ELECTRODYNAMICS
FYS3120: PROBLEM SET 11

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1. Simple Lagrangian Dynamics

A non-relativistic particle, with electric charge q and mass m moves in a magnetic dipole field, given by the vector potential

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi r^3}(\vec{\mu} \times \vec{\mathbf{r}}), \tag{1}$$

where $\vec{\mu}$ is the magnetic dipole moment of a static charge distribution centered at the origin.

1.a. **Lagrangian.** The Lagrangian is given by

$$L = T + q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}}. \tag{2}$$

The kinetic energy is simply $T = \frac{1}{2}m\vec{\mathbf{v}}^2$ while the potential is

$$\begin{aligned} q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}} &= \frac{q\mu_0}{4\pi r^3}\vec{\mathbf{v}} \cdot (\vec{\mu} \times \vec{\mathbf{r}}) \\ &= \frac{q\mu_0}{4\pi r^3}\vec{\mu} \cdot (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \\ &= \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{\ell}, \end{aligned}$$

using the cyclic invariance of the vector triple product and $\vec{\ell} = m\vec{\mathbf{r}} \times \vec{\mathbf{v}}$. Inserting the parts into 2 the Lagrangican becomes

$$L = \frac{1}{2}m\vec{\mathbf{v}}^2 + \frac{q\mu_0}{4\pi m r^3}\vec{\mu} \cdot \vec{\ell}. \tag{3}$$

1.b. **Alternative Lagrangian.** We now make the assumption that the magnetic dipole moment is oriented along the z-axis and that the particle moves in the (x, y)-plane. In the following, $r = |\vec{\mathbf{r}}|$ and the angle ϕ between the x-axis and the position vector var are chosen as generalised coordinates.

With the magnetic dipole moment oriented along the z-axis,

$$\vec{\mu} \cdot \vec{\ell} = |\vec{\mu}|\ell_z = |\vec{\mu}|(\vec{\mathbf{r}} \times \vec{\mathbf{p}})_z = |\vec{\mu}|m(x\dot{y} - y\dot{x}),$$

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where $x = r\cos\phi$ and $y = r\sin\phi$. This gives

$$
\begin{aligned}
x\dot{y} - y\dot{x} &= r\cos\phi(\dot{r}\sin\phi + r\dot{\phi}\cos\phi) \\
&\quad - r\sin\phi(\dot{r}\cos\phi - r\dot{\phi}\sin\phi) \\
&= r^2\phi\cos^2\phi + r^2\phi\sin^2\phi = r^2\phi,
\end{aligned}
$$

similarly

$$
\begin{aligned}
\dot{x} &= \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\
\dot{y} &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\
\dot{x}^2 &= \dot{r}^2\cos^2\phi - 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\sin^2\phi \\
\dot{y}^2 &= \dot{r}^2\sin^2\phi + 2r\dot{r}\dot{\phi}\cos\phi\sin\phi + r^2\dot{\phi}^2\cos^2\phi \\
\dot{v}^2 &= \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2.
\end{aligned}
$$

The Lagrangian with generalised coordinates becomes

$$
L = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\phi}^2) + \frac{q\mu_0}{4\pi m r^3}|\vec{\mu}|mr^2\dot{\phi} = \frac{1}{2}m(\dot{r}^2 + \dot{r}^2\dot{\phi}^2) + \lambda\frac{\dot{\phi}}{r}, \tag{4}
$$

where $\lambda \equiv q\mu_0|\vec{\mu}|/4\pi$.

The canonical momentum p_ϕ conjugate to ϕ becomes

$$
p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\dot{\phi} + \frac{\lambda}{r} \tag{5}
$$

ϕ is a cyclic coordinate, because the Lagrangian in equation 4 does not explicityly depend on ϕ. This implies that the conjugate momentum p_ϕ is constant.

