Problem Sheet 6
FYS3110

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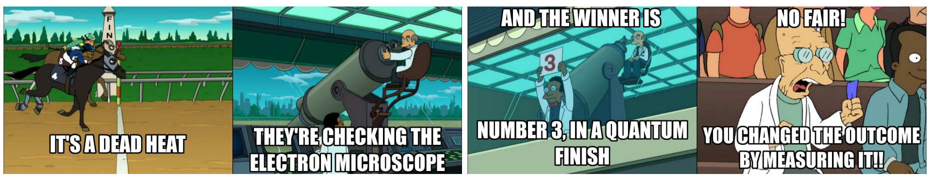

Figure 1: Quantum joke from Futurama.

Problem 6.1

s

An electron which has spin-1/2 is in the states

$$|\psi\rangle = \sqrt{\frac{2}{5}}\,|3,2,1\rangle \otimes |{\downarrow_z}\rangle + \sqrt{\frac{3}{5}}\,|2,1,1\rangle \otimes |{\uparrow_z}\rangle\,, \tag{1}$$

of the hydrogen atom. The state with quantum numbers n, l, m and spin $s_z = \{\uparrow_z = \hbar/2, \quad \downarrow_z = -\hbar/2\}$ along the z-axis is denoted $|n,l,m\rangle \otimes |s_z\rangle$.

a)

The probability that the electron is measured to be in the spin up state along the z-axis can be calculated in a very difficult manner by computing $\langle\psi| (\mathbb{1} \otimes |{\uparrow_z}\rangle \langle{\uparrow_z}|) |\psi\rangle$. However, one can simply look at the coefficients in the superposition represesenation of the state in equation 1 and realize that

$$P(\uparrow_z) = \frac{3}{5}, \quad P(\downarrow_z) = \frac{2}{5}. \tag{2}$$

The probabilities add up to one, implying that the state is normalized.

b)

To find which values can be measured for L^2 and for what probabilities one need simply to apply the $\hat{L}^2$ operator to the state of the electron in equation 1.

$$\begin{aligned}
\hat{L}^2 |\psi\rangle &= \sqrt{\frac{2}{5}}\hat{L}^2 |3,2,1\rangle \otimes |{\downarrow_z}\rangle + \sqrt{\frac{3}{5}}\hat{L}^2 |2,1,1\rangle \otimes |{\uparrow_z}\rangle \\
&= \sqrt{\frac{2}{5}}(\hbar^2 2(2+1) |3,2,1\rangle \otimes |{\downarrow_z}\rangle) + \sqrt{\frac{3}{5}}(\hbar^2 1(1+1) |2,1,1\rangle \otimes |{\uparrow_z}\rangle) \\
&= \sqrt{\frac{2}{5}}(6\hbar |3,2,1\rangle \otimes |{\downarrow_z}\rangle) + \sqrt{\frac{2}{5}}(2\hbar^2 |2,1,1\rangle \otimes |{\uparrow_z}\rangle)
\end{aligned}$$

which means that one measures $6\hbar^2$ with probability $2/5$ and $2\hbar^2$ with probability $3/5$ for L^2.

The corresponding computation for L_z is

$$\begin{aligned}
\hat{L}_z |\psi\rangle &= \sqrt{\frac{2}{5}}(\hat{L}_z |3,2,1\rangle \otimes |{\downarrow_z}\rangle) + \sqrt{\frac{3}{5}}(\hat{L}_z |2,1,1\rangle \otimes |{\uparrow_z}\rangle) \\
&= \sqrt{\frac{2}{5}}(\hbar |3,2,1\rangle \otimes |{\downarrow_z}\rangle) + \sqrt{\frac{3}{5}}(\hbar |2,1,1\rangle \otimes |{\uparrow_z}\rangle)
\end{aligned}$$

which means that one measures $\hbar$ with probability 1 for L_z.

Lastly, the computation for S^2

$$\begin{aligned}
\hat{L}_z |\psi\rangle &= \sqrt{\frac{2}{5}}\left(\hbar^2 \frac{1}{2}\frac{3}{2} |3,2,1\rangle \otimes |{\downarrow_z}\rangle\right) + \sqrt{\frac{3}{5}}\left(\hbar^2 \frac{1}{2}\frac{3}{2} |2,1,1\rangle \otimes |{\uparrow_z}\rangle\right) \\
&= \sqrt{\frac{2}{5}}\left(\frac{3\hbar^2}{4} |3,2,1\rangle \otimes |{\downarrow_z}\rangle\right) + \sqrt{\frac{3}{5}}\left(\frac{3\hbar^2}{4} |3,2,1\rangle \otimes |{\uparrow_z}\rangle\right)
\end{aligned}$$

which means that one measure $\frac{3\hbar^2}{4}$ with probability 1 for S^2.

c)

Now, consider the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ (really $\mathbf{L} \otimes I + I \otimes \mathbf{S}$).

First considering the superposed state from equation 1

$$\begin{aligned}
|\psi\rangle &= \sqrt{\frac{2}{5}} |3,2,1\rangle \otimes |{\downarrow_z}\rangle + \sqrt{\frac{3}{5}} |2,1,1\rangle \otimes |{\uparrow_z}\rangle \\
&= \sqrt{\frac{2}{5}} R_{3,2} Y_2^1 \left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{3}{5}} R_{2,1} Y_1^1 \left|\frac{1}{2}, \frac{1}{2}\right\rangle
\end{aligned}$$

Because the radial wave function is unnecessary to calculate total angular momentum this can be simplified to

$$|\psi\rangle = \sqrt{\frac{2}{5}} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle |2,1\rangle + \sqrt{\frac{3}{2}} \left|\frac{1}{2}, \frac{1}{2}\right\rangle |1,1\rangle \tag{3}$$

Employting Glebsch-Gordan coefficient tables one can find that

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle|2,1\rangle=\sqrt{\frac{2}{5}}\left|\frac{5}{2},\frac{1}{2}\right\rangle+\sqrt{\frac{3}{5}}\left|\frac{3}{5},\frac{1}{2}\right\rangle \tag{4}$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle|1,1\rangle=\left|\frac{1}{2},\frac{1}{2}\right\rangle \tag{5}$$

Inserting 4 and 5 into 3 yields

$$\begin{aligned}|\psi\rangle&=\sqrt{\frac{2}{5}}\left(\sqrt{\frac{2}{5}}\left|\frac{5}{2},\frac{1}{2}\right\rangle+\sqrt{\frac{3}{5}}\left|\frac{3}{2},\frac{1}{2}\right\rangle\right)+\sqrt{\frac{3}{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\\&=\sqrt{\frac{4}{25}}\left|\frac{5}{2},\frac{1}{2}\right\rangle+\sqrt{\frac{15}{25}}\left|\frac{3}{2},\frac{1}{2}\right\rangle+\sqrt{\frac{15}{25}}\left|\frac{1}{2},\frac{1}{2}\right\rangle\end{aligned}$$

One can clearly see that the superposition is normalized, which is good and expected.