Midterm "Take home"-exam
FYS3110

Unable to see candidate no

October 7, 2016

1 Spin-1/2 systems

The following is given:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2, \quad \hat{S}^{\pm} = \hat{S}_x \pm i\hat{S}_y$$
$$|\uparrow\rangle \equiv \left| s = \frac{1}{2}, m_s = \frac{1}{2} \right\rangle, \quad |\downarrow\rangle \equiv \left| s = \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle$$
$$\hat{S}^2 |\uparrow\rangle = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right)|\uparrow\rangle, \quad \hat{S}^2 |\downarrow\rangle = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right)|\downarrow\rangle$$
$$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle, \quad \hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$$
$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

1.1

$$\hat{S}_z\hat{S}^+ |\downarrow\rangle = \hat{S}_z\hat{S}_x |\downarrow\rangle + i\hat{S}_z\hat{S}_y |\downarrow\rangle$$

rewriting commutation relations

$$[\hat{S}_z, \hat{S}_x] = \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z = i\hbar\hat{S}_y \to \hat{S}_z\hat{S}_x = i\hbar\hat{S}_y + \hat{S}_x\hat{S}_z$$
$$[\hat{S}_y, \hat{S}_z] = \hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y = i\hbar\hat{S}_x \to \hat{S}_z\hat{S}_y = \hat{S}_y\hat{S}_z - i\hbar\hat{S}_x,$$

gives

$$\begin{aligned}
\hat{S}_z\hat{S}^+ |\downarrow\rangle &= (i\hbar\hat{S}_y + \hat{S}_x\hat{S}_z + i\hat{S}_y\hat{S}_z + \hbar\hat{S}_x)|\downarrow\rangle \\
&= \left(i\hbar\hat{S}_y - \frac{\hbar}{2}\hat{S}_x - i\frac{\hbar}{2}\hat{S}_y + \hbar\hat{S}_x\right)|\downarrow\rangle \\
&= \left(\frac{\hbar}{2}\hat{S}_x + i\frac{\hbar}{2}\hat{S}_y\right)|\downarrow\rangle = \frac{\hbar}{2}\hat{S}^+ |\downarrow\rangle .
\end{aligned}$$

This means that $\hat{S}^+ |\downarrow\rangle$ is an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$.

1.2

$$
\begin{aligned}
\hat{S}^-\hat{S}^+ &= (\hat{S}_x - i\hat{S}_y)(\hat{S}_x + i\hat{S}_y) \\
&= \hat{S}_x^2 + i\hat{S}_x\hat{S}_y - i\hat{S}_y\hat{S}_x + \hat{S}_y^2 \\
&= \hat{S}_x^2 + \hat{S}_y^2 + i[\hat{S}_x, \hat{S}_y] \\
&= \hat{S}^2 - \hat{S}_z^2 - i\hbar\hat{S}_x
\end{aligned}
$$

This can be uses to compute the norm of $|\psi_1\rangle = \hat{S}^+ |\uparrow\rangle$ and $|\psi_2\rangle = \hat{S}^+ |\uparrow\rangle$.

$$
\begin{aligned}
\langle\psi_1|\psi_1\rangle &= \langle\downarrow|\, \hat{S}^-\hat{S}^+ \,|\downarrow\rangle = \langle\downarrow|\, (\hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z) \,|\downarrow\rangle \\
&= \langle\downarrow|\, \hbar^2 \frac{1}{2}\left(\frac{1}{2} + 1\right) |\downarrow\rangle - \langle\downarrow|\, \frac{\hbar^2}{4} \,|\downarrow\rangle + \langle\downarrow|\, \frac{2\hbar^2}{4} \,|\downarrow\rangle \\
&= \frac{3\hbar^2}{4} - \frac{\hbar^2}{4} + \frac{2\hbar}{4} = \hbar^2
\end{aligned}
$$

which means that $\||\psi_1\rangle\| = \hbar$.

$$
\begin{aligned}
\langle\psi_2|\psi_2\rangle &= \langle\uparrow|\, \hat{S}^-\hat{S}^+ \,|\uparrow\rangle = \langle\uparrow|\, (\hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z) \,|\uparrow\rangle \\
&= \langle\uparrow|\, \hbar^2 \frac{1}{2}\left(\frac{1}{2} + 1\right) |\uparrow\rangle - \langle\uparrow|\, \frac{\hbar^2}{4} \,|\uparrow\rangle - \langle\uparrow|\, \frac{2\hbar^2}{4} \,|\uparrow\rangle \\
&= \frac{3\hbar^2}{4} - \frac{\hbar^2}{4} - \frac{2\hbar^2}{4} = 0
\end{aligned}
$$

which measn that $\||\psi_2\rangle\| = 0$.