MIDTERM EXAM
FYS3120

CANDIDATE 15137

1. A (boring) Lagrangian

A non-relativistic particle (no-potential) of mass m is moving in three dimensions.

1.a. A normal (boring), Cartesian coordinate system will do fine to study this problem in the first instance. One needs three coordinates to accurately describe the particle, and as there are no constraints on the particle, these three coordinates x, y and z are also the generalised coordinates.

The kinetic energy of the particle is given by

$$T = \frac{1}{2}mv^2 \tag{1}$$

where $v = |\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$, such that equation 1 becomes

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{2}$$

there is no potential, so the Lagrangian is simply

$$L = T - V = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mv^2 \tag{3}$$

1.b. The conjugate momenta are

$$p_x = \frac{\partial L}{\partial \partial \dot{x}} = m\dot{x}$$
$$p_y = \frac{\partial L}{\partial \partial \dot{y}} = m\dot{y}$$
$$p_z = \frac{\partial L}{\partial \partial \dot{z}} = m\dot{z},$$

or rather

$$p_v = \frac{\partial L}{\partial \mathbf{v}} = mv, \tag{4}$$

which is exactly the same as the regular mechanical momentum.

Date: March 29, 2017.

1.c. The position of the particle are cyclic coordinates, since

$$\frac{\partial L}{\partial x} = 0$$
$$\frac{\partial L}{\partial y} = 0$$
$$\frac{\partial L}{\partial z} = 0$$

alternatively

$$\frac{\partial L}{\partial \mathbf{r}} = \mathbf{0}. \tag{5}$$

Only the position $\mathbf{r}$ varies with time, but the Lagrangian, interpreted as the physical situation, remains unchanged. This means that the initial value for the position does not determine the path of the particle.

1.d. The Euler-Lagrange for this system equation is

$$\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{r}}} = 0. \tag{6}$$

Since $\partial L/\partial \mathbf{r} = 0$, then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}} = 0,$$

and therefore the conjugate momentum $m\dot{\mathbf{r}} = m\mathbf{v}$ must be a constant of motion.

It now follows that the particle must move in a straight line such that

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t. \tag{7}$$

My hunch is that the angular momentum of the particle must also be conserved,

$$\begin{aligned}
\mathbf{L} = \mathbf{r} \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p} \\
(\mathbf{r}_0 + \mathbf{v}t) \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p} \\
\mathbf{r}_0 \times \mathbf{p} + \mathbf{v}t \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p} \\
\mathbf{r}_0 \times \mathbf{p} &= \mathbf{r}_0 \times \mathbf{p}
\end{aligned}$$

where $\mathbf{v}t \times \mathbf{p} = 0$ because $\mathbf{v}$ and $\mathbf{p}$ are parallel. This means that the system is invariant under a rotation.

In conclusion, the conserved quantities are the momentum $\mathbf{p}$ and the angular momentum $\mathbf{L}$. Said in another way, this system can has both a translational and a rotational symmetry. Because of these two symmetric properties the system must have two corresponding quantities whose values are conserved in time[1].

1.e.

1.f.

[1] This last bit was an informal statement of Emmy Noether's theorem.

1.g. Consider a Lorentz transformation where the Lorentz transformation tensor is given as

$$L^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{8}$$

Any particular Lorentz transformation must leave the line element $ds^2 = dx_\mu dx^\mu$ invariant,

$$\begin{aligned} g_{\mu\nu} dx'^{\mu} dx'^{\nu} = g_{\mu\nu} L^{\mu}{}_{\rho} L^{\mu}{}_{\sigma} dx^{\rho} dx^{\sigma} &= g_{\rho\sigma} dx^{\rho} dx^{\sigma} \\ g_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}_{\sigma} &= g_{\rho\sigma} \end{aligned}$$

To see if the Lorentz transformation in 8 is invariant is must statisfy this requirement

$$\begin{aligned} g_{\rho\sigma} &= q_{\mu\nu} L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} \\ &= g_{\mu\nu} (\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= (\delta_{\nu\rho} + \omega_{\nu\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \\ &= \delta_{\nu\rho}\delta^{\nu}{}_{\sigma} + \delta_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho}\delta^{\nu}{}_{\sigma} + \omega_{\nu\rho}\omega^{\nu}{}_{\sigma} \\ &= g_{\nu\rho}\delta^{\nu}{}_{\sigma} + g_{\nu\rho}\omega^{\nu}{}_{\sigma} + \omega_{\nu\rho} g^{\nu\gamma} g_{\gamma\sigma} + \omega^2_{\rho\sigma} \\ &= \delta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} = g_{\rho\sigma} + g_{\nu\rho}(\omega^{\nu}{}_{\sigma} + \omega_{\sigma}{}^{\nu}), \end{aligned}$$

which only works if ω^{μ}_{ν} is antisymmetric, that is if $\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}$.

