2\int_0^{\infty} \frac{1}{b^2 + a^2t^2}\, dt = 2b^2\int_0^{\infty} \frac{1}{(b/a)^2 + t^2}\, dt.$$

Finally, from the list of standard integrals (see subsection 2.2.3),

$$A = 2b^2\left[\frac{1}{(b/a)}\tan^{-1}\frac{t}{(b/a)}\right]_0^{\infty} = 2ab\left(\frac{\pi}{2} - 0\right) = \pi ab. ◄$$

2.2.12 Integral inequalities

Consider the functions $f(x)$, $\phi_1(x)$ and $\phi_2(x)$ such that $\phi_1(x) \leq f(x) \leq \phi_2(x)$ for all x in the range $a \leq x \leq b$. It immediately follows that

$$\int_a^b \phi_1(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b \phi_2(x)\,dx, \tag{2.39}$$

which gives us a way of estimating an integral that is difficult to evaluate explicitly.

►*Show that the value of the integral*

$$I = \int_0^1 \frac{1}{(1+x^2+x^3)^{1/2}}\,dx$$

lies between 0.810 *and* 0.882.

We note that for x in the range $0 \leq x \leq 1$, $0 \leq x^3 \leq x^2$. Hence

$$(1+x^2)^{1/2} \leq (1+x^2+x^3)^{1/2} \leq (1+2x^2)^{1/2},$$

and so

$$\frac{1}{(1+x^2)^{1/2}} \geq \frac{1}{(1+x^2+x^3)^{1/2}} \geq \frac{1}{(1+2x^2)^{1/2}}.$$

Consequently,

$$\int_0^1 \frac{1}{(1+x^2)^{1/2}}\,dx \geq \int_0^1 \frac{1}{(1+x^2+x^3)^{1/2}}\,dx \geq \int_0^1 \frac{1}{(1+2x^2)^{1/2}}\,dx,$$

from which we obtain

$$\begin{aligned}\left[\ln(x+\sqrt{1+x^2})\right]_0^1 \geq I &\geq \left[\tfrac{1}{\sqrt{2}}\ln\left(x+\sqrt{\tfrac{1}{2}+x^2}\right)\right]_0^1 \\ 0.8814 \geq I &\geq 0.8105 \\ 0.882 \geq I &\geq 0.810.\end{aligned}$$

In the last line the calculated values have been rounded to three significant figures, one rounded up and the other rounded down so that the proved inequality cannot be unknowingly made invalid. ◄

2.2.13 Applications of integration

Mean value of a function

The mean value m of a function between two limits a and b is defined by

$$m = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.40}$$

The mean value may be thought of as the height of the rectangle that has the same area (over the same interval) as the area under the curve $f(x)$. This is illustrated in figure 2.10.

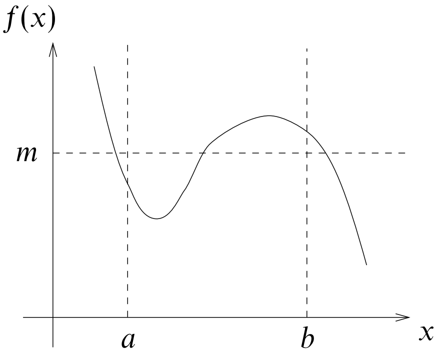

Figure 2.10 The mean value m of a function.

►*Find the mean value m of the function $f(x) = x^2$ between the limits $x = 2$ and $x = 4$.*

Using (2.40),

$$m = \frac{1}{4-2}\int_2^4 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_2^4 = \frac{1}{2}\left(\frac{4^3}{3} - \frac{2^3}{3}\right) = \frac{28}{3}. \blacktriangleleft$$

Finding the length of a curve

Finding the area between a curve and certain straight lines provides one example of the use of integration. Another is in finding the length of a curve. If a curve is defined by $y = f(x)$ then the distance along the curve, Δs, that corresponds to small changes Δx and Δy in x and y is given by

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}; \tag{2.41}$$

this follows directly from Pythagoras' theorem (see figure 2.11). Dividing (2.41) through by Δx and letting $\Delta x \to 0$ we obtain§

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Clearly the total length s of the curve between the points $x = a$ and $x = b$ is then given by integrating both sides of the equation:

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx. \tag{2.42}$$

§ Instead of considering small changes Δx and Δy and letting these tend to zero, we could have derived (2.41) by considering infinitesimal changes dx and dy from the start. After writing $(ds)^2 = (dx)^2 + (dy)^2$, (2.41) may be deduced by using the formal device of dividing through by dx. Although not mathematically rigorous, this method is often used and generally leads to the correct result.

