PROBLEM SHEET 10
FYS3110

SEBASTIAN G. WINTHER-LARSEN

Problem 10.1

The Zeeman correction, when choosing the external field $\mathbf{B}_{ext}$ to lie along the z-axis, can be expressed by the following condensed formula.

$$E_Z^1 = \mu_B g_J B_{ext} j_z, \tag{1}$$

where

$$\mu_B = \frac{e\hbar}{2m} = 5.788 \times 10^{-5} eV/T \tag{2}$$

is the Bohr magneton, and

$$g_J = 1 + \frac{j(j+1) + \frac{3}{4} - l(l+1)}{2j(j+1)} \tag{3}$$

is the Landé g-factor. Adding the fine structure equation

$$E_{nj} = -\frac{13.6eV}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right] \tag{4}$$

to the Zeeman correction (equation 1) yields an equation for total energy in presence of weak-field Zeeman effect

$$E_{nljj_z} = -\frac{13.6eV}{n^2}\left[1 + \frac{\alpha}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right] + \mu_B g_J B_{ext} j_z. \tag{5}$$

The weak-field Zeeman splitting for the first excited state of the hydrogen atom ($n = 2$) is shown in figure 1. The energy E_2 is plotted against $\mu_B B_{ext}$ and we, evidently, get straight lines with slope $g_j j_z$, for every possible value of j_z. The energy splitting is very clear.

Problem 10.2

Assume that the proton of the hydrogen atom has finite size, in the shape of a sphere with radius $b = 1 \times 10^{-15} m$. This would give it the proton an electric potential of

$$V(r) = \begin{cases} \frac{e}{4\pi\epsilon_0 b}\left(\frac{3}{2} - \frac{r^2}{2b^2}\right), & r \leq b \\ \frac{e}{4\pi\epsilon_0 r}, & r > b \end{cases} \tag{6}$$

Date: November 10, 2016.

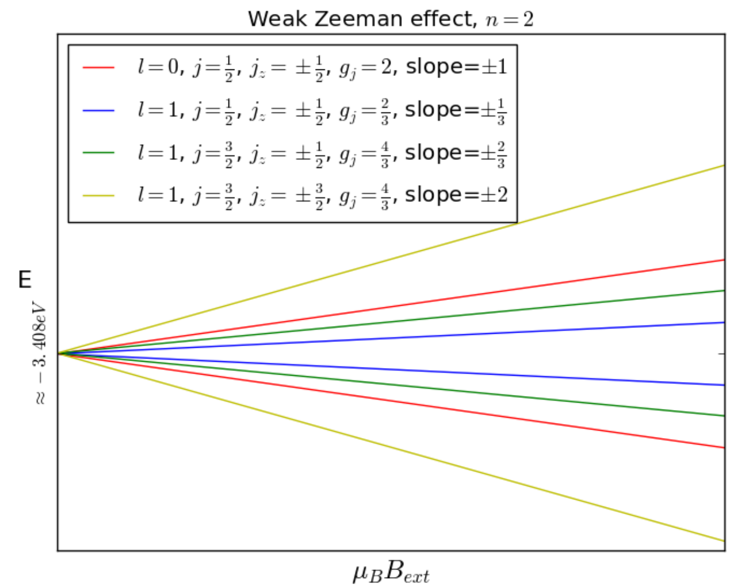

FIGURE 1. Weak-field Zeeman splitting of the first excited state of hydrogen

and the electrostatic potential of an electron present in this potential would be $-eV(r)$. Unperturbed potential is $-\frac{e^2}{4\pi\epsilon_0 r}$ which gives a perturbation of

$$\Delta V(r) = \begin{cases} -\frac{e^2}{4\pi\epsilon_0 b}\left(\frac{3}{2} - \frac{r^2}{2b^2}\right) + \frac{e^2}{4\pi\epsilon_0 r}, & r \leq b \\ 0, & r > b. \end{cases} \tag{7}$$

