Midterm "Take home"-exam
FYS3110

Unable to see candidate no

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1 Spin-$1/2$ systems

The following is given:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2, \quad \hat{S}^{\pm} = \hat{S}_x \pm i\hat{S}_y$$
$$|\uparrow\rangle \equiv \left| s = \frac{1}{2}, m_s = \frac{1}{2} \right\rangle, \quad |\downarrow\rangle \equiv \left| s = \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle$$
$$\hat{S}^2 |\uparrow\rangle = \hbar^2 \frac{1}{2} \left(\frac{1}{2} + 1 \right) |\uparrow\rangle, \quad \hat{S}^2 |\downarrow\rangle = \hbar^2 \frac{1}{2} \left(\frac{1}{2} + 1 \right) |\downarrow\rangle$$
$$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle, \quad \hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$$
$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

1.1

$$\hat{S}_z\hat{S}^+ |\downarrow\rangle = \hat{S}_z\hat{S}_x |\downarrow\rangle + i\hat{S}_z\hat{S}_y |\downarrow\rangle$$

rewriting commutation relations

$$[\hat{S}_z, \hat{S}_x] = \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z = i\hbar\hat{S}_y \to \hat{S}_z\hat{S}_x = i\hbar\hat{S}_y + \hat{S}_x\hat{S}_z$$
$$[\hat{S}_y, \hat{S}_z] = \hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y = i\hbar\hat{S}_x \to \hat{S}_z\hat{S}_y = \hat{S}_y\hat{S}_z - i\hbar\hat{S}_x,$$

gives

$$\begin{aligned}
\hat{S}_z\hat{S}^+ |\downarrow\rangle &= (i\hbar\hat{S}_y + \hat{S}_x\hat{S}_z + i\hat{S}_y\hat{S}_z + \hbar\hat{S}_x) |\downarrow\rangle \\
&= \left(i\hbar\hat{S}_y - \frac{\hbar}{2}\hat{S}_x - i\frac{\hbar}{2}\hat{S}_y + \hbar\hat{S}_x \right) |\downarrow\rangle \\
&= \left(\frac{\hbar}{2}\hat{S}_x + i\frac{\hbar}{2}\hat{S}_y \right) |\downarrow\rangle = \frac{\hbar}{2}\hat{S}^+ |\downarrow\rangle .
\end{aligned}$$

This means that $\hat{S}^+ |\downarrow\rangle$ is an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$.

1.2

$$
\begin{aligned}
\hat{S}^-\hat{S}^+ &= (\hat{S}_x - i\hat{S}_y)(\hat{S}_x + i\hat{S}_y) \\
&= \hat{S}_x^2 + i\hat{S}_x\hat{S}_y - i\hat{S}_y\hat{S}_x + \hat{S}_y^2 \\
&= \hat{S}_x^2 + \hat{S}_y^2 + i[\hat{S}_x, \hat{S}_y] \\
&= \hat{S}^2 - \hat{S}_z^2 - i\hbar\hat{S}_z
\end{aligned}
$$

This can be uses to compute the norm of $|\psi_1\rangle = \hat{S}^+ |\uparrow\rangle$ and $|\psi_2\rangle = \hat{S}^+ |\uparrow\rangle$.

$$
\begin{aligned}
\langle\psi_1|\psi_1\rangle &= \langle\downarrow| \hat{S}^-\hat{S}^+ |\downarrow\rangle = \langle\downarrow| (\hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z) |\downarrow\rangle \\
&= \langle\downarrow| \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right) |\downarrow\rangle - \langle\downarrow| \frac{\hbar^2}{4} |\downarrow\rangle + \langle\downarrow| \frac{2\hbar^2}{4} |\downarrow\rangle \\
&= \frac{3\hbar^2}{4} - \frac{\hbar^2}{4} + \frac{2\hbar}{4} = \hbar^2
\end{aligned}
$$

which means that $\||\psi_1\rangle\| = \hbar$.

