(1) a)

$$x = a\theta\cos(\theta+\omega t)$$

$$y = a\theta\sin(\theta+\omega t)$$

$$\dot{x} = a\dot{\theta}\cos(\theta+\omega t) - a\theta\sin(\theta+\omega t)(\dot{\theta}+\omega)$$

$$\dot{y} = a\dot{\theta}\sin(\theta+\omega t) + a\theta\cos(\theta+\omega t)(\dot{\theta}+\omega)$$

$$\dot{x}^2 = \boxed{a^2\dot{\theta}^2}\cos^2(\theta+\omega t) - \cancel{2a^2\dot{\theta}\theta(\dot{\theta}+\omega)\cos(\theta+\omega t)\sin(\theta+\omega t)}$$
$$+\boxed{a^2\theta^2(\dot{\theta}+\omega)^2}\sin^2(\theta+\omega t)$$

$$\dot{y}^2 = \boxed{a^2\dot{\theta}^2}\sin^2(\theta+\omega t) + \cancel{2a^2\dot{\theta}\theta(\dot{\theta}+\omega)\cos(\theta+\omega t)\sin(\theta+\omega t)}$$
$$+\boxed{a^2\theta^2(\dot{\theta}+\omega)^2}\cos^2(\theta+\omega t)$$

$$\dot{x}^2+\dot{y}^2 = a^2\dot{\theta}^2 + a^2\theta^2(\dot{\theta}+\omega)^2$$
$$= a^2\left[\dot{\theta}^2 + \theta^2(\dot{\theta}^2 + 2\dot{\theta}\omega + \omega^2)\right]$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}ma^2\left[\dot{\theta}^2 + \theta^2(\dot{\theta}^2 + 2\dot{\theta}\omega + \omega^2)\right]$$

$$V = 0.$$

$$L = T - V = T = \frac{1}{2}ma^2\left[\dot{\theta}^2 + \theta^2(\dot{\theta}^2 + 2\dot{\theta}\omega + \omega^2)\right]$$

b) $L = \frac{1}{2} m a^2 \left[\dot{\theta}^2 + \theta^2 (\dot{\theta} + \omega)^2 \right]$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{\cancel{2}} m a^2 \left[\cancel{2}\dot{\theta} + \cancel{2}\theta^2 (\dot{\theta} + \omega) \right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) = m a^2 \left[\ddot{\theta} + 2\theta\dot{\theta}(\dot{\theta} + \omega) + \theta^2 \ddot{\theta} \right]$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{\cancel{2}} m a^2 \left[\cancel{2}\theta (\dot{\theta} + \omega)^2 \right] = m a^2 \left[\theta (\dot{\theta} + \omega)^2 \right]$$

Lagrange eqn:

$$\cancel{m a^2} \left[\ddot{\theta} + 2\theta\dot{\theta}(\dot{\theta} + \omega) + \theta^2 \ddot{\theta} \right] - \cancel{m a^2} \left[\theta (\dot{\theta} + \omega)^2 \right] = 0$$

$$\ddot{\theta} + \theta^2 \ddot{\theta} = (1 + \ddot{\theta})\theta^2 = \ddot{\theta}\theta^2, \quad \theta \gg 1$$

$$\ddot{\theta}\theta^2 + 2\theta\dot{\theta}^2 + 2\theta\dot{\theta}\omega - \theta(\dot{\theta}^2 + 2\dot{\theta}\omega + \omega^2) = 0$$

$$\ddot{\theta}\theta^2 + \cancel{2}\theta\dot{\theta}^2 + \cancel{2\theta\dot{\theta}\omega} - \theta\dot{\theta}^2 - \cancel{2\theta\dot{\theta}\omega} - \theta\omega^2 = 0$$

$$\ddot{\theta}\theta^2 + \theta\dot{\theta}^2 - \theta\omega^2 = 0$$

$$\theta\ddot{\theta} + \dot{\theta}^2 - \omega^2 = 0 \qquad \text{Q.E.D.}$$

c) $r = a\theta \to \theta = \frac{r}{a}, \dot{\theta} = \frac{\dot{r}}{a}, \ddot{\theta} = \frac{\ddot{r}}{a}$

$$\theta\ddot{\theta} + \dot{\theta}^2 - \omega^2 = 0 \quad (*)$$

$$r(t) = \sqrt{At^2 + Bt + C}$$

$$\dot{r}(t) = \frac{1}{2}\frac{1}{\sqrt{At^2+Bt+C}}(2At+B) = \frac{1}{2}(At^2+Bt+C)^{-\frac{1}{2}}(2At+B)$$

$$\ddot{r}(t) = -\frac{1}{4}(At^2+Bt+C)^{-\frac{3}{2}}(2At+B)^2 + \frac{1}{2}(At^2+Bt+C)^{-\frac{1}{2}}2A$$

$$\theta\ddot{\theta} = \frac{r\ddot{r}}{a^2}$$

$$= \frac{1}{a^2}\left(\cancel{(At^2+Bt+C)^{\frac{1}{2}}}\left(-\frac{1}{4}(At^2+Bt+C)^{-\cancel{\frac{3}{2}}-1}(2At+B)^2 + A\cancel{(At^2+Bt+C)^{-\frac{1}{2}}}\right)\right)$$

$$= \frac{1}{a^2}\left(A - \frac{(2At+B)^2}{4(At^2+Bt+C)}\right)$$

$$\dot{\theta}^2 = \frac{\dot{r}^2}{a^2} = \frac{1}{a^2}\left(\frac{(2At+B)^2}{4(At^2+Bt+C)}\right)$$

