1.1

The bra-vector is defined from the inner product: if $|u\rangle = u_0|0\rangle + u_1|1\rangle$ and $|v\rangle = v_0|0\rangle + v_1|1\rangle$, then

$$\langle u|v\rangle = \sum_{i,j} u_i^* v_j \langle i|j\rangle = \sum_{i,j} u_i^* v_j \delta_{ij} = u_0^* v_0 + u_1^* v_1. \tag{1}$$

Where the last two equalities follow since the basis is orthonormal. Since one frequently has to evaluate inner products in quantum mechanics, physicists use the notation

$$\langle u| = \sum_i u_i^* \langle i| = u_0^* \langle 0| + u_1^* \langle 1|. \tag{2}$$

Then $\langle u|v\rangle$ can be evaluated by "multiplying" the bra-vector $\langle u|$ with $|v\rangle$ with the knowledge of the inner products of the basis vectors $\langle i|j\rangle$. In quantum mechanics, the basis vectors are usually chosen orthonormal (some formulas depend on this!). Because the inner product gives a number as a result, a bra-vector is something that gives a number when evaluated with an ordinary vector, i.e. a ket $|\psi\rangle$. The evaluation is the rule $\langle\phi|\psi\rangle$ for a bra vector $\langle\phi|$.

a)

From (2), we get

$$\langle\psi| = c^*\big(\sqrt{5}\langle 0| - \mathrm{i}\langle 1|\big). \tag{3}$$

Using the above definitions,

$$\langle\psi|\psi\rangle = cc^*\Big(\sqrt{5}\langle 0| - \mathrm{i}\langle 1|\Big)\Big(\sqrt{5}|0\rangle + \mathrm{i}|1\rangle\Big) \tag{4}$$

$$= |c|^2\Big(5\langle 0|0\rangle + \langle 1|1\rangle\Big) \tag{5}$$

$$= 6|c|^2, \tag{6}$$

where the cross terms $\langle i|j\rangle$ vanish because of orthonormality of the basis, as before. Normalizing to unit norm, $\langle\psi|\psi\rangle = 1$, gives $|c|^2 = 1/3$ so

$$|c| = \sqrt{\frac{1}{6}}. \tag{7}$$

For simplicity, let us choose the real option, $c = 1/\sqrt{6}$.

b)

In the explicit orthonormal basis indicated,

$$|\psi\rangle \simeq \vec{\psi} = \sqrt{5}c\begin{pmatrix}1\\0\end{pmatrix} + \mathrm{i}c\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{5}\\ \mathrm{i}\end{pmatrix}. \tag{8}$$

Here $|\psi\rangle$ is represented by a column vector in the particular orthonormal basis chosen.
A linear operator $\hat{O}$ on a vector space is determined by its action on the (ordered) basis vectors $\{|i\rangle\}_{i=0,1,2,\ldots}$, assumed to be orthonormal. Let $|\alpha\rangle = \sum_i \alpha_i|i\rangle$ and $|\beta\rangle = \sum_j \beta_j|j\rangle$ be two vectors

such that $\hat{O}\,|\alpha\rangle = |\beta\rangle$. Then

$$\sum_i \alpha_i \hat{O}\,|i\rangle = \sum_j \beta_j\,|j\rangle \quad \overset{\langle k|}{\Rightarrow} \quad \sum_i \alpha_i \langle k|\hat{O}|i\rangle = \sum_j \beta_j\,\langle k|j\rangle = \sum_j \delta_{kj}\beta_j \text{ or,}$$

$$\sum_i O_{ki}\alpha_i = \beta_k, \tag{9}$$

where $O_{ij} = \langle i|\hat{O}|j\rangle$. This is a matrix equation for the vectors $|\alpha\rangle = (\alpha_0, \alpha_1, \alpha_2, \dots)$ and $|\beta\rangle = (\beta_0, \beta_1, \beta_2, \dots)$ with the *matrix elements* $O_{ij} = \langle i|\hat{O}|j\rangle$ corresponding to $\hat{O}$ wrt the basis $\{|i\rangle\}_{i=0,1,2,\dots}$. Thus $\hat{A}$ is represented by the matrix

$$\hat{A} \simeq A = \begin{pmatrix} \langle 0|\hat{A}|0\rangle & \langle 0|\hat{A}|1\rangle \\ \langle 1|\hat{A}|0\rangle & \langle 1|\hat{A}|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \tag{10}$$

where the known action of $\hat{A}$ on the basis vectors was used.

c)

$\langle\psi|\hat{A}|\psi\rangle$ means the inner product of the vectors $\hat{A}\,|\psi\rangle$ and $|\psi\rangle$. The inner product between column vectors $\vec{\psi}$ and $\vec{\phi}$ is $\langle\phi|\psi\rangle = \vec{\phi}^\dagger\vec{\psi}$. Now

$$\hat{A}\,|\psi\rangle \simeq A\vec{\psi} = c\begin{pmatrix} -1 \\ -\sqrt{5}\mathrm{i} \end{pmatrix} \tag{11}$$

so

$$\vec{\psi}^\dagger A\vec{\psi} = |c|^2 \begin{pmatrix} \sqrt{5} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} -1 \\ -\sqrt{5}\mathrm{i} \end{pmatrix} = \frac{1}{6}(-\sqrt{5} - \sqrt{5}) = -\frac{\sqrt{5}}{3} \tag{12}$$

Similarly, we can read off $\hat{A}\,|\psi\rangle$ from the action of $\hat{A}$ on $|0\rangle, |1\rangle$. Then

$$\langle\psi|\hat{A}|\psi\rangle = c^*\Big(\sqrt{5}\,\langle 0| - \mathrm{i}\,\langle 1|\Big)\hat{A}c\Big(\sqrt{5}\,|0\rangle + \mathrm{i}\,|1\rangle\Big) \tag{13}$$

$$= \frac{1}{6}\Big(\sqrt{5}\,\langle 0| - \mathrm{i}\,\langle 1|\Big)\Big(- \sqrt{5}\mathrm{i}\,|1\rangle - |0\rangle\Big) \tag{14}$$

$$= \frac{1}{2}\left(-\sqrt{5}\,\langle 0|0\rangle + \sqrt{5}\mathrm{i}^2\,\langle 1|1\rangle\right) = -\frac{\sqrt{5}}{3}. \tag{15}$$

as expected. The point is that we could compute without the choice of any explicit orthonormal basis, as long as $\hat{A}$ is defined (by its action on the basis vectors).

