FYS3120 Classical Mechanics and Electrodynamics

Problem set 11

April 24, 2017

Problem 1 A non-relativistic particle, with electric charge q and mass m moves in a magnetic dipole field, given by the vector potential

$$\vec{A} = \frac{\mu_0}{4\pi r^3}(\vec{\mu} \times \vec{r}), \tag{1}$$

where $\vec{\mu}$ is the magnetic dipole moment of a static charge distribution centered at the origin. (We use the notation $\vec{\mu}$ for the dipole moment to avoid confusion with the particle mass m).

a) Show that the Lagrangian is

$$L = \frac{1}{2}m\vec{v}^2 + \frac{q\mu_0}{4\pi m r^3}\,\vec{\mu}\cdot\vec{\ell}, \tag{2}$$

where $\vec{\ell} = m\,\vec{r} \times \vec{v}$ is the particle's orbital angular momentum.

We now make the assumption that the magnetic dipole moment is oriented along the z-axis and that the particle moves in the (x, y)-plane. Choose in the following $r = |\vec{r}|$ and the angle ϕ between the x-axis and the position vector $\vec{r}$ as coordinates.

b) Show that the Lagrangian of the particle, when expressed in terms of r, ϕ, and their time derivatives, takes the form

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \lambda\frac{\dot{\phi}}{r}, \tag{3}$$

with $\lambda \equiv q\mu_0|\vec{\mu}|/4\pi$. Find the canonical momentum p_ϕ conjugate to ϕ, and give a physical interpretation of this quantity. Also comment on the consequence of L having no explicit time dependence.

c) Write Lagrange's equation for the coordinate r, expressed in terms of r, $\ddot{r}$ and p_ϕ, and use the equation to find $\dot{r}^2$ as a function of r and p_ϕ. Compare the expression with that of the particle's kinetic energy.

Problem 2 Figure 1 shows a rectangular current loop ABCD. In the loop's rest frame, S, the loop has length a in the x direction and width b in the y direction, the current is I and the charge density is zero. We remind you of the following general definitions of the electric dipole moment $\vec{p}$, and the magnetic dipole moment $\vec{m}$, for a given current distribution:

$$\vec{p} = \int \vec{r}\rho(\vec{r})\,d^3r\,, \quad \vec{m} = \frac{1}{2}\int (\vec{r} \times \vec{j}(\vec{r}))\,d^3r. \tag{4}$$

a) Show that in the rest frame the loop's electric dipole moment is zero and the magnetic moment is $\vec{m} = I\vec{a} \times \vec{b}$, where $I = j\Delta$ with j as the current density and Δ as the cross section area of the current wire.

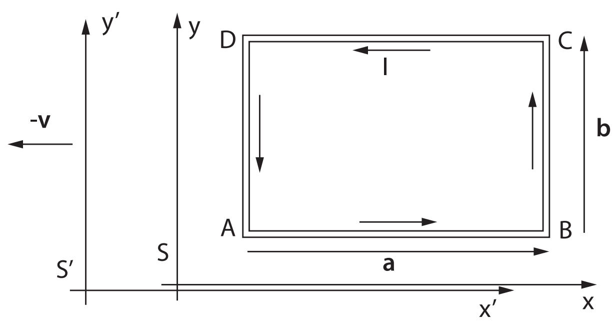

Figure 1: Illustration of current loop.

In the following we will examine how the loop is observed in a reference frame S', where the loop is moving with velocity $\vec{v}$ to the right ($\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$). The Lorentz transformation formulas for charge and current denisities may be useful when solving the problems below.

b) What is the length and width of the loop in S'?

c) Show that the parts AB and CD of the loop have charge $\pm aIv/c^2$ in S'.

d) Show that in S' the loop's electric dipole moment is $\vec{p}' = -\frac{1}{c^2}\vec{m} \times \vec{v}$, and the magnetic dipole moment is $\vec{m}' = (1 - \beta^2/2)\vec{m}$.

e) Show that the current is $I\gamma$ in the AB and CD and I/γ in BC and DA.

f) Show that the result in **e)** is consistent with charge conservation.

Problem 3 An electric point charge q is moving with constant velocity $\vec{v}$ along the x-axis of the inertial frame S, as illustrated in Fig. 2. Assume it passes the origin of S at $t = 0$.

a) Give the expression for the scalar potential ϕ' and the vector potential $\vec{A}'$ set up by the charge in its rest frame S'. In the relativistic description the scalar and vector potentials define the four potential A'^{μ}, with the time component related to the scalar potential as $A'^{\mu} = \phi'/c$. Make use of the transformation properties of the four potential to determine its components A^{μ} in reference frame S as functions of the coordinates (ct, x, y, z) in the same frame.

b) Determine (the components of) the electric field $\vec{E}$ in the reference frame S, as functions of (ct, x, y, z).

c) Determine similarly the magnetic field $\vec{B}$ in reference frame S.

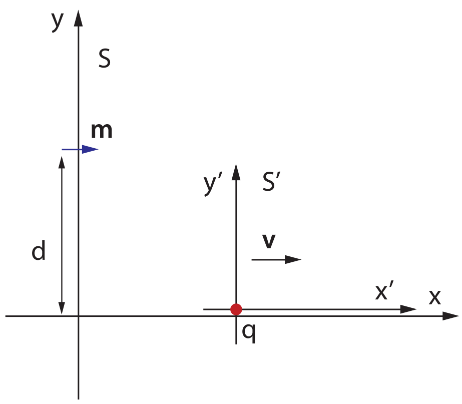

Figure 2: Charge on the move.

A magnetic dipole, with dipole moment $\vec{m}$, is at rest in S, at the position $(x, y, z) = (0, d, 0)$. The dipole vector $\vec{m}$ points in the x-direction.

d) The field from the moving charge acts with a time dependent torque on the dipole, $\vec{M} = \vec{m} \times \vec{B}$. Find the expression for the torque.

e) Assuming the magnetic dipole can be viewed as a small current loop, the force on the dipole from the field produced by the moving charge is $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$. Determine the force.