HAMILTONIAN DYNAMICS OF THE BRACHISTOCHRONE PROBLEM

PROBLEM SHEET 5: FYS3120

SEBASTIAN G. WINTHER-LARSEN

1. Coriolis and Centrifugal Forces

A particle with mass m moves freely on a horizontal plane. There are no constraints, but in the following we will consider the free motion described in a rotating reference frame. We refer to the Cartesian coordinates of a fixed frame as (x, y) and the coordinates of the rotating frame as (ξ, η). They are related by the standard expressions

$$x = \xi \cos \omega t - \eta \sin \omega t, \tag{1}$$

$$y = \xi \sin \omega t + \eta \cos \omega t, \tag{2}$$

where ω is the angular velocity of the rotation.

1.a. **Lagrangian.** First we need $\dot{x}$ and $\dot{y}$;

$$\begin{aligned}
\dot{x} &= \dot{\xi} \cos \omega t - \omega\xi \sin \omega t - \dot{\eta} \sin \omega t - \omega\eta \cos \omega t \\
&= (\dot{\xi} - \omega\eta) \cos \omega t - (\omega\xi + \dot{\eta}) \sin \omega t, \\
\dot{y} &= \dot{\xi} \sin \omega t + \omega\xi \cos \omega t + \dot{\eta} \cos \omega t - \omega\eta \sin \omega t \\
&= (\dot{\xi} - \omega\eta) \sin \omega t + (\omega\xi + \dot{\eta}) \cos \omega t,
\end{aligned}$$

we also need their squares

$$\dot{x}^2 = (\dot{\xi} - \omega\eta)^2 \cos^2 \omega t - 2(\dot{\xi} - \omega\eta)(\omega\xi + \dot{\eta}) \cos \omega t \sin \omega t + (\omega\xi + \dot{\eta})^2 \sin^2 \omega t,$$

$$\dot{y}^2 = (\dot{\xi} - \omega\eta)^2 \sin^2 \omega t + 2(\dot{\xi} - \omega\eta)(\omega\xi + \dot{\eta}) \sin \omega t \cos \omega t + (\omega\xi + \dot{\eta})^2 \cos^2 \omega t,$$

the sum of the squares is

$$\dot{x}^2 + \dot{y}^2 = (\dot{\xi} - \omega\eta)^2 + (\omega\xi + \dot{\eta})^2 = dot\xi^2 - 2\dot{\xi}\omega\eta + \omega^2\eta^2 + \omega^2\xi^2 + 2\omega\xi\dot{\eta} + \dot{\eta}^2,$$

which can now be used to find the Lagrangian

$$L = T = \frac{1}{2} m[\dot{\xi}^2 + \dot{\eta}^2 + \omega^2(\xi^2 + \eta^2) + 2\omega(\xi\dot{\eta} - \dot{\xi}\eta)]. \tag{3}$$

As there is no gravity there is no potential, V.

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1.b. **Equations of Motion.** The general Lagrange equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{4}$$

and can be found for every generalised coordinate.

1.b.1. *Lagrange's equation for ξ.* Start by finding all parts of equation 4

$$\begin{aligned}\frac{\partial L}{\partial \xi} &= m\omega^2\xi + m\omega\dot{\eta},\\ \frac{\partial L}{\partial \dot{\xi}} &= m\dot{\xi} - m\omega\eta,\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) &= m\ddot{\xi} - m\omega\dot{\eta}.\end{aligned}$$

Combining all these gives Lagrange's equation for ξ

$$m\ddot{\xi} = m\omega^2\xi + 2m\omega\dot{\eta}. \tag{5}$$

1.b.2. *Lagrange's equation for η.* I am going to make an implicit symmetry argument here[1] and simply write down the Lagrange equation for η

$$m\ddot{\eta} = m\omega^2\eta - 2m\omega\dot{\xi} \tag{6}$$

1.b.3. *Analysis.* Notice that I did not write down the Lagrange equations for ξ and η, given by equations 5 and 6 respectively, in the conventional way dictated by equation 4. The reason for this is that the Lagrange equations found in this problem are incredibly simlilar to Newton's second law for rotational coordinates,

