HAMILTONIAN DYNAMICS OF THE BRACHISTOCHRONE PROBLEM

PROBLEM SHEET 5: FYS3120

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1. Coriolis and Centrifugal Forces

A particle with mass m moves freely on a horizontal plane. There are no constraints, but in the following we will consider the free motion described in a rotating reference frame. We refer to the Cartesian coordinates of a fixed frame as (x, y) and the coordinates of the rotating frame as (ξ, η). They are related by the standard expressions

$$x = \xi \cos \omega t - \eta \sin \omega t, \tag{1}$$

$$y = \xi \sin \omega t + \eta \cos \omega t, \tag{2}$$

where ω is the angular velocity of the rotation.

1.a. **Lagrangian.** First we need $\dot{x}$ and $\dot{y}$;

$$\begin{aligned}
\dot{x} &= \dot{\xi} \cos \omega t - \omega \xi \sin \omega t - \dot{\eta} \sin \omega t - \omega \eta \cos \omega t \\
&= (\dot{\xi} - \omega \eta) \cos \omega t - (\omega \xi + \dot{\eta}) \sin \omega t, \\
\dot{y} &= \dot{\xi} \sin \omega t + \omega \xi \cos \omega t + \dot{\eta} \cos \omega t - \omega \eta \sin \omega t \\
&= (\dot{\xi} - \omega \eta) \sin \omega t + (\omega \xi + \dot{\eta}) \cos \omega t,
\end{aligned}$$

we also need their squares

$$\begin{aligned}
\dot{x}^2 &= (\dot{\xi} - \omega \eta)^2 \cos^2 \omega t - 2(\dot{\xi} - \omega \eta)(\omega \xi + \dot{\eta}) \cos \omega t \sin \omega t + (\omega \xi + \dot{\eta})^2 \sin^2 \omega t, \\
\dot{y}^2 &= (\dot{\xi} - \omega \eta)^2 \sin^2 \omega t + 2(\dot{\xi} - \omega \eta)(\omega \xi + \dot{\eta}) \sin \omega t \cos \omega t + (\omega \xi + \dot{\eta})^2 \cos^2 \omega t,
\end{aligned}$$

the sum of the squares is

$$\dot{x}^2 + \dot{y}^2 = (\dot{\xi} - \omega \eta)^2 + (\omega \xi + \dot{\eta})^2 = dot\xi^2 - 2\dot{\xi} \omega \eta + \omega^2 \eta^2 + \omega^2 \xi^2 + 2\omega \xi \dot{\eta} + \dot{\eta}^2,$$

which can now be used to find the Lagrangian

$$L = T = \frac{1}{2} m[\dot{\xi}^2 + \dot{\eta}^2 + \omega^2(\xi^2 + \eta^2) + 2\omega(\xi \dot{\eta} - \dot{\xi} \eta)]. \tag{3}$$

As there is no gravity there is no potential, V.

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1.b. **Equations of Motion.** The general Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{4}$$

and can be found for every generalised coordinate.

1.b.1. *Lagrange's equation for* ξ. Start by finding all parts of equation 4

$$\begin{aligned} \frac{\partial L}{\partial \xi} &= m\omega^2\xi + m\omega\dot{\eta}, \\ \frac{\partial L}{\partial \dot{\xi}} &= m\dot{\xi} - m\omega\eta, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) &= m\ddot{\xi} - m\omega\dot{\eta}. \end{aligned}$$