$$
\begin{aligned}
\langle\psi_2|\psi_2\rangle &= \langle\uparrow| \hat{S}^-\hat{S}^+ |\uparrow\rangle = \langle\uparrow| (\hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z) |\uparrow\rangle \\
&= \langle\uparrow| \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right) |\uparrow\rangle - \langle\uparrow| \frac{\hbar^2}{4} |\uparrow\rangle - \langle\uparrow| \frac{2\hbar^2}{4} |\uparrow\rangle \\
&= \frac{3\hbar^2}{4} - \frac{\hbar^2}{4} - \frac{2\hbar^2}{4} = 0
\end{aligned}
$$

which measn that $\||\psi_2\rangle\| = 0$.

1.3

Phases are chosen sucht that the following relations hold

$$
\hat{S}^+ |\downarrow\rangle = \hbar |\uparrow\rangle, \quad \hat{S}^- |\uparrow\rangle = \hbar |\downarrow\rangle .
$$

Introducing a new state

$$
|\phi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + e^{i\theta} |\downarrow\rangle)
$$

where θ is a real number. We wish to compute the "uncertainty" product $\sigma_{sx}^2\sigma_{sy}^2$ where

$$
\begin{aligned}
\sigma_{sx}^2 &= \langle\phi| (\hat{S}_x - \langle\phi| \hat{S}_x |\phi\rangle)^2 |\phi\rangle \\
\sigma_{sy}^2 &= \langle\phi| (\hat{S}_y - \langle\phi| \hat{S}_y |\phi\rangle)^2 |\phi\rangle .
\end{aligned}
$$

First we need to find expressions for $\hat{S}_x$ and $\hat{S}_y$

$$
\hat{S}^+ + \hat{S}^- = (\hat{S}_x + i\hat{S}_y) + (\hat{S}_x - i\hat{S}_y) = 2\hat{S}_x \rightarrow \hat{S}_x = \frac{1}{2}(\hat{S}^+ + \hat{S}^-) \tag{1}
$$

$$
\hat{S}^+ - \hat{S}^- = (\hat{S}_x + i\hat{S}_y) - (\hat{S}_x - i\hat{S}_y) = 2i\hat{S}_y \rightarrow \hat{S}_y = \frac{1}{2i}(\hat{S}^+ - \hat{S}^-) \tag{2}
$$

It will also make things easier to calculate $\hat{S}_x |\uparrow\rangle$, $\hat{S}_x |\downarrow\rangle$, $\hat{S}_y |\uparrow\rangle$ and $\hat{S}_y |\downarrow\rangle$. These values can be found using equations 1 and 2.

$$\begin{aligned}
\hat{S}_x |\uparrow\rangle &= \frac{\hbar}{2} |\downarrow\rangle & \hat{S}_x^2 |\uparrow\rangle &= \frac{\hbar^2}{4} |\uparrow\rangle \\
\hat{S}_x |\downarrow\rangle &= \frac{\hbar}{2} |\uparrow\rangle & \hat{S}_x^2 |\downarrow\rangle &= \frac{\hbar^2}{4} |\downarrow\rangle \\
\hat{S}_y |\uparrow\rangle &= -\frac{\hbar}{2i} |\downarrow\rangle & \hat{S}_y^2 |\uparrow\rangle &= \frac{\hbar^2}{4} |\uparrow\rangle \\
\hat{S}_y |\downarrow\rangle &= \frac{\hbar}{2i} |\uparrow\rangle & \hat{S}_y^2 |\downarrow\rangle &= \frac{\hbar^2}{4} |\downarrow\rangle
\end{aligned}$$

