# ELECTRODYNAMICS
## FYS3120: PROBLEM SET 11

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### 1. Simple Lagrangian Dynamics

A non-relativistic particle, with electric charge $q$ and mass $m$ moves in a magnetic dipole field, given by the vector potential

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi r^3}(\vec{\mu} \times \vec{\mathbf{r}}), \tag{1}$$

where $\vec{\mu}$ is the magnetic dipole moment of a static charge distribution centered at the origin.

1.a. **Lagrangian.** The Lagrangian is given by

$$L = T + q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}}. \tag{2}$$

The kinetic energy is simply $T = \frac{1}{2}m\vec{\mathbf{v}}^2$ while the potential is

$$\begin{aligned} q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}} &= \frac{q\mu_0}{4\pi r^3}\vec{\mathbf{v}} \cdot (\vec{\mu} \times \vec{\mathbf{r}}) \\ &= \frac{q\mu_0}{4\pi r^3}\vec{\mu} \cdot (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \\ &= \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{\ell}, \end{aligned}$$

using the cyclic invariance of the vector triple product and $\vec{\ell} = m\vec{\mathbf{r}} \times \vec{\mathbf{v}}$. Inserting the parts into 2 the Lagrangican becomes

$$L = \frac{1}{2}m\vec{\mathbf{v}}^2 + \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{\ell}. \tag{3}$$

1.b. **Alternative Lagrangian.** We now make the assumption that the magnetic dipole moment is oriented along the $z$-axis and that the particle moves in the $(x, y)$-plane. In the following, $r = |\vec{\mathbf{r}}|$ and the angle $\phi$ between the $x$-axis and the position vector $var$ are chosen as generalised coordinates.

With the magnetic dipole moment oriented along the z-axis,

$$\vec{\mu} \cdot \vec{\ell} = |\vec{\mu}|\ell_z = |\vec{\mu}|(\vec{\mathbf{r}} \times \vec{\mathbf{p}})_z = |\vec{\mu}|m(x\dot{y} - y\dot{x}),$$

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where $x = r\cos\phi$ and $y = r\sin\phi$. This gives

$$\begin{aligned} x\dot{y} - y\dot{x} &= r\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) \\ &\quad - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) \\ &= r^2\phi\cos^2\phi + r^2\phi\sin^2\phi = r^2\phi, \end{aligned}$$

similarly

$$\begin{aligned} \dot{x} &= \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{y} &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{x}^2 &= \dot{r}^2\cos^2\phi - 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\sin^2\phi \\ \dot{y}^2 &= \dot{r}^2\sin^2\phi + 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\cos^2\phi \\ \dot{v}^2 &= \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2. \end{aligned}$$

The Lagrangian with generalised coordinates becomes

$$L = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\phi}^2) + \frac{q\mu_0}{4\pi m r^3}|\vec{\mu}|mr^2\dot{\phi} = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\phi}^2) + \lambda\frac{\dot{\phi}}{r}, \tag{4}$$

where $\lambda \equiv q\mu_0|\vec{\mu}|/4\pi$.

The canonical momentum $p_\phi$ conjugate to $\phi$ becomes

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi} + \frac{\lambda}{r} \tag{5}$$

$\phi$ is a cyclic coordinate, because the Lagrangian in equation 4 does not explicityly depend on $\phi$. This implies that the conjugate momentum $p_\phi$ is constant.

The Lagrangian in equation 4 does not depend exlicitly on time $t$. This means that the Hamiltonian must be conserved

$$H = \dot{r}p_r + \dot{\phi}p_\phi - L = m\dot{r}^2 + mr^2\dot{\phi}^2 + \lambda\frac{\dot{\phi}}{r} - L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = T. \tag{6}$$

Since the Hamiltonian equals the kinetic energy and the Hamiltonian is conserved, the kinetic energy is conserved by the magnetic field.

