MIDTERM EXAM
FYS3120

CANDIDATE 15137

1. A (boring) Lagrangian

A non-relativistic particle (no-potential) of mass m is moving in three dimensions.

1.a.

1.b.

1.c.

1.d.

1.e.

1.f.

1.g. Consider a Lorentz transformation where the Lorentz transformation tensor is given as

$$L^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{1}$$

Any particular Lorentz transformation must leave the line element $ds^2 = dx_{\mu}dx^{\mu}$ invariant,

$$g_{\mu\nu}dx'^{\mu}dx'^{\nu} = g_{\mu\nu}L^{\mu}{}_{\rho}L^{\mu}{}_{\sigma}dx^{\rho}dx^{\sigma} = g_{\rho\sigma}dx^{\rho}dx^{\sigma}$$
$$g_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}_{\sigma} = g_{\rho\sigma}$$

To see if the Lorentz transformation in 1 is invariant is must statisfy this requirement

$$\begin{aligned}
g_{\rho\sigma} &= q_{\mu\nu}L^{\mu}{}_{\rho}L^{\nu}{}_{\sigma} \\
&= g_{\mu\nu}(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\
&= (\delta_{\nu\rho} + \omega_{\nu\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\
&= \delta_{\nu\rho}\delta^{\nu}{}_{\sigma} + \delta_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho}\delta^{\nu}{}_{\sigma} + \omega_{\nu\rho}\omega^{\nu}{}_{\sigma} \\
&= g_{\nu\rho}\delta^{\nu}{}_{\sigma} + g_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho}g^{\nu\gamma}g_{\gamma\sigma} + \omega^{2}_{\rho\sigma} \\
&= \delta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} = g_{\rho\sigma} + g_{\nu\rho}(\omega^{\nu}{}_{\sigma} + \omega_{\sigma}{}^{\nu}),
\end{aligned}$$

which only works if ω^{μ}_{ν} is antisymmetric, that is if $\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}$.

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1.h. A small Lorentz transformation between two reference frames changes the path $x^\mu(\tau)$ of a particle according to

$$\delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau) = \omega^\mu{}_\nu x^\nu(\tau). \tag{2}$$

This corresponds to a perturbation in the Lagrangian.

The variation of the Lagrangian is

$$\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial U^\mu}\delta U^\mu$$

inserting for $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ from equation 2 and

$$\delta U^\mu = \delta \frac{dx^\mu}{dt} = \frac{d}{d\tau}(\delta x^\mu) = \omega^\mu{}_\nu U^\nu$$

yields

$$\delta L = \left(\frac{\partial L}{\partial x^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}U^\nu\right)x^\mu{}_\nu, \tag{3}$$

which is the change in the Lagrangian as a consequence of the change in path.

1.i. The Euler-Lagrange equations states

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}\right) = \frac{\partial L}{\partial x^\mu}. \tag{4}$$

Inserting 4 into 3 gives

$$\delta L = \left(\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right) + \frac{\partial L}{\partial U^\mu}\frac{d}{d\tau}x^\nu\right)\omega^\mu{}_\nu \tag{5}$$

using the product rule for derivation backwards gives

$$\delta L = \frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu = \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \tag{6}$$

and finally "letting everything run it's course"

$$\begin{aligned}
\delta L &= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\omega^\mu{}_\nu - \frac{\delta L}{\delta U^\mu}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial g^{\rho\mu}U_\rho}x^\nu\omega^\mu{}_\nu - \frac{\delta L}{\delta g^{\sigma\mu}U_\sigma}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu g_{\rho\mu}\omega^\mu{}_\nu - \frac{\delta L}{\delta U_\sigma}x^\nu g_{\sigma\mu}\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu\omega_{\rho\nu} - \frac{\delta L}{\delta U_\sigma}x^\nu\omega_{\nu\sigma}\right)
\end{aligned}$$

changing indices back, writing μ instead of ρ, σ, and moving x^ν to the left of derivatives gives

$$\delta L = \frac{1}{2}\frac{d}{d\tau}\left(x^\nu \frac{\partial L}{\partial U_\mu}\omega_{\mu\nu} - x^\nu \frac{\delta L}{\delta U_\mu}\omega_{\nu\mu}\right).$$

Switch indices of first term inside the parenthesis[1], and one ends up with

$$\delta L = \frac{1}{2}\omega_{\nu\mu}\frac{d}{d\tau}\left(x^\mu \frac{\delta L}{\delta U_\nu} - x^\nu \frac{\partial L}{\partial U_\mu}\right) \tag{7}$$

2. Relativistics

3. Finding the Shortest Way

The shortest path between two points on a sphere. At some contstant radius r, some small movement in some direction on the sphere is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta\phi^2 \tag{8}$$

inserting for $d\phi = (d\phi/d\theta)d\theta = \dot{\phi}d\theta$ gives

$$ds = r\sqrt{1+\sin^2\theta\dot{\phi}^2}d\theta \tag{9}$$

A path is given by

$$S = \int ds = r\int_{\theta_A}^{\theta_B}\sqrt{1+\sin^2\theta\dot{\phi}^2}d\theta \tag{10}$$

where the integrand $F(\theta,\phi,\dot{\phi}) = \sqrt{1+\sin^2\theta\dot{\phi}^2}$ does not depend explicitly on ϕ. This implies that $\partial F/\partial\dot{\phi}$ is constant, yielding

$$\frac{\partial F}{\partial\dot{\phi}} = \frac{2\sin^2\theta\dot{\phi}}{\sqrt{1+\sin^2\theta\dot{\phi}^2}} = C' \to \frac{\sin^2\theta\dot{\phi}}{\sqrt{1+\sin^2\theta\dot{\phi}^2}} = C \tag{11}$$

This can be rearranged

$$\begin{aligned}
C^2 &= \frac{\sin^4\theta\dot{\phi}^2}{1+\sin^2\theta\dot{\theta}^2}\\
C^2 + C\sin^2\theta\dot{\phi}^2 &= \sin^4\theta\dot{\phi}^2\\
C^2 &= (\sin^4\theta - C\sin^2\theta)\dot{\phi}^2\\
\dot{\phi}^2 &= \frac{C^2}{(\sin^4\theta - C\sin^2\theta)}\\
\dot{\phi} &= \frac{C}{\sin\theta\sqrt{\sin^2 - C}}
\end{aligned}$$

INTEGRATE!!

[1]This is okay because if one were to move ∂U_μ up from underneath the dividing line the index μ would change to an upstairs variant. This is the same as saying $\sum_i\sum_j x^i \frac{\partial L}{\partial U_j}\omega_{ji} = \sum_j\sum_i x^i \frac{\partial L}{\partial U_i}\omega_{ij}$