Combining all these gives Lagrange's equation for ξ

$$m\ddot{\xi} = m\omega^2\xi + 2m\omega\dot{\eta}. \tag{5}$$

1.b.2. *Lagrange's equation for* η. I am going to make an implicit symmetry argument here[1] and simply write down the Lagrange equation for η

$$m\ddot{\eta} = m\omega^2\eta - 2m\omega\dot{\xi} \tag{6}$$

1.b.3. *Analysis.* Notice that I did not write down the Lagrange equations for ξ and η, given by equations 5 and 6 respectively, in the conventional way dictated by equation 4. The reason for this is that the Lagrange equations found in this problem are incredibly simlilar to Newton's second law for rotational coordinates,

$$\mathbf{F} = m\ddot{\rho} = \mathbf{F}_{\text{imp}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{Euler}}, \tag{7}$$

where $\mathbf{F}_{\text{imp}}$ are the forces impressed on the system ($= 0$ here), $\mathbf{F}_{\text{centrifugal}} = -m\omega \times (\omega \times \rho)$ is the centrifugal force, $\mathbf{F}_{\text{Coriolis}} = -2m\omega \times \dot{\rho}$ is the Coriolis force and $\mathbf{F}_{\text{Euler}}$ is the Euler force, felt in reaction to any acceleration (also $= 0$ here). In all forces ρ is the position vector in the rotating frame. Equation 7 becomes

$$m\ddot{\rho} = -m\omega \times (\omega \times \rho) - 2m\omega \times \dot{\rho}, \tag{8}$$

which is very similar to the equations of motion in 5 and 6.

2. The Brachistochrone Challenge

The problem as posed to Isaac Newton, amongst others, in 1696 can be formulated in the following way,

> Given two points A and B in a vertical plane, what is the curve traced out by a body acted on only by gravity, which starts at A and reaches B in the shortest time.

[1] Did you notice it?

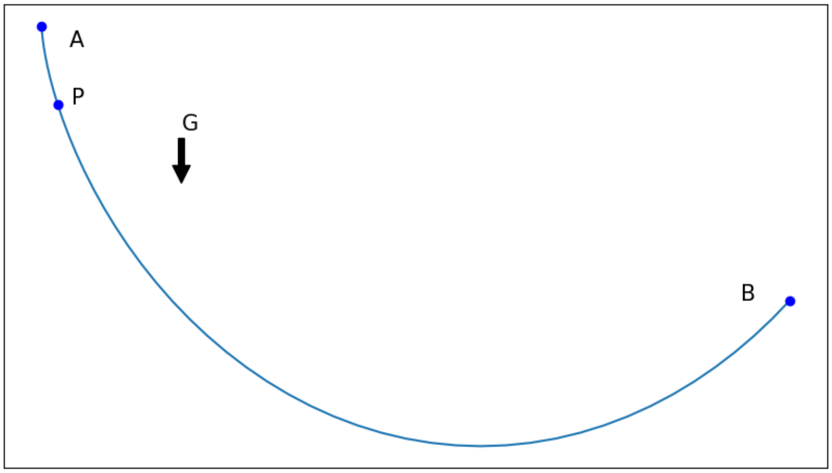

FIGURE 1. Illustration of the Brachistochrone problem

It is said that Newton had a solution already the following day. Herein the challenge is the following, can the problem be solved using the correspondence between the variational problem and the Lagrange equation? The body P is treated as a point particle of mass m and the path is represented by a function $y(x)$ with x as the horizontal axis and y as the vertical axis. The boundary conditions, which fix the positions of point A and B, are specified as $y(x_A) = y_A$, $y(x_b) = y_b$. A simplifications is to assume that $x_A = y_A$.

2.a. **The Period of the Motion.** The total time the body spends moving along the curve is given by a simple integral of infinitesimal time steps

$$T = \int_{t_a}^{t_b} dt. \tag{9}$$

Using the velocity and displacement relations $vdt = ds$ gives

$$T = \int_{A}^{B} \frac{1}{v} ds. \tag{10}$$

One such infinitesimal displacement can be decomposed in a Pythagorean manner into x and y parts, $ds = \sqrt{dx^2 + dy^2}$. Inserting into 10 yields

$$T = \int_{A}^{B} \frac{1}{v} \sqrt{dx^2 + dy^2} = \int_{x_a}^{x_b} \frac{1}{v} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{11}$$

