Midterm “Take home”-exam
FYS3110

Unable to see candidate no

October 13, 2016

1 Spin-$1/2$ systems

The following is given:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2, \quad \hat{S}^{\pm} = \hat{S}_x \pm i\hat{S}_y$$
$$\left|\uparrow\right\rangle \equiv \left|s = \frac{1}{2}, m_s = \frac{1}{2}\right\rangle, \quad \left|\downarrow\right\rangle \equiv \left|s = \frac{1}{2}, m_s = -\frac{1}{2}\right\rangle$$
$$\hat{S}^2\left|\uparrow\right\rangle = \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right)\left|\uparrow\right\rangle, \quad \hat{S}^2\left|\downarrow\right\rangle = \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right)\left|\downarrow\right\rangle$$
$$\hat{S}_z\left|\uparrow\right\rangle = \frac{\hbar}{2}\left|\uparrow\right\rangle, \quad \hat{S}_z\left|\downarrow\right\rangle = -\frac{\hbar}{2}\left|\downarrow\right\rangle$$
$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$$

1.1

$$\hat{S}_z\hat{S}^+\left|\downarrow\right\rangle = \hat{S}_z\hat{S}_x\left|\downarrow\right\rangle + i\hat{S}_z\hat{S}_y\left|\downarrow\right\rangle$$

rewriting commutation relations

$$[\hat{S}_z, \hat{S}_x] = \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z = i\hbar\hat{S}_y \rightarrow \hat{S}_z\hat{S}_x = i\hbar\hat{S}_y + \hat{S}_x\hat{S}_z$$
$$[\hat{S}_y, \hat{S}_z] = \hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y = i\hbar\hat{S}_x \rightarrow \hat{S}_z\hat{S}_y = \hat{S}_y\hat{S}_z - i\hbar\hat{S}_x,$$

gives

$$\begin{aligned}\hat{S}_z\hat{S}^+\left|\downarrow\right\rangle &= (i\hbar\hat{S}_y + \hat{S}_x\hat{S}_z + i\hat{S}_y\hat{S}_z + \hbar\hat{S}_x)\left|\downarrow\right\rangle \\ &= \left(i\hbar\hat{S}_y - \frac{\hbar}{2}\hat{S}_x - i\frac{\hbar}{2}\hat{S}_y + \hbar\hat{S}_x\right)\left|\downarrow\right\rangle \\ &= \left(\frac{\hbar}{2}\hat{S}_x + i\frac{\hbar}{2}\hat{S}_y\right)\left|\downarrow\right\rangle = \frac{\hbar}{2}\hat{S}^+\left|\downarrow\right\rangle.\end{aligned}$$

This means that $\hat{S}^+\left|\downarrow\right\rangle$ is an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$.

1.2

$$
\begin{aligned}
\hat{S}^-\hat{S}^+ &= (\hat{S}_x - i\hat{S}_y)(\hat{S}_x + i\hat{S}_y) \\
&= \hat{S}_x^2 + i\hat{S}_x\hat{S}_y - i\hat{S}_y\hat{S}_x + \hat{S}_y^2 \\
&= \hat{S}_x^2 + \hat{S}_y^2 + i[\hat{S}_x, \hat{S}_y] \\
&= \hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z
\end{aligned}
$$

This can be uses to compute the norm of $|\psi_1\rangle = \hat{S}^+ |\uparrow\rangle$ and $|\psi_2\rangle = \hat{S}^+ |\uparrow\rangle$.

$$
\begin{aligned}
\langle\psi_1|\psi_1\rangle &= \langle\downarrow| \hat{S}^-\hat{S}^+ |\downarrow\rangle = \langle\downarrow| (\hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z) |\downarrow\rangle \\
&= \langle\downarrow| \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) |\downarrow\rangle - \langle\downarrow| \frac{\hbar^2}{4} |\downarrow\rangle + \langle\downarrow| \frac{2\hbar^2}{4} |\downarrow\rangle \\
&= \frac{3\hbar^2}{4} - \frac{\hbar^2}{4} + \frac{2\hbar}{4} = \hbar^2
\end{aligned}
$$

which means that $\||\psi_1\rangle\| = \hbar$.