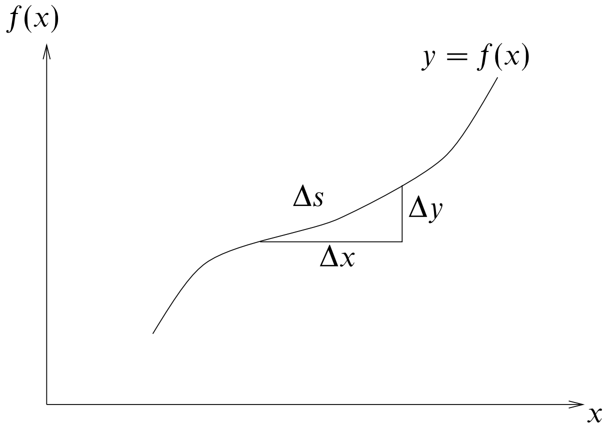

Figure 2.11 The distance moved along a curve, Δs, corresponding to the small changes Δx and Δy.

In plane polar coordinates,

$$ds = \sqrt{(dr)^2 + (r\,d\phi)^2} \qquad \Rightarrow \qquad s = \int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\phi}{dr}\right)^2}\, dr. \tag{2.43}$$

►*Find the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 2$.*

Using (2.42) and noting that $dy/dx = \frac{3}{2}\sqrt{x}$, the length s of the curve is given by

$$\begin{aligned} s &= \int_0^2 \sqrt{1 + \tfrac{9}{4}x}\, dx \\ &= \left[\tfrac{2}{3}\left(\tfrac{4}{9}\right)\left(1 + \tfrac{9}{4}x\right)^{3/2}\right]_0^2 = \tfrac{8}{27}\left[\left(1 + \tfrac{9}{4}x\right)^{3/2}\right]_0^2 \\ &= \tfrac{8}{27}\left[\left(\tfrac{11}{2}\right)^{3/2} - 1\right]. \end{aligned}$$ ◄

Surfaces of revolution

Consider the surface S formed by rotating the curve $y = f(x)$ about the x-axis (see figure 2.12). The surface area of the 'collar' formed by rotating an element of the curve, ds, about the x-axis is $2\pi y\, ds$, and hence the total surface area is

$$S = \int_a^b 2\pi y\, ds.$$

Since $(ds)^2 = (dx)^2 + (dy)^2$ from (2.41), the total surface area between the planes $x = a$ and $x = b$ is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{2.44}$$

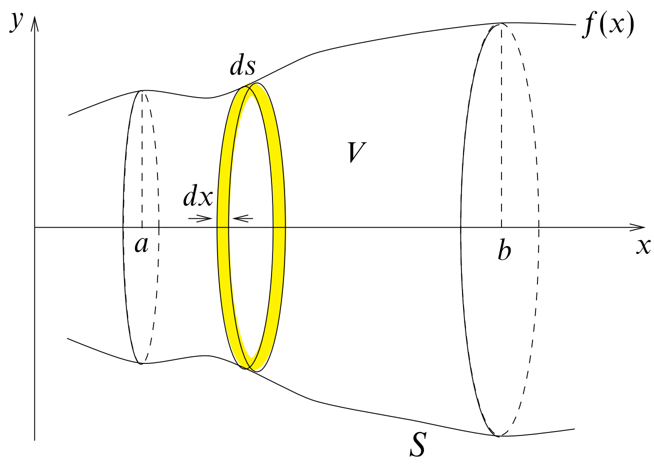

Figure 2.12 The surface and volume of revolution for the curve $y = f(x)$.