1.h. A small Lorentz transformation between two reference frames changes the path $x^\mu(\tau)$ of a particle according to

$$\delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau) = \omega^{\mu}{}_{\nu} x^\nu(\tau). \tag{9}$$

This corresponds to a perturbation in the Lagrangian.

The variation of the Lagrangian is

$$\delta L = \frac{\partial L}{\partial x^\mu} \delta x^\mu + \frac{\partial L}{\partial U^\mu} \delta U^\mu$$

inserting for $\delta x^\mu = \omega^{\mu}{}_{\nu} x^\nu$ from equation 9 and

$$\delta U^\mu = \delta \frac{dx^\mu}{dt} = \frac{d}{d\tau}(\delta x^\mu) = \omega^{\mu}{}_{\nu} U^\nu$$

yields

$$\delta L = \left(\frac{\partial L}{\partial x^\mu} x^\nu + \frac{\partial L}{\partial U^\mu} U^\nu \right) x^{\mu}{}_{\nu}, \tag{10}$$

which is the change in the Lagrangian as a consequence of the change in path.

1.i. The Euler-Lagrange equations states

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}\right) = \frac{\partial L}{\partial x^\mu}. \tag{11}$$

Inserting 11 into 10 gives

$$\delta L = \left(\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right) + \frac{\partial L}{\partial U^\mu}\frac{d}{d\tau}x^\nu\right)\omega^\mu{}_\nu \tag{12}$$

using the product rule for derivation backwards gives

$$\delta L = \frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu = \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \tag{13}$$

and finally "letting everything run it's course"

$$\begin{aligned}
\delta L &= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu + \frac{\partial L}{\partial U^\mu}x^\nu\right)\omega^\mu{}_\nu \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U^\mu}x^\nu\omega^\mu{}_\nu - \frac{\delta L}{\delta U^\mu}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial g^{\rho\mu}U_\rho}x^\nu\omega^\mu{}_\nu - \frac{\delta L}{\delta g^{\sigma\mu}U_\sigma}x^\nu\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu g_{\rho\mu}\omega^\mu{}_\nu - \frac{\delta L}{\delta U_\sigma}x^\nu g_{\sigma\mu}\omega_\nu{}^\mu\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\rho}x^\nu\omega_{\rho\nu} - \frac{\delta L}{\delta U_\sigma}x^\nu\omega_{\nu\sigma}\right)
\end{aligned}$$

changing indices back, writing μ instead of ρ, σ, and moving x^ν to the left of derivatives gives

$$\delta L = \frac{1}{2}\frac{d}{d\tau}\left(x^\nu\frac{\partial L}{\partial U_\mu}\omega_{\mu\nu} - x^\nu\frac{\delta L}{\delta U_\mu}\omega_{\nu\mu}\right).$$

Switch indices of first term inside the parenthesis[2], and one ends up with an alternative expression for δL,

$$\delta L = \frac{1}{2}\omega_{\nu\mu}\frac{d}{d\tau}\left(x^\mu\frac{\delta L}{\delta U_\nu} - x^\nu\frac{\partial L}{\partial U_\mu}\right) \tag{14}$$

2. Relativistics

Two particles with mass m and a photon is sent out from a source at the same time and in the positive x-direction in rest frame S of the source. The massive particles are moving with constant velocity v_1 and $v_2 > v_1$ in this frame. Figure 1 shows a Minkowski space-time diagram of the two particles, the photon and the source in the rest frame of the source S and that of the slowest of the particles S'.

[2]This is okay because if one were to move ∂U_μ up from underneath the dividing line the index μ would change to an upstairs variant. This is the same as saying $\sum_i\sum_j x^i\frac{\partial L}{\partial U_j}\omega_{ji} = \sum_j\sum_i x^i\frac{\partial L}{\partial U_i}\omega_{ij}$

