MIDTERM EXAM
FYS3120

CANDIDATE 15137

1. A (boring) Lagrangian

A non-relativistic particle (no-potential) of mass m is moving in three dimensions. black!20gree

1.a.

1.b.

1.c.

1.d.

1.e.

1.f.

1.g. Consider a Lorentz transformation where the Lorentz transformation tensor is given as

$$L^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{1}$$

Any particular Lorentz transformation must leave the line element $ds^2 = dx_\mu dx^\mu$ invariant,

$$\begin{aligned} g_{\mu\nu} dx'^{\mu} dx'^{\nu} = g_{\mu\nu} L^{\mu}{}_{\rho} L^{\mu}{}_{\sigma} dx^{\rho} dx^{\sigma} = g_{\rho\sigma} dx^{\rho} dx^{\sigma} \\ g_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}_{\sigma} = g_{\rho\sigma} \end{aligned}$$

To see if the Lorentz transformation in 1 is invariant is must statisfy this requirement

$$\begin{aligned} g_{\rho\sigma} &= q_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} \\ &= g_{\mu\nu} (\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= (\delta_{\nu\rho} + \omega_{\nu\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= \delta_{\nu\rho}\delta^{\nu}{}_{\sigma} + \delta_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho}\delta^{\nu}{}_{\sigma} + \omega_{\nu\rho}\omega^{\nu}{}_{\sigma} \\ &= g_{\nu\rho}\delta^{\nu}{}_{\sigma} + g_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho} g^{\nu\gamma} g_{\gamma\sigma} + \omega^2_{\rho\sigma} \\ &= \delta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} = g_{\rho\sigma} + g_{\nu\rho}(\omega^{\nu}{}_{\sigma} + \omega_{\sigma}{}^{\nu}), \end{aligned}$$

which only works if ω^{μ}_{ν} is antisymmetric, that is if $\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}$.

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1.h. A small Lorentz transformation between two reference frames changes the path $x^\mu(\tau)$ of a particle according to

$$\delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau) = \omega^\mu{}_\nu x^\nu(\tau). \tag{2}$$

This corresponds to a perturbation in the Lagrangian.

The variation of the Lagrangian is

$$\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial U^\mu}\delta U^\mu$$

inserting for $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ from equation 2 and

$$\delta U^\mu = \delta \frac{dx^\mu}{dt} = \frac{d}{d\tau}(\delta x^\mu) = \omega^\mu{}_\nu U^\nu$$

yields

$$\delta L = \left(\frac{\partial L}{\partial x^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}U^\nu\right) x^\mu{}_\nu, \tag{3}$$

which is the change in the Lagrangian as a consequence of the change in path.

1.i. The Euler-Lagrange equations states

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}\right) = \frac{\partial L}{\partial x^\mu}. \tag{4}$$

Inserting 4 into 3 gives

$$\delta L = \left(\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right) + \frac{\partial L}{\partial U^\mu}\frac{d}{d\tau}x^\nu\right)\omega^\mu{}_\nu \tag{5}$$

using the product rule for derivation backwards gives

$$\delta L = \frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu = \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \tag{6}$$

and finally "letting everything run it's course"

$$\begin{aligned}
\delta L &= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\omega^\mu{}_\nu - \frac{\delta L}{\delta U^\mu}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial g^{\rho\mu}U_\rho}x^\nu\omega^\mu{}_\nu - \frac{\delta L}{\delta g^{\sigma\mu}U_\sigma}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu g_{\rho\mu}\omega^\mu{}_\nu - \frac{\delta L}{\delta U_\sigma}x^\nu g_{\sigma\mu}\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu\omega_{\rho\nu} - \frac{\delta L}{\delta U_\sigma}x^\nu\omega_{\nu\sigma}\right)
\end{aligned}$$

changing indices back, writing μ instead of ρ, σ, and moving x^ν to the left of derivatives gives

$$\delta L = \frac{1}{2}\frac{d}{d\tau}\left(x^\nu \frac{\partial L}{\partial U_\mu}\omega_{\mu\nu} - x^\nu \frac{\delta L}{\delta U_\mu}\omega_{\nu\mu}\right).$$

Switch indices of first term inside the parenthesis[1], and one ends up with an alternative expression for δL,

$$\delta L = \frac{1}{2}\omega_{\nu\mu}\frac{d}{d\tau}\left(x^\mu \frac{\delta L}{\delta U_\nu} - x^\nu \frac{\partial L}{\partial U_\mu}\right) \tag{7}$$

2. Relativistics

Two particles with mass m and a photon is sent out from a source at the same time and in the positive x-direction in rest frame S of the source. The massive particles are moving with constant velocity v_1 and $v_2 > v_1$ in this frame. Figure 1 shows a Minkowski space-time diagram of the two particles, the photon and the source in the rest frame of the source S and that of the slowest of the particles S'.