►*Find the surface area of a cone formed by rotating about the x-axis the line $y = 2x$ between $x = 0$ and $x = h$.*

Using (2.44), the surface area is given by

$$\begin{aligned} S &= \int_0^h (2\pi)2x\sqrt{1+\left[\frac{d}{dx}(2x)\right]^2}\,dx \\ &= \int_0^h 4\pi x\left(1+2^2\right)^{1/2}\,dx = \int_0^h 4\sqrt{5}\pi x\,dx \\ &= \left[2\sqrt{5}\pi x^2\right]_0^h = 2\sqrt{5}\pi(h^2-0) = 2\sqrt{5}\pi h^2. \blacktriangleleft \end{aligned}$$

We note that a surface of revolution may also be formed by rotating a line about the y-axis. In this case the surface area between $y = a$ and $y = b$ is

$$S = \int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy. \qquad (2.45)$$

Volumes of revolution

The volume V enclosed by rotating the curve $y = f(x)$ about the x-axis can also be found (see figure 2.12). The volume of the disc between x and $x + dx$ is given by $dV = \pi y^2\,dx$. Hence the total volume between $x = a$ and $x = b$ is

$$V = \int_a^b \pi y^2\,dx. \qquad (2.46)$$

►*Find the volume of a cone enclosed by the surface formed by rotating about the x-axis the line $y = 2x$ between $x = 0$ and $x = h$.*

Using (2.46), the volume is given by

$$V = \int_0^h \pi(2x)^2\, dx = \int_0^h 4\pi x^2\, dx$$
$$= \left[\tfrac{4}{3}\pi x^3\right]_0^h = \tfrac{4}{3}\pi(h^3 - 0) = \tfrac{4}{3}\pi h^3. \blacktriangleleft$$

As before, it is also possible to form a volume of revolution by rotating a curve about the y-axis. In this case the volume enclosed between $y = a$ and $y = b$ is

$$V = \int_a^b \pi x^2\, dy. \qquad (2.47)$$

2.3 Exercises

2.1 Obtain the following derivatives from first principles:

(a) the first derivative of $3x + 4$;
(b) the first, second and third derivatives of $x^2 + x$;
(c) the first derivative of $\sin x$.

2.2 Find from first principles the first derivative of $(x+3)^2$ and compare your answer with that obtained using the chain rule.

2.3 Find the first derivatives of

(a) $x^2 \exp x$, (b) $2 \sin x \cos x$, (c) $\sin 2x$, (d) $x \sin ax$,
(e) $(\exp ax)(\sin ax) \tan^{-1} ax$, (f) $\ln(x^a + x^{-a})$,
(g) $\ln(a^x + a^{-x})$, (h) x^x.

2.4 Find the first derivatives of

(a) $x/(a + x)^2$, (b) $x/(1 - x)^{1/2}$, (c) $\tan x$, as $\sin x/ \cos x$,
(d) $(3x^2 + 2x + 1)/(8x^2 - 4x + 2)$.

2.5 Use result (2.12) to find the first derivatives of

(a) $(2x + 3)^{-3}$, (b) $\sec^2 x$, (c) $\operatorname{cosech}^3 3x$, (d) $1/ \ln x$, (e) $1/[\sin^{-1}(x/a)]$.

2.6 Show that the function $y(x) = \exp(-|x|)$ defined by

$$y(x) = \begin{cases} \exp x & \text{for } x < 0, \\ 1 & \text{for } x = 0, \\ \exp(-x) & \text{for } x > 0, \end{cases}$$

is *not* differentiable at $x = 0$. Consider the limiting process for both $\Delta x > 0$ and $\Delta x < 0$.

2.7 Find dy/dx if $x = (t - 2)/(t + 2)$ and $y = 2t/(t + 1)$ for $-\infty < t < \infty$. Show that it is always non-negative, and make use of this result in sketching the curve of y as a function of x.

2.8 If $2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$, show that $dy/dx = 16$ when $x = 1$.

2.9 Find the second derivative of $y(x) = \cos[(\pi/2) - ax]$. Now set $a = 1$ and verify that the result is the same as that obtained by first setting $a = 1$ and simplifying $y(x)$ before differentiating.

2.10 The function $y(x)$ is defined by $y(x) = (1 + x^m)^n$.

(a) Use the chain rule to show that the first derivative of y is $nmx^{m-1}(1 + x^m)^{n-1}$.
(b) The binomial expansion (see section 1.5) of $(1 + z)^n$ is

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}z^r + \cdots .$$

Keeping only the terms of zeroth and first order in dx, apply this result twice to derive result (a) from first principles.
(c) Expand y in a series of powers of x before differentiating term by term. Show that the result is the series obtained by expanding the answer given for dy/dx in (a).

