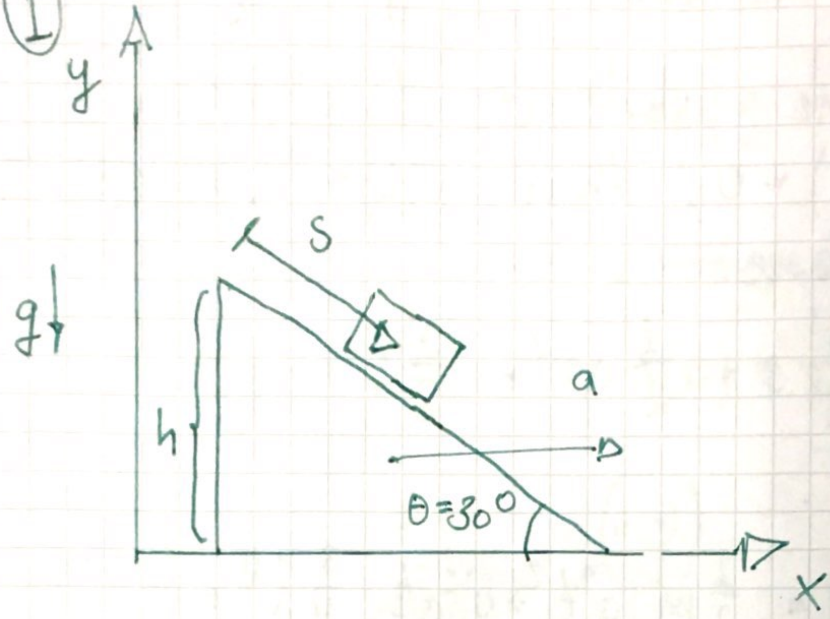
FYS3120 - Problem set 3. Sebastian G. Winther-Larsen

①



a) $\cos\theta = \frac{x}{s}$ (a, v is vanishing)

$$x = s\cos\theta = s\cos 30^\circ = s\cos\frac{\pi}{6} = \underline{\underline{\frac{\sqrt{3}}{2}s}}$$

$$y = h - s\sin\theta = h - s\sin\frac{\pi}{6} = \underline{\underline{h - s/2}}$$

$$\dot{x} = \frac{\sqrt{3}}{2}\dot{s} \qquad \dot{y} = -\frac{\dot{s}}{2}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)^2 = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$= \frac{1}{2}m\left(\frac{3}{4}\dot{s}^2 + \frac{1}{4}\dot{s}^2\right) = \underline{\frac{1}{2}m\dot{s}^2}.$$

$$V = mgy = mg\left(h - \frac{s}{2}\right)$$

$$\rightarrow L = T - V = \underline{\underline{\frac{1}{2}m\dot{s}^2 + \frac{1}{2}mgs - mgh}}$$

b) (a is constant, non-vanishing)

$y = h - \frac{s}{2}$ – the same! $\Rightarrow \dot{y} = -\frac{\dot{s}}{2}$

assuming plane starts at $v(t=0) = 0$ and $x(t=0) = 0$, then $v(t) = at$, $x(t) = \frac{1}{2}at^2$ for the plane.

$$x = x_{PLANE} + \frac{\sqrt{3}}{2}s = \frac{1}{2}at^2 + \frac{\sqrt{3}}{2}s$$

$$\dot{x} = at + \frac{\sqrt{3}}{2}\dot{s}.$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left(a^2t^2 + \sqrt{3}\dot{s}at + \frac{3}{4}\dot{s}^2 + \frac{1}{4}\dot{s}^2\right)$$

$$= \frac{1}{2}m(\dot{s}^2 + \sqrt{3}at\dot{s} + a^2t^2)$$

V does not change in this situation.