For an infinitesimal change in position coordinates we have

$$\begin{aligned} dx' &= \gamma(dx - v_1 dt) = \gamma(u_2 - v_1)dt \\ dt' &= \gamma(dt - \frac{v_1}{c^2}dx) = \gamma(1 - \frac{v_2 v_1}{c^2}) \end{aligned}$$

and from this follows that

$$v_2' = \frac{dx'}{dt'} = \frac{v_2 - v_1}{1 - \frac{v_2 v_1}{c^2}} \tag{8}$$

The difference in rapidity of the two massive particles in the two different rest frames are

$$S: \quad \Delta\chi = \tanh^{-1}\left(\frac{v_2}{c}\right) - \tanh^{-1}\left(\frac{v_1}{c}\right) \tag{9}$$

$$S': \quad \Delta\chi' = \tanh^{-1}\left(\frac{v_2'}{c}\right) - \tanh^{-1}\left(\frac{v_1'}{c}\right) = \tanh^{-1}\left(\frac{v_2'}{c}\right) \tag{10}$$

Rapidity differences should be unchanged by boosts no matter the reference frames, so

$$\begin{aligned} \tanh^{-1}\left(\frac{v_2}{c}\right) - \tanh^{-1}\left(\frac{v_1}{c}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right) \\ \tanh^{-1}\left(\frac{\frac{v_2}{c} - \frac{v_1}{c}}{1 - \frac{v_2 v_1}{c^2}}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right) \\ \tanh^{-1}\left(\frac{1}{c}\frac{v_2 - v_1}{1 - \frac{v_2 v_1}{c^2}}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right), \end{aligned}$$

[1]This is okay because if one were to move ∂U_μ up from underneath the dividing line the index μ would change to an upstairs variant. This is the same as saying $\sum_i \sum_j x^i \frac{\partial L}{\partial U_j}\omega_{ji} = \sum_j \sum_i x^i \frac{\partial L}{\partial U_i}\omega_{ij}$

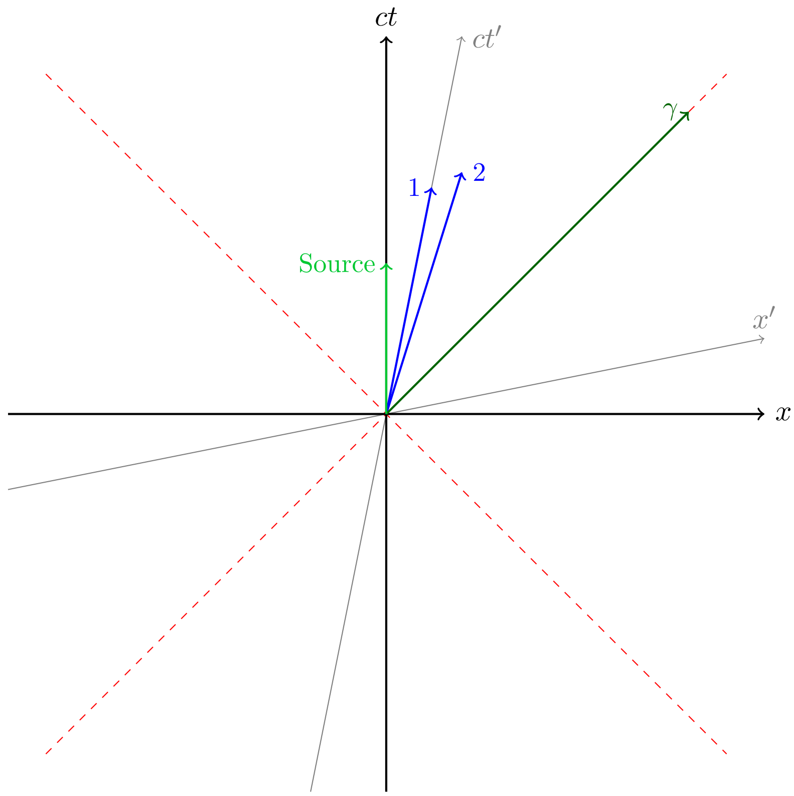

FIGURE 1. Minkowski space-time diagram of two massive particles (velocities v_1 and $v_2 > v_1$) and a photon (γ) sent out from a source at origin in rest frame S. Rest frame S' is that of particle 1.

inserting 8 gives

$$\tanh^{-}1\left(\frac{v_2'}{c}\right) = \tanh^{-}1\left(\frac{v_2'}{c}\right)$$
$$\chi = \chi'.$$

In conclusion, the rapidity difference is the same in the two rest frames S and S.

3. Finding the Shortest Way

The shortest path between two points on a sphere. At some contstant radius r, some small movement in some direction on the sphere is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta\phi^2 \tag{11}$$

inserting for $d\phi = (d\phi/d\theta)d\theta = \dot{\phi}d\theta$ gives

$$ds = r\sqrt{1 + \sin^2\theta\dot{\phi}^2}d\theta \tag{12}$$

A path is given by

$$S = \int ds = r \int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta \dot{\phi}^2} d\theta \tag{13}$$

where the integrand $F(\theta, \phi, \dot{\phi}) = \sqrt{1 + \sin^2\theta \dot{\phi}^2}$ does not depend explicitly on ϕ. This implies that $\partial F / \partial \dot{\phi}$ is constant, yielding

$$\frac{\partial F}{\partial \dot{\phi}} = \frac{2\sin^2\theta \dot{\phi}}{\sqrt{1 + \sin^2\theta \dot{\phi}^2}} = C' \rightarrow \frac{\sin^2\theta \dot{\phi}}{\sqrt{1 + \sin^2\theta \dot{\phi}^2}} = C \tag{14}$$

This can be rearranged

$$\begin{aligned}
C^2 &= \frac{\sin^4\theta \dot{\phi}^2}{1 + \sin^2\theta \dot{\theta}^2} \\
C^2 + C\sin^2\theta \dot{\phi}^2 &= \sin^4\theta \dot{\phi}^2 \\
C^2 &= (\sin^4\theta - C\sin^2\theta)\dot{\phi}^2 \\
\dot{\phi}^2 &= \frac{C^2}{(\sin^4\theta - C\sin^2\theta)} \\
\dot{\phi} &= \frac{C}{\sin\theta\sqrt{\sin^2 - C}}
\end{aligned}$$

INTEGRATE!!