RELATIVISTIC KINEMATICS

FYS3120: PROBLEM SET 8

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1. Mirror Mirror on the (Moving) Wall

A monocromatic light source is at rest in the laboratory and sends photons with frequency ν_0 towards a mirror which has its reflective surface perpendicular to the beam direction. The mirror moves away from the light source with velocity v. The transformation formula for four-momentum is given by $p^\mu = (E/c, \mathbf{p})$ and the Planck relation is $E = h\nu$.

1.a. **Light Frequency in Rest Frame of Mirror.** The relativistic energy of a moving particle is

$$E = \sqrt{p^2c^2 + m^2c^4}. \tag{1}$$

Because a photon is without mass, the energy of a photon according to the formula above is

$$E = pc, \tag{2}$$

which can be inserted into Planck relation yielding

$$p = \frac{h\nu_0}{c}. \tag{3}$$

This provides an expression for the four vector

$$p^\mu = \left(\frac{E}{c}, \mathbf{p}\right) = \left(\frac{h\nu_0}{c}\right) = (p, p, 0, 0). \tag{4}$$

To get from emitted frequency ν_0 in lab reference frame S, to frequency ν in mirror reference frame S' one needs to take the Lorentz transform

$$p'^\mu = L^\mu{}_\rho p^\rho, \tag{5}$$

because the mirror reference frame is just a boost along the x-axis, relative to the lab reference frame.

$$\begin{pmatrix} p'^0 \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = \gamma(1-\beta) \begin{pmatrix} p \\ p \\ 0 \\ 0 \end{pmatrix}, \tag{6}$$

so

$$p' = \gamma(1-\beta)p. \tag{7}$$

Date: March 20, 2017.

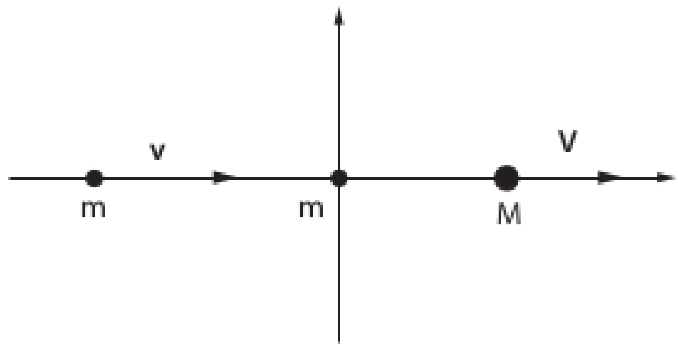

FIGURE 1. Collision between two particles of mass m resulting in a particle with mass M

The de Broglie relations gives

$$p = \frac{h}{\lambda} = \frac{h\nu}{c}, \tag{8}$$

so the frequency of the emitted and reflected light in the rest frame of the mirror must be

$$\nu' = \gamma(1-\beta)\nu. \tag{9}$$

The frequency of the emitted and reflected light must necessarily be the same, due to conservation of momentum.

1.b. **Frequency of Reflected Light in Lab System.** Denoting frequency of reflected light as ν_R and frequency of emitted light as ν_0, we already have that

$$\nu'_R = \gamma(1-\beta)\nu_0, \tag{10}$$

in the mirror rest frame. Similarly, the frequency of reflected light in laboratory rest frame is

$$\nu_R = \gamma(1-\beta)\nu'_R. \tag{11}$$

Inserting 10 into 11 yields

$$\nu_R = \gamma^2(1-\beta)^2\nu_0 = \frac{(1-\beta)^2}{1-\beta^2}\nu_0 = \frac{(1-\beta)^2}{(1+\beta)(1-\beta)}\nu_0 = \frac{1-\beta}{1+\beta}\nu_0 \tag{12}$$

2. Relativistic Collision

Figure 1 shows a particle with mass m and (relativistic) kinetic energy V in the laboratory frame S. The particles is moving towards another particle, with the same mass m, which is at rest in S.

2.a. **Velocity of First Particle.** Relativistic kinetic energy is given by

$$T = (\gamma - 1)mc^2. \tag{13}$$

Introducing the dimensionless quantity $\alpha = T/mc^2$,

$$\begin{aligned}
\alpha = \frac{T}{mc^2} = \frac{(\gamma-1)mc^2}{mc^2} = (\gamma - 1) &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \\
\alpha + 1 &= \frac{1}{\sqrt{1-\beta^2}} \\
1 - \beta^2 &= \frac{1}{(\alpha+1)^2} \\
\beta &= \pm\sqrt{1 - \frac{1}{(\alpha+1)^2}} \\
v &= \pm c\sqrt{1-(\alpha+1)^{-2}},
\end{aligned}$$

yields an expression for the velocity of the moving particle.