$$\underline{\underline{L = T - V = \frac{1}{2}m(\dot{s}^2 + \sqrt{3}at\dot{s} + a^2t^2) + \frac{1}{2}mgs - mgh}}$$

c) Lagrange equation. Ass'n: at $t=0$, $s=0$ zero vel. rel. to plane.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

$$\frac{\partial L}{\partial s} = \frac{1}{2}mg \qquad \frac{\partial L}{\partial \dot{s}} = m\dot{s} + \frac{\sqrt{3}}{2}mat$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s} + \frac{\sqrt{3}}{2}ma$$

gives: $m\ddot{s} + \frac{\sqrt{3}}{2}ma - \frac{1}{2}mg = 0 \Rightarrow \ddot{s} + \frac{\sqrt{3}}{2}a - \frac{1}{2}g = 0$

$$\ddot{s} = \frac{1}{2}g \quad \frac{\sqrt{3}}{2}a \,.$$

$$\dot{s} = \int_0^t \left(\frac{1}{2}g - \frac{\sqrt{3}}{2}a\right)dt = \left(\frac{1}{2}g - \frac{\sqrt{3}}{2}a\right)\int_0^t 1\, dt$$

same procedure (ish)

$$\dot{s} = \frac{1}{2}(g - \sqrt{3}a)t \rightarrow s(t) = \frac{1}{4}(g - \sqrt{3}a)t^2$$

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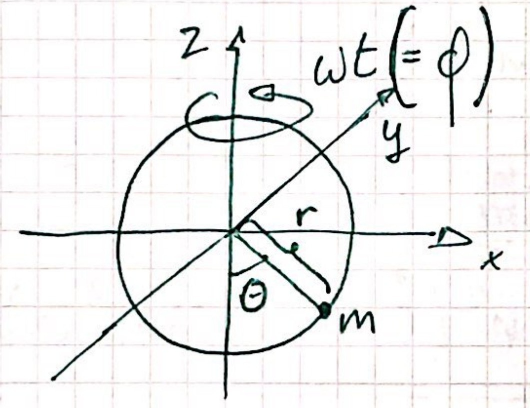


rotating hoop, a particle slides along it.

a) spherical coordinate conversion gives the cartesian coordinates;

$x = r\sin\theta\cos\omega t, \quad y = r\sin\theta\sin\omega t, \quad z = -r\cos\theta$

$\dot{x} = r\cos\theta\cos\omega t\cdot\dot{\theta} - r\sin\theta\sin\omega t\cdot\omega$

$\dot{y} = r\cos\theta\sin\omega t\cdot\dot{\theta} + r\sin\theta\cos\omega t\cdot\omega$

$\dot{z} = r\sin\theta\cdot\dot{\theta}$

$T = \frac{1}{2}m\boxed{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}$ — need this.

$$\dot{x}^2 = r^2\left(\dot\theta^2\cos^2\theta\cos^2\omega t - 2\dot\theta\omega\cos\theta\sin\theta\cos\omega t\sin\omega t + \omega^2\sin^2\theta\sin^2\omega t\right)$$

$$\dot{y}^2 = r^2\left(\dot\theta^2\cos^2\theta\sin^2\omega t + 2\dot\theta\omega\cos\theta\sin\theta\cos\omega t\sin\omega t + \omega^2\sin^2\theta\cos^2\omega t\right)$$

$\cos^2 x + \sin^2 x = 1$ the same, opposite signs. $\cos^2 x + \sin^2 x = 1$

$$\dot{z}^2 = r^2\dot\theta^2\sin^2\theta$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2\left(\dot\theta^2\cos^2\theta + \omega^2\sin\theta + \dot\theta^2\sin^2\theta\right)$$

$$= r^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right)$$

$$T = \frac{1}{2}mr^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right)$$

$V = 0$ no gravity

$$L = T - V = T = \underline{\underline{\frac{1}{2}mr^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right)}}$$