$$
\begin{aligned}
\langle\psi_2|\psi_2\rangle &= \langle\uparrow| \hat{S}^-\hat{S}^+ |\uparrow\rangle = \langle\uparrow| (\hat{S}^2 - \hat{S}_z^2 - \hbar\hat{S}_z) |\uparrow\rangle \\
&= \langle\uparrow| \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) |\uparrow\rangle - \langle\uparrow| \frac{\hbar^2}{4} |\uparrow\rangle - \langle\uparrow| \frac{2\hbar^2}{4} |\uparrow\rangle \\
&= \frac{3\hbar^2}{4} - \frac{\hbar^2}{4} - \frac{2\hbar^2}{4} = 0
\end{aligned}
$$

which means that $\||\psi_2\rangle\| = 0$. Useful in the following problems..

1.3

Phases are chosen sucht that the following relations hold

$$
\hat{S}^+ |\downarrow\rangle = \hbar |\uparrow\rangle, \quad \hat{S}^- |\uparrow\rangle = \hbar |\downarrow\rangle .
$$

Introducing a new state

$$
|\phi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + e^{i\theta} |\downarrow\rangle)
$$

where θ is a real number. We wish to compute the "uncertainty" product $\sigma_{sx}^2\sigma_{sy}^2$ where

$$
\begin{aligned}
\sigma_{sx}^2 &= \langle\phi| (\hat{S}_x - \langle\phi| \hat{S}_x |\phi\rangle)^2 |\phi\rangle \\
\sigma_{sy}^2 &= \langle\phi| (\hat{S}_y - \langle\phi| \hat{S}_y |\phi\rangle)^2 |\phi\rangle .
\end{aligned}
$$

First we need to find expressions for $\hat{S}_x$ and $\hat{S}_y$

$$
\hat{S}^+ + \hat{S}^- = (\hat{S}_x + i\hat{S}_y) + (\hat{S}_x - i\hat{S}_y) = 2\hat{S}_x \to \hat{S}_x = \frac{1}{2}(\hat{S}^+ + \hat{S}^-) \tag{1}
$$

$$
\hat{S}^+ - \hat{S}^- = (\hat{S}_x + i\hat{S}_y) - (\hat{S}_x - i\hat{S}_y) = 2i\hat{S}_y \to \hat{S}_y = \frac{1}{2i}(\hat{S}^+ - \hat{S}^-) \tag{2}
$$

It will also make things easier to calculate $\hat{S}_x\left|\uparrow\right\rangle$, $\hat{S}_x\left|\downarrow\right\rangle$, $\hat{S}_y\left|\uparrow\right\rangle$ and $\hat{S}_y\left|\downarrow\right\rangle$. These values can be found using equations 1 and 2.

$$\begin{aligned}
\hat{S}_x\left|\uparrow\right\rangle &= \frac{\hbar}{2}\left|\downarrow\right\rangle & \hat{S}_x^2\left|\uparrow\right\rangle &= \frac{\hbar^2}{4}\left|\uparrow\right\rangle \\
\hat{S}_x\left|\downarrow\right\rangle &= \frac{\hbar}{2}\left|\uparrow\right\rangle & \hat{S}_x^2\left|\downarrow\right\rangle &= \frac{\hbar^2}{4}\left|\downarrow\right\rangle \\
\hat{S}_y\left|\uparrow\right\rangle &= -\frac{\hbar}{2i}\left|\downarrow\right\rangle & \hat{S}_y^2\left|\uparrow\right\rangle &= \frac{\hbar^2}{4}\left|\uparrow\right\rangle \\
\hat{S}_y\left|\downarrow\right\rangle &= \frac{\hbar}{2i}\left|\uparrow\right\rangle & \hat{S}_y^2\left|\downarrow\right\rangle &= \frac{\hbar^2}{4}\left|\downarrow\right\rangle
\end{aligned}$$

