Problem Sheet 1
FYS3110

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Problem 1.1

a)

$$
\begin{aligned}
|\Psi\rangle &= c(\sqrt{5}\,|0\rangle + i\,|1\rangle) \\
\langle\Psi|\Psi\rangle &= c^*(\sqrt{5}\,\langle 0| - i\,\langle 1|)c(\sqrt{5}\,|0\rangle + i\,|1\rangle) \\
&= c^*c(5\,\langle 0|0\rangle + \sqrt{5}i\,\langle 0|1\rangle - \sqrt{5}\,\langle 1|0\rangle + \langle 1|1\rangle) \\
&= |c|(5+1) \to |c| = \frac{1}{6}
\end{aligned}
$$

b)

The following representation is used for the basis vectors

$$
\begin{aligned}
|0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
|1\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
|\Psi\rangle &= c(\sqrt{5}\,|0\rangle + i\,|1\rangle) \\
&= c\left(\sqrt{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix} i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\
&= c\begin{pmatrix} \sqrt{5} \\ i \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\langle\Psi| &= c^*(\sqrt{5}\,\langle 0| - i\,\langle 1|) \\
&= c^*(\sqrt{5}(1 \quad 0) - i(0 \quad 1)) \\
&= c^*(\sqrt{5} \quad -i)
\end{aligned}
$$

$$
\langle\Psi|\Psi\rangle = |c|6 \rightarrow |c| = \frac{1}{6}
$$

Furthermore, $\hat{A}$ is an operator defined thusly

$$
\hat{A}\,|0\rangle = -i\,|1\rangle\,, \quad \hat{A}\,|1\rangle = i\,|0\rangle
$$

It is quite easy to see that

$$
\hat{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
$$

$\hat{A}$ applied to $|0\rangle$:

$$
\hat{A}\,|0\rangle = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix} = -i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\,|1\rangle
$$

and $\hat{A}$ applied to $|1\rangle$:

$$
\hat{A}\,|1\rangle = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix} = i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\,|0\rangle
$$

c)

There are now two ways to calculate $\langle\Psi|\,\hat{A}\,|\Psi\rangle$

$$
\begin{aligned}
\langle\Psi|\,\hat{A}\,|\Psi\rangle &= c^* \begin{pmatrix} \sqrt{5} & -i \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} c \begin{pmatrix} \sqrt{5} \\ i \end{pmatrix} \\
&= |c| \begin{pmatrix} \sqrt{5} & -i \end{pmatrix} \begin{pmatrix} -1 \\ -\sqrt{5}i \end{pmatrix} \\
&= \frac{1}{6}(-\sqrt{5} - \sqrt{5}) = -\frac{\sqrt{5}}{3}
\end{aligned}
$$

$$
\begin{aligned}
\langle\Psi|\,\hat{A}\,|\Psi\rangle &= c^*(\sqrt{5}\,\langle 0|\ \langle 1|)\hat{A}c(\sqrt{5}\,|0\rangle + i\,|1\rangle) \\
&= |c|(\sqrt{5}\,\langle 0| - i\,\langle 1|)(-\sqrt{5}i\,|1\rangle - |0\rangle) \\
&= \frac{1}{6}(-5i\,\langle 0|1\rangle - \sqrt{5}\,\langle 0|0\rangle - \sqrt{5}\,\langle 1|1\rangle + i\,\langle 1|0\rangle) \\
&= \frac{1}{6}(-2\sqrt{5}) = -\frac{\sqrt{5}}{3}
\end{aligned}
$$

Problem 1.2

A complex matrix is given as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are complex numbers.

a)

$$U^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$
$$U^\dagger = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}^T = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

b)

U is *Hermittian* if $U = U^\dagger$, which implies that $a = a^*$, $b = c^*$, $c = b^*$, and $d = d*$. This means that a and d are real.

c)

Eigenvalues λ for matrix A satisfy $|A - \lambda I| = 0$.

$$|U - \lambda I| = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc$$

$$\left|U^\dagger - \lambda^\dagger I\right| = \begin{vmatrix} a^* - \lambda^\dagger & c^* \\ b^* & d^* - \lambda^\dagger \end{vmatrix} = (a^* - \lambda^\dagger)(d^* - \lambda^\dagger) - b^* c^*$$

Because $a = a^*$, $b = c^*$, $c = b^*$, and $d = d^*$, one sees that λ must be equal to $\lambda^\dagger$, which implies that λ is real. Here follows a more general proof that eigenvalues of a Hermitian matrix ($U = U^\dagger$) are real

$$
\begin{aligned}
U\mathbf{v} &= \lambda\mathbf{v} \\
(U\mathbf{v})^\dagger &= (\lambda\mathbf{v})^\dagger \\
\mathbf{v}^\dagger U^\dagger &= \lambda^\dagger \mathbf{v}^\dagger \\
\mathbf{v}^\dagger U^\dagger \mathbf{v} &= \lambda^\dagger \mathbf{v}^\dagger \mathbf{v} \\
(U &= U^\dagger) \\
\mathbf{v}^\dagger U \mathbf{v} &= \lambda^\dagger \mathbf{v}^\dagger \mathbf{v} \\
\mathbf{v}^\dagger \lambda \mathbf{v} &= \lambda^\dagger \mathbf{v}^\dagger \mathbf{v} \\
\lambda \mathbf{v}^\dagger \mathbf{v} &= \lambda^\dagger \mathbf{v}^\dagger \mathbf{v} \\
\lambda &= \lambda^\dagger
\end{aligned}
$$

Which must mean that the eigenvalues are real. A note on notation: the adjoint is to en operator what the complex conjugate is to numbers. $\lambda^\dagger$ is the complex conjugate of the eigenvalue.

d)

An operator U is *unitary* if $UU^\dagger = I$.

$$
UU^\dagger = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ a^*c + b^*d & cc^* + dd^* \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}
$$

This implies the following conditions:

- $a^2 + bc = 1$
- $b(a + d) = 0$
- $c(a + d) = 0$
- $d^2 + bc = 1$

The trivial case is when $b = c = 0$ and $a, d = \pm 1$. If instead $a + d = 0$ and $b, c \neq 0$ there are other, more interesting solutions to the problem, for instance

$$
\begin{pmatrix} \cos\theta & i\sin\theta \\ -i\sin\theta & -\cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi]
$$

d)

The determinant of a unitary matrix must be 1, because $\det(U)\det\left(U^T\right) = \det\left(UU^T\right) = \det(1) = 1$. Moreover,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 1$$

then

$$|U - \lambda I| = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - (a+d)\lambda + 1 = 0$$

From above, $-(a+d)$ can have three possible values -2, 0, 2, which gives the possible eigenvalues of ± 1 and $\pm i$. Only real, eigenvalues are accepted if the matrix is Hermitian. If the matrix is both Hermitian and unitary, the matrix must therefore have unit egenvalues.