2.11 Show by differentiation and substitution that the differential equation

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 + 3)y = 0$$

has a solution of the form $y(x) = x^n \sin x$, and find the value of n.

2.12 Find the positions and natures of the stationary points of the following functions:

(a) $x^3 - 3x + 3$; (b) $x^3 - 3x^2 + 3x$; (c) $x^3 + 3x + 3$;
(d) $\sin ax$ with $a \neq 0$; (e) $x^5 + x^3$; (f) $x^5 - x^3$.

2.13 Show that the lowest value taken by the function $3x^4 + 4x^3 - 12x^2 + 6$ is -26.

2.14 By finding their stationary points and examining their general forms, determine the range of values that each of the following functions $y(x)$ can take. In each case make a sketch-graph incorporating the features you have identified.

(a) $y(x) = (x - 1)/(x^2 + 2x + 6)$.
(b) $y(x) = 1/(4 + 3x - x^2)$.
(c) $y(x) = (8 \sin x)/(15 + 8 \tan^2 x)$.

2.15 Show that $y(x) = xa^{2x} \exp x^2$ has no stationary points other than $x = 0$, if $\exp(-\sqrt{2}) < a < \exp(\sqrt{2})$.

2.16 The curve $4y^3 = a^2(x + 3y)$ can be parameterised as $x = a\cos 3\theta$, $y = a\cos\theta$.

(a) Obtain expressions for dy/dx (i) by implicit differentiation and (ii) in parameterised form. Verify that they are equivalent.
(b) Show that the only point of inflection occurs at the origin. Is it a stationary point of inflection?
(c) Use the information gained in (a) and (b) to sketch the curve, paying particular attention to its shape near the points $(-a, a/2)$ and $(a, -a/2)$ and to its slope at the 'end points' (a, a) and $(-a, -a)$.

2.17 The parametric equations for the motion of a charged particle released from rest in electric and magnetic fields at right angles to each other take the forms

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta).$$

Show that the tangent to the curve has slope $\cot(\theta/2)$. Use this result at a few calculated values of x and y to sketch the form of the particle's trajectory.

2.18 Show that the maximum curvature on the catenary $y(x) = a\cosh(x/a)$ is $1/a$. You will need some of the results about hyperbolic functions stated in subsection 3.7.6.

2.19 The curve whose equation is $x^{2/3} + y^{2/3} = a^{2/3}$ for positive x and y and which is completed by its symmetric reflections in both axes is known as an astroid. Sketch it and show that its radius of curvature in the first quadrant is $3(axy)^{1/3}$.

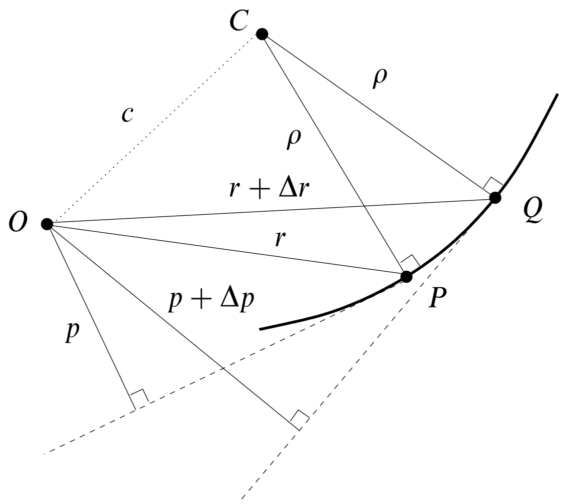

Figure 2.13 The coordinate system described in exercise 2.20.

2.20 A two-dimensional coordinate system useful for orbit problems is the tangential-polar coordinate system (figure 2.13). In this system a curve is defined by r, the distance from a fixed point O to a general point P of the curve, and p, the perpendicular distance from O to the tangent to the curve at P. By proceeding as indicated below, show that the radius of curvature, ρ, at P can be written in the form $\rho = r\,dr/dp$.

Consider two neighbouring points, P and Q, on the curve. The normals to the curve through those points meet at C, with (in the limit $Q \to P$) $CP = CQ = \rho$. Apply the cosine rule to triangles OPC and OQC to obtain two expressions for c^2, one in terms of r and p and the other in terms of $r + \Delta r$ and $p + \Delta p$. By equating them and letting $Q \to P$ deduce the stated result.

