FYS3120 Classical Mechanics and Electrodynamics

Problem set 7

March 5, 2017

Problem 1 A small body with mass m is constrained to move (without friction) along a spiral-shaped channel, which is etched in a flat, circular disk, see Fig. 1. The disk rotates in the horizontal plane about an axis through the center of the disk, with constant angular velocity ω. The points on the spiral are characterized by polar coordinates (r, θ), with $r = a\theta$, where a is a constant. The angle variable θ is measured relative to a frame which rotates with the disk. In the expression for r the angle θ is chosen to take positive values and is not restricted to be less than 2π. The radius of the disk we refer to as R.

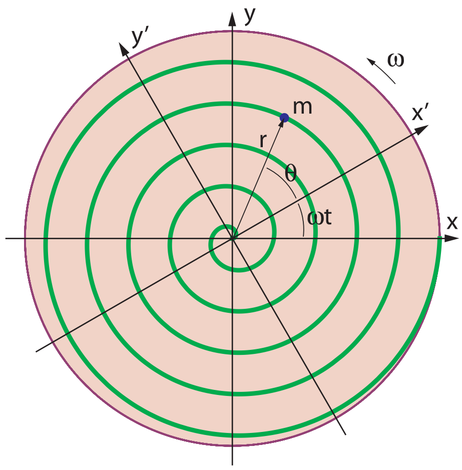

Figure 1: Rotating spiral.

a) Find the Cartesian coordinates of the moving body (x, y), measured relative to the non-rotating reference frame, expressed in terms of the (generalized) coordinate θ. Use this to determine the Lagrangian L as a function of $\dot{\theta}$ and θ.

b) Find Lagrange's equation, expressed as a differential equation for θ. Assuming the condition $\theta \gg 1$ to be satisfied, we make in the following the approximation $(1+\theta^2)\ddot{\theta} \approx \theta^2\ddot{\theta}$. Show that Lagrange's equation is then simplified to

$$\theta\ddot{\theta} + \dot{\theta}^2 - \omega^2 = 0. \tag{1}$$

c) Assume as initial conditions $r(0) = r_0$ and $\dot{r}(0) = 0$, with the above condition satisfied. Show that the (simplified) equation has solution of the form

$$r(t) = \sqrt{At^2 + Bt + C}, \tag{2}$$

and determine the constants A, B and C.

d) Determine the time T, which the body takes to reach the edge of the disk, expressed as a function of the initial coordinate r_0.

Problem 2

a) Below we have four equations that involve tensors of different ranks. Clearly the consistency rules for covariant equations are not satisfied in all places. Show where there are errors in each equation, and show how the equations can be modified to bring them to correct covariant form (there will be multiple alternative solutions but we prefer the simple ones).

$$C^\mu = T^\mu{}_\nu \, A^\mu \,, \quad D_\nu = T^\mu{}_\nu \, A_\mu \,, \quad E_{\mu\nu\rho} = T_{\mu\nu} \, S^\nu{}_\rho \,, \quad G = S_{\mu\nu} \, T^\nu{}_\alpha \, A^\alpha . \tag{3}$$

b) Assume A^μ and B^μ to be four-vectors and $T^{\mu\nu}$ to be a rank-2 tensor. Show that by making products of these and by lowering and contracting indices, one can form several new four-vectors and scalars.

c) We have defined the following four tensor fields as functions of the space-time coordinates $x = (x^0, x^1, x^2, x^3)$,

$$f(x) = x_\mu x^\mu \,, \quad g^\mu(x) = x^\mu \,, \quad b^{\mu\nu}(x) = x^\mu x^\nu \,, \quad h^\mu(x) = \frac{x^\mu}{x_\nu x^\nu} . \tag{4}$$

Calculate the following derivatives,

$$\partial_\mu f(x) \,, \quad \partial_\mu g^\mu(x) \,, \quad \partial_\mu b^{\mu\nu}(x) \,, \quad \partial_\mu h^\mu(x), \tag{5}$$

where the differential operator ∂_μ is defined by

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} . \tag{6}$$

Hint: If you are uncertain about results for tensors, a convenient way to check these is always to specify the index values explicitly, *e.g.*, in the first case, by choosing $\mu = 1$, which gives $\partial_\mu = \partial_x$, and writing $f(x) = (ct)^2 - (x^2 + y^2 + z^2)$.

Problem 3 Modified version of final-exam question in 2006

An electron, with charge e, moves in a constant electric field $\vec{E}$. The motion is determined by the relativistic Newton's equation

$$\frac{d}{dt}\vec{p} = e\vec{E}, \tag{7}$$

where $\vec{p}$ denotes the relativistic momentum $\vec{p} = m_e \gamma \vec{v}$, with m_e as the electron rest mass, $\vec{v}$ as the velocity and $\gamma = 1/\sqrt{1 - (v/c)^2}$ as the relativistic gamma factor. We assume the electron to move along the field lines, that is, there is no velocity component orthogonal to $\vec{E}$. *Hint:* We remind you that the relativistic energy can be written

$$E = \gamma m_e c^2 = \sqrt{p^2c^2 + m_e^2c^4}. \tag{8}$$

a) Show that if $v = 0$ at time $t = 0$, then γ depends on time t as

$$\gamma = \sqrt{1 + \kappa^2 t^2}, \tag{9}$$

and determine κ.

b) The proper time τ is related to the coordinate time t by the formula $\frac{dt}{d\tau} = \gamma$. Show that if we write $\gamma = \cosh \kappa\tau$ then τ satisfies the above condition.

c) For linear motion we have the following relation between the proper acceleration a_0 and the acceleration a measured in a fixed inertial reference frame, $a_0 = \gamma^3 a$. Use this to show that the electron has a constant proper acceleration, given by $\vec{a}_0 = e\vec{E}/m_e$. *Hint:* You will need to find the time-derivative of γ. (A look in the lecture notes might be helpful.) As a reminder we also give the following functional relations:

$$\cosh^2 x - \sinh^2 x = 1\,, \quad \frac{d}{dx}\cosh x = \sinh x\,, \quad \frac{d}{dx}\sinh x = \cosh x. \tag{10}$$