Problem Sheet 4

FYS3110

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The hamiltonian of particle with mass m in a one-dimensional oscillator potetial having a characteristic frequency ω is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{X}^2 \tag{1}$$

The ladder operators for the harmonic oscillator potential are

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) \qquad \text{(lowering operator)} \tag{2}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) \qquad \text{(raising operator)} \tag{3}$$

Problem 4.1

a)

I want to find an expression for $\hat{X}$ in terms of $\hat{a}_{nm}$ and $\hat{a}^\dagger_{nm}$. This can be done by first rewriting equation 2

$$\begin{aligned}\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) &= \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P} \\ \rightarrow \hat{X} &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{\sqrt{2\hbar m\omega}}\hat{P},\end{aligned}$$

and then equation 3

$$\begin{aligned}\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) &= \sqrt{\frac{2\hbar}{m\omega}}\hat{X} + \frac{i}{2\hbar m\omega}\hat{P} \\ \rightarrow \hat{P} &= \frac{2\hbar m\omega}{i}\sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a}^\dagger.\end{aligned}$$

Now putting the latter equation into the former yields

$$\begin{aligned}\hat{X} &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{2\hbar m\omega}\left(\frac{2\hbar m\omega}{i}\sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a}\right) \\ &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} + \sqrt{\frac{2\hbar}{m\omega}}\hat{a}^\dagger - \sqrt{\frac{2\hbar}{m\omega}}\sqrt{\frac{m\omega}{2\hbar}}\hat{x},\end{aligned}$$

which simplifies to

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \tag{4}$$

It will be necessary to know what $[\hat{a}, \hat{H}]$ is. This is easiest to compute if one knows how $\hat{a}$ and $\hat{a}^\dagger$ is related to $\hat{H}$. By looking at the expressions for $\hat{a}$ and $\hat{a}^\dagger$ one is tempted to compute the following

$$\hat{a}\hat{a}^\dagger = \frac{m\omega}{2\hbar}\hat{X}^2 + \frac{1}{2m\omega\hbar}\hat{P}^2 + \frac{i}{2\hbar}[\hat{X}, \hat{P}],$$

where $[X, P] = i\hbar$, which follows from $\hat{X} \to x$ and $\hat{P} \to i\hbar(d/dx)$, but is independent of basis. So we see that

$$\hat{H} = (\hat{a}\hat{a}^\dagger + \frac{1}{2})\hbar\omega. \tag{5}$$

Then we have that

$$[\hat{a}, \hat{H}] = [\hat{a}, \hat{a}^\dagger\hat{a} + 1/2] = [\hat{a}, , \hat{a}^\dagger\hat{a}] = \hat{a}, \tag{6}$$

if we measure the eigenvalues in units of $\hbar\omega$. Similarly, $[\hat{a^\dagger}, \hat{H}] = -\hat{a}^\dagger$.

The utility of $\hat{a}$ and $\hat{a}^\dagger$ stems from the fact that given an eigenstate of $\hat{H}$, they generate others. Consider

$$\hat{H}a\,|E\rangle = (\hat{a}\hat{H} - [\hat{a}, \hat{H}])\,|E\rangle = (\hat{a}\hat{H} - \hat{a})\,|E\rangle = \tag{7}$$

where ε is the energy measured in units of $\hbar\omega$.

Now my idea is to find matrix elements for a_{nm} and $a^\dagger_{nm}$, plug these into equation 4 to get the matrix elements of X_{nm}.