ELECTRODYNAMICS
FYS3120: PROBLEM SET 11

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1. Simple Lagrangian Dynamics

A non-relativistic particle, with electric charge q and mass m moves in a magnetic dipole field, given by the vector potential

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi r^3}(\vec{\mu} \times \vec{\mathbf{r}}), \tag{1}$$

where $\vec{\mu}$ is the magnetic dipole moment of a static charge distribution centered at the origin.

1.a. **Lagrangian.** The Lagrangian is given by

$$L = T + q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}}. \tag{2}$$

The kinetic energy is simply $T = \frac{1}{2}m\vec{\mathbf{v}}^2$ while the potential is

$$\begin{aligned} q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}} &= \frac{q\mu_0}{4\pi r^3}\vec{\mathbf{v}} \cdot (\vec{\mu} \times \vec{\mathbf{r}}) \\ &= \frac{q\mu_0}{4\pi r^3}\vec{\mu} \cdot (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \\ &= \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{\ell}, \end{aligned}$$

using the cyclic invariance of the vector triple product and $\vec{\ell} = m\vec{\mathbf{r}} \times \vec{\mathbf{v}}$. Inserting the parts into 2 the Lagrangican becomes

$$L = \frac{1}{2}m\vec{\mathbf{v}}^2 + \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{\ell}. \tag{3}$$

1.b. **Alternative Lagrangian.** We now make the assumption that the magnetic dipole moment is oriented along the z-axis and that the particle moves in the (x, y)-plane. In the following, $r = |\vec{\mathbf{r}}|$ and the angle ϕ between the x-axis and the position vector var are chosen as generalised coordinates.

With the magnetic dipole moment oriented along the z-axis,

$$\vec{\mu} \cdot \vec{\ell} = |\vec{\mu}|\ell_z = |\vec{\mu}|(\vec{\mathbf{r}} \times \vec{\mathbf{p}})_z = |\vec{\mu}|m(x\dot{y} - y\dot{x}),$$

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where $x = r\cos\phi$ and $y = r\sin\phi$. This gives

$$\begin{aligned} x\dot{y} - y\dot{x} &= r\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) \\ &\quad - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) \\ &= r^2\phi\cos^2\phi + r^2\phi\sin^2\phi = r^2\phi, \end{aligned}$$

similarly

$$\begin{aligned} \dot{x} &= \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{y} &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{x}^2 &= \dot{r}^2\cos^2\phi - 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\sin^2\phi \\ \dot{y}^2 &= \dot{r}^2\sin^2\phi + 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\cos^2\phi \\ \dot{v}^2 &= \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2. \end{aligned}$$

The Lagrangian with generalised coordinates becomes

$$L = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\phi}^2) + \frac{q\mu_0}{4\pi m r^3}|\vec{\mu}|mr^2\dot{\phi} = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\phi}^2) + \lambda\frac{\dot{\phi}}{r}, \tag{4}$$

where $\lambda \equiv q\mu_0|\vec{\mu}|/4\pi$.

The canonical momentum p_ϕ conjugate to ϕ becomes

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi} + \frac{\lambda}{r} \tag{5}$$

ϕ is a cyclic coordinate, because the Lagrangian in equation 4 does not explicityly depend on ϕ. This implies that the conjugate momentum p_ϕ is constant.

The Lagrangian in equation 4 does not depend exlicitly on time t. This means that the Hamiltonian must be conserved

$$H = \dot{r}p_r + \dot{\phi}p_\phi - L = m\dot{r}^2 + mr^2\dot{\phi}^2 + \lambda\frac{\dot{\phi}}{r} - L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = T. \tag{6}$$

Since the Hamiltonian equals the kinetic energy and the Hamiltonian is conserved, the kinetic energy is conserved by the magnetic field.

1.c. **Kinetic Energy Conservation.** Lagrange's equation for r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = \dot{p}_r - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\phi}^2 + \lambda\frac{\dot{\phi}}{r^2} = 0. \tag{7}$$

Here one can eliminate $\dot{\phi}$ by inserting $\dot{\phi} = \frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3}$ found from equation 5. This yields

$$
\begin{aligned}
& m\ddot{r} - mr\left(\frac{p_\phi}{mr^2} + \frac{\lambda}{mr^3}\right) - \frac{\lambda}{r^2}\left(\frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3}\right) \\
=& m\ddot{r} - mr\left(\frac{p_\phi^2}{m^2r^4} - \frac{2p_\phi\lambda}{m^2r^5} + \frac{\lambda^2}{m^2r^6}\right) + \frac{p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} \\
=& m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{2p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} + \frac{p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} = 0
\end{aligned}
$$

$$\rightarrow m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{3p_\phi\lambda}{mr^4} - \frac{2\lambda^2}{mr^5} = 0. \tag{8}$$

