FYS3120 Classical Mechanics and Electrodynamics

Problem set 4

February 9, 2017

Problem 1 The figure shows a rod of length b and mass m, with the mass evenly distributed along the rod. One endpoint of the rod is constrained to move along a horisontal line and the other endpoint along a vertical line. The two lines are in the same plane. There is no friction and the acceleration due to gravity is g. The set-up is illustrated in Fig. 1.

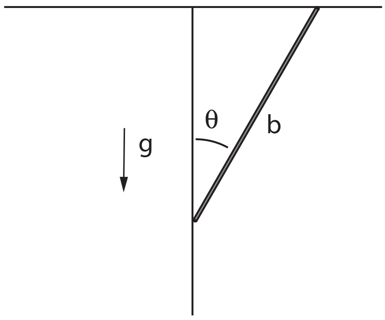

Figure 1: Constrained rod.

a) Find the Lagrangian L with the angle θ as coordinate, and show that Lagrange's equation gives

$$\ddot{\theta} + \frac{3g}{2b} \sin\theta = 0. \tag{1}$$

Hint: For the moment of inertia of the rod, see Problem 2 in Set 2.

b) What is the stable equilibrium position of the rod? Find the period T_0 for small oscillations about equilibrium.

c) Since L has no explicit time dependence, there is a corresponding constant of motion. What is the expression for this constant and what is the physical interpretation? Comment on how the expression is related to the equation of motion.

d) Assume the rod oscillates about the equilibrium position with a maximum angle θ_0, with $0 < \theta_0 \leq \pi/2$. Show that the period T of the oscillations is generally expressed by the integral

$$T = T_0 \frac{\sqrt{2}}{\pi} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \tag{2}$$

Determine the ratio T/T_0 for the maximum amplitude $\theta_0 = \pi/2$. *Hint:* In Rottman you will find a general formula, which can be used to express the integral in terms of the Euler gamma-functions.

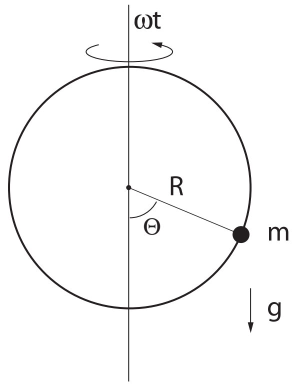

Figure 2: Rotating pendulum.

Problem 2 Midterm Exam 2014

Note: The following is really a repeat of Problem 2 in Set 3, but now with the effect of gravity included.

A circular hoop is rotating with constant angular velocity ω around a symmetry axis with vertical orientation, as shown in Fig. 2. Inside the hoop a planar pendulum can perform free oscillations, while the plane of the pendulum rotates with the hoop. The mass of the pendulum bob is m, the length of the pendulum rod is R and the gravitational acceleration is g. The pendulum rod is considered massless. As generalized coordinate we use the angle θ of the pendulum relative to the vertical axis.

a) Express the Cartesian coordinates of the pendulum bob as functions of θ and ω. Find the Lagrangian of the pendulum, and show that it can be written in the form

$$L = \frac{1}{2} m R^2 \dot{\theta}^2 - W(\theta), \tag{3}$$

with $W(\theta)$ as an effective potential. $W(\theta)$ has an extra contribution in addition to the potential energy due to gravity. Do you have a physical interpretation for this term?

b) Derive Lagrange's equation for the system, and find the oscillation frequency Ω for small oscillations about the equilibrium point $\theta = 0$.

c) Show that $\theta = 0$ is a *stable* equilibrium only for $\omega < \omega_{cr}$ and determine ω_{cr}. Show that for $\omega > \omega_{cr}$ there are two new equilibrium points $\theta_\pm \neq 0, \pi$ and determine these points as functions of ω.

d) Show that $\theta_\pm$ are points of *stable* equilibrium (for $\omega > \omega_{cr}$). Make a plot of the function $W(\theta)$, for two values for ω, one smaller and one larger than ω_{cr} (for example $\omega = 0.5\,\omega_{cr}$ and $\omega = 1.5\,\omega_{cr}$).

e) Find the Hamiltonian H of the system as function of θ and its conjugate momentum p_θ, and explain why H is a constant of motion. (A complete proof is not needed.)

f) The Hamiltonian $H(\theta, p_\theta)$ can be considered as a potential function in phase space, with θ and p_θ as coordinates. Make a two-dimensional phase-space plot, which shows the equipotential lines of $H(\theta, p_\theta)$, for the case $\omega = 1.5\,\omega_{cr}$. Use a convenient choice of scales of the two coordinate axes.

Give a short description of the different types of motion shown by the plot, and indicate in the diagram the location of the *separatrices*, which are the curves that separate the different types of motion. Indicate also the direction of motion of the system in the diagram.