

ELECTRODYNAMICS
FYS3120: PROBLEM SET 11

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1. SIMPLE LAGRANGIAN DYNAMICS

A non-relativistic particle, with electric charge q and mass m moves in a magnetic dipole field, given by the vector potential

$$(1) \quad \vec{\mathbf{A}} = \frac{\mu_0}{4\pi r^3} (\vec{\mu} \times \vec{\mathbf{r}}),$$

where $\vec{\mu}$ is the magnetic dipole moment of a static charge distribution centered at the origin.

1.a. **Lagrangian.** The Lagrangian is given by

$$(2) \quad L = T + q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}}.$$

The kinetic energy is simply $T = \frac{1}{2}m\vec{\mathbf{v}}^2$ while the potential is

$$\begin{aligned} q\vec{\mathbf{v}} \cdot \vec{\mathbf{A}} &= \frac{q\mu_0}{4\pi r^3} \vec{\mathbf{v}} \cdot (\vec{\mu} \times \vec{\mathbf{r}}) \\ &= \frac{q\mu_0}{4\pi r^3} \vec{\mu} \cdot (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \\ &= \frac{q\mu_0}{4\pi m r^3} \vec{\mu} \cdot \vec{\ell}, \end{aligned}$$

using the cyclic invariance of the vector triple product and $\vec{\ell} = m\vec{\mathbf{r}} \times \vec{\mathbf{v}}$. Inserting the parts into 2 the Lagrangian becomes

$$(3) \quad L = \frac{1}{2}m\vec{\mathbf{v}}^2 + \frac{q\mu_0}{4\pi m r^3} \vec{\mu} \cdot \vec{\ell}.$$

1.b. **Alternative Lagrangian.** We now make the assumption that the magnetic dipole moment is oriented along the z -axis and that the particle moves in the (x, y) -plane. In the following, $r = |\vec{\mathbf{r}}|$ and the angle ϕ between the x -axis and the position vector var are chosen as generalised coordinates.

With the magnetic dipole moment oriented along the z -axis,

$$\vec{\mu} \cdot \vec{\ell} = |\vec{\mu}| \ell_z = |\vec{\mu}| (\vec{\mathbf{r}} \times \vec{\mathbf{p}})_z = |\vec{\mu}| m(xy\dot{-} y\dot{x}),$$

where $x = r \cos \phi$ and $y = r \sin \phi$. This gives

$$\begin{aligned} x\dot{y} - y\dot{x} &= r \cos \phi (\dot{r} \sin \phi + r \dot{\phi} \cos \phi) \\ &\quad - r \sin \phi (\dot{r} \cos \phi - r \dot{\phi} \sin \phi) \\ &= r^2 \dot{\phi} \cos^2 \phi + r^2 \dot{\phi} \sin^2 \phi = r^2 \dot{\phi}, \end{aligned}$$

similarly

$$\begin{aligned} \dot{x} &= \dot{r} \cos \phi - r \dot{\phi} \sin \phi \\ \dot{y} &= \dot{r} \sin \phi + r \dot{\phi} \cos \phi \\ \dot{x}^2 &= \dot{r}^2 \cos^2 \phi - 2r\dot{r}\dot{\phi} \cos \phi \sin \phi + r^2 \dot{\phi}^2 \sin^2 \phi \\ \dot{y}^2 &= \dot{r}^2 \sin^2 \phi + 2r\dot{r}\dot{\phi} \cos \phi \sin \phi + r^2 \dot{\phi}^2 \cos^2 \phi \\ v^2 &= \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\phi}^2. \end{aligned}$$

The Lagrangian with generalised coordinates becomes

$$(4) \quad L = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{q\mu_0}{4\pi m r^3} |\vec{\mu}| m r^2 \dot{\phi} = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\phi}^2) + \lambda \frac{\dot{\phi}}{r},$$

where $\lambda \equiv q\mu_0 |\vec{\mu}| / 4\pi$.

The canonical momentum p_ϕ conjugate to ϕ becomes

$$(5) \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi} + \frac{\lambda}{r}$$

ϕ is a cyclic coordinate, because the Lagrangian in equation 4 does not explicitly depend on ϕ . This implies that the conjugate momentum p_ϕ is constant.

The Lagrangian in equation 4 does not depend explicitly on time t . This means that the Hamiltonian must be conserved

$$(6) \quad H = \dot{r}p_r + \dot{\phi}p_\phi - L = m\dot{r}^2 + m r^2 \dot{\phi}^2 + \lambda \frac{\dot{\phi}}{r} - L = \frac{1}{2}m(\dot{r}^2 + r^2 \dot{\phi}^2) = T.$$

Since the Hamiltonian equals the kinetic energy and the Hamiltonian is conserved, the kinetic energy is conserved by the magnetic field.