2.21 Use Leibnitz' theorem to find

(a) the second derivative of $\cos x \sin 2x$,
(b) the third derivative of $\sin x \ln x$,
(c) the fourth derivative of $(2x^3 + 3x^2 + x + 2)\exp 2x$.

2.22 If $y = \exp(-x^2)$, show that $dy/dx = -2xy$ and hence, by applying Leibnitz' theorem, prove that for $n \geq 1$

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0.$$

2.23 Use the properties of functions at their turning points to do the following:

(a) By considering its properties near $x = 1$, show that $f(x) = 5x^4 - 11x^3 + 26x^2 - 44x + 24$ takes negative values for some range of x.
(b) Show that $f(x) = \tan x - x$ cannot be negative for $0 \leq x < \pi/2$, and deduce that $g(x) = x^{-1}\sin x$ decreases monotonically in the same range.

2.24 Determine what can be learned from applying Rolle's theorem to the following functions $f(x)$: (a) e^x; (b) $x^2 + 6x$; (c) $2x^2 + 3x + 1$; (d) $2x^2 + 3x + 2$; (e) $2x^3 - 21x^2 + 60x + k$. (f) If $k = -45$ in (e), show that $x = 3$ is one root of $f(x) = 0$, find the other roots, and verify that the conclusions from (e) are satisfied.

2.25 By applying Rolle's theorem to $x^n \sin nx$, where n is an arbitrary positive integer, show that $\tan nx + x = 0$ has a solution α_1 with $0 < \alpha_1 < \pi/n$. Apply the theorem a second time to obtain the nonsensical result that there is a real α_2 in $0 < \alpha_2 < \pi/n$, such that $\cos^2(n\alpha_2) = -n$. Explain why this incorrect result arises.

2.26 Use the mean value theorem to establish bounds in the following cases.

(a) For $-\ln(1-y)$, by considering $\ln x$ in the range $0 < 1-y < x < 1$.
(b) For $e^y - 1$, by considering $e^x - 1$ in the range $0 < x < y$.

2.27 For the function $y(x) = x^2 \exp(-x)$ obtain a simple relationship between y and dy/dx and then, by applying Leibnitz' theorem, prove that

$$xy^{(n+1)} + (n + x - 2)y^{(n)} + ny^{(n-1)} = 0.$$

2.28 Use Rolle's theorem to deduce that, if the equation $f(x) = 0$ has a repeated root x_1, then x_1 is also a root of the equation $f'(x) = 0$.

(a) Apply this result to the 'standard' quadratic equation $ax^2 + bx + c = 0$, to show that a necessary condition for equal roots is $b^2 = 4ac$.
(b) Find all the roots of $f(x) = x^3 + 4x^2 - 3x - 18 = 0$, given that one of them is a repeated root.
(c) The equation $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2 = 0$ has a repeated integer root. How many real roots does it have altogether?

2.29 Show that the curve $x^3 + y^3 - 12x - 8y - 16 = 0$ touches the x-axis.

2.30 Find the following indefinite integrals:

(a) $\int (4 + x^2)^{-1}\, dx$; (b) $\int (8 + 2x - x^2)^{-1/2}\, dx$ for $2 \le x \le 4$;
(c) $\int (1 + \sin\theta)^{-1}\, d\theta$; (d) $\int (x\sqrt{1-x})^{-1}\, dx$ for $0 < x \le 1$.

2.31 Find the indefinite integrals J of the following ratios of polynomials:

(a) $(x + 3)/(x^2 + x - 2)$;
(b) $(x^3 + 5x^2 + 8x + 12)/(2x^2 + 10x + 12)$;
(c) $(3x^2 + 20x + 28)/(x^2 + 6x + 9)$;
(d) $x^3/(a^8 + x^8)$.

2.32 Express $x^2(ax + b)^{-1}$ as the sum of powers of x and another integrable term, and hence evaluate

$$\int_0^{b/a} \frac{x^2}{ax + b}\, dx.$$

2.33 Find the integral J of $(ax^2 + bx + c)^{-1}$, with $a \neq 0$, distinguishing between the cases (i) $b^2 > 4ac$, (ii) $b^2 < 4ac$ and (iii) $b^2 = 4ac$.