Lagrange's eq'n: $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$

$$\frac{\partial L}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{1}{2}mr^2\omega^2\sin^2\theta\right) = mr^2\omega^2\sin\theta\cos\theta$$

$$\frac{\partial L}{\partial\dot\theta} = mr^2\dot\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = mr^2\ddot\theta$$

$$mr^2\ddot\theta - mr^2\omega^2\sin\theta\cos\theta = 0$$

$$\underline{\underline{\ddot\theta - \omega^2\sin\theta\cos\theta = 0}}$$

b) The potential of a particle moving as described can be found by way of the fictious centrifugal force

$$F_{centrifugal} = \omega \times (\omega \times r)$$

$$\Phi_{centr.} = -\int_0^r [\omega \times (\omega \times r)] \cdot dr$$

$$= -\int_0^r \omega^2 r dr = -\frac{1}{2}\omega^2 r^2$$

$$= -\frac{1}{2}\omega^2 r^2 \sin^2\theta$$

including the mass of the particle;

$$V(\theta) = -\frac{1}{2} m \omega^2 r^2 \sin^2\theta$$

$$L = \frac{1}{2} m r^2 (\dot{\theta}^2 + \omega^2 \sin^2\theta)$$

which gives the corresponding

$$T = \frac{1}{2} m r^2 \dot{\theta}^2,$$

could translate to a kinetic energy, when the hoop is <u>not</u> moving ($\omega = 0$)

The optimal T's can be found in the usual "high school way", by differentiating $V(\theta)$ wrt. θ, but one might as well be smart and realise that the <u>stable</u> equilibria is when the particle is farthest away from the rotational axis, at $\theta = \frac{\pi}{2}$, $\Theta = \frac{3\pi}{2}$. These are minima. Unstable equilibria are when the particle is <u>on</u> the rotational axis, i.e. $\theta = 0$, $\theta = \pi$.

c) θ_0 is an eq'm point. Small deviation $\varphi = \theta - \theta_0$. Small angle form?

$\Rightarrow \theta = \theta_0 + \varphi$ pick a point: $\theta_0 = \frac{\pi}{2}$.

gives $\theta = \frac{\pi}{2} + \varphi$, $\dot{\theta} = \dot{\varphi}$, $\ddot{\theta} = \ddot{\varphi}$

inserting into equation from a)

$$\ddot{\theta} - \omega^2 \sin\theta \cos\theta = \ddot{\varphi} - \omega^2 \sin\left(\frac{\pi}{2} + \varphi\right) \cos\left(\frac{\pi}{2} + \varphi\right) = 0$$

$\sin\left(\frac{\pi}{2} + x\right) = \cos x$, $\cos\left(\frac{\pi}{2} + \varphi\right) = -\sin x$

$$\Rightarrow \ddot{\varphi} + \omega^2 \cos\varphi \sin\varphi = 0$$

as $\varphi \to 0$, $\cos\varphi \to 1$ and $\sin\varphi \to \varphi$.

which gives $\ddot{\varphi} + \omega^2 \varphi = 0$ <u>HARMONIC OSCILLATOR!</u>

sol'n: $\varphi \sim e^{\lambda t}$. substitute: $\frac{\partial}{\partial t}\left(e^{\lambda t}\right) + \omega^2 e^{\lambda t} = 0$

gives: $\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0 \Rightarrow (\omega^2 + \lambda^2) e^{\lambda t} = 0$

$e^{\lambda t} \neq 0$. $\omega^2 + \lambda^2 = 0 \Rightarrow \lambda = i\omega \vee \lambda = -i\omega$

$\varphi_1 = c_1 e^{-i\omega t}$, $\varphi_2 = c_2 e^{+i\omega t}$. $\varphi = \varphi_1 + \varphi_2$

$\varphi = c_1 e^{-i\omega t} + c_2 e^{i\omega t} = c_1(\cos\omega t - i\sin\omega t) + c_2(\cos\omega t - \sin\omega t)$

$\varphi = (c_1 + c_2)\cos\omega t + i(c_1 + c_2)\sin\omega t$.