2.b. **Compound Particle of Perfectly Inelastic Collision.** Assuming that the collision is completely inelastic, they will "stick together" after the collision, and form a new compounded particle. The momentum is conserved so that

$$\left(\frac{E_1}{c}, \mathbf{p}_1\right) + \left(\frac{E_2}{c}, \mathbf{p}_2\right) = \left(\frac{E_3}{c}, \mathbf{p}_3\right), \tag{14}$$

where $E_1 = \gamma(v_1)mc^2$ and $E_2 = \gamma(v_2)mc^2$. Since $v_2 = 0$, $\gamma(v_2) = 1$ and $\mathbf{p}_2 = \mathbf{0}$. This gives a relation between the time elements of the four-momenta, which yields the energy of the compounded particle.

$$\begin{aligned}
\frac{E_1 + E_2}{c} &= \frac{E_3}{c} \\
E_1 + E_2 &= E_3 \\
E = E_3 = \gamma mc^2 + mc^2 &= (1+\gamma)mc^2. \qquad (15)
\end{aligned}$$

Similarly, the momentum of the compounded particle must be

$$P = \mathbf{p}_3 = \mathbf{p}_1 = \gamma m \mathbf{v}_1 = \gamma m v \tag{16}$$

To find the mass M of the compounded particle, one can use the energy-momentum relations

$$\begin{aligned} E^2 &= (pc)^2 + (Mc^2)^2 \\ \to M^2c^4 &= E^2 - p^2c^2 = (1+\gamma)^2m^2c^4 - \gamma^2m^2v^2c^2 \\ M^2 &= (1+2\gamma+\gamma^2)m^2 - \gamma^2m^2v^2c^{-2} \\ M^2 &= \left[2\gamma + 1 + \gamma^2\left(1 - \frac{v^2}{c^2}\right)\right] \\ M^2 &= m^2(2\gamma+2) = m^2 2(\gamma+1) \end{aligned}$$

$$M = m\sqrt{2(\gamma+1)}. \tag{17}$$

Notice that in the non-relativistic case, where $v << c$, $\gamma \to 1$ and $M = 2m$.

The velocity V of the compounded particle can be found by the inverse of the specific Lorentz factor

$$\gamma_V = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}, \;\to\; V = c\sqrt{1 - \gamma_V^2}. \tag{18}$$

Now to find $\gamma(V)$ expressed in some other way. The goal is to find the velocity V as a function of variables that describe the first particle. There are two expressions for the energy of the compounded particle

$$\begin{aligned} E &= \gamma_V Mc^2 \\ E &= (\gamma+1)mc^2, \end{aligned}$$

combining these, and inserting the expression for M from equation 17 yields

$$\gamma_V m(2(\gamma+1)) = (\gamma+1)mc^2 \;\to\; \gamma_V = \frac{\gamma+1}{\sqrt{2(\gamma+1)}}. \tag{19}$$

This can now be inserted into 18

$$V = c\sqrt{1 - \gamma_V^{-2}} = c\sqrt{1 - \frac{2}{\gamma+1}} = c\sqrt{\frac{\gamma-1}{\gamma+1}}. \tag{20}$$

This result is a somewhat complicated function of v. One way to get a clearer picture of how the velocity of the compounded particle depends on the velocity of the first particle in the non-relativistic limit would be to do some sort of series expansion[1].

[1] After a bit of dabbling with Symbolic Python, it appears that $V \approx \frac{1}{2}v$ in the non-relativistic limit.

The initial kinetic energy of the system is $T_1 = (\gamma - 1)mc^2$ and the final kinetic energy is $T_2 = E - Mc^2$. The change in kinetic energy is

$$\begin{aligned}\Delta T = T_1 - T_2 &= (\gamma - 1)mc^2 - (E - Mc^2) \\ &= (\gamma - 1)mc^2 - (\gamma + 1)mc^2 + Mc^2 \\ &= (M - 2m)c^2 \\ &= (\sqrt{2(\gamma + 1)} - 2)mc^2 \end{aligned} \tag{21}$$

2.c. **Elastic Collision.** In the rest of this exercise the assumption will be that the particles collide elastically instead. This means that the particles don't form a compounded particle after the collision, but remain the same with no change in their masses. The collision happens in such a way that the particles after the collision make the same angle, θ, with the x-axis in the lab frame S.

