HAMILTONIAN VENTURE
PROBLEM SHEET 4: FYS3120

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1. Constrained rod

Figure 1 shows a rod of lenth b and evenly distributed mass m. One endpoint of the rod is constrained to move along a horizontal line, and the other endpoint is constrained to move along a vertical line. The two lines are in the same plane. There is no friction and the acceleration due to gravity is g.

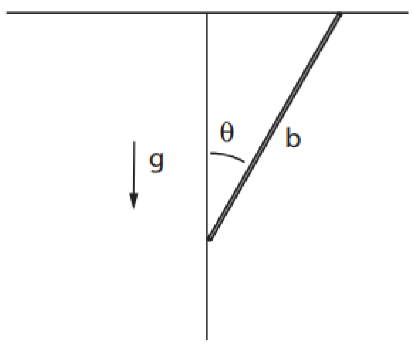

Figure 1. Constrained rod

1.a. **The Lagrangian and Lagrange's equation.** To fully describe the rod one needs one needs two translational coordinates and one rotational coordinate. The system has two constraints, so it is sufficient with one generalised coordinate, θ, as the system only has one degree of freedom. The position of the left and right endpoint of the rod in terms of θ is

$$(0, -b\cos\theta) \text{ and } (b\sin\theta, 0)$$

respectively. The position of the rods centre of mass must therefore be

$$\mathbf{r} = (\frac{b}{2}\sin\theta, -\frac{b}{2}\cos\theta). \tag{1}$$

It follows that

$$\dot{x} = \frac{b}{2}\dot{\theta}\cos\theta, \quad \dot{y} = \frac{b}{2}\dot{\theta}\sin\theta,$$

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which gives

$$\dot{x}+\dot{y}=\frac{b^2}{4}\dot{\theta}^2. \tag{2}$$

The kinetic energy is the sum of translational and rotational energy[1]

$$T=\frac{1}{2}m(\dot{x}+\dot{y})+\frac{1}{2}I\dot{\theta}^2=\frac{1}{2}m\frac{b^2}{4}\dot{\theta}^2+\frac{1}{2}\frac{mb^2}{12}=\frac{mb^2}{8}\dot{\theta}^2+\frac{mb^2}{24}\dot{\theta}^2=\frac{mb^2}{6}\dot{\theta}^2. \tag{3}$$

The potential energy is

$$V=mgy=-\frac{1}{2}mg\cos\theta. \tag{4}$$

The Lagrangian becomes

$$L=T-V=\frac{1}{6}mb^2\dot{\theta}^2+\frac{1}{2}mgb\cos\theta \tag{5}$$

Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0. \tag{6}$$

Each part can be computed separately

$$\begin{aligned}\frac{\partial L}{\partial\theta}&=-\frac{1}{2}mgb\sin\theta\\ \frac{\partial L}{\partial\dot{\theta}}&=\frac{1}{3}mb^2\dot{\theta}\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)&=\frac{1}{3}mb^2\ddot{\theta},\end{aligned}$$

and Lagrange's equation becomes

$$\frac{1}{3}mb^2\ddot{\theta}+\frac{1}{2}mgb\sin\theta=0\to\ddot{\theta}+\frac{3g}{2b}\sin\theta=0 \tag{7}$$

1.b. **Equilibrium of the rod.** As usual, the system will tend towards a configuration where the potential, V, is as low as possible. This point can be found by setting $\frac{\partial V}{\partial\theta}=0$, but it is easy to see that it must be when $\theta=0$.

When $\theta\to0$, then $\sin\theta\to\theta$. Inserting this small-angle approximation into the Lagrange equation yields

$$\ddot{\theta}+\frac{3g}{2b}\theta=0, \tag{8}$$

this equation corresponds to a harmonic oscillator with angular frequency $\omega=\sqrt{\frac{3g}{2b}}$. The period of an oscillation around the equilibrium orientation must then be

$$T_0=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{2b}{3g}} \tag{9}$$

[1] The moment of intertia for a rod rotating around its centre of mass is $I=\frac{mL^2}{12}$.

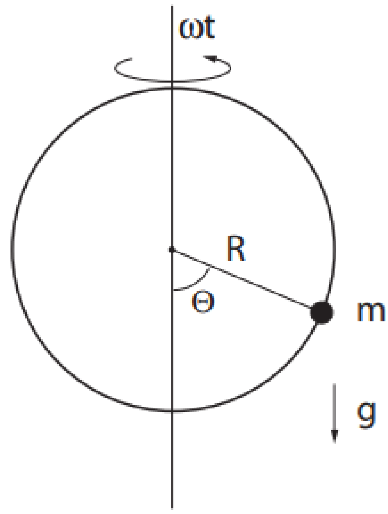

FIGURE 2. Rotating pendulum

2. Rotating pendulum