1.c. **Kinetic Energy Conservation.** Lagrange's equation for $r$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = \dot{p}_r - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\phi}^2 + \lambda\frac{\dot{\phi}}{r^2} = 0. \tag{7}$$

Here one can eliminate $\dot{\phi}$ by inserting $\dot{\phi} = \frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3}$ found from equation 5. This yields

$$\begin{aligned}
&m\ddot{r} - mr\left(\frac{p_\phi}{mr^2} + \frac{\lambda}{mr^3}\right) - \frac{\lambda}{r^2}\left(\frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3}\right) \\
=&m\ddot{r} - mr\left(\frac{p_\phi^2}{m^2r^4} - \frac{2p_\phi\lambda}{m^2r^5} + \frac{\lambda^2}{m^2r^6}\right) + \frac{p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} \\
=&m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{2p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} + \frac{p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} = 0
\end{aligned}$$

$$\to m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{3p_\phi\lambda}{mr^4} - \frac{2\lambda^2}{mr^5} = 0. \tag{8}$$

We are interested in the behaviour of $\dot{r}^2$ one can multiply the expression in 8 with $\dot{r}$. This gives

$$m\ddot{r}\dot{r} = \frac{p_\phi^2}{mr^3}\dot{r} - \frac{3p_\phi\lambda}{mr^4}\dot{r} + \frac{2\lambda^2}{mr^5}\dot{r} \tag{9}$$

using $\dot{r}\ddot{r} = \frac{1}{2}\frac{d}{dt}(\dot{r}^2)$ and $\dot{r}dt = \frac{dr}{dt}dt = dr$

$$\begin{aligned}
\frac{1}{2}m\frac{d}{dt}(\dot{r}^2) &= \left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)\dot{r} \\
d(\dot{r}^2) &= \frac{2}{m}\left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)dr
\end{aligned}$$

now to integrate from $r_0$ to $r(t)$

$$\begin{aligned}
\dot{r}(t)^2 - \dot{r}(0)^2 &= \frac{2}{m}\int_{r(0)}^{r(t)}\left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)dr \\
&= -\frac{p_\phi^2}{m^2}(r^{-2} - r_0^{-2}) + \frac{2p_\phi\lambda}{m^2}(r^{-3} - r_o^{-3}) \\
&\quad - \frac{\lambda^2}{m^2}(r^{-4} - r_0^{-4}) \\
&= \frac{1}{m^2r_0^2}\left(p_\phi - \frac{\lambda}{r_0}\right)^2 - \frac{1}{m^2r^2}\left(p_\phi - \frac{\lambda}{r}\right)
\end{aligned}$$

from equation 5 we have $\dot{\phi}mr^2 = \left(p_\phi - \frac{\lambda}{r}\right)$, inserting in the expression above gives

$$\dot{r}^2 - \dot{r}_0^2 = r_0^2\dot{\phi}_0^2 - r^2\dot{\phi}^2$$

which can be rearranged to

$$\dot{r}^2 + r^2\dot{\phi}^2 = \dot{r}_0^2 + r_0^2\dot{\phi}_0^2. \tag{10}$$

We see again that the kinetic energy is conserved.

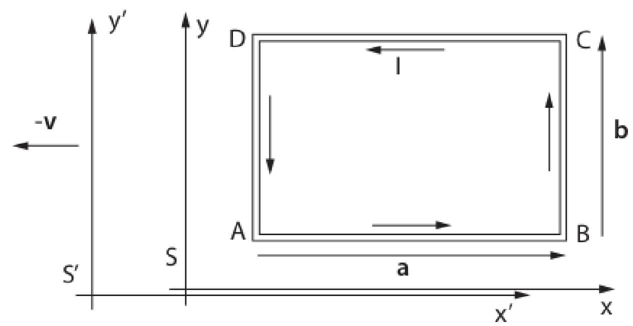


FIGURE 1. Illustration of current loop.

## 2. Rectangular Current Loop

Figure 1 shows a rectangular current loop ABCD. In the loop's rest frame, $S$, the loop as length $a$ in the $x$-direction and width $b$ in $y$-direction. The current is $I$ and the charge density is zero. The electric dipole moment $\vec{\mathbf{p}}$ and the magnetic dipole moment $\vec{\mathbf{m}}$ for a given current distribution is defined by the following

$$\vec{\mathbf{p}} = \int \vec{\mathbf{r}}\rho(\vec{\mathbf{r}})d^3r, \quad \vec{\mathbf{m}} = \frac{1}{2}\int (\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}))d^3r \tag{11}$$

2.a. **Electric and Magnetic Dipole Moment in $S$.** Since the charge density in rest frame $S$ is zero, $\rho(\vec{\mathbf{r}}) = 0$, the electric dipole moment must also be zero, $\vec{\mathbf{p}} = 0$.