The momentum of the first particle after the collision will be

$$\mathbf{p}_1 = p_1(\cos\theta, \sin\theta), \tag{22}$$

and the momentum of the second particle after the collision will be

$$\mathbf{p}_2 = p_2(\cos\theta, -\sin\theta). \tag{23}$$

Conservation of momentum gives

$$(p_1, 0) = ((p_1 + p_2)\cos\theta,\ (p_1 - p2)\sin\theta). \tag{24}$$

The second component gives

$$0 = (p_1 - p_2)\sin\theta \to \begin{cases} p_1 = p_2 \\ \sin\theta = 0 \end{cases} \tag{25}$$

of which the first case seems more likely, as $\sin\theta = 0$ would imply no angle between the particles at all. Therefore, the magnitude of the momenta of the two particles after the collision must be the same.

The momenta of the particles is the same after the collision,

$$\gamma_1 m v_1 = \gamma_2 m v_2. \tag{26}$$

This means that both velocities and Lorentz factors are the same, ergo

$$E_1 = \gamma_1 mc^2 = \gamma_2 mc^2 = E_2. \tag{27}$$

2.d. **Energy and Momentum.** The energy of the two particles (equation 15) should be the same as the combined energy of the two particles, due to conservation of four-momentum.

$$E_{Compound} = 2E = (\gamma + 1)mc^2 \to E = \frac{1}{2}(\gamma + 1)mc^2. \tag{28}$$

Making sure the three other elements of the four-momentum vectors are conserved as well gives the momentum of the two particles after the collision

$$\mathbf{p}_1 = (2\cos\theta, 0) \to \gamma m v = 2p\cos\theta \to p = \frac{\gamma m v}{2\cos\theta} \tag{29}$$

2.e. **Collision Angle.** Starting with the Energy-momentum relation

$$E^2 = pc^2 + m^2c^4,$$

inserting energy from 28 and momentum from 29 gives

$$\begin{aligned}
\frac{1}{4}(\gamma+1)^2 m^2c^4 &= \frac{\gamma^2 m^2 v^2 c^2}{4\cos^2\theta} + m^2c^4 \\
\frac{\gamma^2 m^2 v^2 c^2}{\cos^2\theta} &= (\gamma+1)^2 m^2 c^4 - 4m^2c^4 \\
\frac{\gamma^2 m^2 v^2 c^2}{\cos^2\theta} &= ((\gamma+1)-4)m^2c^4 \\
\cos^2\theta &= \frac{v^2}{c^2}\frac{\gamma^2}{(\gamma+1)^2-4},
\end{aligned}$$

inserting expressions for $\beta = v/c$ and α from section 2.a gives

$$\begin{aligned}
\cos^2\theta &= \left(1 - \frac{1}{(\alpha+1)^2}\right)\frac{(\alpha+1)^2}{(\alpha+2)^2-4} \\
&= \frac{(\alpha+1)^2-1}{(\alpha+2)^2-4} = \frac{\alpha^2+2\alpha}{\alpha^2+4\alpha} = \frac{\alpha+2}{\alpha+4} \\
\rightarrow \theta &= \arccos\sqrt{\frac{\alpha+2}{\alpha+4}}.
\end{aligned}$$

And lastly,

$$\lim_{\alpha\to 0}\theta = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4} \tag{30}$$

3. The Final Frontier

The federation starship USS Enterprise NCC-1701 leaves Earth at $t = 0$. I heads for the planet Vulcan, $d = 4.2$ light years away. initial velocity is $v = 0$ relative to Earth. During the first part of the journey, the space ship has constant acceleration $a = g = 9.81ms^{-2}$. The second part of the journey, starting at time τ_0, measured in proper time of the Enterprise, acceleration is reversed. The Enterprise reaches Vulcan, with final velocity $v = 0$. After picking up Ambassador Spock, the USS Enterprise returns in the same fashion, which can be considered part three and four of the journey. The time spent on Vulcan is negligible.

