ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Homework I, Advanced Algorithms 2019

Due on Friday March 29 at 17:00 (upload one solution per group on moodle). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups of up to three students; it is sufficient to hand in one solution per group.

1 *(20 pts)* Let $G = (V, E)$ be an undirected graph where every vertex $v \in V$ has at most $\Delta \in \mathbb{N}$ neighbors. We will analyze the following randomized algorithm that assigns a "color" $c(v) \in \{1, 2, \dots, \Delta\}$ to every vertex:

> For each vertex v, select $c(v) \in \{1, 2, \dots, \Delta\}$ uniformly at random and independently of other vertices. In other words, v receives a color $i \in \{1, 2, \dots, \Delta\}$ with probability $1/\Delta$.

1a *(5 pts)* For a coloring $c : V \to \{1, 2, \dots, \Delta\}$, we say that an edge $\{u, v\} \in E$ is *monochromatic* if $c(u) = c(v)$. Show that the expected number of monochromatic edges in the coloring c output by our randomized algorithm equals $\frac{|E|}{\Delta}$.

1b *(15 pts)* For a vertex v, define $N(v) = \{u \in V : \{u, v\} \in E\}$ to be the set of neighbors of v. For a coloring $c : V \to \{1, 2, \dots, \Delta\}$, we say that a vertex v is *good* if $c(v) \neq c(u)$ for all neighbors $u \in N(V)$. Show that the expected number of good vertices in the coloring c output by our randomized algorithm is at least $\left(1 - \frac{1}{\Delta}\right)^{\Delta} |V|$.

2 *(20 pts)* Consider a general (not necessarily bipartite) graph $G = (V, E)$. Let $(V, \mathcal{I})$ be the matroid with ground set V and

$$\mathcal{I} = \{U \subseteq V : \ G \text{ has a matching in which every vertex of } U \text{ is matched}\}.$$

Recall that we say that a vertex v is matched by a matching M if there is an edge in M incident to v. Show that $(V, \mathcal{I})$ is indeed a matroid by verifying the two axioms.

3 *(20 pts)* **Primal-dual algorithm for the weighted 3-Uniform Vertex Cover problem.** A k-uniform hypergraph is defined by a tuple (V, E) where V is the set of vertices and every hyperedge $e \in E$ is a subset of V of cardinality k. We remark that a 2-uniform hypergraph is simply a graph. Here we will consider the weighted vertex cover problem on 3-uniform hypergraphs:

Input: A 3-uniform hypergraph $G = (V, E)$ with vertex weights $w : V \to \mathbb{R}_+$.

Output: A vertex cover $C \subseteq V$ minimizing the weight $w(C) = \sum_{v \in C} w(v)$ subject to that every edge $e \in E$ is covered, i.e., $e \cap C \neq \emptyset$.

The LP relaxation and its dual is similar to that of the vertex cover problem on graphs:

(Primal) LP Relaxation	(Dual)
$$\begin{aligned} \textbf{minimize} \quad & \sum_{v \in V} w(v) x_v \\ \textbf{subject to} \quad & x_u + x_v + x_w \geq 1 \quad \text{for } \{u, v, w\} \in E \\ & x_v \geq 0 \quad \text{for } v \in V \end{aligned}$$	$$\begin{aligned} \textbf{maximize} \quad & \sum_{e \in E} y_e \\ \textbf{subject to} \quad & \sum_{e \in E : v \in e} y_e \leq w(v) \text{ for } v \in V \\ & y_e \geq 0 \quad \text{for } e \in E \end{aligned}$$

The Vertex Cover problem (also on 3-uniform hypergraphs) is NP-hard and therefore we do not expect an efficient (polynomial-time) algorithm that finds exact solutions. In this problem, we will analyze and implement a simple and very fast primal-dual algorithm that achieves the best-known guarantees.

The primal-dual algorithm works as follows. It will maintain a feasible dual solution y that initially is set to $y_e = 0$ for every $e \in E$. It will then iteratively improve the dual solution until the set $C = \{v \in V : \sum_{e \in E : v \in e} y_e = w(v)\}$ forms a vertex cover. Note that C consists of those vertices whose constraints in the dual are tight. The formal description of the algorithm is as follows:

1. Initialize the dual solution y to be $y_e = 0$ for every $e \in E$.
2. While $C = \{v \in V : \sum_{e \in E : v \in e} y_e = w(v)\}$ is not a vertex cover:
   - Select an edge $\{u, v, w\} \in E$ that is not covered by C, i.e., $C \cap \{u, v, w\} = \emptyset$.
   - Increase $y_{\{u,v,w\}}$ until one of the dual constraints (corresponding to u, v or w) becomes tight.
3. Return $C = \{v \in V : \sum_{e \in E : v \in e} y_e = w(v)\}$.

Show that the primal-dual algorithm has an approximation guarantee of 3. That is, show that the returned vertex cover C has weight $\sum_{v \in C} w(v)$ at most three times the weight of an optimal solution.

4 *(20 pts)* **Interval packing.** Given a set of unit intervals $[a_i, a_i + 1]$ for each $1 \leq i \leq n$ and a weight function on intervals, the *maximum weight unit interval packing* problem asks for a subset J of intervals of maximum weight such that no intervals in J overlap.

Formulate a linear program for the maximum weight unit interval packing problem and show that any extreme point of your linear program is integral.

5 *(20 pts)* **Implementation.** The objective of this problem is to successfully solve the problem *Hiking Trails* on our online judge. You will find detailed instructions on how to do this on Moodle.