The current along every edge of the rectangle will be $j\vec{\mathbf{n}}$, where $\vec{\mathbf{n}}$ is a unit vector pointing along the edge in question.

$$\begin{aligned} \text{AB} : \vec{\mathbf{j}} = j\vec{\mathbf{e}}_x \quad \text{BC} : \vec{\mathbf{j}} = j\vec{\mathbf{e}}_y \\ \text{CD} : \vec{\mathbf{j}} = -j\vec{\mathbf{e}}_x \quad \text{DA} : \vec{\mathbf{j}} = -j\vec{\mathbf{e}}_x. \end{aligned}$$

Given a point $\vec{\mathbf{r}}$ along the AB segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_x) = -yj\vec{\mathbf{e}}_z, \tag{12}$$

along the BC segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_y) = xj\vec{\mathbf{e}}_z, \tag{13}$$

along the CD segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_x) = yj\vec{\mathbf{e}}_z, \tag{14}$$

and along the DA segment

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_y) = -xj\vec{\mathbf{e}}_z, \tag{15}$$

It is a reasonable approximation to use the factor $\Delta$, which is the cross-sectional area of the current wire, instead of integrating in directions perpendicular to the direction of the conductor. Then the coordinate $\vec{\mathbf{r}}$ is simply the centre of the conductor. Employing these assumptions/approximations and assigning the lower left corner of the rectangle coordinates $(x_0, y_0)$ and using the results from equations 12 13, 14 and 15 the magnetic dipole moment is

$$\begin{aligned}\vec{\mathbf{m}} &= \frac{1}{2} j\Delta\vec{\mathbf{e}}_z \left( -\int_{x_0}^{y_0+a} y_0 dx + \int_{y_0}^{y_0+b} (x_0+a)dy + \int_{x_0}^{x_0+a} (y_0+b)dx - \int_{y_0}^{y_0+b} x_0 dy \right) \\ &= \frac{1}{2} I(-y_0 a + (x_0+a)b + (y_0+b)a - x_0 b)\vec{\mathbf{e}}_z \\ &= abI\vec{\mathbf{e}}_z.\end{aligned}$$

Since $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = ab\vec{\mathbf{e}}_z$, which gives

$$\vec{\mathbf{m}} = I\vec{\mathbf{a}} \times \vec{\mathbf{b}}. \tag{16}$$

2.b. **Length Contraction.** Reference frame $S'$ moves at velocity $\vec{\mathbf{v}}$ to the right, away from the rest frame $S$. Because the width of the loop is orthogonal to the boost, it remains the same, $b' = b$.

We have the following Lorentz transformations between the positions of the endpoints of the rectangle in the two reference frames

$$\begin{aligned} x_a &= \gamma(-vt'_a + x'_a) \quad & x'_a &= \gamma(vt_a + x_a) \\ x_b &= \gamma(-vt'_b + x'_b) \quad & x'_b &= \gamma(vt_b + x_b). \end{aligned}$$

The two endpoints must be measured at the same time in reference frame $S'$ to compute the correct length

$$\begin{aligned} a = x_b - x_a &= \gamma(x'_b - x'_a) + \cancel{\gamma v(t'_a - t'_b)} \\ &= \gamma(x'_b - x'_a) = \gamma a' \end{aligned}$$

which yields

$$a = \frac{1}{\gamma} a'. \tag{17}$$

2.c. **Charge of Segments AB and CD.** In reference frame $S$, the segment AB will have current four-vector $J^\mu = (0, j, 0, 0)$. Lorentz transform to reference frame $S'$, which has velocity $-v$ relative to $S$ gives

$$\rho' c = J'^0 = L^0{}_\nu J^\nu = \gamma(J^0 + \beta J^1) = \beta\gamma J,$$

where $\beta = v/c$. Assuming uniform charge density of the conductor segment gives

$$\rho' = \frac{J'^0}{c} = \frac{j\gamma v}{c^2}.$$

To find the total charge one needs simply to multiply with the volume of this conductor segment in reference frama $S'$, $|\vec{\mathbf{a}}|\Delta = (1/\gamma)a\Delta$

$$Q'_{AB} = V'_{AB}\rho'_{AB} = \frac{1}{\gamma}a\Delta j\gamma\frac{v}{c^2} = Ia\frac{v}{c^2}. \tag{18}$$

A similiar computation can be made for conductor segment CD, with current four-vector $J^\mu = (0, -j, 0, 0)$.