The Lagrangian in equation 4 does not depend exlicitly on time t. This means that the Hamiltonian must be conserved

$$
H = \dot{r}p_r + \dot{\phi}p_\phi - L = m\dot{r}^2 + mr^2\dot{\phi}^2 + \lambda\frac{\dot{\phi}}{r} - L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = T. \tag{6}
$$

Since the Hamiltonian equals the kinetic energy and the Hamiltonian is conserved, the kinetic energy is conserved by the magnetic field.

1.c. **Kinetic Energy Conservation.** Lagrange's equation for r is

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = \dot{p}_r - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\phi}^2 + \lambda\frac{\dot{\phi}}{r^2} = 0. \tag{7}
$$

Here one can eliminate $\dot{\phi}$ by inserting $\dot{\phi} = \frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3}$ found from equation 5. This yields

$$\begin{aligned}
&m\ddot{r} - mr\left(\frac{p_\phi}{mr^2} + \frac{\lambda}{mr^3}\right) - \frac{\lambda}{r^2}\left(\frac{p_\phi}{mr^2} - \frac{\lambda}{mr^3}\right)\\
=&m\ddot{r} - mr\left(\frac{p_\phi^2}{m^2r^4} - \frac{2p_\phi\lambda}{m^2r^5} + \frac{\lambda^2}{m^2r^6}\right) + \frac{p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5}\\
=&m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{2p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} + \frac{p_\phi\lambda}{mr^4} - \frac{\lambda^2}{mr^5} = 0
\end{aligned}$$

$$\to m\ddot{r} - \frac{p_\phi^2}{mr^3} + \frac{3p_\phi\lambda}{mr^4} - \frac{2\lambda^2}{mr^5} = 0. \tag{8}$$

We are interested in the behaviour of $\dot{r}^2$ one can multiply the expression in 8 with $\dot{r}$. This gives

$$m\ddot{r}\dot{r} = \frac{p_\phi^2}{mr^3}\dot{r} - \frac{3p_\phi\lambda}{mr^4}\dot{r} + \frac{2\lambda^2}{mr^5}\dot{r} \tag{9}$$

using $\dot{r}\ddot{r} = \frac{1}{2}\frac{d}{dt}(\dot{r}^2)$ and $\dot{r}dt = \frac{dr}{dt}dt = dr$

$$\begin{aligned}
\frac{1}{2}m\frac{d}{dt}(\dot{r}^2) &= \left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)\dot{r}\\
d(\dot{r}^2) &= \frac{2}{m}\left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)dr
\end{aligned}$$

now to integrate from r_0 to $r(t)$

$$\begin{aligned}
\dot{r}(t)^2 - \dot{r}(0)^2 &= \frac{2}{m}\int_{r(0)}^{r(t)}\left(\frac{p_\phi^2}{mr^3} - \frac{3p_\phi\lambda}{mr^4} + \frac{2\lambda^2}{mr^5}\right)dr\\
&= -\frac{p_\phi^2}{m^2}(r^{-2} - r_0^{-2}) + \frac{2p_\phi\lambda}{m^2}(r^{-3} - r_o^{-3})\\
&\quad - \frac{\lambda^2}{m^2}(r^{-4} - r_0^{-4})\\
&= \frac{1}{m^2r_0^2}\left(p_\phi - \frac{\lambda}{r_0}\right)^2 - \frac{1}{m^2r^2}\left(p_\phi - \frac{\lambda}{r}\right)
\end{aligned}$$

from equation 5 we have $\dot{\phi}mr^2 = \left(p_\phi - \frac{\lambda}{r}\right)$, inserting in the expression above gives

$$\dot{r}^2 - \dot{r}_0^2 = r_0^2\dot{\phi}_0^2 - r^2\dot{\phi}^2$$

which can be rearranged to

$$\dot{r}^2 + r^2\dot{\phi}^2 = \dot{r}_0^2 + r_0^2\dot{\phi}_0^2. \tag{10}$$

We see again that the kinetic energy is conserved.