We can begin on what is the real task at hand

$$\begin{aligned}
\langle\phi| \hat{S}_x |\phi\rangle &= \frac{1}{2}(\langle\uparrow| + e^{-i\theta} \langle\downarrow|)\hat{S}_x(|\uparrow\rangle + e^{i\theta} |\downarrow\rangle) \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta} \langle\downarrow|) \left(\frac{\hbar}{2} |\downarrow\rangle + e^{i\theta}\frac{\hbar}{2} |\uparrow\rangle\right) \\
&= \frac{\hbar}{4}\left(e^{i\theta} + e^{-i\theta}\right) \\
&= \frac{\hbar}{4}(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta) \\
&= \frac{\hbar}{2}\cos\theta \\
\langle\phi| \hat{S}_y |\phi\rangle &= \frac{1}{2}(\langle\uparrow| + e^{-i\theta} \langle\downarrow|)\hat{S}_y(|\uparrow\rangle + e^{i\theta} |\downarrow\rangle) \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta} \langle\downarrow|) \left(-\frac{\hbar}{2i} |\downarrow\rangle + e^{i\theta}\frac{\hbar}{2i} |\uparrow\rangle\right) \\
&= \frac{\hbar}{4i}\left(e^{i\theta} - e^{-i\theta}\right) \\
&= \frac{\hbar}{4i}(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta) \\
&= \frac{\hbar}{2}\sin\theta
\end{aligned}$$

$$
\begin{aligned}
\sigma_{sx}^2 &= \langle\phi| (\hat{S}_x - \frac{\hbar}{2}\cos\theta)^2 |\phi\rangle \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|)(\hat{S}_x^2 - \hbar\cos\theta\hat{S}_x + \frac{\hbar^2}{4}\cos^2\theta)(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle) \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|) \\
&\quad \left(\frac{\hbar^2}{4}|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}|\downarrow\rangle - \frac{\hbar^2}{2}\cos\theta|\downarrow\rangle - \frac{\hbar^2}{2}e^{i\theta}\cos\theta|\uparrow\rangle + \frac{\hbar^2}{4}\cos^2\theta|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}\cos^2\theta|\uparrow\rangle\right) \\
&= \frac{\hbar^2}{8} - \frac{\hbar^2}{4}e^{i\theta}\cos\theta + \frac{\hbar^2}{8}\cos^2\theta + \frac{\hbar^2}{8}e^{i\theta}\cos^2\theta + \frac{\hbar^2}{8} - \frac{\hbar^2}{4}e^{-i\theta}\cos\theta \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}(\cos^2\theta + i\sin\theta\cos\theta) + \frac{\hbar^2}{8}\cos^2\theta + \frac{\hbar^2}{8}(\cos^3\theta + i\sin\theta\cos^2\theta) - \frac{\hbar^2}{4}(\cos^2\theta - i\sin\theta\cos\theta) \\
&= \frac{\hbar^2}{4} - \frac{3\hbar^2}{8}\cos^2\theta + \frac{\hbar^2}{8}(\cos^3\theta + i\sin\theta\cos^2\theta) \\
\sigma_{sy}^2 &= \langle\phi| (\hat{S}_y - \frac{\hbar}{2}\sin\theta)^2 |\phi\rangle \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|)(\hat{S}_y^2 - \hbar\sin\theta\hat{S}_y + \frac{\hbar^2}{4}\sin^2\theta)(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle) \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|) \\
&\quad \left(\frac{\hbar^2}{4}|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}|\downarrow\rangle + \frac{\hbar^2}{2i}\sin\theta|\downarrow\rangle - \frac{\hbar^2}{2i}e^{i\theta}\sin\theta|\uparrow\rangle + \frac{\hbar^2}{4}\sin^2\theta|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}\sin^2\theta|\uparrow\rangle\right) \\
&= \frac{\hbar^2}{8} - \frac{\hbar^2}{4i}e^{i\theta}\sin\theta + \frac{\hbar^2}{8}\sin^2\theta + \frac{\hbar^2}{8}e^{i\theta}\sin^2\theta + \frac{\hbar^2}{8} + \frac{\hbar^2}{4i}e^{-i\theta}\sin\theta \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{4i}(\sin\theta\cos\theta + i\sin^2\theta) + \frac{\hbar^2}{8}\sin^2\theta + \frac{\hbar^2}{8}(\sin^2\theta\cos\theta + i\sin^3\theta) + \frac{\hbar^2}{4i}(\sin\theta\cos\theta - i\sin^2\theta) \\
&= \frac{\hbar^2}{4} - \frac{3\hbar^2}{8}\sin^2\theta + \frac{\hbar^2}{8}(\sin^2\theta\cos\theta + i\sin^3\theta)
\end{aligned}
$$