2.34 Use logarithmic integration to find the indefinite integrals J of the following:

(a) $\sin 2x/(1 + 4\sin^2 x)$;
(b) $e^x/(e^x - e^{-x})$;
(c) $(1 + x\ln x)/(x \ln x)$;
(d) $[x(x^n + a^n)]^{-1}$.

2.35 Find the derivative of $f(x) = (1 + \sin x)/\cos x$ and hence determine the indefinite integral J of $\sec x$.

2.36 Find the indefinite integrals, J, of the following functions involving sinusoids:

(a) $\cos^5 x - \cos^3 x$;
(b) $(1 - \cos x)/(1 + \cos x)$;
(c) $\cos x \sin x/(1 + \cos x)$;
(d) $\sec^2 x/(1 - \tan^2 x)$.

2.37 By making the substitution $x = a\cos^2\theta + b\sin^2\theta$, evaluate the definite integrals J between limits a and b ($> a$) of the following functions:

(a) $[(x - a)(b - x)]^{-1/2}$;
(b) $[(x - a)(b - x)]^{1/2}$;

(c) $[(x-a)/(b-x)]^{1/2}$.

2.38 Determine whether the following integrals exist and, where they do, evaluate them:

(a) $\int_0^{\infty} \exp(-\lambda x)\,dx$; (b) $\int_{-\infty}^{\infty} \frac{x}{(x^2+a^2)^2}\,dx$;

(c) $\int_1^{\infty} \frac{1}{x+1}\,dx$; (d) $\int_0^1 \frac{1}{x^2}\,dx$;

(e) $\int_0^{\pi/2} \cot\theta\,d\theta$; (f) $\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx$.

2.39 Use integration by parts to evaluate the following:

(a) $\int_0^y x^2 \sin x\,dx$; (b) $\int_1^y x \ln x\,dx$;

(c) $\int_0^y \sin^{-1} x\,dx$; (d) $\int_1^y \ln(a^2+x^2)/x^2\,dx$.

2.40 Show, using the following methods, that the indefinite integral of $x^3/(x+1)^{1/2}$ is

$$J = \tfrac{2}{35}(5x^3 - 6x^2 + 8x - 16)(x+1)^{1/2} + c.$$

(a) Repeated integration by parts.
(b) Setting $x+1 = u^2$ and determining dJ/du as $(dJ/dx)(dx/du)$.

2.41 The gamma function $\Gamma(n)$ is defined for all $n > -1$ by

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}\,dx.$$

Find a recurrence relation connecting $\Gamma(n+1)$ and $\Gamma(n)$.

(a) Deduce (i) the value of $\Gamma(n+1)$ when n is a non-negative integer, and (ii) the value of $\Gamma\left(\frac{7}{2}\right)$, given that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
(b) Now, taking factorial m for *any* m to be defined by $m! = \Gamma(m+1)$, evaluate $\left(-\frac{3}{2}\right)!$.

2.42 Define $J(m,n)$, for non-negative integers m and n, by the integral

$$J(m,n) = \int_0^{\pi/2} \cos^m\theta \sin^n\theta\,d\theta.$$

(a) Evaluate $J(0,0)$, $J(0,1)$, $J(1,0)$, $J(1,1)$, $J(m,1)$, $J(1,n)$.
(b) Using integration by parts, prove that, for m and n both > 1,

$$J(m,n) = \frac{m-1}{m+n}J(m-2,n) \quad \text{and} \quad J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) Evaluate (i) $J(5,3)$, (ii) $J(6,5)$ and (iii) $J(4,8)$.

2.43 By integrating by parts twice, prove that I_n as defined in the first equality below for positive integers n has the value given in the second equality:

$$I_n = \int_0^{\pi/2} \sin n\theta \cos\theta\,d\theta = \frac{n - \sin(n\pi/2)}{n^2-1}.$$

2.44 Evaluate the following definite integrals:

(a) $\int_0^{\infty} xe^{-x}\,dx$; (b) $\int_0^1 \left[(x^3+1)/(x^4+4x+1)\right]\,dx$;

(c) $\int_0^{\pi/2} [a + (a-1)\cos\theta]^{-1}\,d\theta$ with $a > \frac{1}{2}$; (d) $\int_{-\infty}^{\infty} (x^2+6x+18)^{-1}\,dx$.