We are interested in the behaviour of $\dot{r}^2$ one can multiply the expression in 8 with $\dot{r}$. This gives

$$m\ddot{r}\dot{r} = \frac{p_\phi^2}{mr^3}\dot{r} - \frac{3p_\phi\lambda}{mr^4}\dot{r} + \frac{2\lambda^2}{mr^5}\dot{r} \tag{9}$$

using $\dot{r}\ddot{r} = \frac{1}{2}\frac{d}{dt}(\dot{r}^2)$ and $\dot{r}dt = \frac{dr}{dt}dt = dr$

$$
\begin{aligned}
\frac{1}{2}m\frac{d}{dt}(\dot{r}^2) &= \left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)\dot{r} \\
d(\dot{r}^2) &= \frac{2}{m}\left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)dr
\end{aligned}
$$

now to integrate from r_0 to $r(t)$

$$
\begin{aligned}
\dot{r}(t)^2 - \dot{r}(0)^2 &= \frac{2}{m}\int_{r(0)}^{r(t)}\left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)dr \\
&= -\frac{p_\phi^2}{m^2}(r^{-2} - r_0^{-2}) + \frac{2p_\phi\lambda}{m^2}(r^{-3} - r_o^{-3}) \\
&\quad - \frac{\lambda^2}{m^2}(r^{-4} - r_0^{-4}) \\
&= \frac{1}{m^2r_0^2}\left(p_\phi - \frac{\lambda}{r_0}\right)^2 - \frac{1}{m^2r^2}\left(p_\phi - \frac{\lambda}{r}\right)
\end{aligned}
$$

from equation 5 we have $\dot{\phi}mr^2 = \left(p_\phi - \frac{\lambda}{r}\right)$, inserting in the expression above gives

$$\dot{r}^2 - \dot{r}_0^2 = r_0^2\dot{\phi}_0^2 - r^2\dot{\phi}^2$$

which can be rearranged to

$$\dot{r}^2 + r^2\dot{\phi}^2 = \dot{r}_0^2 + r_0^2\dot{\phi}_0^2. \tag{10}$$

We see again that the kinetic energy is conserved.

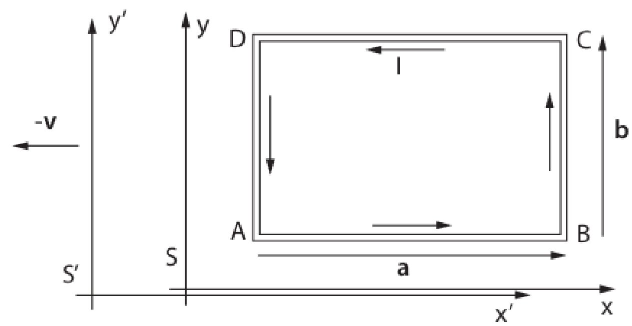

FIGURE 1. Illustration of current loop.

2. Rectangular Current Loop

Figure 1 shows a rectangular current loop ABCD. In the loop's rest frame, S, the loop as length a in the x-direction and width b in y-direction. The current is I and the charge density is zero. The electric dipole moment $\vec{\mathbf{p}}$ and the magnetic dipole moment $\vec{\mathbf{m}}$ for a given current distribution is defined by the following

$$\vec{\mathbf{p}} = \int \vec{\mathbf{r}}\rho(\vec{\mathbf{r}})d^3r, \quad \vec{\mathbf{m}} = \frac{1}{2}\int (\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}))d^3r \tag{11}$$

2.a. **Electric and Magnetic Dipole Moment in S.** Since the charge density in rest frame S is zero, $\rho(\vec{\mathbf{r}}) = 0$, the electric dipole moment must also be zero, $\vec{\mathbf{p}} = 0$.

The current along every edge of the rectangle will be $j\vec{\mathbf{n}}$, where $\vec{\mathbf{n}}$ is a unit vector pointing along the edge in question.

$$\text{AB} : \vec{\mathbf{j}} = j\vec{\mathbf{e}}_x \quad \text{BC} : \vec{\mathbf{j}} = j\vec{\mathbf{e}}_y$$
$$\text{CD} : \vec{\mathbf{j}} = -j\vec{\mathbf{e}}_x \quad \text{DA} : \vec{\mathbf{j}} = -j\vec{\mathbf{e}}_x.$$

Given a point $\vec{\mathbf{r}}$ along the AB segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_x) = -yj\vec{\mathbf{e}}_z, \tag{12}$$

along the BC segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_y) = xj\vec{\mathbf{e}}_z, \tag{13}$$

along the CD segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_x) = yj\vec{\mathbf{e}}_z, \tag{14}$$

and along the DA segment

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_y) = -xj\vec{\mathbf{e}}_z, \tag{15}$$

It is a reasonable approximation to use the factor Δ, which is the cross-sectional area of the current wire, instead of integrating in directions perpendicular to the direction of the conductor. Then the coordinate $\vec{\mathbf{r}}$ is simply the centre of the conductor. Employing these assumptions/approximations and assigning the lower left corner of the rectangle coordinates (x_0, y_0) and using the results from equations 12 13, 14 and 15 the magnetic dipole moment is

$$\begin{aligned}\vec{\mathbf{m}} &= \frac{1}{2} j\Delta\vec{\mathbf{e}}_z \left(-\int_{x_0}^{y_0+a} y_0 dx + \int_{y_0}^{y_0+b} (x_0+a)dy + \int_{x_0}^{x_0+a} (y_0+b)dx - \int_{y_0}^{y_0+b} x_0 dy \right) \\ &= \frac{1}{2} I(-y_0 a + (x_0+a)b + (y_0+b)a - x_0 b)\vec{\mathbf{e}}_z \\ &= abI\vec{\mathbf{e}}_z.\end{aligned}$$

Since $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = ab\vec{\mathbf{e}}_z$, which gives

$$\vec{\mathbf{m}} = I\vec{\mathbf{a}} \times \vec{\mathbf{b}}. \tag{16}$$