MIDTERM EXAM

FYS3120

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1. A (boring) Lagrangian

A non-relativistic particle (no-potential) of mass m is moving in three dimensions.

1.a. A normal (boring), Cartesian coordinate system will do fine to study this problem in the first instance. One needs three coordinates to accurately describe the particle, and as there are no constraints on the particle, these three coordinates x, y and z are also the generalised coordinates.

The kinetic energy of the particle is given by

$$T = \frac{1}{2}mv^2 \tag{1}$$

where $v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$, such that equation 1 becomes

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{2}$$

there is no potential, so the Lagrangian is simply

$$L = T - V = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mv^2 \tag{3}$$

1.b. The conjugate momenta are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \partial \dot{x}} = m\dot{x} \\ p_y &= \frac{\partial L}{\partial \partial \dot{y}} = m\dot{y} \\ p_z &= \frac{\partial L}{\partial \partial \dot{z}} = m\dot{z}, \end{aligned}$$

or rather

$$p_v = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v}, \tag{4}$$

which is exactly the same as the regular mechanical momentum.

Date: March 31, 2017.

1.c. The position of the particle are cyclic coordinates, since

$$\frac{\partial L}{\partial x} = 0$$
$$\frac{\partial L}{\partial y} = 0$$
$$\frac{\partial L}{\partial z} = 0$$

alternatively

$$\frac{\partial L}{\partial \mathbf{r}} = \mathbf{0}. \tag{5}$$

The position $\mathbf{r}$ varies with time, but the Lagrangian, interpreted as the physical situation, remains unchanged. This means that the initial value for the position does not determine the path of the particle.

1.d. The Euler-Lagrange equations for this system equation is

$$\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{r}}} = 0. \tag{6}$$

Since $\partial L/\partial \mathbf{r} = 0$, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}} = 0,$$

and therefore the conjugate momentum $m\dot{\mathbf{r}} = m\mathbf{v}$ must be a constant of motion.

It follows that the particle must move in a straight line, as the particle has constant velocity and is not acted upon by a force, such that

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t. \tag{7}$$

My hunch is that the angular momentum of the particle must also be conserved,

$$\begin{aligned} \mathbf{L} = \mathbf{r} \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p} \\ (\mathbf{r}_0 + \mathbf{v}t) \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p} \\ \mathbf{r}_0 \times \mathbf{p} + \mathbf{v}t \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p} \\ \mathbf{r}_0 \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p} \end{aligned}$$

where $\mathbf{v}t \times \mathbf{p} = 0$ because $\mathbf{v}$ and $\mathbf{p}$ are parallel[1]. This means that the system is invariant under a rotation.

In conclusion, the some of conserved quantities are the momentum $\mathbf{p}$ and the angular momentum $\mathbf{L}$. Said in another way, this system can has both a translational and a rotational symmetry. Because of these two symmetric properties the system must have two corresponding quantities whose values

[1] This proof holds when the particle is viewed from any position *not* on then path of the particle, such that $\mathbf{r} \nparallel \mathbf{p}$.

are conserved in time[2]. Other conserved quantities are, but may not be limited to; mass, energy, charge etc.

1.e. For any mechanical system there exist a certain integral S, called the action, which has minimum value for the actual motion, so that its variation is zero: $\delta S = 0$. To determine the action for a free particle, the integral must not depend on choice of reference system, because it must be invariant under Lorentz transformations. It follows that it must depend on a scalar. The action is

$$S = -k \int_a^b ds, \tag{8}$$

where k is some constant, $\int_a^b$ is the integral along the world line of the particles between two points a and b, and ds is a small displacement, and also a scalar of the right kind - being Lorentz invariant. The integral has a negative sign because this is the obvious way to ensure it has a minimum.

The action integral can be represented as an integral with respect to time instead

$$S = \int_{t_1}^{t_2} L dt, \tag{9}$$

where L is the Lagrangian of the mechanical system.

Now, a small detour. The the invariance of intervals gives

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2,$$

from which

$$dt' = dt\sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}}.$$

Furthermore,

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = v^2,$$

therefore

$$dt' = \frac{ds}{c} = dt\sqrt{1 - \frac{v^2}{c^2}} \tag{10}$$

Equation 10 can be insterted into 8 to give

$$S = -\int_{t_1}^{t_2} kc\sqrt{1 - \frac{v^2}{c^2}} dt. \tag{11}$$

Consequently, the Lagrangian of the free particle is $L = -kc\sqrt{1 - v^2/c^2}$. One can expand L in powers of v/c, ignoring higher order terms.

$$L = -kc\sqrt{1 - \frac{v^2}{c^2}} \approx -kc + \frac{kv^2}{2c}.$$

[2]This last bit was an informal statement of Emmy Noether's theorem.