redefine $c_1 = c_1 + c_2$ $c_2 = i(c_2 - c_1)$

$$\boxed{\varphi(t) = c_1 \cos\omega t + c_2 \sin\omega t.}$$

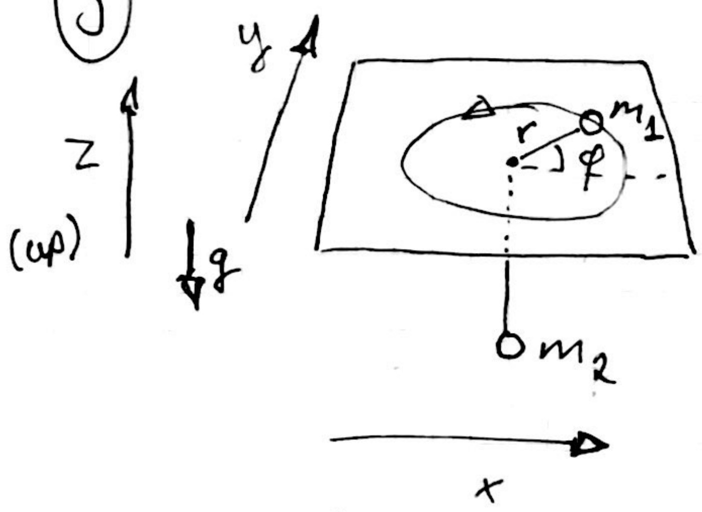
some sort of oscillation aroun the equilibrium point of the particle. The other stable eq'm point should yield the same result

$$\ddot{\theta} - \omega^2 \sin\theta \cos\theta = \ddot{\varphi} - \omega^2 \sin\left(\frac{3\pi}{2} + \varphi\right)\cos\left(\frac{3\pi}{2} + \varphi\right)$$

$$\left.\begin{array}{l} \sin\left(\frac{3\pi}{2} + x\right) = -\cos(x) \\ \cos\left(\frac{3\pi}{2} + x\right) = \sin(x) \end{array}\right\} \Rightarrow = \ddot{\varphi} + \omega^2 \cos\varphi \sin\varphi = 0$$

yup! same equation.

(3)



$m_1 = m$

$m_2 = m$

l is length of rope.

origin is the hole.

find Lagrange's eq'ns of motion in polar coordinates.

$x_1 = r\cos\varphi \qquad y_1 = r\sin\varphi \qquad z_1 = 0$

$x_2 = 0 \qquad y_2 = 0 \qquad z_2 = r - l.$

$\dot{x}_1 = -\dot{\varphi} r \sin\varphi + \dot{r}\cos\varphi$

$\dot{y}_1 = \dot{\varphi} r \cos\varphi + \dot{r}\sin\varphi$

$\dot{z}_2 = \dot{r}$

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{z}_2^2) = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_2^2)$$

$$= \frac{1}{2}m(r^2\dot{\varphi}^2\sin^2\varphi - \cancel{2r\dot{r}\dot{\varphi}\sin\varphi\cos\varphi} + \dot{r}^2\cos^2\varphi$$
$$+ r^2\dot{\varphi}^2\cos^2\varphi + \cancel{2r\dot{r}\dot{\varphi}\cos\varphi\sin\varphi} + \dot{r}^2\sin^2\varphi + \dot{r}^2)$$

$$= \frac{1}{2}m(r^2\dot{\varphi}^2 + 2\dot{r}^2)$$

$$V = mgz_1 + mgz_2 = 0 + mg(r-1) = mg(r-1)$$

$$L = T - V = \frac{1}{2}m(r^2\dot{\varphi}^2 + 2\dot{r}^2) - mg(r-1)$$

Lagrange's eq'ns:

$$\frac{\partial L}{\partial r} = mr\dot{\varphi}^2 - mg \qquad \frac{\partial L}{\partial \dot{r}} = 2m\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 2m\ddot{r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 2m\ddot{r} - m\dot{\varphi}r + mg = 0$$