We can begin on what is the real task at hand

$$\begin{aligned}
\left\langle\phi\right|\hat{S}_x\left|\phi\right\rangle &= \frac{1}{2}(\left\langle\uparrow\right| + e^{-i\theta}\left\langle\downarrow\right|)\hat{S}_x(\left|\uparrow\right\rangle + e^{i\theta}\left|\downarrow\right\rangle) \\
&= \frac{1}{2}(\left\langle\uparrow\right| + e^{-i\theta}\left\langle\downarrow\right|)\left(\frac{\hbar}{2}\left|\downarrow\right\rangle + e^{i\theta}\frac{\hbar}{2}\left|\uparrow\right\rangle\right) \\
&= \frac{\hbar}{4}\left(e^{i\theta} + e^{-i\theta}\right) \\
&= \frac{\hbar}{4}(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta) \\
&= \frac{\hbar}{2}\cos\theta \\
\left\langle\phi\right|\hat{S}_y\left|\phi\right\rangle &= \frac{1}{2}(\left\langle\uparrow\right| + e^{-i\theta}\left\langle\downarrow\right|)\hat{S}_y(\left|\uparrow\right\rangle + e^{i\theta}\left|\downarrow\right\rangle) \\
&= \frac{1}{2}(\left\langle\uparrow\right| + e^{-i\theta}\left\langle\downarrow\right|)\left(-\frac{\hbar}{2i}\left|\downarrow\right\rangle + e^{i\theta}\frac{\hbar}{2i}\left|\uparrow\right\rangle\right) \\
&= \frac{\hbar}{4i}\left(e^{i\theta} - e^{-i\theta}\right) \\
&= \frac{\hbar}{4i}(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta) \\
&= \frac{\hbar}{2}\sin\theta
\end{aligned}$$

$$
\begin{aligned}
\sigma_{sx}^2 &= \langle\phi|\,(\hat{S}_x - \frac{\hbar}{2}\cos\theta)^2\,|\phi\rangle \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|)(\hat{S}_x^2 - \hbar\cos\theta\hat{S}_x + \frac{\hbar^2}{4}\cos^2\theta)(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle) \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|) \\
&\quad \left(\frac{\hbar^2}{4}|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}|\downarrow\rangle - \frac{\hbar^2}{2}\cos\theta|\downarrow\rangle - \frac{\hbar^2}{2}e^{i\theta}\cos\theta|\uparrow\rangle + \frac{\hbar^2}{4}\cos^2\theta|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}\cos^2\theta|\uparrow\rangle\right) \\
&= \frac{\hbar^2}{8} - \frac{\hbar^2}{4}e^{i\theta}\cos\theta + \frac{\hbar^2}{8}\cos^2\theta + \frac{\hbar^2}{8}e^{i\theta}\cos^2\theta + \frac{\hbar^2}{8} - \frac{\hbar^2}{4}e^{-i\theta}\cos\theta \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{4}(\cos^2\theta + i\sin\theta\cos\theta) + \frac{\hbar^2}{8}\cos^2\theta + \frac{\hbar^2}{8}(\cos^3\theta + i\sin\theta\cos^2\theta) - \frac{\hbar^2}{4}(\cos^2\theta - i\sin\theta\cos\theta) \\
&= \frac{\hbar^2}{4} - \frac{3\hbar^2}{8}\cos^2\theta + \frac{\hbar^2}{8}(\cos^3\theta + i\sin\theta\cos^2\theta) \\
\sigma_{sy}^2 &= \langle\phi|\,(\hat{S}_y - \frac{\hbar}{2}\sin\theta)^2\,|\phi\rangle \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|)(\hat{S}_y^2 - \hbar\sin\theta\hat{S}_y + \frac{\hbar^2}{4}\sin^2\theta)(|\uparrow\rangle + e^{i\theta}|\downarrow\rangle) \\
&= \frac{1}{2}(\langle\uparrow| + e^{-i\theta}\langle\downarrow|) \\
&\quad \left(\frac{\hbar^2}{4}|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}|\downarrow\rangle + \frac{\hbar^2}{2i}\sin\theta|\downarrow\rangle - \frac{\hbar^2}{2i}e^{i\theta}\sin\theta|\uparrow\rangle + \frac{\hbar^2}{4}\sin^2\theta|\uparrow\rangle + \frac{\hbar^2}{4}e^{i\theta}\sin^2\theta|\uparrow\rangle\right) \\
&= \frac{\hbar^2}{8} - \frac{\hbar^2}{4i}e^{i\theta}\sin\theta + \frac{\hbar^2}{8}\sin^2\theta + \frac{\hbar^2}{8}e^{i\theta}\sin^2\theta + \frac{\hbar^2}{8} + \frac{\hbar^2}{4i}e^{-i\theta}\sin\theta \\
&= \frac{\hbar^2}{4} - \frac{\hbar^2}{4i}(\sin\theta\cos\theta + i\sin^2\theta) + \frac{\hbar^2}{8}\sin^2\theta + \frac{\hbar^2}{8}(\sin^2\theta\cos\theta + i\sin^3\theta) + \frac{\hbar^2}{4i}(\sin\theta\cos\theta - i\sin^2\theta) \\
&= \frac{\hbar^2}{4} - \frac{3\hbar^2}{8}\sin^2\theta + \frac{\hbar^2}{8}(\sin^2\theta\cos\theta + i\sin^3\theta)
\end{aligned}
$$

