

ELECTRODYNAMICS

FYS3120: PROBLEM SET 11

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1. SIMPLE LAGRANGIAN DYNAMICS

A non-relativistic particle, with electric charge q and mass m moves in a magnetic dipole field, given by the vector potential

$$(1) \quad \vec{\mathbf{A}} = \frac{\mu_0}{4\pi r^3} (\vec{\mu} \times \vec{\mathbf{r}}),$$

where $\vec{\mu}$ is the magnetic dipole moment of a static charge distribution centered at the origin.

1.a. **Lagrangian.** The Lagrangian is given by

$$(2) \quad L = T + q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}}.$$

The kinetic energy is simply $T = \frac{1}{2}m\vec{\mathbf{v}}^2$ while the potential is

$$\begin{aligned} q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}} &= \frac{q\mu_0}{4\pi r^3} \vec{\mathbf{v}} \cdot (\vec{\mu} \times \vec{\mathbf{r}}) \\ &= \frac{q\mu_0}{4\pi r^3} \vec{\mu} \cdot (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \\ &= \frac{q\mu_0}{4\pi m r^3} \vec{\mu} \cdot \vec{\ell}, \end{aligned}$$

using the cyclic invariance of the vector triple product and $\vec{\ell} = m\vec{\mathbf{r}} \times \vec{\mathbf{v}}$. Inserting the parts into 2 the Lagrangian becomes

$$(3) \quad L = \frac{1}{2}m\vec{\mathbf{v}}^2 + \frac{q\mu_0}{4\pi m r^3} \vec{\mu} \cdot \vec{\ell}.$$

1.b. **Alternative Lagrangian.** We now make the assumption that the magnetic dipole moment is oriented along the z -axis and that the particle moves in the (x, y) -plane. In the following, $r = |\vec{\mathbf{r}}|$ and the angle ϕ between the x -axis and the position vector var are chosen as generalised coordinates.

With the magnetic dipole moment oriented along the z -axis,

$$\vec{\mu} \cdot \vec{\ell} = |\vec{\mu}|\ell_z = |\vec{\mu}|(\vec{\mathbf{r}} \times \vec{\mathbf{p}})_z = |\vec{\mu}|m(x\dot{y} - y\dot{x}),$$

where $x = r \cos \phi$ and $y = r \sin \phi$. This gives

$$\begin{aligned} x\dot{y} - y\dot{x} &= r \cos \phi (\dot{r} \sin \phi + r \dot{\phi} \cos \phi) \\ &\quad - r \sin \phi (\dot{r} \cos \phi - r \dot{\phi} \sin \phi) \\ &= r^2 \dot{\phi} \cos^2 \phi + r^2 \dot{\phi} \sin^2 \phi = r^2 \dot{\phi}, \end{aligned}$$

similarly

$$\begin{aligned} \dot{x} &= \dot{r} \cos \phi - r \dot{\phi} \sin \phi \\ \dot{y} &= \dot{r} \sin \phi + r \dot{\phi} \cos \phi \\ \dot{x}^2 &= \dot{r}^2 \cos^2 \phi - 2r\dot{r}\dot{\phi} \cos \phi \sin \phi + r^2 \dot{\phi}^2 \sin^2 \phi \\ \dot{y}^2 &= \dot{r}^2 \sin^2 \phi + 2r\dot{r}\dot{\phi} \cos \phi \sin \phi + r^2 \dot{\phi}^2 \cos^2 \phi \\ \dot{v}^2 &= \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\phi}^2. \end{aligned}$$

The Lagrangian with generalised coordinates becomes

$$(4) \quad L = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{q\mu_0}{4\pi m r^3} |\vec{\mu}| m r^2 \dot{\phi} = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\phi}^2) + \lambda \frac{\dot{\phi}}{r},$$

where $\lambda \equiv q\mu_0 |\vec{\mu}| / 4\pi$.

The canonical momentum p_ϕ conjugate to ϕ becomes

$$(5) \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} + \frac{\lambda}{r}$$

ϕ is a cyclic coordinate, because the Lagrangian in equation 4 does not explicitly depend on ϕ . This implies that the conjugate momentum p_ϕ is constant.

The Lagrangian in equation 4 does not depend explicitly on time t . This means that the Hamiltonian must be conserved

$$(6) \quad H = \dot{r}p_r + \dot{\phi}p_\phi - L = m\dot{r}^2 + m r^2 \dot{\phi}^2 + \lambda \frac{\dot{\phi}}{r} - L = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\phi}^2) = T.$$

Since the Hamiltonian equals the kinetic energy and the Hamiltonian is conserved, the kinetic energy is conserved by the magnetic field.