By assuming conservations of energy $\frac{1}{2}mv^2 + mgy = 0 \rightarrow v = \sqrt{-2gy}$ and setting $y' = \frac{dy}{dx}$ equation 11 becomes

$$T = \int_{x_a}^{x_b} \sqrt{\frac{1+y'^2}{-2gy}} dx. \tag{12}$$

If one were to consider this an action integral, then the integrand must therefore be the Lagrangian for the system with y and y' as generalised coordinates

$$L(y, y') = \sqrt{\frac{1+y'^2}{-2gy}} \tag{13}$$

2.b. **Hamiltonian and Differential Equation for the Problem.** Notice that the Lagrangian (equation 13) does not depend explicitly on x, which has taken the role as t in this problem. It follows from the Lagrangian-Hamiltonian-relationship

$$\frac{dH}{dx} = -\frac{dL}{dx} = 0 \tag{14}$$

that the Hamiltonian, H, is a constant of motion. Then one can use the conjugate momentum as is given by $p = \frac{\partial L}{\partial y'}$ and write the Hamiltonian as $H = py' - L$. The conjugate momentum is

$$\frac{\partial L}{\partial y'} = \frac{1}{\sqrt{-2gy}} \frac{y'}{\sqrt{1+y'^2}}$$

and the Hamiltonian becomes

$$\begin{aligned} H = py' - L &= \frac{1}{\sqrt{-2gy}} \left(\frac{y'^2}{\sqrt{1+y'^2}} + \sqrt{1+y'^2} \right) \\ &= \frac{1}{\sqrt{-2gy}} \frac{1}{\sqrt{1+y'^2}} \left(y'2 - 1 - y'^2 \right) \end{aligned}$$

$$\begin{aligned} \rightarrow (-2gy)(1+y'^2) &= \frac{1}{H^2} \\ (1+y'^2)y &= -\frac{1}{2gH^2} \\ (1+y'^2)y &= -k^2. \end{aligned} \tag{15}$$

$y(x)$ must satisfy a differential equation given by equation 15, where $k = \frac{1}{\sqrt{2g}H}$.

2.c. **Parametric Solution.** The solution to the Brachistochrone probem can be written in parametric form as

$$y = \frac{1}{2}k^2(\theta - \sin\theta), \tag{16}$$

$$x = \frac{1}{2}k^2(\cos\theta - 1), . \tag{17}$$

I am going to show that these indeed is a solution to 15.

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta},$$

where

$$\begin{aligned}
\frac{dx}{d\theta} &= \frac{1}{2}k^2(1 - \cos\theta) = -y, \\
\frac{dy}{d\theta} &= -\frac{1}{2}k^2 \sin\theta, \\
y' &= \frac{1}{2y}k^2 \sin\theta
\end{aligned}$$

This can now be inserted into the left hand side of the differential equation

$$\begin{aligned}
(1 + y'^2)y &= (1 + \frac{1}{4y^2}k^4 \sin^2\theta)y \\
&= \frac{1}{4}\frac{1}{y}(4y^2 + k^4 \sin^2\theta) \\
&= \frac{1}{4}\frac{1}{y}(k^4(\cos\theta - 1)^2 + k^4 \sin^2\theta) \\
&= \frac{1}{4}\frac{1}{y}k^4(\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta) \\
&= \frac{1}{y}k^2\frac{1}{2}k^2(1 - \cos\theta) \\
&= k^2\frac{1}{y}(-y) \\
&= -k^2,
\end{aligned}$$

and one can see that it equates to the right hand side.