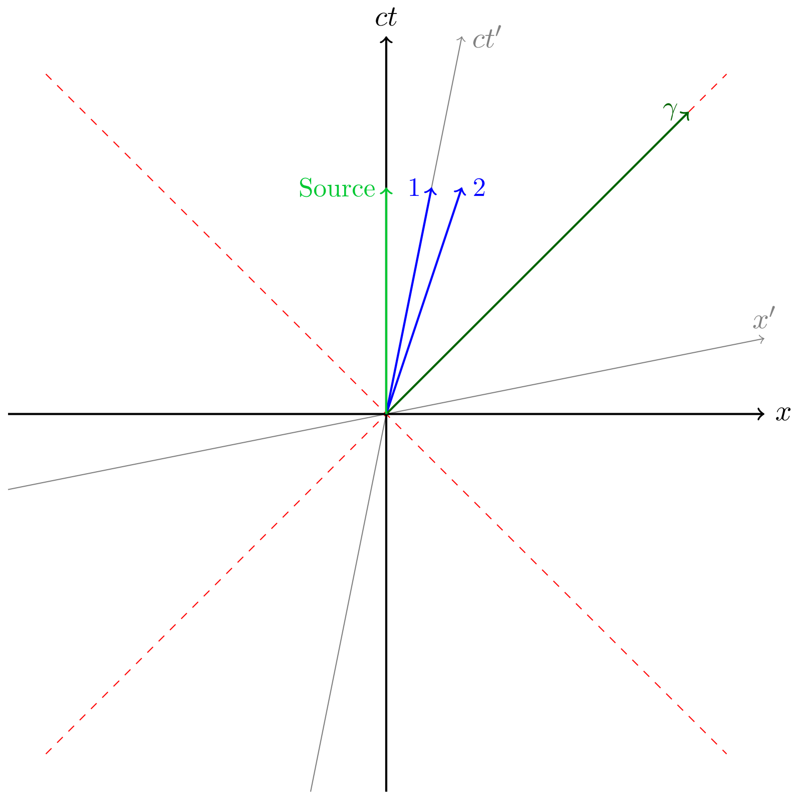

FIGURE 1. Minkowski space-time diagram of two massive particles (velocities v_1 and $v_2 > v_1$) and a photon (γ) sent out from a source at origin in rest frame S. Rest frame S' is that of particle 1.

For an infinitesimal change in position coordinates we have

$$dx' = \gamma(dx - v_1 dt) = \gamma(u_2 - v_1)dt$$
$$dt' = \gamma(dt - \frac{v_1}{c^2}dx) = \gamma(1 - \frac{v_2 v_1}{c^2})$$

and from this follows that

$$v_2' = \frac{dx'}{dt'} = \frac{v_2 - v_1}{1 - \frac{v_2 v_1}{c^2}} \tag{15}$$

The difference in rapidity of the two massive particles in the two different rest frames are

$$S: \quad \Delta\chi = \tanh^{-1}\left(\frac{v_2}{c}\right) - \tanh^{-1}\left(\frac{v_1}{c}\right) \tag{16}$$

$$S': \quad \Delta\chi' = \tanh^{-1}\left(\frac{v_2'}{c}\right) - \tanh^{-1}\left(\frac{v_1'}{c}\right) = \tanh^{-1}\left(\frac{v_2'}{c}\right) \tag{17}$$

Rapidity differences should be unchanged by boosts no matter the reference frames, so

$$\begin{aligned}\tanh^{-1}\left(\frac{v_2}{c}\right) - \tanh^{-1}\left(\frac{v_1}{c}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right)\\ \tanh^{-1}\left(\frac{\frac{v_2}{c} - \frac{v_1}{c}}{1 - \frac{v_2 v_1}{c^2}}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right)\\ \tanh^{-1}\left(\frac{1}{c}\frac{v_2 - v_1}{1 - \frac{v_2 v_1}{c^2}}\right) &= \tanh^{-1}\left(\frac{v_2'}{c}\right),\end{aligned}$$

inserting 15 gives

$$\begin{aligned}\tanh^{-}1\left(\frac{v_2'}{c}\right) &= \tanh^{-}1\left(\frac{v_2'}{c}\right)\\ \chi &= \chi'.\end{aligned}$$

In conclusion, the rapidity difference is the same in the two rest frames S and S.

3. Finding the Shortest Way

The shortest path between two points on a sphere. At some contstant radius r, some small movement in some direction on the sphere is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta\phi^2 \tag{18}$$

inserting for $d\phi = (d\phi/d\theta)d\theta = \dot{\phi}d\theta$ gives

$$ds = r\sqrt{1 + \sin^2\theta\dot{\phi}^2}d\theta \tag{19}$$

A path is given by

$$S = \int ds = r\int_{\theta_A}^{\theta_B} \sqrt{1 + \sin^2\theta\dot{\phi}^2}d\theta \tag{20}$$

where the integrand $F(\theta, \phi, \dot{\phi}) = \sqrt{1 + \sin^2\theta\dot{\phi}^2}$ does not depend explicitly on ϕ. This implies that $\partial F/\partial\dot{\phi}$ is constant, yielding

$$\frac{\partial F}{\partial\dot{\phi}} = \frac{2\sin^2\theta\dot{\phi}}{\sqrt{1 + \sin^2\theta\dot{\phi}^2}} = C' \rightarrow \frac{\sin^2\theta\dot{\phi}}{\sqrt{1 + \sin^2\theta\dot{\phi}^2}} = C \tag{21}$$

This can be rearranged

$$
\begin{aligned}
C^2 &= \frac{\sin^4\theta\dot{\phi}^2}{1+\sin^2\theta\dot{\theta}^2} \\
C^2 + C\sin^2\theta\dot{\phi}^2 &= \sin^4\theta\dot{\phi}^2 \\
C^2 &= (\sin^4\theta - C\sin^2\theta)\dot{\phi}^2 \\
\dot{\phi}^2 &= \frac{C^2}{(\sin^4\theta - C\sin^2\theta)} \\
\dot{\phi} &= \frac{C}{\sin\theta\sqrt{\sin^2 - C}}
\end{aligned}
$$

INTEGRATE!!