$$\begin{aligned}\rho' c = J'^0 &= L^0{}_\nu J^\nu = \gamma(J^0 + \beta J^1) = -\gamma\beta j\\ \rightarrow \rho' &= \frac{J'^0}{c} = -\frac{\gamma\beta j}{c} = -\frac{\gamma v j}{c^2}\end{aligned}$$

$$Q'_{CD} = V'_{CD}\rho'_{CD} = -\frac{1}{\gamma}a\Delta\gamma j\frac{v}{c^2} = -Ia\frac{v}{c^2} \tag{19}$$

2.d. **Electric and Magnetic Dipole Moment in $S'$.** Segments AB and CD always have charge densities $\rho = \pm\frac{I}{\Delta}\frac{v}{c^2}$. Inserting this into the $\vec{\mathbf{p}}$ from 11 and assuming a thin conductor by replacing the cross-sectional integration dimensions with $\Delta$ gives

$$\begin{aligned}\vec{\mathbf{p}}' &= \int \vec{\mathbf{r}}\rho(\vec{\mathbf{r}})d^3r\\ &= \frac{I}{\cancel{\Delta}}\frac{v}{c^2}\left[\cancel{\Delta}\int_{x_0}^{x_0+a}(\cancel{x\vec{\mathbf{e}}_x} + y_0\vec{\mathbf{e}}_y)dx - \cancel{\Delta}\int_{x_0}^{x_0+a}(\cancel{x\vec{\mathbf{e}}_x} + (y_0+b)\vec{\mathbf{e}}_y)dx\right]\\ &= I\frac{v}{c^2}\left(\cancel{y_0 x\Big|_{x_0}^{x_0+a}} - (\cancel{y_0} + b)x\Big|_{x_0}^{x_0+a}\right)\vec{\mathbf{e}}_y\\ &= I\frac{v}{c^2}\left(-b(x_0 + a - x_0)\right)\vec{\mathbf{e}}_y\\ &= -I\frac{v}{c^2}ab\vec{\mathbf{e}}_y\end{aligned}$$

Moreover

$$\begin{aligned}-\frac{1}{c^2}\vec{\mathbf{m}}\times\vec{\mathbf{v}} &= -\frac{1}{c^2}(abI\vec{\mathbf{e}}_z)\times(v\vec{\mathbf{e}}_x)\\ &= -\frac{v}{c^2}Iab(\vec{\mathbf{e}}_z\times\vec{\mathbf{e}}_x)\\ &= -I\frac{v}{c^2}ab\vec{\mathbf{e}}_y,\end{aligned}$$

which implies that

$$\vec{\mathbf{p}}' = -\frac{1}{c^2}\vec{\mathbf{m}}\times\vec{\mathbf{v}} \tag{20}$$

In order to calculate the magnetic dipole moment, one needs the current densities for all conductor segments.

$$\begin{aligned}&\text{AB: } J = (0, j, 0, 0) \quad j' = J'^1 = \beta\gamma J^0 + \gamma J^1 = \gamma j\\ &\text{CD: } J = (0, -j, 0, 0) \quad j' = J'^1 = \beta\gamma J^0 + \gamma J^1 = -\gamma j.\end{aligned}$$

Segments AB and CD both have current densities $j' = \gamma j$ in $x$-direction. Segments BA and DA have current density $j$ unchanged. However, the width of these conductors are Lorentz-contracted, meaning that the area is reduced by a factor $\gamma^{-1}$, so that $\Delta' = \Delta/\gamma$. Now to compute the magnetic dipole moment in the same manner as before

$$\begin{aligned}\vec{\mathbf{m}}' &= \frac{1}{2}\vec{\mathbf{e}}_z\left[\gamma j\Delta\int_{x'_0}^{x'_0+a'}(\cancel{y_0}+b-\cancel{y_0})dx' j\frac{\Delta}{\gamma}\int_{y'_0}^{y'_0+b'}(\cancel{x'_0}+a-\cancel{x'_0})\right]\\ &= \frac{1}{2}j\Delta ab(1+\gamma^{-2})\vec{\mathbf{e}}_z = \frac{1}{2}Iab(2-\beta^2)\vec{\mathbf{e}}_z = Iab(1-\frac{\beta^2}{2})\vec{\mathbf{e}}_z\end{aligned}$$

$$\vec{\mathbf{m}}' = \left(1-\frac{\beta^2}{2}\right)\vec{\mathbf{m}} \tag{21}$$

2.e. **Current in the Different Segments.**