And finally

$$
\sigma_{sx}^2\sigma_{sy}^2 = \frac{\hbar^4}{64}(e^{i\theta}\sin^2\theta - 3\sin^2\theta + 2)(e^{i\theta}\cos^2\theta - 3\cos^2\theta + 2)
$$

1.4

A system has three interacting spin degrees of freedom with the followin hamiltonian

$$
H = \frac{J}{\hbar^2}(\mathbf{S}_1\cdot\mathbf{S}_2 + \mathbf{S}_2\cdot\mathbf{S}_3 + \mathbf{S}_3\cdot\mathbf{S}_1) \tag{3}
$$

where J is a positive number with units of energy. The spin operators are $\mathbf{S}_1 \equiv \mathbf{S}\otimes\mathbb{1}\otimes\mathbb{1}$, $\mathbf{S}_2 \equiv \mathbb{1}\otimes\mathbf{S}\otimes\mathbb{1}$ and $\mathbf{S}_3 \equiv \mathbb{1}\otimes\mathbb{1}\otimes\mathbf{S}$, where $\mathbf{S} = (S_x, S_y, S_z)$. A general state of this three-spin system is a linear combination of product states

$|m_{s1}m_{s2}m_{s3}\rangle \equiv |m_{s1}\rangle \otimes |m_{s2}\rangle \otimes |m_{s3}\rangle$ where m_{si} is hte spin-z quantum number of spin number i, either up ($\frac{1}{2}$) or down ($-\frac{1}{2}$). For example: the product state $|\uparrow\downarrow\uparrow\rangle$ is a state where spin number one is in state $|\uparrow\rangle$, spin number two is in state $|\downarrow\rangle$ and spin number three is in state $|\uparrow\rangle$.

$\mathbf{S}_1 \cdot \mathbf{S}_2$ can be expressed in terms of S_1^+, S_1^-, S_2^+, S_2^-, S_1^z and S_2^z. First we have

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z$$

where

$$\begin{aligned} S_1^x S_2^x &= \frac{1}{4}(S_1^+ S_2^+ + S_1^+ S_2^- + S_1^- S_2^+ + S_1^+ S_2^-) \\ S_1^y S_2^y &= -\frac{1}{4}(S_1^+ S_2^+ - S_1^+ S_2^- - S_1^- S_2^+ + S_1^+ S_2^-) \end{aligned}$$

