PROBLEM SHEET 9
FYS3110

SEBASTIAN G. WINTHER-LARSEN

Problem 6.1

For the harmonic oscillator the potential is $V(x) = \frac{1}{2}kx^2$ and the allowed energies are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \text{ for } n = 0, 1, 2, \dots \tag{1}$$

where $\omega = \sqrt{\frac{k}{m}}$ is the classical angular frequency.

a. The spring constant is increased slightly from k to $(1+\epsilon)k$. The exact new allowed energies are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{(1+\epsilon)k}{m}}. \tag{2}$$

The MacLaurin series[1] of the increased spring constant up to second order is

$$\sqrt{1+\epsilon} \approx 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8}\dots \tag{3}$$

Inserting equation 3 into 2 yields

$$E_n \approx \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{k}{m}}\left(1 + \frac{\epsilon}{2} - \frac{\epsilon}{8}\right) \tag{4}$$

b. Now to calculate the first-order peturbation in the energy

$$E_n^1 = \langle\psi_n^0| H' |\psi_n^0\rangle, \tag{5}$$

where $H' = T + V'$ and $V' = \frac{1+\epsilon}{2}kx^2$. The change in change in energy is

$$H' - H = V' - V = \frac{1+\epsilon}{2}kx^2 - \frac{1}{2}kx^2 = \frac{1}{2}\epsilon kx^2 = \epsilon V,$$

which reduces equation 5 to

$$E_n^1 = \langle\psi_n^0| \epsilon V |\psi_n^0\rangle. \tag{6}$$

[1] Taylor expansion around zero, from which the power series arises.

This equation can be solved quite easily be employing the virial theorem for a stationary state

$$2\langle T\rangle = \left\langle x\frac{dV}{dx}\right\rangle . \tag{7}$$

For the harmonic oscillator

$$\left\langle x\frac{dV}{dx}\right\rangle = k\left\langle x^2\right\rangle \rightarrow \langle T\rangle = k\left\langle x^2\right\rangle \rightarrow \langle T\rangle = \frac{1}{2}k\left\langle x^2\right\rangle = \langle V\rangle = \frac{E_n}{2}.$$

It follows that equation 6 becomes

$$E_n^1 = \frac{\epsilon}{2}E_n^0 = \frac{\epsilon}{2}\left(n+\frac{1}{2}\right)\hbar\omega, \tag{8}$$

which is interesting considering that ω includes the original spring constant.