FYS 3120: Classical Mechanics and Electrodynamics

Formula Collection

1 Analytical Mechanics

The Lagrangian

$$L = L(q, \dot{q}, t) \ , \tag{1}$$

is a function of the *generalized coordinates* $q = \{q_i \, ; \, i = 1, 2, ..., d\}$ of the physical system, and their time derivatives $\dot{q} = \{\dot{q}_i \, ; \, i = 1, 2, ..., d\}$. The Lagrangian may also have an *explicit* dependence of time t.

Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \ , \quad i = 1, 2, .., d. \tag{2}$$

There is one equation for each generalized coordinate.

Generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \ , \quad i = 1, 2, .., d. \tag{3}$$

is also referred to as *canonical* or *conjugate* momentum. There is one generalized momentum p_i conjugate to each generalized coordinate q_i.

The Hamiltonian

$$H(p, q) = \sum_{i=1}^{d} \dot{q}_i p_i - L \tag{4}$$

is usually considered as a function of the generalized coordinates q_i and momenta p_i.

Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \ , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \ , \quad i = 1, 2, .., d \tag{5}$$

$$\tag{6}$$

Standard expressions for L og H

$$\begin{aligned} L &= K - V \\ H &= K + V \end{aligned} \tag{7}$$

with K as kinetic energy and V as potential energy. There are cases where H is *not* the total energy.

Charged particle in electromagnetic field (non-relativistic)

$$\begin{aligned} L = L(\mathbf{r}, \mathbf{v}) &= \frac{1}{2}mv^2 - e\phi + e\mathbf{v} \cdot \mathbf{A} \\ H = H(\mathbf{r}, \mathbf{p}) &= \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + e\phi \end{aligned} \tag{8}$$

2 Relativity

Space-time coordinates

$$(x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{r}) \tag{9}$$

General Lorentz transformation

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu + a^\mu \tag{10}$$

Special Lorentz transformation with velocity v in the x direction

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \end{aligned} \tag{11}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, and x^2 og x^3 are unchanged.

Condition satisfied by Lorentz transformation matrices

$$g_{\mu\nu} L^\mu{}_\rho L^\nu{}_\sigma = g_{\rho\sigma} \tag{12}$$

Invariant line element

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \mathbf{r}^2 = g_{\mu\nu}\, \Delta x^\mu \Delta x^\nu = \Delta x_\mu\, \Delta x^\mu \tag{13}$$

Metric tensor

$$g_{\mu\nu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu = 0 \\ -1, & \mu = \nu \neq 0 \end{cases}$$

Upper and lower index

$$\begin{aligned} & x_\mu = g_{\mu\nu}\, x^\nu, \quad (x^\mu) = (ct, \mathbf{r}), \quad (x_\mu) = (ct, -\mathbf{r}) \\ & x^\mu = g^{\mu\nu} x_\nu\ , \quad g_{\mu\rho}\, g^{\rho\nu} = \delta^\nu_\mu \end{aligned} \tag{14}$$

Proper time - time dilatation

$$d\tau = \frac{1}{c}\sqrt{ds^2} = \frac{1}{\gamma}dt, \tag{15}$$

$d\tau$: proper time interval = time measured in an (instantaneous) rest frame of a moving body (by a co-moving clock)
ds^2: invariant line element of an infinitesimal section of the object's world line
dt: coordinate time interval = time interval measured in arbitrarily chosen inertial system

Length contraction

$$L = \frac{1}{\gamma}L_0 \tag{16}$$

Lengths of a moving body measured in the direction of motion.
L_0: length measured in the rest frame of a moving body
L: length measured (at simultaneity) in an arbitrarily chosen inertial frame.

Four velocity

$$U^\mu = \frac{dx^\mu}{d\tau} = \gamma\,(c, \mathbf{v})\;, \qquad U^\mu U_\mu = c^2 \tag{17}$$

Four acceleration

$$\mathcal{A}^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}\;, \quad \mathcal{A}^\mu U_\mu = 0 \tag{18}$$

Proper acceleration $\mathbf{a_0}$
Acceleration measured in instantaneous rest frame,

$$\mathcal{A}^\mu \mathcal{A}_\mu = -\mathbf{a_0}^2 \tag{19}$$

Four momentum

$$p^\mu = m\,U^\mu = m\gamma(c, \mathbf{v}) = (\frac{E}{c}, \mathbf{p}) \tag{20}$$

with m as the (rest) mass of a moving body.

Relativistic energy

$$E = \gamma mc^2 \tag{21}$$

γm is sometimes referred to as the *relativistic mass* of the moving body.

3 Electrodynamics

Maxwell's equations

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E} &= \mu_0 \mathbf{j} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} &= 0
\end{aligned}
\tag{22}
$$

Maxwell's equations in covariant form

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= \mu_0 j^\nu \ , \quad \partial_\nu \equiv \frac{\partial}{\partial x^\nu} \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0 \ , \qquad \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}
\end{aligned}
\tag{23}
$$

Electromagnetic field tensor

$$
\begin{aligned}
F^{k0} &= \frac{1}{c}E_k \ , \quad F^{ij} = -\epsilon_{ijk} B_k \\
\tilde{F}^{k0} &= B_k \ , \quad \tilde{F}^{ij} = -\frac{1}{c}\epsilon_{ijk} E_k
\end{aligned}
\tag{24}
$$

Four-current density

$$
(j^\mu) = (c\rho, \mathbf{j})
\tag{25}
$$

Charge conservation

$$
\partial_\mu j^\mu = 0 \ , \quad \frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0
\tag{26}
$$

