MIDTERM EXAM
FYS3120

CANDIDATE 15137

1. A (boring) Lagrangian

A non-relativistic particle (no-potential) of mass m is moving in three dimensions.

1.a.

1.b.

1.c.

1.d.

1.e.

1.f.

1.g. Consider a Lorentz transformation where the Lorentz transformation tensor is given as

$$L^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{1}$$

Any particular Lorentz transformation must leave the line element $ds^2 = dx_\mu dx^\mu$ invariant,

$$\begin{aligned} g_{\mu\nu} dx'^{\mu} dx'^{\nu} = g_{\mu\nu} L^{\mu}{}_{\rho} L^{\mu}{}_{\sigma} dx^{\rho} dx^{\sigma} &= g_{\rho\sigma} dx^{\rho} dx^{\sigma} \\ g_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}_{\sigma} &= g_{\rho\sigma} \end{aligned}$$

To see if the Lorentz transformation in 1 is invariant is must statisfy this requirement

$$\begin{aligned} g_{\rho\sigma} &= q_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} \\ &= g_{\mu\nu} (\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= (\delta_{\nu\rho} + \omega_{\nu\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= \delta_{\nu\rho}\delta^{\nu}{}_{\sigma} + \delta_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho}\delta^{\nu}{}_{\sigma} + \omega_{\nu\rho}\omega^{\nu}{}_{\sigma} \\ &= g_{\nu\rho}\delta^{\nu}{}_{\sigma} + g_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho} g^{\nu\gamma} g_{\gamma\sigma} + \omega^2_{\rho\sigma} \\ &= \delta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} = g_{\rho\sigma} + g_{\nu\rho}(\omega^{\nu}{}_{\sigma} + \omega_{\sigma}{}^{\nu}), \end{aligned}$$

which only works if ω^{μ}_{ν} is antisymmetric, that is if $\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}$.

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1.h. A small Lorentz transformation between two reference frames changes the path $x^\mu(\tau)$ of a particle according to

$$\delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau) = \omega^\mu{}_\nu x^\nu(\tau). \tag{2}$$

This corresponds to a perturbation in the Lagrangian.

The variation of the Lagrangian is

$$\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial U^\mu}\delta U^\mu$$

inserting for $\delta x^\mu = \omega^\mu{}_\nu x^\nu$ from equation 2 and

$$\delta U^\mu = \delta \frac{dx^\mu}{dt} = \frac{d}{d\tau}(\delta x^\mu) = \omega^\mu{}_\nu U^\nu$$

yields

$$\delta L = \left(\frac{\partial L}{\partial x^\mu} x^\nu + \frac{\partial L}{\partial U^\mu} U^\nu \right) x^\mu{}_\nu, \tag{3}$$

which is the change in the Lagrangian as a consequence of the change in path.

2. Relativistics

3. Finding the Shortest Way

The shortest path between two points on a sphere. At some contstant radius r, some small movement in some direction on the sphere is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta \phi^2 \tag{4}$$

inserting for $d\phi = (d\phi/d\theta)d\theta = \dot{\phi} d\theta$ gives

$$ds = r\sqrt{1 + \sin^2\theta\dot{\phi}^2}d\theta \tag{5}$$

A path is given by

$$S = \int ds = r\int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta\dot{\phi}^2}d\theta \tag{6}$$

where the integrand $F(\theta, \phi, \dot{\phi}) = \sqrt{1 + \sin^2\theta\dot{\phi}^2}$ does not depend explicitly on ϕ. This implies that $\partial F/\partial\dot{\phi}$ is constant, yielding

$$\frac{\partial F}{\partial \dot{\phi}} = \frac{2\sin^2\theta\dot{\phi}}{\sqrt{1 + \sin^2\theta\dot{\phi}^2}} = C' \to \frac{\sin^2\theta\dot{\phi}}{\sqrt{1 + \sin^2\theta\dot{\phi}^2}} = C \tag{7}$$

This can be rearranged

$$\begin{aligned}
C^2 &= \frac{\sin^4\theta\dot{\phi}^2}{1+\sin^2\theta\dot{\theta}^2}\\
C^2 + C\sin^2\theta\dot{\phi}^2 &= \sin^4\theta\dot{\phi}^2\\
C^2 &= (\sin^4\theta - C\sin^2\theta)\dot{\phi}^2\\
\dot{\phi}^2 &= \frac{C^2}{(\sin^4\theta - C\sin^2\theta)}\\
\dot{\phi} &= \frac{C}{\sin\theta\sqrt{\sin^2 - C}}
\end{aligned}$$

INTEGRATE!!