During the first part of the journey (part one), the coordinates of the federation starship Enterprise in the Earth fixed reference frame S, define a hyperbolic space-time orbit

$$x - x_I = \frac{c^2}{a}\cosh\left(\frac{a}{c}(\tau-\tau_I)\right), \quad t - t_I = \frac{c}{a}\sinh\left(\frac{a}{c}(\tau-\tau_I)\right), \tag{31}$$

where x_I, t_I and τ_I are constants. Departure space-time event is at $x = t = \tau = 0$.

3.a. **Proper Time and Acceleration.** Proper time is characterised by the relation $ds^2 = -c^2 d\tau^2$. Now to check if τ is indeed the proper time of the starship Enterprise,

$$\begin{aligned} dx &= \frac{a}{c}\frac{c^2}{a}\sinh\left(\frac{a}{c}(\tau - \tau_I)\right) d\tau = c\sinh\left(\frac{a}{c}(\tau - \tau_I)\right) d\tau \\ dt &= \cosh\left((\tau - \tau_I)\right) d\tau, \end{aligned}$$

now

$$\begin{aligned} ds^2 &= dx^2 - d(ct)^2 = dx^2 - c^2 dt^2 \\ &= c^2 \sinh^2\left(\frac{a}{c}(\tau - \tau_I)\right) d\tau^2 - c^2 \cosh^2\left(\frac{a}{c}(\tau - \tau_I)\right) d\tau^2 \\ &= -c^2 d\tau^2, \end{aligned}$$

which means that τ satisfies the condition for proper time.

Now to find the four-velocity as a function of proper time τ. Four-velocity is given by

$$\mathbf{U} = \frac{dx}{d\tau} = \frac{d}{d\tau}(ct, x, y, z), \tag{32}$$

where the elements are

$$\begin{aligned} \frac{d}{d\tau}ct &= c\frac{d}{dt}t = c\cosh\left(\frac{a}{c}(\tau - \tau_I)\right) \\ \frac{d}{d\tau}x &= c\sinh\left(\frac{a}{c}(\tau - \tau_I)\right), \end{aligned}$$

which gives

$$\mathbf{U} = c(\cosh\left(\frac{a}{c}(\tau - \tau_I)\right),\ \sinh\left(\frac{a}{c}(\tau - \tau_I)\right),\ 0,\ 0). \tag{33}$$

The four-acceleration is found in a similar manner

$$\mathbf{A} = \frac{d\mathbf{U}}{d\tau}, \tag{34}$$

where the components are

$$\begin{aligned} A^0 &= c\frac{a}{c}\sinh\left(\frac{a}{c}(\tau - \tau_I)\right) = a\sinh\left(\frac{a}{c}(\tau - \tau_I)\right) \\ A^1 &= c\frac{a}{c}\cosh\left(\frac{a}{c}(\tau - \tau_I)\right) = a\cosh\left(\frac{a}{c}(\tau - \tau_I)\right), \end{aligned}$$

giving

$$\mathbf{A} = a(\sinh\left(\frac{a}{c}(\tau - \tau_I)\right),\ \cosh\left(\frac{a}{c}(\tau - \tau_I)\right),\ 0,\ 0) \tag{35}$$

Now to check that the acceleration defined by the path in equations 31, is a. This is quite straight-forward,

$$\begin{aligned} (a^0)^2 &= (A^0)^2 - (A^0)^2 \\ &= a^2 \cosh^2\left(\frac{a}{c}(\tau - \tau_I)\right) - \sinh^2\left(\frac{a}{c}(\tau - \tau_I)\right) = a^2. \end{aligned}$$

$a^0 = a$ as expected.

The space-time-path described by equations 31 is hyperbolic because the equation for the distance between two space-time events described by these equation will from a hyperbola,

$$\Delta s^2 = (x - x_I)^2 - c^2(t - t_I) = \frac{c^4}{a^2}. \tag{36}$$

3.b. **Constants for the Journey.** We must have that $x(\tau = 0) = t(\tau = 0) = 0$ and $v(t = 0) = 0$, this gives

$$x_I = -\frac{c^2}{a}\cosh\left(\frac{a}{c}(-\tau_I)\right) \tag{37}$$

$$t_I = -\frac{c}{a}\sinh\left(\frac{a}{c}(-\tau_I)\right) \tag{38}$$

$$v = \frac{dx}{dt} = \frac{c\sinh\left(\frac{a}{c}(\tau - \tau_I)\right)d\tau}{\cosh\left(\frac{a}{c}(\tau - \tau_I)\right)d\tau}$$