1.c. Kinetic Energy Conservation. Lagrange's equation for r is

$$(7) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \dot{p}_r - \frac{\partial L}{\partial r} = m\ddot{r} - m r \dot{\phi}^2 + \lambda \frac{\dot{\phi}}{r^2} = 0.$$

Here one can eliminate $\dot{\phi}$ by inserting $\dot{\phi} = \frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3}$ found from equation 5. This yields

$$\begin{aligned}
 & m\ddot{r} - mr \left(\frac{p_\phi}{mr^2} + \frac{\lambda}{mr^3} \right) - \frac{\lambda}{r^2} \left(\frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3} \right) \\
 &= m\ddot{r} - mr \left(\frac{p_\phi^2}{m^2 r^4} - \frac{2p_\phi \lambda}{m^2 r^5} + \frac{\lambda^2}{m^2 r^6} \right) + \frac{p_\phi \lambda}{mr^4} - \frac{\lambda^2}{mr^5} \\
 &= m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{2p_\phi \lambda}{mr^4} - \frac{\lambda^2}{mr^5} + \frac{p_\phi \lambda}{mr^4} - \frac{\lambda^2}{mr^5} = 0 \\
 (8) \quad & \rightarrow m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{3p_\phi \lambda}{mr^4} - \frac{2\lambda^2}{mr^5} = 0.
 \end{aligned}$$

We are interested in the behaviour of \dot{r}^2 one can multiply the expression in 8 with \dot{r} . This gives

$$(9) \quad m\ddot{r}\dot{r} = \frac{p_\phi^2}{mr^3}\dot{r} - \frac{3p_\phi \lambda}{mr^4}\dot{r} + \frac{2\lambda^2}{mr^5}\dot{r}$$

using $\dot{r}\ddot{r} = \frac{1}{2} \frac{d}{dt}(\dot{r}^2)$ and $\dot{r}dt = \frac{dr}{dt}dt = dr$

$$\begin{aligned}
 \frac{1}{2}m \frac{d}{dt}(\dot{r}^2) &= \left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi \lambda}{mr^4} + \frac{2\lambda^2}{mr^5} \right) \dot{r} \\
 d(\dot{r}^2) &= \frac{2}{m} \left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi \lambda}{mr^4} + \frac{2\lambda^2}{mr^5} \right) dr
 \end{aligned}$$

now to integrate from r_0 to $r(t)$

$$\begin{aligned}
 \dot{r}(t)^2 - \dot{r}(0)^2 &= \frac{2}{m} \int_{r(0)}^{r(t)} \left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi \lambda}{mr^4} + \frac{2\lambda^2}{mr^5} \right) dr \\
 &= -\frac{p_\phi^2}{m^2} (r^{-2} - r_0^{-2}) + \frac{2p_\phi \lambda}{m^2} (r^{-3} - r_0^{-3}) \\
 &\quad - \frac{\lambda^2}{m^2} (r^{-4} - r_0^{-4}) \\
 &= \frac{1}{m^2 r_0^2} \left(p_\phi - \frac{\lambda}{r_0} \right)^2 - \frac{1}{m^2 r^2} \left(p_\phi - \frac{\lambda}{r} \right)
 \end{aligned}$$

from equation 5 we have $\dot{\phi}mr^2 = (p_\phi - \frac{\lambda}{r})$, inserting in the expression above gives

$$\dot{r}^2 - \dot{r}_0^2 = r_0^2 \dot{\phi}_0^2 - r^2 \dot{\phi}^2$$

which can be rearranged to

$$(10) \quad \dot{r}^2 + r^2 \dot{\phi}^2 = \dot{r}_0^2 + r_0^2 \dot{\phi}_0^2.$$

We see again that the kinetic energy is conserved.

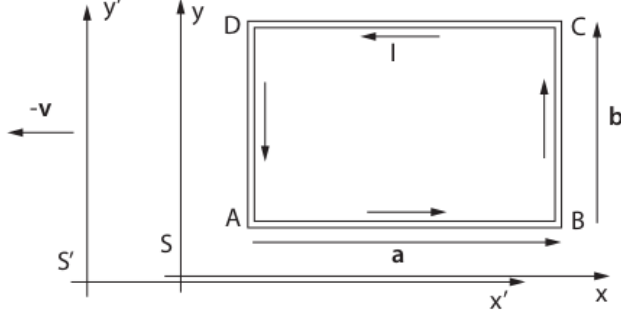


FIGURE 1. Illustration of current loop.