Lagrange eq'n in r: $\ddot{r} = \frac{1}{2}\dot{\varphi}r - \frac{1}{2}g$

$$\frac{\partial L}{\partial \varphi} = 0 \qquad \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 2m\dot{\varphi}r\dot{r} + mr^2\ddot{\varphi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 2m\dot{\varphi}r\dot{r} + mr^2\ddot{\varphi} - 0 = 0$$

$$2\cancel{m}\dot{\varphi}\cancel{r}\dot{r} + \cancel{m}r^{\cancel{2}}\ddot{\varphi} = 0$$

and for φ: $\ddot{\varphi} = -\frac{2}{r}\dot{\varphi}\dot{r}$

b) The angular momentum of a mass about a point, as the one on the plane is $L = r^2 m\omega$, where $\omega = \dot{\varphi}$. This is constant, and therefore

$$\frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = k \quad (\text{constant})$$

$$\downarrow$$

$$\dot{\varphi} = \frac{k}{m r^2}$$

this can be inserted into the Lagrange eq'n for r:

$$2m\ddot{r} - m\dot{\varphi}^2 r + mg = 0$$

$$2m\ddot{r} - m\left(\frac{k}{mr^2}\right)^2 r + mg = 0$$

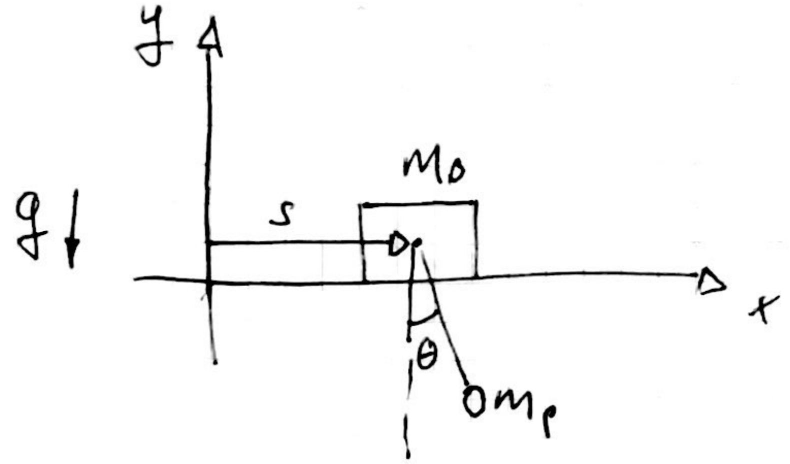
$$2m\ddot{r} - \frac{k^2}{mr^3} + mg = 0$$

$$\ddot{r} = \frac{1}{2}\frac{k^2}{m^2 r^3} - \frac{1}{2}g \, .$$

Describes the "battle" between the pull of gravity and the pull of the centrifuge. If $k \neq 0$, then the particle on top of the plane is spinning and counteracting gravity. If r gets below zero the velocity will rapidly increase. I think the equation represents some sort of non-harmonic oscillation.

④ A pendulum anchored to a box.

$m_B = m_p = m$



at $t=0$, pendulum angle is θ_0, box and pendulum at zero velocity.

a) Box: $x_B = s$ $\quad y_p = 0$

PENDULUM: $x_p = s + d\sin\theta \quad y_p = -d\cos\theta$.

$$T = \frac{1}{2} m_b (\dot{x}_B^2 + \dot{y}_B^2) + \frac{1}{2} m_p (\dot{x}_p^2 + \dot{y}_p^2)$$

$$= \frac{1}{2} m (\dot{x}_B^2 + \dot{x}_p^2 + \dot{y}_p^2)$$

$$= \frac{1}{2} m (\dot{s}^2 + (\dot{s} + d\dot{\theta}\cos\theta)^2 + (d\dot{\theta}\sin\theta)^2)$$

$$= \frac{1}{2} m (\dot{s}^2 + \dot{s}^2 + 2d\dot{s}\dot{\theta}\cos\theta + d^2\dot{\theta}^2 \cos^2\theta + d^2\dot{\theta}^2 \sin^2\theta)$$

$$= \frac{1}{2} m (2\dot{s} + 2d\dot{s}\dot{\theta}\cos\theta + d^2\dot{\theta}^2)$$

$$V = \underbrace{\cancel{mg y_b}}_{=0} + mg y_p = -mgd\cos\theta$$

$$L = T - V = \frac{1}{2} m (2\dot{s}^2 + 2d\dot{s}\dot{\theta}\cos\theta + d^2\dot{\theta}^2) + mgd\cos\theta$$

$$= m\dot{s} + md\dot{s}\dot{\theta}\cos\theta + \frac{1}{2} md^2\dot{\theta}^2 + mgd\cos\theta$$

b) L does not depend on s, thus

$$\frac{\partial L}{\partial s} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \dot{s}} = k \text{ (constant)}$$

one calls $p_s = \frac{\partial L}{\partial \dot{s}}$ the conjugate momentum.