Constant terms in the Lagrangian do not affect the equations of motion and can be omitted. Compared with the classical expression $L = mv^2/2$, the constant must be $k = mc$.

In conclusion, a Lorentz invariant Lagrangian for a free particle is

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} = -\frac{mc^2}{\gamma}. \tag{12}$$

1.f. In the relativistic Lagrangian in equation 12, the position does not appear explicitly. Consequently, from the Lagrange-Euler equation

$$\frac{\partial L}{\partial \mathbf{r}} = 0 \to \frac{d}{dt}\frac{\partial L}{\partial \mathbf{r}} = 0.$$

one may gather that the conjugate momentum is a constant of motion. The conjugate momentum will take the expected form

$$\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(-mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}\right) = mc^2\frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}}\left(-\frac{v}{c^2}\right) = m\mathbf{v}\gamma, \tag{13}$$

which is the usual way to write relativistic momentum.

The four-vector momentum is given by

$$\underline{\mathbf{P}} = (\frac{E}{c}, \mathbf{p}) = (\gamma mc, \gamma m\mathbf{v}). \tag{14}$$

Since the energy of a freely moving particle relates to momentum by $E^2 = c^2\mathbf{p}^2 + m^2c^2$, the four-vector momentum contains only conserved quantities and is a constant of motion.

1.g. Consider a Lorentz transformation where the Lorentz transformation tensor is given as

$$L^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{15}$$

Any particular Lorentz transformation must leave the line element $ds^2 = dx_\mu dx^\mu$ invariant,

$$\begin{aligned} g_{\mu\nu}dx'^{\mu}dx'^{\nu} = g_{\mu\nu}L^{\mu}{}_{\rho}L^{\mu}{}_{\sigma}dx^{\rho}dx^{\sigma} &= g_{\rho\sigma}dx^{\rho}dx^{\sigma} \\ &\to g_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}_{\sigma} = g_{\rho\sigma} \end{aligned}$$

For the Lorentz transformation in equation 15 to be invariant is must satisfy the requirement above

$$\begin{aligned} g_{\rho\sigma} &= q_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}{}_{\sigma} \\ &= g_{\mu\nu}(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= (\delta_{\nu\rho} + \omega_{\nu\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= \delta_{\nu\rho}\delta^{\nu}{}_{\sigma} + \delta_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho}\delta^{\nu}{}_{\sigma} + \omega_{\nu\rho}\omega^{\nu}{}_{\sigma} \\ &= g_{\nu\rho}\delta^{\nu}{}_{\sigma} + g_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho}g^{\nu\gamma}g_{\gamma\sigma} + \cancel{\omega^2_{\rho\sigma}} \\ &= \delta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} = g_{\rho\sigma} + g_{\nu\rho}(\omega^{\nu}{}_{\sigma} + \omega_{\sigma}{}^{\nu}), \end{aligned}$$

which only works if $\omega^{\mu}{}_{\nu}$ is antisymmetric, that is if $\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}$.

1.h. A small Lorentz transformation between two reference frames changes the path $x^\mu(\tau)$ of a particle according to

$$\delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau) = \omega^\mu{}_\nu x^\nu(\tau). \tag{16}$$

This corresponds to a perturbation in the Lagrangian.

The variation of the Lagrangian is

$$\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial U^\mu}\delta U^\mu$$

inserting for $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ from equation 16 and

$$\delta U^\mu = \delta \frac{dx^\mu}{dt} = \frac{d}{d\tau}(\delta x^\mu) = \omega^\mu{}_\nu U^\nu,$$

which yields

$$\delta L = \left(\frac{\partial L}{\partial x^\mu} x^\nu + \frac{\partial L}{\partial U^\mu} U^\nu \right) x^\mu{}_\nu. \tag{17}$$

This is the change in the Lagrangian as a consequence of the change in path.