If the ladder operators does not commute then

$$S_1^x S_2^x + S_1^y S_2^y = \frac{1}{2}(S_1^+ S_2^- + S_2^+ S_1^-)$$

and we end up with

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}(S_1^+ S_2^- + S_2^+ S_1^-) + S_1^z S_2^z \tag{4}$$

Computing $H|\uparrow\downarrow\downarrow\rangle$ should now be quite straight-forward.

$$\begin{aligned} H|\uparrow\downarrow\downarrow\rangle = \frac{J}{\hbar^2}\big(&\frac{1}{2}(S_1^+ S_2^- + S_2^+ S_1^-)|\uparrow\downarrow\downarrow\rangle + S_1^z S_2^z|\uparrow\downarrow\downarrow\rangle \\ &+\frac{1}{2}(S_2^+ S_3^- + S_3^+ S_2^-)|\uparrow\downarrow\downarrow\rangle + S_2^z S_3^z|\uparrow\downarrow\downarrow\rangle \\ &+\frac{1}{2}(S_3^+ S_1^- + S_1^+ S_3^-)|\uparrow\downarrow\downarrow\rangle + S_3^z S_1^z|\uparrow\downarrow\downarrow\rangle\big) \\ =&\frac{J}{\hbar^2}\big(\frac{\hbar^2}{2}|\downarrow\uparrow\downarrow\rangle + \frac{\hbar^2}{2}|\downarrow\downarrow\uparrow\rangle - \frac{\hbar^2}{4}|\uparrow\downarrow\downarrow\rangle\big) \\ =&J(\frac{1}{2}|\downarrow\uparrow\downarrow\rangle + \frac{1}{2}|\downarrow\downarrow\uparrow\rangle - \frac{1}{4}|\uparrow\downarrow\downarrow\rangle) \end{aligned}$$

This result is confirmed by the python script in appendix A. $|\uparrow\downarrow\downarrow\rangle$ is not an eigen state of H.

1.5

It is realtively easy to show with matrices or algebra or a script or anything that

$$[H, S_{tot}^z] = 0 \tag{5}$$

The eigenvalues of S_{tot}^z are easy enough to compute

$$S_{tot}^z|\uparrow\uparrow\uparrow\rangle = \frac{\hbar}{2}|\uparrow\uparrow\uparrow\rangle + \frac{\hbar}{2}|\uparrow\uparrow\uparrow\rangle + \frac{\hbar}{2}|\uparrow\uparrow\uparrow\rangle = \frac{3\hbar}{2}|\uparrow\uparrow\uparrow\rangle$$

the rest are

$$
\begin{aligned}
S^z_{tot}\left|\uparrow\uparrow\downarrow\right\rangle &= \frac{\hbar}{2}\left|\uparrow\uparrow\downarrow\right\rangle \\
S^z_{tot}\left|\uparrow\downarrow\uparrow\right\rangle &= \frac{\hbar}{2}\left|\uparrow\downarrow\uparrow\right\rangle \\
S^z_{tot}\left|\downarrow\uparrow\downarrow\right\rangle &= \frac{\hbar}{2}\left|\downarrow\uparrow\downarrow\right\rangle \\
S^z_{tot}\left|\downarrow\downarrow\uparrow\right\rangle &= -\frac{\hbar}{2}\left|\downarrow\downarrow\uparrow\right\rangle \\
S^z_{tot}\left|\downarrow\uparrow\downarrow\right\rangle &= -\frac{\hbar}{2}\left|\downarrow\uparrow\downarrow\right\rangle \\
S^z_{tot}\left|\uparrow\downarrow\downarrow\right\rangle &= -\frac{\hbar}{2}\left|\uparrow\downarrow\downarrow\right\rangle \\
S^z_{tot}\left|\downarrow\downarrow\downarrow\right\rangle &= -\frac{3\hbar}{2}\left|\downarrow\downarrow\downarrow\right\rangle
\end{aligned}
$$

The “general” rule appears to be

$$
S^z_{tot}\left|m_{s1}m_{s2}m_{s3}\right\rangle = \hbar(m_{s1}+m_{s2}+m_{s3})\left|m_{s1}m_{s2}m_{s3}\right\rangle \tag{6}
$$

