Problem 1.1

A two-dimensional Hilbert-space has a set of basis vectors $|0\rangle$ and $|1\rangle$ that are orthonormal, that is $\langle i|j\rangle = \delta_{ij}$ where $i, j = 0, 1$. In this space there is a ket-vector

$$|\psi\rangle = c\left(\sqrt{5}|0\rangle + i|1\rangle\right)$$

where c is a complex number.

a) Find the corresponding bra-vector $\langle\psi|$, and determine the modulus(absolute value) of c such that $|\psi\rangle$ is normalized to one.

An operator $\hat{A}$ is defined as: $\hat{A}|0\rangle = -i|1\rangle$, $\hat{A}|1\rangle = i|0\rangle$

b) Find the column vector representation of $|\psi\rangle$ using the following representation of the basis vectors: $|0\rangle \simeq \binom{1}{0}$ and $|1\rangle \simeq \binom{0}{1}$ ($\simeq$ means here "represented by"). Find also the corresponding matrix representation of $\hat{A}$.

c) Compute $\langle\psi|\hat{A}|\psi\rangle$ in two ways. First by using the representation in b), and then directly from the definitions of $|\psi\rangle$ and $\hat{A}$, using linearity of the inner product and orthonormality of the basis vectors.

Problem 1.2

The intention of this problem is for you to brush up your knowledge of linear algebra. A complex matrix is given as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are complex numbers.

a) Find its transpose U^T and its hermitian conjugate $U^\dagger$ (adjoint)

b) What are the conditions on a, b, c, d for U to be a hermitian matrix?

c) Compute the eigenvalues of U and show, for this matrix, that they are real when U is hermitian.

d) For which values of a, b, c, d is U both hermitian *and* unitary?

e) What are the possible eigenvalues of U when U is both hermitian and unitary?

Problem 1.3 (optional)

Suppose that the Hamiltonian $\hat{H}$ (energy operator) is linear and acts as follows

$$\hat{H}|\psi\rangle = g|\phi\rangle, \quad \hat{H}|\phi\rangle = g^*|\psi\rangle, \quad \hat{H}|\gamma_n\rangle = 0$$

where g is an arbitrary complex number, $|\psi\rangle$ and $|\phi\rangle$ is a pair of linearly independent states, both normalized to unity, but not necessarily orthogonal, and $|\gamma_n\rangle$, $n \in [1, N]$ are all states orthogonal to both $|\psi\rangle$ and $|\phi\rangle$. What are the conditions that $|\psi\rangle$ and $|\phi\rangle$ must satisfy for $\hat{H}$ to be hermitian?