1.i. The Euler-Lagrange equations states

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}\right) = \frac{\partial L}{\partial x^\mu}. \tag{18}$$

Inserting 18 into 17 gives

$$\delta L = \left(\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu} x^\nu\right) + \frac{\partial L}{\partial U^\mu}\frac{d}{d\tau}x^\nu \right)\omega^\mu{}_\nu \tag{19}$$

using the product rule for derivation backwards gives

$$\delta L = \frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu = \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \tag{20}$$

and finally "letting everything run its course"

$$\begin{aligned}
\delta L &= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\omega^\mu{}_\nu - \frac{\partial L}{\partial U^\mu}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial g^{\rho\mu}U_\rho}x^\nu\omega^\mu{}_\nu - \frac{\partial L}{\partial g^{\rho\mu}U_\rho}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu g_{\rho\mu}\omega^\mu{}_\nu - \frac{\partial L}{\partial U_\rho}x^\nu g_{\rho\mu}\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu\omega_{\rho\nu} - \frac{\partial L}{\partial U_\rho}x^\nu\omega_{\nu\rho}\right)
\end{aligned}$$

changing indices back, writing μ instead of ρ, and moving x^ν to the left of the derivatives gives

$$\delta L = \frac{1}{2}\frac{d}{d\tau}\left(x^\nu\frac{\partial L}{\partial U_\mu}\omega_{\mu\nu} - x^\nu\frac{\delta L}{\delta U_\mu}\omega_{\nu\mu}\right).$$

Switch indices of first term inside the parenthesis[3], and one ends up with an alternative expression for δL

$$\delta L = \frac{1}{2}\omega_{\nu\mu}\frac{d}{d\tau}\left(x^{\mu}\frac{\partial L}{\partial U_{\nu}} - x^{\nu}\frac{\partial L}{\partial U_{\mu}}\right) \tag{21}$$

1.j. For the path change to be invariant, there must, according to Hamilton's principle, be no change in the action

$$\delta S = 0. \tag{22}$$

This means that

$$\delta S = \int_{\tau_1}^{\tau_2}\delta L d\tau = \int_{\tau_1}^{\tau_2}\frac{1}{2}\omega_{\nu\mu}\frac{d}{d\tau}\left(x^{\mu}\frac{\partial L}{\partial U_{\nu}} - x^{\nu}\frac{\partial L}{\partial U_{\mu}}\right)d\tau = 0, \tag{23}$$

which is true if

$$\delta L = \frac{1}{2}\omega_{\nu\mu}\frac{d}{d\tau}\left(x^{\mu}\frac{\partial L}{\partial U_{\nu}} - x^{\nu}\frac{\partial L}{\partial U_{\mu}}\right) = 0, \tag{24}$$

or alternatively if

$$x^{\mu}\frac{\partial L}{\partial U_{\nu}} - x^{\nu}\frac{\partial L}{\partial U_{\mu}} = C \tag{25}$$

where C is a constant. Since $\frac{\partial L}{\partial U^{\mu}} = p^{\mu}$, then equation 25 is the tensor form of the angular momentum

$$\ell^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}, \tag{26}$$

leading one to conclude that the angular momentum is conserved because of the invariance under Lorentz transformation.

2. Relativistics

Two particles with mass m and a photon is sent out from a source at the same time and in the positive x-direction in rest frame S of the source. The massive particles are moving with constant velocity v_1 and $v_2 > v_1$ in this frame. Figure 1 shows a Minkowski space-time diagram of the two particles, the photon and the source in the rest frame of the source S and that of the slowest of the particles S'.

The relativistic formula for transition between two inertial frames is given by

$$x' = \gamma(x - vt), \quad t' = \gamma(t - \frac{v}{c^2}x), \tag{27}$$

[3]This is okay because if one were to move ∂U_{μ} up from underneath the dividing line the index μ would change to an upstairs variant. This is the same as saying $\sum_i\sum_j x^i\frac{\partial L}{\partial U_j}\omega_{ji} = \sum_j\sum_i x^i\frac{\partial L}{\partial U_i}\omega_{ij}$