1.6

Finding eigenvalues of H. The trick is first to express H in terms of S^2_{tot}.

$$
\begin{aligned}
S^2_{tot} &= S_1^2 + S_2^2 + S_3^2 + 2S_1\cdot S_2 + 2S_2\cdot S_3 + 2S_3\cdot S_1 \\
H &= \frac{J}{2\hbar^2}(2S_1\cdot S_2 + 2S_2\cdot S_3 + 2S_3\cdot S_1) \\
&= \frac{J}{2\hbar^2}(S^2_{tot} - (S_1^2 + S_2^2 + S_3^3))
\end{aligned}
$$

We know from the previous problem that total spin angular momentum quantum number s_{tot} must be 3/2 or 1/2. The general formula for the energy eigenvalue of total spin quantum number squared is $S^2_{tot}\left|\psi\right\rangle = s_{tot}(1 + s_{tot})\hbar^2\left|\psi\right\rangle$. The eigenvalue energy for $s_{tot} = 3/2$ is therefore given by

$$
\begin{aligned}
H\left|\psi\right\rangle =& \frac{J}{2\hbar^2}(S^2_{tot}\left|\psi\right\rangle - (S_1^2+S_2^2+S_3^2)\left|\psi\right\rangle) = \frac{J}{2\hbar^2}\left(\frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2 + 3\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\right) \\
=& \frac{J}{2\hbar^2}\left(\frac{15\hbar^2}{4}\left|\psi\right\rangle - 3\frac{3\hbar^2}{4}\left|\psi\right\rangle\right) = J\frac{3}{4}\left|\psi\right\rangle,
\end{aligned}
$$

and for $s_{tot} = 1/2$

$$
\begin{aligned}
H\left|\psi\right\rangle =& \frac{J}{2\hbar^2}(S^2_{tot}\left|\psi\right\rangle - (S_1^2+S_2^2+S_3^2)\left|\psi\right\rangle) = \frac{J}{2\hbar^2}\left(\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 + 3\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2\right) \\
=& \frac{J}{2\hbar^2}\left(\frac{3\hbar^2}{4}\left|\psi\right\rangle - 3\frac{3\hbar^2}{4}\left|\psi\right\rangle\right) = -J\frac{3}{4}\left|\psi\right\rangle.
\end{aligned}
$$

The eigenvalues of H is $\pm\frac{3}{4}J$.

1.7

In order to write down the normalized eigenstates of S_{tot} of total spin angular momentum quantum number $s_{tot} = \frac{1}{2}$ one must employ Clebsch-Gordan coefficient tables. First, combine two of the spins, and then the result with the third spin.

$$|s_1 m_{s1} s_2 m_{s2} s_3 m_{s3}\rangle = |s_1 m_{s1} s_2 m_{s2}\rangle \otimes |s_3 m_{s3}\rangle = |s_1 m_{s1}\rangle \otimes |s_2 m_{s2}\rangle \otimes |s_3 m_{s3}\rangle$$

A system of two spin-1/2 particles can have $s_{tot} = 0$ and $s_{tot} = 1$. The former case, the singlet, has only one possible linear combination of $s = 1/2$ kets

$$|0,0\rangle = \frac{1}{\sqrt{2}} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, +\frac{1}{2}\right\rangle, \tag{7}$$

while if $s = 1$ we have the triplet

$$|1,-1\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \tag{8}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}} \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \frac{1}{\sqrt{2}} \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \tag{9}$$

$$|1,+1\rangle = \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \tag{10}$$