2. RECTANGULAR CURRENT LOOP

Figure 1 shows a rectangular current loop ABCD. In the loop's rest frame, S , the loop has length a in the x -direction and width b in y -direction. The current is I and the charge density is zero. The electric dipole moment $\vec{\mathbf{p}}$ and the magnetic dipole moment $\vec{\mathbf{m}}$ for a given current distribution is defined by the following

$$(11) \quad \vec{\mathbf{p}} = \int \vec{\mathbf{r}} \rho(\vec{\mathbf{r}}) d^3r, \quad \vec{\mathbf{m}} = \frac{1}{2} \int (\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}})) d^3r$$

2.a. Electric and Magnetic Dipole Moment in S . Since the charge density in rest frame S is zero, $\rho(\vec{\mathbf{r}}) = 0$, the electric dipole moment must also be zero, $\vec{\mathbf{p}} = 0$.

The current along every edge of the rectangle will be $j\vec{\mathbf{n}}$, where $\vec{\mathbf{n}}$ is a unit vector pointing along the edge in question.

$$\begin{aligned} \text{AB} : \vec{\mathbf{j}} &= j\vec{\mathbf{e}}_x & \text{BC} : \vec{\mathbf{j}} &= j\vec{\mathbf{e}}_y \\ \text{CD} : \vec{\mathbf{j}} &= -j\vec{\mathbf{e}}_x & \text{DA} : \vec{\mathbf{j}} &= -j\vec{\mathbf{e}}_y. \end{aligned}$$

Given a point $\vec{\mathbf{r}}$ along the AB segment,

$$(12) \quad \vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_x) = -yj\vec{\mathbf{e}}_z,$$

along the BC segment,

$$(13) \quad \vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_y) = xj\vec{\mathbf{e}}_z,$$

along the CD segment,

$$(14) \quad \vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_x) = yj\vec{\mathbf{e}}_z,$$

and along the DA segment

$$(15) \quad \vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_y) = -xj\vec{\mathbf{e}}_z,$$

It is a reasonable approximation to use the factor Δ , which is the cross-sectional area of the current wire, instead of integrating in directions perpendicular to the direction of the conductor. Then the coordinate \vec{r} is simply the centre of the conductor. Employing these assumptions/approximations and assigning the lower left corner of the rectangle coordinates (x_0, y_0) and using the results from equations 12 13, 14 and 15 the magnetic dipole moment is

$$\begin{aligned}\vec{m} &= \frac{1}{2}j\Delta\vec{e}_z \left(-\int_{x_0}^{y_0+a} y_0 dx + \int_{y_0}^{y_0+b} (x_0+a) dy + \int_{x_0}^{x_0+a} (y_0+b) dx - \int_{y_0}^{y_0+b} x_0 dy \right) \\ &= \frac{1}{2}I(-y_0a + (x_0+a)b + (y_0+b)a - x_0b)\vec{e}_z \\ &= abI\vec{e}_z.\end{aligned}$$

Since \vec{a} and \vec{b} are orthogonal $\vec{a} \times \vec{b} = ab\vec{e}_z$, which gives

$$(16) \quad \vec{m} = I\vec{a} \times \vec{b}.$$

2.b. Length Contraction. Reference frame S' moves at velocity \vec{v} to the right, away from the rest frame S . Because the width of the loop is orthogonal to the boost, it remains the same, $b' = b$.

We have the following Lorentz transformations between the positions of the endpoints of the rectangle in the two reference frames

$$\begin{aligned}x_a &= \gamma(-vt'_a + x'_a) & x'_a &= \gamma(vt_a + x_a) \\ x_b &= \gamma(-vt'_b + x'_b) & x'_b &= \gamma(vt_b + x_b).\end{aligned}$$

The two endpoints must be measured at the same time in reference frame S' to compute the correct length

$$\begin{aligned}a &= x_b - x_a = \gamma(x'_b - x'_a) + \cancel{\gamma v(t'_a - t'_b)} \\ &= \gamma(x'_b - x'_a) = \gamma a'\end{aligned}$$

which yields

$$(17) \quad a = \frac{1}{\gamma}a'.$$

2.c. Charge of Segments AB and CD. In reference frame S , the segment AB will have current four-vector $J^\mu = (0, j, 0, 0)$. Lorentz transform to reference frame S' , which has velocity $-v$ relative to S gives

$$\rho'c = J'^0 = L^0_\nu J^\nu = \gamma(J^0 + \beta J^1) = \beta\gamma J,$$

where $\beta = v/c$. Assuming uniform charge density of the conductor segment gives

$$\rho' = \frac{J'^0}{c} = \frac{j\gamma v}{c^2}.$$

To find the total charge one needs simply to multiply with the volume of this conductor segment in reference frame S' , $|\vec{a}|\Delta = (1/\gamma)a\Delta$

$$(18) \quad Q'_{AB} = V'_{AB}\rho'_{AB} = \frac{1}{\gamma}a\Delta j\gamma\frac{v}{c^2} = Ia\frac{v}{c^2}.$$