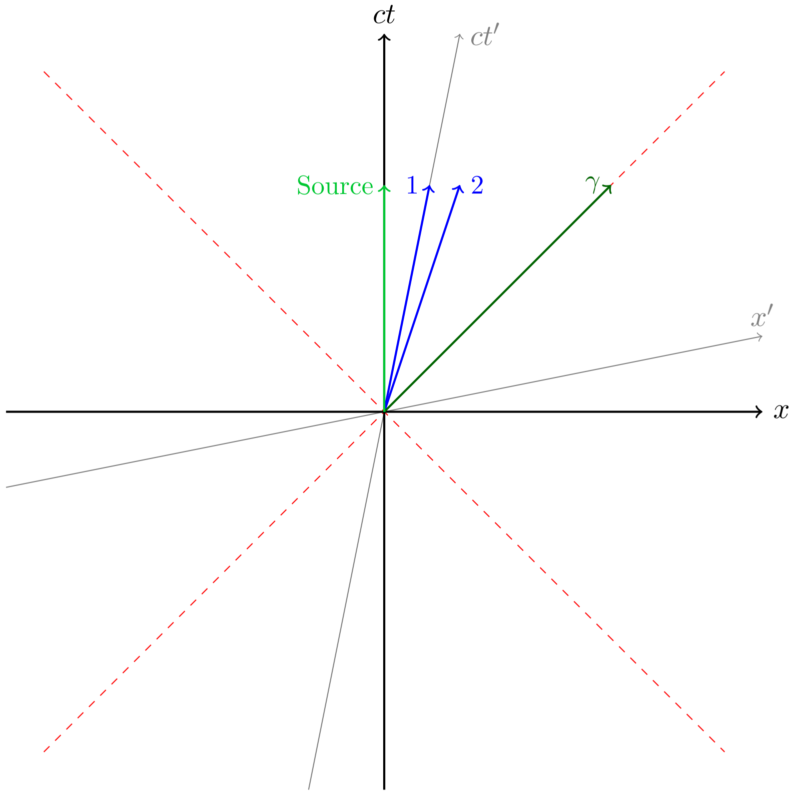

Figure 1. Minkowski space-time diagram of two massive particles (velocities v_1 and $v_2 > v_1$) and a photon (γ) sent out from a source at origin in rest frame S. Rest frame S' is that of particle 1.

therefore, for an infinitesimal change in position coordinates we have

$$dx' = \gamma(dx - vdt) = \gamma(u - v)dt$$
$$dt' = \gamma(dt - \frac{v}{c^2}dx) = \gamma(1 - \frac{uv}{c^2})$$

and from this follows that

$$u' = \frac{dx'}{dt'} = \frac{u - v}{1 - \frac{uv}{c^2}},$$

or specifically to this situation

$$v_2' = \frac{dx'}{dt'} = \frac{v_2 - v_1}{1 - \frac{v_2 v_1}{c^2}}. \tag{28}$$

The difference in rapidity of the two massive particles in the two different rest frames are

$$S: \quad \Delta\chi = \tanh^{-1}\left(\frac{v_2}{c}\right) - \tanh^{-1}\left(\frac{v_1}{c}\right) \tag{29}$$

$$S': \quad \Delta\chi' = \tanh^{-1}\left(\frac{v_2'}{c}\right) - \tanh^{-1}\left(\frac{v_1'}{c}\right) = \tanh^{-1}\left(\frac{v_2'}{c}\right) \tag{30}$$

Rapidity differences should be unchanged by boosts no matter the reference frames, so

$$\begin{aligned} \tanh^{-1}\left(\frac{v_2}{c}\right) - \tanh^{-1}\left(\frac{v_1}{c}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right) \\ \tanh^{-1}\left(\frac{\frac{v_2}{c} - \frac{v_1}{c}}{1 - \frac{v_2 v_1}{c^2}}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right) \\ \tanh^{-1}\left(\frac{1}{c}\frac{v_2 - v_1}{1 - \frac{v_2 v_1}{c^2}}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right), \end{aligned}$$

inserting 28 gives

$$\begin{aligned} \tanh^{-}1\left(\frac{v_2'}{c}\right) &= \tanh^{-}1\left(\frac{v_2'}{c}\right) \\ \chi &= \chi'. \end{aligned}$$

In conclusion, the rapidity difference is the same in the two rest frames S and S.

3. Finding the Shortest Way

The shortest path between two points on a sphere. At some contstant radius r, some small movement in some direction on the sphere is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{31}$$

inserting for $d\phi = (d\phi/d\theta)d\theta = \dot{\phi}d\theta$ gives

$$ds = r\sqrt{1 + \sin^2\theta\dot{\phi}^2}d\theta \tag{32}$$

A path on the sphere is given by

$$S = \int ds = r\int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta\dot{\phi}^2}d\theta \tag{33}$$

where the integrand $F(\theta, \phi, \dot{\phi}) = \sqrt{1 + \sin^2\theta\dot{\phi}^2}$ does not depend explicitly on ϕ. This implies that $\partial F/\partial\dot{\phi}$ is constant, according to the Lagrange equations or Noether's theorem, yielding

$$\frac{\partial F}{\partial\dot{\phi}} = \frac{2\sin^2\theta\dot{\phi}}{\sqrt{1 + \sin^2\theta\dot{\phi}^2}} = C' \to \frac{\sin^2\theta\dot{\phi}}{\sqrt{1 + \sin^2\theta\dot{\phi}^2}} = C \tag{34}$$