$$\mathbf{F} = m\ddot{\rho} = \mathbf{F}_{\text{imp}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{Euler}}, \tag{7}$$

where $\mathbf{F}_{\text{imp}}$ are the forces impressed on the system (= 0 here), $\mathbf{F}_{\text{centrifugal}} = -m\omega \times (\omega \times \rho)$ is the centrifugal force, $\mathbf{F}_{\text{Coriolis}} = -2m\omega \times \dot{\rho}$ is the Coriolis force and $\mathbf{F}_{\text{Euler}}$ is the Euler force, felt in reaction to any acceleration (also = 0 here). In all forces ρ is the position vector in the rotating frame. Equation 7 becomes

$$m\ddot{\rho} = -m\omega \times (\omega \times \rho) - 2m\omega \times \dot{\rho}, \tag{8}$$

which is very similar to the equations of motion in 5 and 6.

2. The Brachistochrone Challenge

The problem as posed to Isaac Newton, amongst others, in 1696 can be formulated in the following way,

> Given two points A and B in a vertical plane, what is the curve traced out by a body acted on only by gravity, which starts at A and reaches B in the shortest time.

[1] Did you notice it?

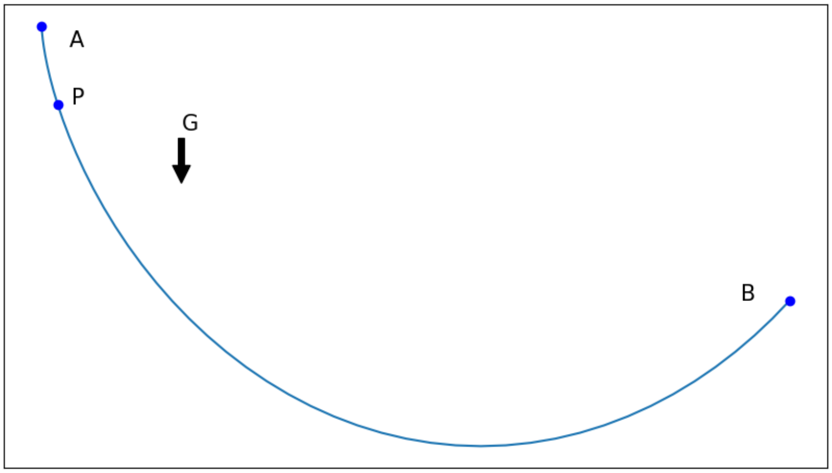

FIGURE 1. Illustration of the Brachistochrone problem

It is said that Newton had a solution already the following day. Herein the challenge is the following, can the problem be solved using the correspondence between the variational problem and the Lagrange equation? The body P is treated as a point particle of mass m and the path is represented by a function $y(x)$ with x as the horizontal axis and y as the vertical axis. The boundary conditions, which fix the positions of point A and B, are specified as $y(x_A) = y_A$, $y(x_b) = y_b$. A simplifications is to assume that $x_A = y_A$.

2.a. **The Period of the Motion.** The total time the body spends moving along the curve is given by a simple integral of infinitesimal time steps

$$T = \int_{t_a}^{t_b} dt. \tag{9}$$

Using the velocity and displacement relations $vdt = ds$ gives

$$T = \int_{A}^{B} \frac{1}{v} ds. \tag{10}$$

One such infinitesimal displacement can be decomposed in a Pythagorean manner into x and y parts, $ds = \sqrt{dx^2 + dy^2}$. Inserting into 10 yields

$$T = \int_{A}^{B} \frac{1}{v} \sqrt{dx^2 + dy^2} = \int_{x_a}^{x_b} \frac{1}{v} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx. \tag{11}$$

By assuming conservations of energy $\frac{1}{2}mv^2 + mgy = 0 \rightarrow v = \sqrt{-2gy}$ and setting $y' = \frac{dy}{dx}$ equation 11 becomes

$$T = \int_{x_a}^{x_b} \sqrt{\frac{1 + y'^2}{-2gy}} dx. \tag{12}$$

If one were to consider this an action integral, then the integrand must therefore be the Lagrangian for the system with y and y' as generalised coordinates

$$L(y, y') = \sqrt{\frac{1 + y'^2}{-2gy}} \tag{13}$$