A similar computation can be made for conductor segment CD, with current four-vector $J^\mu = (0, -j, 0, 0)$.

$$\begin{aligned} \rho'c &= J'^0 = L^0_\nu J^\nu = \gamma(J^0 + \beta J^1) = -\gamma\beta j \\ \rightarrow \rho' &= \frac{J'^0}{c} = -\frac{\gamma\beta j}{c} = -\frac{\gamma v j}{c^2} \end{aligned}$$

$$(19) \quad Q'_{CD} = V'_{CD}\rho'_{CD} = -\frac{1}{\gamma}a\Delta\gamma j\frac{v}{c^2} = -Ia\frac{v}{c^2}$$

2.d. Electric and Magnetic Dipole Moment in S' . Segments AB and CD always have charge densities $\rho = \pm \frac{I}{\Delta} \frac{v}{c^2}$. Inserting this into the \vec{p} from 11 and assuming a thin conductor by replacing the cross-sectional integration dimensions with Δ gives

$$\begin{aligned} \vec{p}' &= \int \vec{r}\rho(\vec{r})d^3r \\ &= \frac{I}{\Delta} \frac{v}{c^2} \left[\Delta \int_{x_0}^{x_0+a} (x\vec{e}_x + y_0\vec{e}_y)dx - \Delta \int_{x_0}^{x_0+a} (x\vec{e}_x + (y_0+b)\vec{e}_y)dx \right] \\ &= I \frac{v}{c^2} \left(y_0x \Big|_{x_0}^{x_0+a} - (y_0+b)x \Big|_{x_0}^{x_0+a} \right) \vec{e}_y \\ &= I \frac{v}{c^2} (-b(x_0+a-x_0)) \vec{e}_y \\ &= -I \frac{v}{c^2} ab\vec{e}_y \end{aligned}$$

Moreover

$$\begin{aligned} -\frac{1}{c^2} \vec{m} \times \vec{v} &= -\frac{1}{c^2} (abI\vec{e}_z) \times (v\vec{e}_x) \\ &= -\frac{v}{c^2} Iab(\vec{e}_z \times \vec{e}_x) \\ &= -I \frac{v}{c^2} ab\vec{e}_y, \end{aligned}$$

which implies that

$$(20) \quad \vec{p}' = -\frac{1}{c^2} \vec{m} \times \vec{v}$$

In order to calculate the magnetic dipole moment, one needs the current densities for all conductor segments.

$$\begin{aligned} \text{AB: } J &= (0, j, 0, 0) \quad j' = J'^1 = \beta\gamma J^0 + \gamma J^1 = \gamma j \\ \text{CD: } J &= (0, -j, 0, 0) \quad j' = J'^1 = \beta\gamma J^0 + \gamma J^1 = -\gamma j. \end{aligned}$$

Segments AB and CD both have current densities $j' = \gamma j$ in x -direction. Segments BA and DA have current density j unchanged. However, the width of these conductors are Lorentz-contracted, meaning that the area is reduced by a factor γ^{-1} , so that $\Delta' = \Delta/\gamma$. Now to compute the magnetic dipole moment in the same manner as before

$$\begin{aligned}
 \vec{\mathbf{m}}' &= \frac{1}{2} \vec{\mathbf{e}}_z \left[\gamma j \Delta \int_{x'_0}^{x'_0+a'} (y'_0 + b - y'_0) dx' j \frac{\Delta}{\gamma} \int_{y'_0}^{y'_0+b'} (x'_0 + a - x'_0) \right] \\
 &= \frac{1}{2} j \Delta ab (1 + \gamma^{-2}) \vec{\mathbf{e}}_z = \frac{1}{2} I ab (2 - \beta^2) \vec{\mathbf{e}}_z = I ab \left(1 - \frac{\beta^2}{2}\right) \vec{\mathbf{e}}_z \\
 (21) \quad \vec{\mathbf{m}}' &= \left(1 - \frac{\beta^2}{2}\right) \vec{\mathbf{m}}
 \end{aligned}$$

2.e. Current in the Different Segments. The current density is $j' = \gamma j$ in AB and CD, while the cross-sectional area is unchanged. This makes the current

$$(22) \quad I' = \Delta j^1 = \gamma \Delta j = \gamma I.$$

For BC and DA $j' = j$, but $\Delta' = \Delta/\gamma$, so

$$(23) \quad I' = \Delta' j = \frac{\Delta}{\gamma} j = \frac{I}{\gamma}$$