The unperturbed ground state wave function for hydrogen is

$$|100\rangle = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \tag{8}$$

and the first order energy correction is

$$\begin{aligned} \Delta E = \langle 100|\, \Delta V(r)\, |100\rangle &= \frac{1}{\pi a^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{b} \Delta V e^{-2r/a} r^2 \sin\phi dr dr \theta d\phi \\ &= \frac{4\pi}{\pi a^3} \int_o^b \Delta V(r) e^{-2r/a} r^2 dr \end{aligned}$$

if $b << a$ then $e^{-2r/a}$, and we are left with

$$\Delta E = \frac{4}{a^3}\int_o^b \Delta V(r) r^2 dr = \frac{4}{a^3}\int_o^b -\frac{e^2 r^2}{4\pi\epsilon b}\left(\frac{3}{2} - \frac{r^2}{2b^2}\right) + \frac{e^2 r^2}{4\pi\epsilon_0 r} dr$$

$$= \frac{4}{a^3}\frac{e^2}{4\pi\epsilon_0}\int_0^b -\frac{3r^2}{2b} + \frac{r^4}{2b^3} + r dr = \frac{4}{a^3}\frac{e^2}{4\pi\epsilon_0}\left(-\left[\frac{1}{2b}r^3\right]_0^b + \left[\frac{1}{10b^3}r^5\right]_0^b + \left[\frac{1}{2}r^2\right]_0^b\right)$$

$$= \frac{4}{a^3}\frac{e^2}{4\pi\epsilon_0}\left(-\frac{1}{2}b^2 + \frac{1}{10}b^2 + \frac{1}{2}b^2\right) = \frac{4}{5}\left(\frac{b}{a}\right)^2 \frac{e^2}{2(4\pi\epsilon_0)a}.$$

Inserting for the Bohr radius, a, in the last fraction yields

(9)

$$\Delta E = \frac{4}{5}\left(\frac{b}{a}\right)^2 \frac{m_e e^4}{2(4\pi\epsilon_0)^2\hbar^2} = \frac{4}{5}\left(\frac{b}{a}\right)^2 Ry \approx 2.85\times 10^{-10} Ry \approx 3.87\times 10^{-9} eV$$

For comparison, the fine structure shift is (for $n = 1$ and $j = \frac{1}{2}$)

$$\Delta E_{fs} = 1.8088 \times 10^{-4} eV$$

and the hyperfine structure shift is

$$\Delta E_{hs} = 5.88 \times 10^{-6} eV$$

One can see that the shift due to the perturbation of a finite-size proton is smaller than both the fine structure and hyperfine structure shift.

Problem 10.3

The wave function for helium is the product of two hydrogen-like electron wave functions in 1s orbitals

$$\psi_0(\mathbf{r}_1, \mathbf{r}_2) = \psi_{100}(\mathbf{r}_1)\psi_{100}(\mathbf{r}_2) = \frac{Z^3}{\pi a^3} e^{-Z(r_1+r_2)/a}, \tag{10}$$

and the helium Hamlitonian is

(11)

$$H = \frac{\hbar^2}{2m_e}(-\nabla_1^2 - \nabla_2^2)\psi(\mathbf{r}_1, \mathbf{r}_2) - \frac{2e^2}{4\pi\epsilon_0}\left(\frac{1}{\mathbf{r}_1} + \frac{1}{\mathbf{r_2}}\right) + \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_2 - \mathbf{r}_1|} = H^0 + H'$$

The first order perturbative correction to the energy level is, as always

$$E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle \tag{12}$$

which, by inserting

$$H' = \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_2 - \mathbf{r}_1|},$$

gives

$$E_n^1 = \frac{e^2}{4\pi\epsilon_0}\left(\frac{Z^3}{\pi a^3}\right)^2 \int \frac{e^{-2Z(r_1+r_2)}}{|\mathbf{r}_2 - \mathbf{r}_1|} d^3\mathbf{r}_1 d^3\mathbf{r}_2 = \frac{5Z}{8a}\left(\frac{e^2}{4\pi\epsilon_0}\right) = -\frac{5Z}{4}E_h \tag{13}$$

Thus the ground state with first order perturbation correction is

$$E_0 = E_0^0 + E_0^1 = -4E_h + \frac{5}{8}ZE_h = -\frac{11}{4}E_h = -74.8 eV \tag{14}$$

when inserting $Z = 2$. Variational theory gives the same result.