The Hamiltonian of a particle with mass m in a one-dimensional harmonic oscillator potential having a charateristic frequency ω is

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2.$$

The expressions for the lowering and raising operators, $\hat{a}$ and $\hat{a}^\dagger$, are

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P})$$

Problem 4.1

a) Compute the matrix elements $X_{nm} = \langle n|\hat{X}|m\rangle$, where $\{|n\rangle\}, n = 0, 1, 2, ...$ is the energy eigenbasis $\hat{H}|n\rangle = \hbar\omega(n + 1/2)|n\rangle$, using the lowering and raising operators $\hat{a}$, $\hat{a}^\dagger$.

b) Let $|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$. Find the condition on the c_n's such that $|\psi(0)\rangle$ has unit norm. Find an expression for the time-dependent state $|\psi(t)\rangle$.

c) Compute the expectation values $\langle\psi(t)|\hat{H}|\psi(t)\rangle$ and $\langle\psi(t)|\hat{X}|\psi(t)\rangle$.

d) Set $c_0 = e^{-\alpha^2/2}$ and $c_n = c_0\alpha^n/\sqrt{n!}$ for $n \geq 1$. α is a positive real number. Compute $\langle\psi(t)|\hat{X}|\psi(t)\rangle$ in this case, and compare the answer to that of the position of a classical harmonic oscillator with amplitude A. Use $\langle\psi(t)|\hat{H}|\psi(t)\rangle = kA^2/2$ to define the classical amplitude A.

Problem 4.2

Griffiths Chapter 4, problem 38.

Problem 4.3 (optional)

a) Use the equation $\hat{a}|0\rangle = 0$ to find the expression for the ground state wavefunction $\psi_0(p) \equiv \langle p|0\rangle$ in the momentum representation where $\hat{P} \simeq p$ and $\hat{X} \simeq i\hbar\frac{\partial}{dp}$ ($\simeq$ means represented by). $|p\rangle$ is an eigenket of the momentum operator having the eigenvalue p.

b) Use $a^\dagger$ and your answer in a) to find the explicit expression for the first excited wave function of the harmonic oscillator in the momentum representation.