1.2

In this problem we consider the 2×2 complex matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

a)

The elements of the transpose matrix are defined as $(U^T)_{ij} = U_{ji}$ (interchange of rows and columns), so

$$U^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \tag{16}$$

The elements of the hermitian conjugate matrix are defined as $(U^\dagger)_{ij} = U^*_{ji}$ (transpose and complex conjugation),

$$U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}. \tag{17}$$

b)

A matrix is hermitean if $U^\dagger = U$, so

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}. \tag{18}$$

This gives that a hermitean matrix U_H is given by

$$U_H = \begin{pmatrix} a_0 & b \\ b^* & d_0 \end{pmatrix}, \quad a_0, d_0 \in \mathbb{R}. \tag{19}$$

This gives U_H with 4 real parameters: the real numbers a_0, d_0 plus the complex number b.

c)

The eigenvalue equation is $U\vec{x} = \lambda\vec{x}$. This has a solution $\vec{x}$ iff $\det(\lambda I - U) = 0$. In this case,

$$\det(\lambda I - U) = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \lambda \mathrm{tr}(U) + \det(U) = 0. \tag{20}$$

This gives two eigenvalues as solutions of the quadratic equation:

$$\lambda_\pm = \frac{\mathrm{tr}(U) \pm \sqrt{\mathrm{tr}(U)^2 - 4\det(U)}}{2}. \tag{21}$$

We know from b) that $Tr(U) = a + d$ is real. For $\lambda_\pm$ to be real, the expression in the square root must then be positive, $\mathrm{tr}(U)^2 - 4\det(U) \geq 0$. For the hermitean matrix in **b)**, we have

$$\det(U)^2 - 4\mathrm{tr}(U) = (a_0 + d_0)^2 - 4(a_0 d_0 - |b|^2) = (a_0 - d_0)^2 + 4\,|b|^2 \geq 0. \tag{22}$$

In fact any hermitian matrix has real eigenvalues, not just the 2×2 ones.

d)

A matrix is unitary if $U^\dagger U = I$ so

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{23}$$

If the matrix is simultaneously hermitean, we have from **b)** that $a^* = a$, $d^* = d, c^* = b$, and so Eq.(23) gives the conditions

$$
\begin{aligned}
a^2 + |b|^2 &= 1 && (24)\\
d^2 + |b|^2 &= 1 && (25)\\
b(a+d) &= 0 && (26)\\
b^*(a+d) &= 0. && (27)
\end{aligned}
$$

From this we see that $a^2 = d^2$, and so there are two possibilities: $a = d$ (in which case $b = 0$), or $a = -d$ (with $a^2 + |b|^2$=1). The former thus gives

$$
U = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{28}
$$

The latter case can be parametrized by renaming $a = \cos\theta$, so $b = \pm\sin\theta$ or $b = \pm i \sin\theta$, so the remaining options are

$$
U' = \begin{pmatrix} \cos\theta & \pm\sin\theta \\ \pm\sin\theta & -\cos\theta \end{pmatrix}, \quad U'' = \begin{pmatrix} \cos\theta & \pm i\sin\theta \\ \mp i\sin\theta & -\cos\theta \end{pmatrix} \tag{29}
$$

e)

Inserting the results from **d)** into the expression for the eigenvalues in **c)** we get

$$
\lambda = \pm 1. \tag{30}
$$

1.3 (Optional)

Let us use the definition of a Hermitian operator as given in Griffiths Chapter 3, Eq.[15],

$$
\left\langle \Psi | (\hat{H}\Psi) \right\rangle = \left\langle (\hat{H}\Psi) | \Psi \right\rangle \tag{31}
$$

for an arbitrary state $|\Psi\rangle$. In particular, for the (non-orthogonal) states given in the problem, we must have

$$
\begin{aligned}
\langle\psi| \hat{H} |\psi\rangle &= \left\langle \psi | (\hat{H}\psi) \right\rangle = \langle \psi | (g\phi) \rangle = g \langle \psi|\phi\rangle && (32)\\
&= \left\langle (\hat{H}\psi) | \psi \right\rangle = \langle g\psi)|\psi\rangle = g^* \langle \phi|\psi\rangle = g^* \langle\psi|\phi\rangle^* . && (33)
\end{aligned}
$$

Thus, the condition we get is

$$
g \langle\psi|\phi\rangle = g^* \langle\psi|\phi\rangle^* . \tag{34}
$$

One gets the same relation from considering $\langle\phi| \hat{H} |\phi\rangle$, or any linear combination of the states in the problem. [The remaining states $|\gamma_n\rangle$ are orthogonal to both $|\psi\rangle$ and $|\phi\rangle$, with zero eigenvalue for $\hat{H}$ and drop out of the expressions]. Since both g and the inner product $\langle\psi|\phi\rangle$ are complex numbers, this means that we get a relation between the phase of g and the phase of $\langle\psi|\phi\rangle$. [Reminder: Any complex number can be written on the form $re^{i\theta}$, and we refer to θ as the phase.]