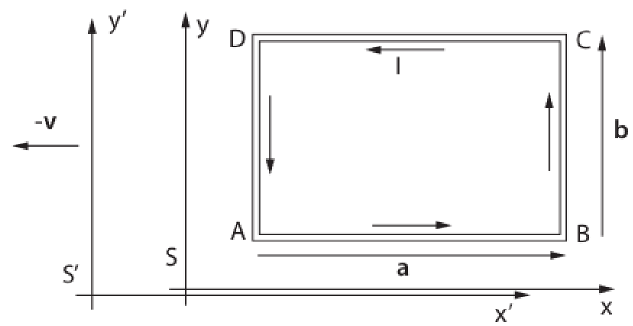

FIGURE 1. Illustration of current loop.

2. Rectangular Current Loop

Figure 1 shows a rectangular current loop ABCD. In the loop's rest frame, S, the loop as length a in the x-direction and width b in y-direction. The current is I and the charge density is zero. The electric dipole moment $\vec{\mathbf{p}}$ and the magnetic dipole moment $\vec{\mathbf{m}}$ for a given current distribution is defined by the following

$$\vec{\mathbf{p}} = \int \vec{\mathbf{r}}\rho(\vec{\mathbf{r}})d^3r, \quad \vec{\mathbf{m}} = \frac{1}{2}\int (\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}))d^3r \tag{11}$$

2.a. **Electric and Magnetic Dipole Moment in S.** Since the charge density in rest frame S is zero, $\rho(\vec{\mathbf{r}}) = 0$, the electric dipole moment must also be zero, $\vec{\mathbf{p}} = 0$.

The current along every edge of the rectangle will be $j\vec{\mathbf{n}}$, where $\vec{\mathbf{n}}$ is a unit vector pointing along the edge in question.

$$\begin{aligned} \text{AB} : \vec{\mathbf{j}} = j\vec{\mathbf{e}}_x \quad \text{BC} : \vec{\mathbf{j}} = j\vec{\mathbf{e}}_y \\ \text{CD} : \vec{\mathbf{j}} = -j\vec{\mathbf{e}}_x \quad \text{DA} : \vec{\mathbf{j}} = -j\vec{\mathbf{e}}_x. \end{aligned}$$

Given a point $\vec{\mathbf{r}}$ along the AB segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_x) = -yj\vec{\mathbf{e}}_z, \tag{12}$$

along the BC segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (j\vec{\mathbf{e}}_y) = xj\vec{\mathbf{e}}_z, \tag{13}$$

along the CD segment,

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_x) = yj\vec{\mathbf{e}}_z, \tag{14}$$

and along the DA segment

$$\vec{\mathbf{r}} \times \vec{\mathbf{j}}(\vec{\mathbf{r}}) = (x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y) \times (-j\vec{\mathbf{e}}_y) = -xj\vec{\mathbf{e}}_z, \tag{15}$$

It is a reasonable approximation to use the factor Δ, which is the cross-sectional area of the current wire, instead of integrating in directions perpendicular to the direction of the conductor. Then the coordinate $\vec{\mathbf{r}}$ is simply the centre of the conductor. Employing these assumptions/approximations and assigning the lower left corner of the rectangle coordinates (x_0, y_0) and using the results from equations 12 13, 14 and 15 the magnetic dipole moment is

$$\begin{aligned}\vec{\mathbf{m}} &= \frac{1}{2} j\Delta\vec{\mathbf{e}}_z\left(-\int_{x_0}^{y_0+a} y_0 dx + \int_{y_0}^{y_0+b}(x_0+a)dy + \int_{x_0}^{x_0+a}(y_0+b)dx - \int_{y_0}^{y_0+b} x_0 dy\right)\\ &= \frac{1}{2} I(-y_0 a + (x_0+a)b + (y_0+b)a - x_0 b)\vec{\mathbf{e}}_z\\ &= abI\vec{\mathbf{e}}_z.\end{aligned}$$

Since $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = ab\vec{\mathbf{e}}_z$, which gives

$$\vec{\mathbf{m}} = I\vec{\mathbf{a}} \times \vec{\mathbf{b}}. \tag{16}$$

2.b. **Length Contraction.** Reference frame S' moves at velocity $\vec{\mathbf{v}}$ to the right, away from the rest frame S. Because the width of the loop is orthogonal to the boost, it remains the same, $b' = b$.