2.45 If J_r is the integral

$$\int_0^\infty x^r \exp(-x^2)\,dx$$

show that

(a) $J_{2r+1} = (r!)/2$,
(b) $J_{2r} = 2^{-r}(2r-1)(2r-3)\cdots(5)(3)(1)\,J_0$.

2.46 Find positive constants a, b such that $ax \le \sin x \le bx$ for $0 \le x \le \pi/2$. Use this inequality to find (to two significant figures) upper and lower bounds for the integral

$$I = \int_0^{\pi/2} (1+\sin x)^{1/2}\,dx.$$

Use the substitution $t = \tan(x/2)$ to evaluate I exactly.

2.47 By noting that for $0 \le \eta \le 1$, $\eta^{1/2} \ge \eta^{3/4} \ge \eta$, prove that

$$\frac{2}{3} \le \frac{1}{a^{5/2}} \int_0^a (a^2 - x^2)^{3/4}\,dx \le \frac{\pi}{4}.$$

2.48 Show that the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, which can be parameterised as $x = a\cos^3\theta$, $y = a\sin^3\theta$, is $6a$.

2.49 By noting that $\sinh x < \frac{1}{2}e^x < \cosh x$, and that $1 + z^2 < (1+z)^2$ for $z > 0$, show that, for $x > 0$, the length L of the curve $y = \frac{1}{2}e^x$ measured from the origin satisfies the inequalities $\sinh x < L < x + \sinh x$.

2.50 The equation of a cardioid in plane polar coordinates is

$$\rho = a(1 - \sin\phi).$$

Sketch the curve and find (i) its area, (ii) its total length, (iii) the surface area of the solid formed by rotating the cardioid about its axis of symmetry and (iv) the volume of the same solid.

2.4 Hints and answers

2.1 (a) 3; (b) $2x+1$, 2, 0; (c) $\cos x$.

2.3 Use: the product rule in (a), (b), (d) and (e)[3 factors]; the chain rule in (c), (f) and (g); logarithmic differentiation in (g) and (h).
(a) $(x^2+2x)\exp x$; (b) $2(\cos^2 x - \sin^2 x) = 2\cos 2x$;
(c) $2\cos 2x$; (d) $\sin ax + ax\cos ax$;
(e) $(a\exp ax)[(\sin ax + \cos ax)\tan^{-1} ax + (\sin ax)(1+a^2x^2)^{-1}]$;
(f) $[a(x^a - x^{-a})]/[x(x^a + x^{-a})]$; (g) $[(a^x - a^{-x})\ln a]/(a^x + a^{-x})$; (h) $(1+\ln x)x^x$.

2.5 (a) $-6(2x+3)^{-4}$; (b) $2\sec^2 x\tan x$; (c) $-9\,\mathrm{cosech}^3 3x\coth 3x$;
(d) $-x^{-1}(\ln x)^{-2}$; (e) $-(a^2-x^2)^{-1/2}[\sin^{-1}(x/a)]^{-2}$.

2.7 Calculate dy/dt and dx/dt and divide one by the other. $(t+2)^2/[2(t+1)^2]$. Alternatively, eliminate t and find dy/dx by implicit differentiation.

2.9 $-\sin x$ in both cases.

2.11 The required conditions are $8n - 4 = 0$ and $4n^2 - 8n + 3 = 0$; both are satisfied by $n = \frac{1}{2}$.

2.13 The stationary points are the zeros of $12x^3 + 12x^2 - 24x$. The lowest stationary value is -26 at $x = -2$; other stationary values are 6 at $x = 0$ and 1 at $x = 1$.

2.15 Use logarithmic differentiation. Set $dy/dx = 0$, obtaining $2x^2 + 2x\ln a + 1 = 0$.

2.17 See figure 2.14.

2.19 $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$; $\quad \dfrac{d^2y}{dx^2} = \dfrac{a^{2/3}}{3x^{4/3}y^{1/3}}$.

