RELATIVISTIC KINEMATICS
FYS3120: PROBLEM SET 8

SEBASTIAN G. WINTHER-LARSEN

1. Mirror Mirror on the (Moving) Wall

A monocromatic light source is at rest in the laboratory and sends photons with frequency ν_0 towards a mirror which has its reflective surface perpendicular to the beam direction. The mirror moves away from the light source with velocity v. The transformation formula for four-momentum is given by $p^\mu = (E/c, \mathbf{p})$ and the Planck relation is $E = h\nu$.

1.a. **Light Frequency in Rest Frame of Mirror.** The relativistic energy of a moving particle is

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{1}$$

Because a photon is without mass, the energy of a photon according to the formula above is

$$E = pc, \tag{2}$$

which can be inserted into Planck relation yielding

$$p = \frac{h\nu_0}{c}. \tag{3}$$

This provides an expression for the four vector

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right) = \left(\frac{h\nu_0}{c}\right) = (p, p, 0, 0). \tag{4}$$

To get from emitted frequency ν_0 in lab reference frame S, to frequency ν in mirror reference frame S' one needs to take the Lorentz transform

$$p'^\mu = L^\mu{}_\rho p^\rho, \tag{5}$$

because the mirror reference frame is just a boost along the x-axis, relative to the lab reference frame.

$$\begin{pmatrix} p'^0 \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = \gamma(1-\beta) \begin{pmatrix} p \\ p \\ 0 \\ 0 \end{pmatrix}, \tag{6}$$

so

$$p' = \gamma(1-\beta)p. \tag{7}$$

Date: March 17, 2017.

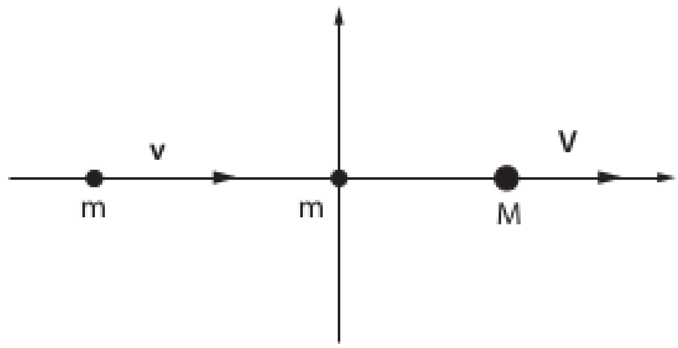

FIGURE 1. Collision between two particles of mass m resulting in a particle with mass M

The de Broglie relations gives

$$p = \frac{h}{\lambda} = \frac{h\nu}{c}, \tag{8}$$

so the frequency of the emitted and reflected light in the rest frame of the mirror must be

$$\nu' = \gamma(1-\beta)\nu. \tag{9}$$

The frequency of the emitted and reflected light must necessarily be the same, due to conservation of momentum.

1.b. **Frequency of Reflected Light in Lab System.** Denoting frequency of reflected light as ν_R and frequency of emitted light as ν_0, we already have that

$$\nu'_R = \gamma(1-\beta)\nu_0, \tag{10}$$

in the mirror rest frame. Similarly, the frequency of reflected light in laboratory rest frame is

$$\nu_R = \gamma(1-\beta)\nu'_R. \tag{11}$$

Inserting 10 into 11 yields

$$\nu_R = \gamma^2(1-\beta)^2\nu_0 = \frac{(1-\beta)^2}{1-\beta^2}\nu_0 = \frac{(1-\beta)^2}{(1+\beta)(1-\beta)}\nu_0 = \frac{1-\beta}{1+\beta}\nu_0 \tag{12}$$

2. Relativistic Collision

Figure 1 shows a particle with mass m and (relativistic) kinetic energy V in the laboratory frame S. The particles is moving towards another particle, with the same mass m, which is at rest in S.

2.a. **Velocity of First Particle.** Relativistic kinetic energy is given by

$$T = (\gamma - 1)mc^2. \tag{13}$$

Introducing the dimensionless quantity $\alpha = T/mc^2$,

$$\alpha = \frac{T}{mc^2} = \frac{(\gamma - 1)mc^2}{mc^2} = (\gamma - 1) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1$$

$$\alpha + 1 = \frac{1}{\sqrt{1 - \beta^2}}$$

$$1 - \beta^2 = \frac{1}{(\alpha + 1)^2}$$

$$\beta = \pm\sqrt{1 - \frac{1}{(\alpha + 1)^2}}$$

$$v = \pm c\sqrt{1 - (\alpha + 1)^{-2}},$$

yields an expression for the velocity of the moving particle.

2.b. **Compound Particle of Perfectly Inelastic Collision.** Assuming that the collision is completely inelastic, they will "stick together" after the collision, and form a new compounded particle. The momentum is conserved so that

$$\left(\frac{E_1}{c}, \mathbf{p}_1\right) + \left(\frac{E_2}{c}, \mathbf{p}_2\right) = \left(\frac{E_3}{c}, \mathbf{p}_3\right), \tag{14}$$

where $E_1 = \gamma(v_1)mc^2$, $E_2 = \gamma(v_2)mc^2$ and $E_3 = \gamma(v_3)mc^2$. The second particle is not moving relative to the reference frame. Since $v_2 = 0$, $\gamma(v_2) = 1$ and $\mathbf{p}_2 = \mathbf{0}$. Equation 14 now reads

$$(\gamma(v_1)mc + mc, \mathbf{p}_1) = (\gamma(v_3)Mc, \mathbf{p}_3). \tag{15}$$

Let's first look at the first elements of the four-vectors in equation 15:

$$(\gamma(v_1) + 1)mc = \gamma(v_3)Mc. \tag{16}$$

This yields the mass of the large particle as

$$M = \frac{\gamma(v_1) + 1}{\gamma(v_3)}m. \tag{17}$$

Of the three elements left in the four-vectors in equation 15 two are zero. What remains is

$$\gamma(v_1)mv_1 = \gamma(v_3)Mv_3, \tag{18}$$

inserting the mass from equation 17 gives

$$\gamma(v_1)mv_1 = (\gamma(v_1) + 1)mv_3. \tag{19}$$

Rearranging this expression gives the velocity of the compounded particle

$$v_3 = \frac{\gamma(v_1)}{\gamma(v_1) + 1}v_1 = \frac{\alpha}{\alpha + 1}v_1 \tag{20}$$