We have the following Lorentz transformations between the positions of the endpoints of the rectangle in the two reference frames

$$\begin{aligned} x_a &= \gamma(-vt'_a + x'_a) \quad & x'_a &= \gamma(vt_a + x_a)\\ x_b &= \gamma(-vt'_b + x'_b) \quad & x'_b &= \gamma(vt_b + x_b).\end{aligned}$$

The two endpoints must be measured at the same time in reference frame S' to compute the correct length

$$\begin{aligned} a = x_b - x_a &= \gamma(x'_b - x'_a) + \cancel{\gamma v(t'_a - t'_b)}\\ &= \gamma(x'_b - x'_a) = \gamma a'\end{aligned}$$

which yields

$$a = \frac{1}{\gamma} a'. \tag{17}$$

2.c. **Charge of Segments AB and CD.** In reference frame S, the segment AB will have current four-vector $J^\mu = (0, j, 0, 0)$. Lorentz transform to reference frame S', which has velocity $-v$ relative to S gives

$$\rho' c = J'^0 = L^0{}_\nu J^\nu = \gamma(J^0 + \beta J^1) = \beta\gamma J,$$

where $\beta = v/c$. Assuming uniform charge density of the conductor segment gives

$$\rho' = \frac{J'^0}{c} = \frac{j\gamma v}{c^2}.$$

To find the total charge one needs simply to multiply with the volume of this conductor segment in reference frama S', $|\vec{\mathbf{a}}|\Delta = (1/\gamma)a\Delta$

$$Q'_{AB} = V'_{AB}\rho'_{AB} = \frac{1}{\gamma}a\Delta j\gamma\frac{v}{c^2} = Ia\frac{v}{c^2}. \tag{18}$$

A similiar computation can be made for conductor segment CD, with current four-vector $J^\mu = (0, -j, 0, 0)$.

$$\begin{aligned}\rho' c = J'^0 &= L^0{}_\nu J^\nu = \gamma(J^0 + \beta J^1) = -\gamma\beta j \\ \rightarrow \rho' &= \frac{J'^0}{c} = -\frac{\gamma\beta j}{c} = -\frac{\gamma v j}{c^2}\end{aligned}$$

$$Q'_{CD} = V'_{CD}\rho'_{CD} = -\frac{1}{\gamma}a\Delta\gamma j\frac{v}{c^2} = -Ia\frac{v}{c^2} \tag{19}$$

2.d. **Electric and Magnetic Dipole Moment in S'.** Segments AB and CD always have charge densities $\rho = \pm\frac{I}{\Delta}\frac{v}{c^2}$. Inserting this into the $\vec{\mathbf{p}}$ from 11 and assuming a thin conductor by replacing the cross-sectional integration dimensions with Δ gives

$$\begin{aligned}\vec{\mathbf{p}}' &= \int \vec{\mathbf{r}}\rho(\vec{\mathbf{r}})d^3r \\ &= \frac{I}{\cancel{\Delta}}\frac{v}{c^2}\left[\cancel{\Delta}\int_{x_0}^{x_0+a}(\cancel{x\vec{\mathbf{e}}_x} + y_0\vec{\mathbf{e}}_y)dx - \cancel{\Delta}\int_{x_0}^{x_0+a}(\cancel{x\vec{\mathbf{e}}_x} + (y_0+b)\vec{\mathbf{e}}_y)dx\right] \\ &= I\frac{v}{c^2}\left(\cancel{y_0 x\Big|_{x_0}^{x_0+a}} - (\cancel{y_0} + b)x\Big|_{x_0}^{x_0+a}\right)\vec{\mathbf{e}}_y \\ &= I\frac{v}{c^2}\left(-b(x_0 + a - x_0)\right)\vec{\mathbf{e}}_y \\ &= -I\frac{v}{c^2}ab\vec{\mathbf{e}}_y\end{aligned}$$