$$v = c\tanh\left(\frac{a}{c}(\tau - \tau_I)\right). \tag{39}$$

Using equation 39,

$$\begin{aligned} v(t = 0) = c\tanh\left(\frac{a}{c}(-\tau_I)\right) &= -c\tanh\left(\tau_I\frac{a}{c}\right) = 0 \\ &\to \tau_I = \frac{c}{s}\arctan(0) = 0 \end{aligned} \tag{40}$$

this can the be inserted into 37 and 38 to get the constants of the space-time path

$$x_I = -\frac{c^2}{a}\cosh\left(\frac{a}{c}(-\tau_I)\right) = -\frac{c^2}{a}\cosh\left(\tau_I\frac{a}{c})\right) = -\frac{c^2}{a} \tag{41}$$

$$t_I = -\frac{c}{a}\sinh\left(\frac{a}{c}(-\tau_I)\right) = -\frac{c^2}{a}\sinh\left(-\tau_I\frac{a}{c})\right) = 0, \tag{42}$$

this in turn gives

$$x = \frac{c^2}{g}\cosh\left(\frac{g}{c}\tau\right) - \frac{c^2}{g} \tag{43}$$

$$t = \frac{c}{g}\sinh\left(\frac{g}{c}\tau\right) \tag{44}$$

See figure 2 for a truly stunning illustration of the space-time path of the starship Enterpise in a Minkowski diagram.

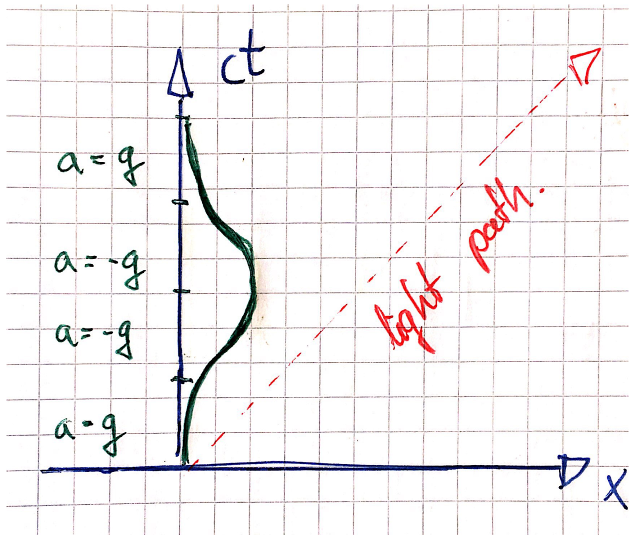

FIGURE 2. The space-time path of the federation starship Enterprise.

3.c. **The Time of the Journey.** τ_0 is at $x = \frac{d}{2}$, this gives

$$\begin{aligned}
\frac{d}{2} &= \frac{c^2}{g} \cosh\left(\frac{g}{c}\tau_0\right) \\
\cosh\left(\frac{g}{c}\tau_0\right) &= \frac{g}{c^2}\left(\frac{d}{2} + \frac{c^2}{g}\right) \\
\tau_0 &= \frac{c}{g} \cosh^{-1}\left(\frac{gd}{2c2} + 1\right) \\
&= \frac{3 \times 10^8}{9.81} \cosh^{-1}\left(\frac{9.81 \cdot 4.2 \cdot 9.5 \times 10^15}{2 \cdot (3 \times 10^8)^2} + 1\right) \\
&= \frac{3 \times 10^8}{9.81} \cdot 1.82 \approx 1.76 \text{ years.}
\end{aligned}$$

The total time aboard the Enterprise is

$$\tau = 4\tau_0 = 7 \text{ years}, \tag{45}$$

and the total time on Earth is

$$t = 4\frac{c}{g} \sinh\left(\frac{g}{c}\tau_0\right) = 11.6 \text{ years}, \tag{46}$$

which is a bit ridiculous, because warp drive was already invented by the time the Enterprise was build. I guess it is being propelled by impulse power only.

3.d. **The Speed of the Starship.** The maximum speed of the federation starship enterprise measured from earth can be found from

$$v_{\max} = c \tanh\left(\frac{g}{c}\tau_0\right) = 0.95c \tag{47}$$