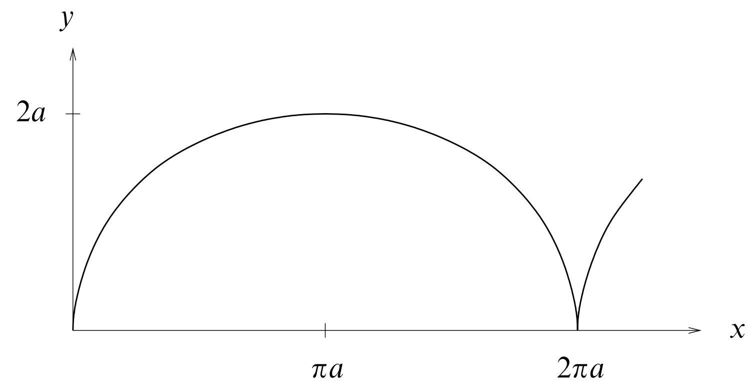

Figure 2.14 The solution to exercise 2.17.

2.21 (a) $2(2-9\cos^2 x)\sin x$; (b) $(2x^{-3}-3x^{-1})\sin x-(3x^{-2}+\ln x)\cos x$; (c) $8(4x^3+30x^2+62x+38)\exp 2x$.

2.23 (a) $f(1)=0$ whilst $f'(1)\neq 0$, and so $f(x)$ must be negative in some region with $x=1$ as an endpoint.
(b) $f'(x)=\tan^2 x>0$ and $f(0)=0$; $g'(x)=(-\cos x)(\tan x-x)/x^2$, which is never positive in the range.

2.25 The false result arises because $\tan nx$ is not differentiable at $x=\pi/(2n)$, which lies in the range $0<x<\pi/n$, and so the conditions for applying Rolle's theorem are not satisfied.

2.27 The relationship is $x\,dy/dx=(2-x)y$.

2.29 By implicit differentiation, $y'(x)=(3x^2-12)/(8-3y^2)$, giving $y'(\pm 2)=0$. Since $y(2)=4$ and $y(-2)=0$, the curve touches the x-axis at the point $(-2,0)$.

2.31 (a) Express in partial fractions; $J=\frac{1}{3}\ln[(x-1)^4/(x+2)]+c$.
(b) Divide the numerator by the denominator and express the remainder in partial fractions; $J=x^2/4+4\ln(x+2)-3\ln(x+3)+c$.
(c) After division of the numerator by the denominator, the remainder can be expressed as $2(x+3)^{-1}-5(x+3)^{-2}$; $J=3x+2\ln(x+3)+5(x+3)^{-1}+c$.
(d) Set $x^4=u$; $J=(4a^4)^{-1}\tan^{-1}(x^4/a^4)+c$.

2.33 Writing b^2-4ac as $\Delta^2>0$, or $4ac-b^2$ as $\Delta'^2>0$:
(i) $\Delta^{-1}\ln[(2ax+b-\Delta)/(2ax+b+\Delta)]+k$;
(ii) $2\Delta'^{-1}\tan^{-1}[(2ax+b)/\Delta']+k$;
(iii) $-2(2ax+b)^{-1}+k$.

2.35 $f'(x)=(1+\sin x)/\cos^2 x=f(x)\sec x$; $J=\ln(f(x))+c=\ln(\sec x+\tan x)+c$.

2.37 Note that $dx=2(b-a)\cos\theta\sin\theta\,d\theta$.
(a) π; (b) $\pi(b-a)^2/8$; (c) $\pi(b-a)/2$.

2.39 (a) $(2-y^2)\cos y+2y\sin y-2$; (b) $[(y^2\ln y)/2]+[(1-y^2)/4]$;
(c) $y\sin^{-1}y+(1-y^2)^{1/2}-1$;
(d) $\ln(a^2+1)-(1/y)\ln(a^2+y^2)+(2/a)[\tan^{-1}(y/a)-\tan^{-1}(1/a)]$.

2.41 $\Gamma(n+1)=n\Gamma(n)$; (a) (i) $n!$, (ii) $15\sqrt{\pi}/8$; (b) $-2\sqrt{\pi}$.

2.43 By integrating twice, recover a multiple of I_n.

2.45 $J_{2r+1}=rJ_{2r-1}$ and $2J_{2r}=(2r-1)J_{2r-2}$.

2.47 Set $\eta=1-(x/a)^2$ throughout, and $x=a\sin\theta$ in one of the bounds.

2.49 $L=\int_0^x\left(1+\frac{1}{4}\exp 2x\right)^{1/2}dx$.