Moreover

$$\begin{aligned}-\frac{1}{c^2}\vec{\mathbf{m}} \times \vec{\mathbf{v}} &= -\frac{1}{c^2}(abI\vec{\mathbf{e}}_z) \times (v\vec{\mathbf{e}}_x) \\ &= -\frac{v}{c^2}Iab(\vec{\mathbf{e}}_z \times \vec{\mathbf{e}}_x) \\ &= -I\frac{v}{c^2}ab\vec{\mathbf{e}}_y,\end{aligned}$$

which implies that

$$\vec{\mathbf{p}}' = -\frac{1}{c^2}\vec{\mathbf{m}} \times \vec{\mathbf{v}} \tag{20}$$

In order to calculate the magnetic dipole moment, one needs the current densities for all conductor segments.

$$\begin{aligned}&\text{AB: } J = (0, j, 0, 0) \quad j' = J'^1 = \beta\gamma J^0 + \gamma J^1 = \gamma j \\ &\text{CD: } J = (0, -j, 0, 0) \quad j' = J'^1 = \beta\gamma J^0 + \gamma J^1 = -\gamma j.\end{aligned}$$

Segments AB and CD both have current densities $j' = \gamma j$ in x-direction. Segments BA and DA have current density j unchanged. However, the width of these conductors are Lorentz-contracted, meaning that the area is reduced by a factor γ^{-1}, so that $\Delta' = \Delta/\gamma$. Now to compute the magnetic dipole moment in the same manner as before

$$\begin{aligned}\vec{\mathbf{m}}' &= \frac{1}{2}\vec{\mathbf{e}}_z\left[\gamma j\Delta\int_{x_0'}^{x_0'+a'}(\cancel{y_0}+b-\cancel{y_0})dx' j\frac{\Delta}{\gamma}\int_{y_0'}^{y_0'+b'}(\cancel{x_0'}+a-\cancel{x_0'})\right]\\ &= \frac{1}{2}j\Delta ab(1+\gamma^{-2})\vec{\mathbf{e}}_z = \frac{1}{2}Iab(2-\beta^2)\vec{\mathbf{e}}_z = Iab(1-\frac{\beta^2}{2})\vec{\mathbf{e}}_z\end{aligned}$$

$$\vec{\mathbf{m}}' = \left(1-\frac{\beta^2}{2}\right)\vec{\mathbf{m}} \tag{21}$$

2.e. **Current in the Different Segments.** The current density is $j' = \gamma j$ in AB and CD, while the cross-sectional area is unchanged. This makes the current

$$I' = \Delta j^1 = \gamma\Delta j = \gamma I. \tag{22}$$

For BC and DA $j' = j$, but $\Delta' = \Delta/\gamma$, so

$$I' = \Delta' j = \frac{\Delta}{\gamma}j = \frac{I}{\gamma} \tag{23}$$

2.f. **Charge Conservation.** To look at how the charge of the rectangle changes, look first at the lower left corner of the rectangle. Consider an enclosed region in the vicinity of this corner. Here, there is a stronger current flowing through the AB segment, than the DA segment. Locally, charge leaves this region at a rate of

$$\frac{dQ}{dt} = I_{AB} - I_{DA} = \left(\gamma - \frac{1}{\gamma}\right)I = \gamma(1-\gamma^{-2})I = \gamma\frac{v^2}{c^2}I. \tag{24}$$

At the same time, the AB segment moves to the right, out of the region here focused upon. If the lengt of the rod contained within this region is l, the region will at some instantaneous moment contain charge

$$Q = \frac{l}{a'}Q_{AB} = \gamma\frac{l}{a}Q_{AB} = \gamma I\frac{v}{c^2}l. \tag{25}$$

The velocity of the length of the AB segment within the region is $dl/dt = v$. Employing this relation gives

$$\frac{dQ}{dt} = \gamma I\frac{v^2}{c^2} \tag{26}$$

Since charge leaves the region at the same time that is entering the region, leaving charge conserved.

3. Moving Point Charge

An electric point charge moves with constant velocity $\vec{\mathbf{v}}$ along the x-axis of intertial frame S, passing the origin at $t = 0$.