

Problem Sheet 4

FYS3110

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The hamiltonian of particle with mass m in a one-dimensional oscillator potential having a characteristic frequency ω is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{X}^2 \quad (1)$$

The ladder operators for the harmonic oscillator potential are

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) \quad (\text{lowering operator}) \quad (2)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) \quad (\text{raising operator}) \quad (3)$$

Problem 4.1

a)

I want to find an expression for \hat{X} in terms of \hat{a}_{nm} and \hat{a}_{nm}^\dagger . This can be done by first rewriting equation 2

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} + i\hat{P}) = \sqrt{\frac{m\omega}{2\hbar}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P} \\ &\rightarrow \hat{X} = \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{\sqrt{2\hbar m\omega}}\hat{P}, \end{aligned}$$

and then equation 3

$$\begin{aligned} \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{X} - i\hat{P}) = \sqrt{\frac{2\hbar}{m\omega}}\hat{X} + \frac{i}{\sqrt{2\hbar m\omega}}\hat{P} \\ &\rightarrow \hat{P} = \frac{2\hbar m\omega}{i}\sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a}^\dagger. \end{aligned}$$

Now putting the latter equation into the former yields

$$\begin{aligned}\hat{X} &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} - \sqrt{\frac{2\hbar}{m\omega}}\frac{i}{2\hbar m\omega} \left(\frac{2\hbar m\omega}{i} \sqrt{\frac{m\omega}{2\hbar}}\hat{X} - \frac{2\hbar m\omega}{i}\hat{a} \right) \\ &= \sqrt{\frac{2\hbar}{m\omega}}\hat{a} + \sqrt{\frac{2\hbar}{m\omega}}\hat{a}^\dagger - \sqrt{\frac{2\hbar}{m\omega}}\sqrt{\frac{m\omega}{2\hbar}}\hat{x},\end{aligned}$$

which simplifies to

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \quad (4)$$

It will be necessary to know what $[\hat{a}, \hat{H}]$ is. This is easiest to compute if one knows how \hat{a} and \hat{a}^\dagger is related to \hat{H} . By looking at the expressions for \hat{a} and \hat{a}^\dagger one is tempted to compute the following

$$\hat{a}\hat{a}^\dagger = \frac{m\omega}{2\hbar}\hat{X}^2 + \frac{1}{2m\omega\hbar}\hat{P}^2 + \frac{i}{2\hbar}[\hat{X}, \hat{P}],$$

where $[X, P] = i\hbar$, which follows from $\hat{X} \rightarrow x$ and $\hat{P} \rightarrow i\hbar(d/dx)$, but is independent of basis. So we see that

$$\hat{H} = \hbar\omega(\hat{a}\hat{a}^\dagger + \frac{1}{2}), \quad (5)$$

or alternatively that $\hat{a}\hat{a}^\dagger = \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$. This combined ladder operator can be referred to by a new name $\hat{a}\hat{a}^\dagger = \hat{N}$, so that equation 5 becomes

$$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2}), \quad (6)$$

Thus, we have the energy eigenbasis which satisfy

$$\hat{H} |n\rangle = \hbar\omega(n - \frac{1}{2}) |n\rangle, \text{ for } n \in 0, 1, 2, \dots \quad (7)$$

We now get

$$\hat{a} |n\rangle = C_n |n-1\rangle, \quad (8)$$

where C_n is a constant which can be found the following way

$$\begin{aligned}\langle n | \hat{a}^\dagger \hat{a} | n \rangle &= |C_n|^2 \langle n-1 | n-1 \rangle \\ &\rightarrow |C_n| = \sqrt{n} = C_n,\end{aligned}$$

by choosing the phase to be zero¹. We land at

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle \quad (9)$$

Similarly for \hat{a}^\dagger

$$\hat{a}^\dagger |n\rangle = D_n |n+1\rangle \quad (10)$$

¹Actually, $C_n = \sqrt{n}e^{i\phi}$ where ϕ is arbitrary, but it is conventional to set $\phi = 0$.

$$\begin{aligned}\langle n | \hat{a} \hat{a}^\dagger | n \rangle &= |D_n|^2 \langle n+1 | n+1 \rangle \\ &\rightarrow |D_n| = \sqrt{n+1} = D_n\end{aligned}$$

$$\hat{a}^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle \quad (11)$$

Now to compute the matrix elements for \hat{a} and \hat{a}^\dagger

$$\langle m | \hat{a} | n \rangle = \sqrt{n} \langle m | n-1 \rangle = \sqrt{n} \delta_{m,n-1} \quad (12)$$

$$\langle m | \hat{a}^\dagger | n \rangle = \sqrt{n+1} \langle m | n+1 \rangle = \sqrt{n+1} \delta_{m,n+1}. \quad (13)$$

The actual matrices for these particular operators will look something like this²

$$a \leftrightarrow \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \\ 0 & 0 & 0 & \sqrt{3} & \\ \vdots & & & & \ddots \end{bmatrix} \quad (14)$$

$$a^\dagger \leftrightarrow \begin{bmatrix} 0 & 0 & 0 & \dots \\ \sqrt{1} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \\ 0 & 0 & \sqrt{3} & \\ \vdots & & & \ddots \end{bmatrix}. \quad (15)$$

Finding the matrix representation of \hat{X} is now an easy matter of employing equation 4

$$X \leftrightarrow \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \ddots \end{bmatrix}, \quad (16)$$

while an algebraic expression will be (also employing equation 4)

$$\langle m | X | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} \langle m | n+1 \rangle + \sqrt{n} \langle m | n-1 \rangle) \quad (17)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1}). \quad (18)$$

²I am perfectly aware that $\sqrt{1} = 1$ but I will keep the root symbol here in order to underline the symmetry.