This can be rearranged

$$
\begin{aligned}
C^2 &= \frac{\sin^4\theta\dot{\phi}^2}{1+\sin^2\theta\dot{\theta}^2} \\
C^2 + C\sin^2\theta\dot{\phi}^2 &= \sin^4\theta\dot{\phi}^2 \\
C^2 &= (\sin^4\theta - C\sin^2\theta)\dot{\phi}^2 \\
\dot{\phi}^2 &= \frac{C^2}{(\sin^4\theta - C\sin^2\theta)} \\
\dot{\phi} &= \frac{C}{\sin\theta\sqrt{\sin^2 - C}}
\end{aligned}
$$

Now this will be integrated from the starting point $(\theta_0, \phi_0) = (\pi/2, 0)$ to the stop point (θ, ϕ)

$$
\begin{aligned}
\phi - \phi_0 &= \int_{\theta_0}^{\theta} \frac{C}{\sin\vartheta\sqrt{\sin^2\vartheta - C^2}} d\vartheta \\
\phi &= \int_{\frac{\pi}{2}}^{\theta} \frac{C}{\sin\vartheta\sqrt{\sin^2\vartheta - C^2}} d\vartheta.
\end{aligned}
$$

The integrand can be simplified by making the substitution[4] $\sin^{-2}\vartheta = \csc^2\vartheta = 1 + \cot^2\vartheta$,

$$
\begin{aligned}
\frac{C}{\sin\vartheta\sqrt{\sin^2\vartheta - C^2}} &= \frac{C}{\sin^2\vartheta\sqrt{1 - \frac{C^2}{\sin^2\vartheta}}} \\
= \frac{C\csc^2\vartheta}{\sqrt{1-(1+\cot^2\vartheta)C^2}} &= \frac{C\csc^2\vartheta}{\sqrt{1 - C^2 - C^2\cot^2\vartheta}}
\end{aligned}
$$

The reason for rewriting the integrand in this manner is to be able to solve the integral by a substitution. Notice that $\partial(A\cot\vartheta)/\partial\vartheta = -A\csc^2\vartheta$. Inserting $u = A\cot\vartheta$ and $du = -A\csc^2\vartheta d\vartheta$ gives

$$
\frac{C\csc^{\vartheta} d\vartheta}{\sqrt{1 - C^2 - C^2\cot^2\vartheta}} = -\frac{Cudu}{A\sqrt{1 - C^2 - C^2\left(\frac{u}{A}\right)^2}},
$$

setting $A = C/\sqrt{1-C^2}$ gives a very nice expression,

$$
\begin{aligned}
-\frac{Cudu}{A\sqrt{1 - C^2 - C^2\left(\frac{u}{A}\right)^2}} &= \frac{-\cancel{C}\sqrt{1-C^2}udu}{\cancel{C}\sqrt{1 - C^2 - \frac{\cancel{C^2}u^2(1-C^2)}{\cancel{C^2}}}} \\
= -\frac{\sqrt{1-C^2}udu}{\sqrt{1 - C^2 - u^2 - u^2C^2}} &= -\frac{\cancel{\sqrt{1-C^2}}udu}{\cancel{\sqrt{1-C^2}}\sqrt{1-u^2}}.
\end{aligned}
$$

[4]These are simply trigonometric identities.

Karl Rottmann can tell me that $\partial(\arccos x)/\partial x = -(1-x^2)^{\frac{1}{2}}$ (p.130), so the proper substitution to make is

$$u = \frac{C}{\sqrt{1-C^2}}\cot\vartheta \leftrightarrow \cot\vartheta = \frac{u}{C}\sqrt{1-C^2}$$
$$\frac{du}{d\vartheta} = -\frac{C}{\sqrt{1-C^2}}\csc^2\vartheta \leftrightarrow \csc^2\vartheta d\vartheta = -\frac{\sqrt{1-C^2}}{C}du,$$

which gives

$$\phi = \int_{\frac{\pi}{2}}^{\theta}\frac{C\csc^2\vartheta d\vartheta}{\sqrt{1-C^2-C^2\cot^2\vartheta}} = -\int_{u(\frac{\pi}{2})}^{u(\theta)}\frac{du}{\sqrt{1-u^2}}$$

$$\phi(\theta) = \arccos(u(\theta)) - \arccos\Big(u(\frac{\pi}{2})\Big) = \arccos\Big(\frac{C}{\sqrt{1-C^2}}\cot\theta\Big), \tag{35}$$

a nice, explicit expression for ϕ as a function of θ.

A special solution is needed where $\theta = \pi/2$ (or $\phi = 0$) because $\cot\theta = \cos\theta/\sin\theta$ and if $\theta = 0$, $\sin\theta = 0$ which would give 0 in the denominator. I have chosen to ignore such special solutions.