

# Problem Sheet 4

## FYS3110

Sebastian G. Winther-Larsen

September 24, 2016

The hamiltonian of particle with mass  $m$  in a one-dimensional oscillator potential having a characteristic frequency  $\omega$  is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{X}^2 \quad (1)$$

The ladder operators for the harmonic oscillator potential are

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) \quad (\text{lowering operator}) \quad (2)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) \quad (\text{raising operator}) \quad (3)$$

### Problem 4.1

a)

I want to find an expression for  $\hat{X}$  in terms of  $\hat{a}_{nm}$  and  $\hat{a}_{nm}^\dagger$ . This can be done by first rewriting equation 2

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P} \\ &\rightarrow \hat{X} = \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{\sqrt{2\hbar m\omega}}\hat{P}, \end{aligned}$$

and then equation 3

$$\begin{aligned} \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) = \sqrt{\frac{2\hbar}{m\omega}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P} \\ &\rightarrow \hat{P} = \frac{2\hbar m\omega}{i}\sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a}^\dagger. \end{aligned}$$

Now putting the latter equation into the former yields

$$\begin{aligned}\hat{X} &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{2\hbar m\omega} \left( \frac{2\hbar m\omega}{i} \sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a} \right) \\ &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} + \sqrt{\frac{2\hbar}{m\omega}}\hat{a}^\dagger - \sqrt{\frac{2\hbar}{m\omega}}\sqrt{\frac{m\omega}{2\hbar}}\hat{x},\end{aligned}$$

which simplifies to

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \quad (4)$$

It will be necessary to know what  $[\hat{a}, \hat{H}]$  is. This is easiest to compute if one knows how  $\hat{a}$  and  $\hat{a}^\dagger$  is related to  $\hat{H}$ . By looking at the expressions for  $\hat{a}$  and  $\hat{a}^\dagger$  one is tempted to compute the following

$$\hat{a}\hat{a}^\dagger = \frac{m\omega}{2\hbar}\hat{X}^2 + \frac{1}{2m\omega\hbar}\hat{P}^2 + \frac{i}{2\hbar}[\hat{X}, \hat{P}],$$

where  $[X, P] = i\hbar$ , which follows from  $\hat{X} \rightarrow x$  and  $\hat{P} \rightarrow i\hbar(d/dx)$ , but is independent of basis. So we see that

$$\hat{H} = \hbar\omega(\hat{a}\hat{a}^\dagger + \frac{1}{2}), \quad (5)$$

or alternatively that  $\hat{a}\hat{a}^\dagger = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$ . This combined ladder operator can be referred to by a new name  $\hat{a}\hat{a}^\dagger = \hat{N}$ , so that equation 5 becomes

$$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2}), \quad (6)$$

Thus, we have the energy eigenbasis which satisfy

$$\hat{H} |n\rangle = \hbar\omega(n - \frac{1}{2}) |n\rangle, \text{ for } n = 0, 1, 2, \dots \quad (7)$$

We now get

$$\hat{a} |n\rangle = C_n |n-1\rangle, \quad (8)$$

where  $C_n$  is a constant which can be found the following way

$$\begin{aligned}\langle n | \hat{a}^\dagger \hat{a} | n \rangle &= |C_n|^2 \langle n-1 | n-1 \rangle \\ &\rightarrow |C_n| = \sqrt{n} = C_n,\end{aligned}$$

by choosing the phase to be zero<sup>1</sup>. We land at

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (9)$$

Similarly for  $\hat{a}^\dagger$

$$\hat{a}^\dagger |n\rangle = D_n |n+1\rangle \quad (10)$$

---

<sup>1</sup>Actually,  $C_n = \sqrt{n}e^{i\phi}$  where  $\phi$  is arbitrary, but it is conventional to set  $\phi = 0$ .

$$\begin{aligned}
\langle n | \hat{a} \hat{a}^\dagger | n \rangle &= |D_n|^2 \langle n+1 | n+1 \rangle \\
&\rightarrow |D_n| = \sqrt{n+1} = D_n \\
\hat{a}^\dagger | n \rangle &= \sqrt{n+1} | n+1 \rangle
\end{aligned} \tag{11}$$

Now to compute the matrix elements for  $\hat{a}$  and  $\hat{a}^\dagger$

$$\langle m | \hat{a} | n \rangle = \sqrt{n} \langle m | n-1 \rangle = \sqrt{n} \delta_{m,n-1} \tag{12}$$

$$\langle m | \hat{a}^\dagger | n \rangle = \sqrt{n+1} \langle m | n+1 \rangle = \sqrt{n+1} \delta_{m,n+1}. \tag{13}$$

The actual matrices for these particular operators will look something like this<sup>2</sup>

$$a \leftrightarrow \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ \vdots & & & & \ddots \end{bmatrix} \tag{14}$$

$$a^\dagger \leftrightarrow \begin{bmatrix} 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \\ 0 & 0 & \sqrt{3} & \\ \vdots & & & \ddots \end{bmatrix}. \tag{15}$$

Finding the matrix representation of  $\hat{X}$  is now an easy matter of employing equation 4

$$X \leftrightarrow \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \tag{16}$$

while an algebraic expression will be (also employing equation 4)

$$\langle m | X | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} \langle m | n+1 \rangle + \sqrt{n} \langle m | n-1 \rangle) \tag{17}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1}). \tag{18}$$

**b)**

Let  $|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ .

---

<sup>2</sup>I am perfectly aware that  $\sqrt{1} = 1$  but I will keep the root symbol here in order to underline the symmetry.