And finally

$$
\sigma_{sx}^2\sigma_{sy}^2 = \frac{\hbar^4}{64}(e^{i\theta}\sin^2\theta - 3\sin^2\theta + 2)(e^{i\theta}\cos^2\theta - 3\cos^2\theta + 2)
$$

1.4

A system has three interacting spin degrees of freedom with the followin hamiltonian

$$
H = \frac{J}{\hbar^2}(\mathbf{S}_1\cdot\mathbf{S}_2 + \mathbf{S}_2\cdot\mathbf{S}_3 + \mathbf{S}_3\cdot\mathbf{S}_1) \tag{3}
$$

where J si a positive number with units of energy. The spin operators are $\mathbf{S}_1 \equiv \mathbf{S}\otimes\mathbb{1}\otimes\mathbb{1}$, $\mathbf{S}_2 \equiv \mathbb{1}\otimes\mathbf{S}\otimes\mathbb{1}$ and $\mathbf{S}_3 \equiv \mathbb{1}\otimes\mathbb{1}\otimes\mathbf{S}$, where $\mathbf{S} = (S_x, S_y, S_z)$. A general state of this three-spin system is a linear combination of product states

$|m_{s1}m_{s2}m_{s3}\rangle \equiv |m_{s1}\rangle \otimes |m_{s2}\rangle \otimes |m_{s3}\rangle$ where m_{si} is hte spin-z quantum number of spin number i, either up ($\frac{1}{2}$) or down ($-\frac{1}{2}$). For example: the product state $|\uparrow\downarrow\uparrow\rangle$ is a state where spin number one is in state $|\uparrow\rangle$, spin number two is in state $|\downarrow\rangle$ and spin number three is in state $|\uparrow\rangle$.

$\mathbf{S}_1 \cdot \mathbf{S}_2$ can be expressed in terms of S_1^+, S_1^-, S_2^+, S_2^-, S_1^z and S_2^z. First we have

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z$$

where

$$\begin{aligned} S_1^x S_2^x &= \frac{1}{4}(S_1^+ S_2^+ + S_1^+ S_2^- + S_1^- S_2^+ + S_1^+ S_2^+) \\ S_1^y S_2^y &= -\frac{1}{4}(S_1^+ S_2^+ - S_1^+ S_2^- - S_1^- S_2^+ + S_1^+ S_2^+) \end{aligned}$$

then

$$S_1^x S_2^x + S_1^y S_2^y = S_1^+ S_2^-$$

assuming that the lowering and raising operators of different spins commutes[1], i.e. $[S_i^+, S_j^j] = 0$. We end up with

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = S_1^+ S_2^- + S_1^z S_2^z \tag{4}$$

if the ladder operators does not commute then

$$S_1^x S_2^x + S_1^y S_2^y = \frac{1}{2}(S_1^+ S_2^- + S_2^+ S_1^-)$$

and we end up with

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}(S_1^+ S_2^- + S_2^+ S_1^-) + S_1^z S_2^z \tag{5}$$

which seems more reasonable.

[1]Ladder operator for same spin/state commute: $[S^+, S^-] = (S^x + iS^y)(S^x - iS^y) - (S^x - iS^y)(S^x + iS^y) = S_x^2 + S_y^2 - S_x^2 - S_y^2 = 0$