Problem Sheet 4
FYS3110

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The hamiltonian of particle with mass m in a one-dimensional oscillator potetial having a characteristic frequency ω is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{X}^2 \tag{1}$$

The ladder operators for the harmonic oscillator potential are

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) \qquad \text{(lowering operator)} \tag{2}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) \qquad \text{(raising operator)} \tag{3}$$

Problem 4.1

a)

I want to find an expression for $\hat{X}$ in terms of $\hat{a}_{nm}$ and $\hat{a}^\dagger_{nm}$. This can be done by first rewriting equation 2

$$\begin{aligned}\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) &= \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P} \\ \rightarrow \hat{X} &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{\sqrt{2\hbar m\omega}}\hat{P},\end{aligned}$$

and then equation 3

$$\begin{aligned}\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) &= \sqrt{\frac{2\hbar}{m\omega}}\hat{X} + \frac{i}{2\hbar m\omega}\hat{P} \\ \rightarrow \hat{P} &= \frac{2\hbar m\omega}{i}\sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a}^\dagger.\end{aligned}$$

Now putting the latter equation into the former yields

$$\begin{aligned}\hat{X} &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{2\hbar m\omega}\left(\frac{2\hbar m\omega}{i}\sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a}\right)\\ &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} + \sqrt{\frac{2\hbar}{m\omega}}\hat{a}^\dagger - \sqrt{\frac{2\hbar}{m\omega}}\sqrt{\frac{m\omega}{2\hbar}}\hat{x},\end{aligned}$$

which simplifies to

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \tag{4}$$

It will be necessary to know what $[\hat{a}, \hat{H}]$ is. This is easiest to compute if one knows how $\hat{a}$ and $\hat{a}^\dagger$ is related to $\hat{H}$. By looking at the expressions for $\hat{a}$ and $\hat{a}^\dagger$ one is tempted to compute the following

$$\hat{a}\hat{a}^\dagger = \frac{m\omega}{2\hbar}\hat{X}^2 + \frac{1}{2m\omega\hbar}\hat{P}^2 + \frac{i}{2\hbar}[\hat{X}, \hat{P}],$$

where $[X, P] = i\hbar$, which follows from $\hat{X} \to x$ and $\hat{P} \to i\hbar(d/dx)$, but is independent of basis. So we see that

$$\hat{H} = \hbar\omega(\hat{a}\hat{a}^\dagger + \frac{1}{2}), \tag{5}$$

or alternatively that $\hat{a}\hat{a}^\dagger = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$. This combined ladder operator can be referred to by a new name $\hat{a}\hat{a}^\dagger = \hat{N}$, so that equation 5 becomes

$$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2}), \tag{6}$$

Thus, we have the energy eigenbasis which satisfy

$$\hat{H}\left|n\right\rangle = \hbar\omega(n - \frac{1}{2})\left|n\right\rangle, \text{ for } i \in 0, 1, 2 \ldots \tag{7}$$

We now get

$$\hat{a}\left|n\right\rangle = C_n\left|n\right\rangle, \tag{8}$$

where C_n is a constant which can be found the following way

$$\begin{aligned}\langle n|\,\hat{a}^\dagger\hat{a}\,|n\rangle =& |C_n|^2\,\langle n-1|n-1\rangle\\ \to& |C_n| = \sqrt{n} = C_n,\end{aligned}$$

by choosing the phase to be zero[1]. We land at

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \tag{9}$$

Similarly for $\hat{a}^\dagger$

$$\hat{a}^\dagger\left|n\right\rangle = D_n\left|n+1\right\rangle \tag{10}$$

[1]Actually, $C_n = \sqrt{n}e^{i\phi}$ where ϕ is arbitrary, but is is conventional to set $\phi = 0$.

$$
\begin{aligned}
\langle n|\,\hat{a}\hat{a}^{\dagger}\,|n\rangle =& |D_n|^2 \langle n+1|n+1\rangle \\
&\rightarrow |D_n| = \sqrt{n+1} = D_n
\end{aligned}
$$

$$\hat{a}^{\dagger}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{11}$$

Now to compute the matrix elements for $\hat{a}$ and $\hat{a}^{\dagger}$

$$\langle m|\,\hat{a}\,|n\rangle = \sqrt{n}\,\langle m|n-1\rangle = \sqrt{n}\delta_{m,n-1} \tag{12}$$

$$\langle m|\,\hat{a}^{\dagger}\,|n\rangle = \sqrt{n-1}\,\langle m|n+1\rangle = \sqrt{n-1}\delta_{m,n+1}. \tag{13}$$

The actual matrices for these particular operators will look something like this[2]

$$a \leftrightarrow \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ \vdots & & & & \ddots \end{bmatrix} \tag{14}$$

$$a^{\dagger} \leftrightarrow \begin{bmatrix} 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \\ 0 & 0 & \sqrt{3} & \\ \vdots & & & \ddots \end{bmatrix}. \tag{15}$$

Finding the matrix representation of $\hat{X}$ is now an easy matter of employing equation 4

$$X \leftrightarrow \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \tag{16}$$

while an algebraic expression will be (also employing equation 4)

$$\langle m|\,X\,|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\,\langle m|n+1\rangle + \sqrt{n}\,\langle m|n-1\rangle) \tag{17}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}). \tag{18}$$

[2]I am perfectly aware that $\sqrt{1} = 1$ but I will keep the root symbol here in order to underline the symmetry.

b)

Let $|\psi(0)\rangle = \sum_{x=0}^{n} c_n |n\rangle$. It follows that

$$\begin{aligned}
\langle\psi(0)|\psi(0)\rangle &= \sum_n |c_n|^2 \langle n|n\rangle + 2\sum_{n\neq n'} c_n c_{n'} \langle n|n'\rangle \\
&= \sum_n |c_n|^2 \delta_{n,n} + 2\sum_{n\neq n'} c_n c_{n'} \delta_{n,n'},
\end{aligned}$$

where the first Kronecker delta will always be 1 and the second Kronecker delta will always be 0. The condition on the c_n's for $|\psi(0)\rangle$ to have unit norm is therefore

$$\sum_n |c_n|^2 = 1 \tag{19}$$