Electromagnetic potentials

$$
\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A} \ , \quad \mathbf{B} = \nabla \times \mathbf{A}
\tag{27}
$$

Four potential

$$
F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ , \quad (A^\mu) = (\frac{1}{c}\phi, \mathbf{A})
\tag{28}
$$

Lorentz force
Force from the electromagnetic field on a point particle with charge q

$$
\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\tag{29}
$$

Potentials from charge and current distributions
in Lorentz gauge, $\partial_\mu A^\mu = 0$:

$$
\begin{aligned}
\phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV' \\
\mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV'
\end{aligned}
\tag{30}
$$

Retarded time

$$t' = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'| \tag{31}$$

Electric dipole moment

$$\mathbf{p} = \int \mathbf{r}\rho(\mathbf{r})dV \tag{32}$$

Electric dipole potential (dipole at the origin)

$$\phi = \frac{\mathbf{n}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2}\ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{33}$$

Force and torque (about the origin)

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}\ , \quad \mathbf{M} = \mathbf{p}\times\mathbf{E} \tag{34}$$

Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2}\int \mathbf{r}\times\mathbf{j}(\mathbf{r})\ dV \tag{35}$$

Magnetic dipole potential (dipole at the origin)

$$\mathbf{A} = \frac{\mu_0\ \mathbf{m}\times\mathbf{n}}{4\pi r^2}\ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{36}$$

Force and torque (about the origin)

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})\ (\text{current loop})\ , \quad \mathbf{M} = \mathbf{m}\times\mathbf{B} \tag{37}$$

Lorentz transformation of the electromagnetic field

$$F'^{\mu\nu} = L^{\mu}_{\ \rho}L^{\nu}_{\ \sigma}F^{\rho\sigma} \tag{38}$$

Lorentz invariants

$$\begin{aligned} \mathbf{E}^2 - c^2\mathbf{B}^2 &= -\frac{c^2}{2}F_{\mu\nu}F^{\mu\nu} \\ \mathbf{E}\cdot\mathbf{B} &= \frac{c}{4}\tilde{F}_{\mu\nu}F^{\mu\nu} \end{aligned} \tag{39}$$

Special Lorentz transformations

$$\begin{aligned} \mathbf{E}'_{||} &= \mathbf{E}_{||}\ , \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}) \\ \mathbf{B}'_{||} &= \mathbf{B}_{||}\ , \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v}\times\mathbf{E}/c^2) \end{aligned} \tag{40}$$

The fields are decomposed in a parallel component ($||$), in the direction of transformation velocity $\mathbf{v}$, and a perpendicular component ($\perp$), orthogonal to $\mathbf{v}$.

Electromagnetic field energy density

$$u = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) = \frac{\epsilon_0}{2}(E^2 + c^2B^2) \tag{41}$$

Electromagnetic energy current density (Poynting's vector)

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \tag{42}$$

Monochromatic plane waves, plane polarized

$$\begin{aligned}
&\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \ , \quad \mathbf{E}_0 = E_0 \mathbf{e}_1 \\
&\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \, ; \quad \mathbf{B}_0 = B_0 \mathbf{e}_2 \\
&\mathbf{E}_0 \cdot \mathbf{k} = \mathbf{B}_0 \cdot \mathbf{k} = 0 \ , \quad \mathbf{B}_0 = \frac{1}{c}\mathbf{n} \times \mathbf{E}_0 \ , \quad \mathbf{n} = \frac{\mathbf{k}}{k}
\end{aligned} \tag{43}$$

Monochromatic plane waves, circular polarized

$$\begin{aligned}
&\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left(\mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]\right) \ , \quad \mathbf{E}_0 = E_0 \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2) \\
&\mathbf{B}(\mathbf{r}, t) = \mathrm{Re}\left(\mathbf{B}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]\right) \ , \quad \mathbf{B}_0 = B_0 \frac{1}{\sqrt{2}}(\mathbf{e}_2 \mp i\mathbf{e}_1)
\end{aligned} \tag{44}$$

Polarization vectors

$$\mathbf{e}_1 \cdot \mathbf{k} = \mathbf{e}_2 \cdot \mathbf{k} = 0 \ , \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \ , \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1 \tag{45}$$

Four-wave vector

$$(k^\mu) = (\frac{\omega}{c}, \mathbf{k}) \ , \quad \omega = ck \tag{46}$$

Radiation fields, in the wave zone ($r >> r', \lambda$)

$$\begin{aligned}
&\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c}\frac{\mathbf{n}}{r} \times \frac{d}{dt}\int \mathbf{j}(\mathbf{r}', t') dV' \ , \quad \mathbf{n} = \frac{\mathbf{r}}{r} \\
&\mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n}
\end{aligned} \tag{47}$$

Electric dipole radiation

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c}\frac{\mathbf{n}}{r} \times \ddot{\mathbf{p}}(t - r/c) \ , \quad \mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n} \tag{48}$$

Radiation from accelerated, charged particle

$$\begin{aligned}
&\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi c r}[\mathbf{a} \times \mathbf{n}]_{ret} \, , \quad \mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n}_{ret} \\
&\mathbf{n} = \mathbf{R}/R \, , \quad \mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t)
\end{aligned} \tag{49}$$

with $\mathbf{r}(t)$ as the particle's position vector.

Radiated power, Larmor's formula

$$P = \frac{\mu_0 q^2}{6\pi c}\mathbf{a}^2 \tag{50}$$