$$\frac{\partial L}{\partial \dot{s}} = 2m\dot{s} + md\dot{\theta}\cos\theta = m\dot{s} + m\dot{s} + md\dot{\theta}\cos\theta$$

$$= m\dot{s} + m(\dot{s} + d\dot{\theta}\cos\theta)$$

total momentum in x-direction!

$$= m\dot{x}_B + m\dot{x}_P$$

one can express $\dot{s}$ as

$$\dot{s} = \frac{1}{2m}\left(p_s - md\dot{\theta}\cos\theta\right) = \frac{p_s}{2m} - \frac{1}{2}d\dot{\theta}\cos\theta .$$

and insert into Lagrangian:

$$L = m\left(\frac{p_s}{2m} - \frac{1}{2}d\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}md^2\dot{\theta}^2$$

$$+ md\left(\frac{p_s}{2m} - \frac{1}{2}d\dot{\theta}\cos\theta\right)\dot{\theta}\cos\theta + mgd\cos\theta$$

$$= \frac{p_s^2}{4m} - \cancel{\frac{p_s}{2}d\dot{\theta}\cos\theta} + \frac{1}{4}md^2\dot{\theta}^2\cos^2\theta$$

$$+ \frac{1}{2}md^2\dot{\theta}^2 + \cancel{\frac{p_s}{2}d\dot{\theta}\cos\theta} - \frac{1}{2}md^2\dot{\theta}^2\cos^2\theta$$

$$+ mgd\cos\theta$$

$$= \frac{p_s^2}{4m} + \frac{1}{2}md^2\dot{\theta}^2\left(1 - \frac{1}{2}\cos^2\theta\right) + mgd\cos\theta$$

c) $\dfrac{\partial L}{\partial \theta} = \dfrac{1}{2} m d^2 \dot\theta^2 \cos\theta \sin\theta - mgd \sin\theta$

$\dfrac{\partial L}{\partial \dot\theta} = m d^2 \dot\theta \left(1 - \dfrac{1}{2}\cos^2\theta\right)$

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\theta}\right) = m d^2 \ddot\theta - \dfrac{1}{2} m d^2 \ddot\theta \cos^2\theta + m d^2 \dot\theta^2 \cos\theta \sin\theta$

$= m d^2 \ddot\theta \left(1 - \dfrac{1}{2}\cos^2\theta\right) + m d^2 \dot\theta^2 \cos\theta \sin\theta$

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot\theta}\right) - \dfrac{\partial L}{\partial \theta} = 0$

$\cancel{md^2}\ddot\theta\left(1 - \dfrac{1}{2}\cos^2\theta\right) + \cancel{md^2}\underline{\dot\theta^2 \cos\theta \sin\theta}$

$-\dfrac{1}{2}\cancel{md^2}\underline{\dot\theta^2 \cos\theta \sin\theta} + \cancel{mgd}^{\,-1} \sin\theta = 0$

$\left(1 - \dfrac{1}{2}\cos^2\theta\right)\ddot\theta + \dfrac{1}{2}\sin\theta\cos\theta\,\dot\theta^2 + \dfrac{g}{d}\sin\theta = 0$

d) approximations to first order:

$\cos\theta \simeq 1 \qquad \sin\theta \simeq \theta$.

the Lagrange equation becomes:

$\dfrac{1}{2}\ddot\theta + \dfrac{1}{2}\theta \cdot \dot\theta^2 + \dfrac{g}{d}\theta = 0$

$\ddot\theta + \theta \cdot \dot\theta^2 + \dfrac{2g}{d}\theta = 0$

$\theta \cdot \dot\theta^2$ is hopefully very small.

$\rightarrow$ SHO with $\omega = \sqrt{\dfrac{2g}{d}}$