1.8

At time $t = 0$ the system is in state $|\uparrow\downarrow\downarrow\rangle$. After som time t the system will be in state $\hat{U}(t,t_0)\,|\uparrow\downarrow\downarrow\rangle$, where $\hat{U}(t,t_0)$ is the time evolution operator (or propagator). The propagator satisfy three important properties. First, it does nothing when $t = 0$

$$\lim_{t \to t_0} \hat{U}(t, t0) = 1. \tag{11}$$

Second, it is unitary ($\hat{U}^\dagger \hat{U} = 1$), and as a consequence preserves the norm of the states

$$\langle\psi|\psi\rangle = \langle\psi(t)|\psi(t)\rangle = \langle\psi(t)|\,\hat{U}^\dagger(t,t_0) U(t,t_0)\,|\psi(t)\rangle \tag{12}$$

Third, it satisfies the composition property

$$\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\hat{U}(t_1, t_0) \tag{13}$$

One can see from the simplest form of Scrhdinger's equation that the Hamiltonian H generates the time evolution of quantum states. if $|\psi(t)\rangle$ is the state of the system at time t, then

$$H\,|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}\,|\psi(t)\rangle\,. \tag{14}$$

Given the state at some initial time ($t = 0$) one can solve Schrdinger's equation in order to obtain the state at any subsequent time. Particularly, if H is independent of time, then

$$|\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(0)\rangle\,. \tag{15}$$

This exponential operator is usually defined by the corresponding power series.

$$U(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{it}{\hbar} \right) H^k = e^{-iHt/\hbar} \tag{16}$$

If a spin system is initially in state $\left|\uparrow\downarrow\downarrow\right\rangle$ at $t = t_0 = 0$ then the expression $\left| \left\langle\uparrow\downarrow\downarrow\right| \hat{U}(t,0) \left|\uparrow\downarrow\downarrow\right\rangle \right|^2$ gives the probability that the system is still in that state.

To make things a little simpler, I am

2

TAYLOR THEN TRUNCATE?

A Numerical computation of $H\left|\uparrow\downarrow\downarrow\right\rangle$

```
'''
TAKE HOME MIDTERM EXAM, Quantum Mechanics FYS3110
The first part of this script is to check the computation
in problem 1.4.
'''

import numpy as np
import scipy.linalg

up = np.array([[1], [0]])
dn = np.array([[0], [1]])

S_plus  = np.array([[0, 1], [0, 0]])
S_minus = np.array([[0, 0], [1, 0]])
Sz = (1.0/2)*np.array([[1, 0], [0, -1]])

S1z = np.kron(Sz, np.kron(np.eye(2), np.eye(2)))
S2z = np.kron(np.eye(2), np.kron(Sz, np.eye(2)))
S3z = np.kron(np.eye(2), np.kron(np.eye(2), Sz))

Sztot = S1z + S2z + S3z

S1_plus = np.kron(S_plus, np.kron(np.eye(2), np.eye(2)))
S2_plus = np.kron(np.eye(2), np.kron(S_plus, np.eye(2)))
S3_plus = np.kron(np.eye(2), np.kron(np.eye(2), S_plus))

S1_minus = np.kron(S_minus, np.kron(np.eye(2), np.eye(2)))
S2_minus = np.kron(np.eye(2), np.kron(S_minus, np.eye(2)))
S3_minus = np.kron(np.eye(2), np.kron(np.eye(2), S_minus))

# Hamilton operator w/o (J/hbar^2) factor
def Hamilton(state):
        return \
        (1.0/2)*\
                (np.dot(S1_plus, np.dot(S2_minus, state)) +\
                np.dot(S2_plus, np.dot(S1_minus,state))) +\
                np.dot(S1z, np.dot(S2z, state)) +\
        (1.0/2)*\
                (np.dot(S2_plus, np.dot(S3_minus, state)) +\
                np.dot(S3_plus, np.dot(S2_minus,state))) +\
                np.dot(S2z, np.dot(S3z, state)) + \
        (1.0/2)*\
                (np.dot(S3_plus, np.dot(S1_minus, state)) +\
```

```
                    np.dot(S1_plus, np.dot(S3_minus,state))) +\
                    np.dot(S3z, np.dot(S1z, state))


updndn = np.kron(up, np.kron(dn, dn))
print("Hamiltonian(up down down) = ")
print(Hamilton(updndn))
```