It is worth checking how the boundary conditions are taken care of in this situation,

$$\begin{aligned}
x_a &= 0, \\
\rightarrow \theta_a - \sin\theta_a = 0 \rightarrow \theta_a &= 0, \\
\rightarrow \cos\theta_a &= 1, \\
\rightarrow y_a = \frac{1}{2}k^2(\cos\theta - 1) &= 0.
\end{aligned}$$

So point A is fine. Now to look at point B where the boundary condition is satisfied if the following equations are satisfied,

$$x_b = \frac{1}{2}k^2(\theta_b - \sin\theta_b),$$
$$y_b = \frac{1}{2}k^2(\cos\theta_b - 1).$$

This means that these two equations determine k and θ_b and that k is no longer an arbitrary constant.

2.d. **Cycloid.** The solution to the brachistrochrone problem, as given by the parameter equations 16 and 17 form a cycloid. This is the curve formed by a point of a rolling circle. I have already plotted such a curve in figure 1, but I have additionally made an animation in JavaScript which visualises the nature of a cycloid much better: `http://folk.uio.no/sebastwi/FYS3120/cycloid/`. A "normal" cycloid is a convex function formed by a circle rolling on the ground, while the solution of this problem is a concave cycloid formed by a circle rolling on the ceiling.

2.e. **Optimal endpoint.** Assume the endpoint B is the lowest point on the cycloid. This is the point where the rolling circle that constructs the cycloid has completed a half revolution, or $\theta_b = \pi$. The arc length is the path the circle has travelled given by $x_b = r\theta_b = r\pi$. The point that draws the cycloid must have moved from the top of the circle to the bottom, $y_b = -2r$. The two Cartesian coordinates for point B (lowest point) must therefore have the following relation,

$$y_b = -\frac{2}{\pi}x_b \tag{18}$$

The time it takes for the body to get to this point is

$$T = \int_0^{x_b} \sqrt{\frac{1+y'^2}{-2gy}}dx = \int_0^{\pi} \sqrt{\frac{1+y'^2}{-2gy}}\frac{dx}{d\theta}d\theta$$

from previous calculations I know that $1-y'^2 = -k^2/y$ and that $dx/d\theta = -y$

$$T = \int_0^{\pi} \sqrt{\frac{k^2}{2gy^2}}(-y)d\theta = \frac{k}{\sqrt{2g}}\int_0^{\pi} d\theta = \frac{\pi k}{\sqrt{2g}}. \tag{19}$$

There appears to be a sign error here, but bear in mind that $y < 0$, which means that $\sqrt{y^2} = -y$.

It would be nice to compare this with something, like a straight line. A straight line from A to B is given by

$$s = \sqrt{x_b^2 + y_b^2} = \sqrt{\frac{1}{4}k^4\pi^2 - k^4} = k^2\sqrt{\frac{\pi^2}{4} - 1}. \tag{20}$$

At constant acceleration a

$$s = \frac{1}{2} a T'^2 \rightarrow T' = \sqrt{\frac{2s}{a}}$$

where $a = g \cos \alpha$, and α is the angle between y-axis and the straight line, given by

$$\cos \alpha = \frac{|y_b|}{s} = \frac{\left|-k^2\right|}{k^2 \sqrt{\frac{\pi^2}{4} - 1}} = \frac{1}{\sqrt{\frac{\pi^2}{4} - 1}}.$$

Putting everything together,

$$T' = \sqrt{\frac{2s}{a}} = \sqrt{\frac{2}{g} k^2 \sqrt{\frac{\pi^2}{4} - 1} \sqrt{\frac{\pi^2}{4} - 1}} = k \sqrt{\frac{2}{g} \left(\frac{\pi^2}{4} - 1 \right)}. \tag{21}$$

In relation to the period for the cycloid path in equation 19 this is

$$\frac{T'}{T} = \frac{\frac{\pi k}{\sqrt{2g}}}{k \sqrt{k \frac{2}{g} \left(\frac{\pi^2}{4} + 1 \right)}} = \frac{\pi}{g} \left(\frac{\pi^2}{4} + 1 \right) \approx 1.11. \tag{22}$$

In conclusion, the straight line path takes 1.11 times longer time than the cycloid path.