(*) becomes:

$$\frac{1}{a^2}\left(A - \cancel{\frac{(2At+B)^2}{4(At^2+Bt+C)}} + \cancel{\frac{(2At+B)^2}{4(At^2+Bt+C)}}\right) - \omega^2 = 0$$

$$\frac{A}{a^2} - \omega^2 = 0 .$$

Is a solution if $\omega^2 = \frac{A}{a^2}$

This gives an expression for A,

$$A = \frac{\omega^2}{a^2} .$$

$$r(t) = \sqrt{\frac{\omega^2}{a^2} t^2 + Bt + C}$$

$$r(0) = r_0 \rightarrow \sqrt{C} = r_0 \rightarrow C = r_0^2 .$$

$$r(t) = \sqrt{\frac{\omega^2}{a^2} + Bt + r_0^2}$$

$$\dot{r}(t) = \frac{1}{2}(At^2 + Bt + C)^{-\frac{1}{2}}(2At + B)$$

$$\dot{r}(0) = 0 \rightarrow \frac{\sqrt{C}\, B}{2} = 0$$

gives $B = 0$.

the solution becomes:

$$r(t) = \sqrt{\frac{\omega^2}{a^2} t^2 + r_0^2} .$$

d) $r(T) = R \rightarrow \sqrt{\frac{\omega^2}{a^2} T^2 + r_0^2} = R$

$$\frac{\omega^2}{a^2} T^2 = R^2 - r_0^2$$

$$T = \frac{a}{\omega} \sqrt{R^2 - r_0^2}$$

time it takes for the body to reach the edge of the disk as a function of initial coordinate r_0.

② a) correcting equations in accordance with consistency rules.

(I) $C^{\mu} = T^{\mu}{}_{\nu} A^{\mu}$ $\longrightarrow$ $C_{\nu} = T^{\mu}{}_{\nu} A_{\mu}$

too many μ's upstairs

Rule: on same side, the same indices should be <u>upstairs</u> <u>and</u> <u>downstairs</u>. On opposite side of equal sign: <u>either</u> <u>upstairs</u> or <u>downstairs</u>.

(II) $D_{\nu} = T^{\mu}{}_{\nu} A_{\mu}$ this one is fine.

(III) $E_{\mu\nu\rho} = T_{\mu\nu} S^{\nu}{}_{\rho}$ ν is alredy used here.

$\longrightarrow$ $E_{\mu\rho} = T_{\mu\nu} S^{\nu}{}_{\rho}$

a dimension too many

(IV) $G = S_{\mu\nu} T^{\nu}{}_{\alpha} A^{\alpha}$ A-OKAY!

missing a μ $\longrightarrow$ $G_{\mu} = S_{\mu\nu} T^{\nu}{}_{\alpha} A^{\alpha}$.

b) A^μ an B^μ are four-vectors
$T^{\mu\nu}$ is a rank 2 tensor.
(Raising/lowering: $x_\mu = g_{\mu\nu} x^\nu$, $x^\mu = g^{\mu\nu} x_\nu$)
From the constructs above, new things can be made:

scalars: $a = A^\mu (g_{\mu\nu} B^\nu) = A^\mu B_\mu$

$b = T^{\mu\nu} T_{\mu\nu}$, $c = A^\mu B^\nu T_{\mu\nu}$ etc...

four-vectors: $C^\nu = T^{\mu\nu}(g_{\mu\nu} A^\nu) = T^{\mu\nu} A_\mu$

$A_\mu = g_{\mu\nu} A^\mu$, etc...

c) The following tensor fields are function of space-time coordinates $x = (x^0, x^1, x^2, x^3)$.

$f(x) = x_\mu x^\mu$, $g^\mu(x) = x^\mu$, $b^{\mu\nu}(x) = x^\mu x^\nu$, $h^\mu(x) = \frac{x^\mu}{x_\nu x^\nu}$

the differential operator is defined as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

$$\partial_\mu f(x) = \partial_\mu (x_\nu x^\nu) = (\partial_\mu x_\nu) x^\nu + x_\nu (\partial_\mu x^\nu)$$

$$\partial_\mu x^\nu = \frac{\partial x^\nu}{\partial x_\mu} = \delta_\mu^\nu$$

$$\partial_\mu x_\nu = \partial_\mu (g_{\rho\nu} x^\rho) = g_{\rho\nu} \partial_\mu x^\rho$$

$$= g_{\rho\nu} \delta_\mu^\rho = g_{\mu\nu}.$$

similarly: $\partial^\mu x^\nu = g^{\mu\nu}$

$$= g_{\mu\nu} x^\nu + x_\nu g_\mu^\nu = x_\mu + x_\mu = \underline{2x_\mu}$$

$$\partial_\mu g^\mu(x) = \partial_\mu (x^\mu) = \delta^\mu_\mu = \underline{\underline{4.}}$$

$$\partial_\mu b^{\mu\nu}(x) = \partial_\mu (x^\mu x^\nu) = (\partial_\mu x^\mu) x^\nu + x^\mu (\partial_\mu x^\nu)$$

$$= \delta^\mu_\mu x^\nu + x^\mu \delta^\nu_\mu$$

$$= (1+1+1+1) x^\nu + (x^0 \delta^\nu_0 + x^1 \delta^\nu_1 + x^2 \delta^\nu_2 + x^3 \delta^\nu_3)$$

$$= 4x^\nu + x^\nu = \underline{\underline{5x^\nu}}.$$

$$\partial_\mu h^\mu(x) = \partial_\mu \left(\frac{x^\mu}{x_\nu x^\nu} \right)$$

$$= \frac{\partial_\mu (x^\mu)(x_\nu x^\nu) - x^\mu (\partial_\mu x_\nu x^\nu)}{(x_\nu x^\nu)^2}$$

$$= \frac{\delta^\mu_\mu (x_\nu x^\nu) - x^\mu (2 x_\mu)}{(x_\nu x^\nu)^2}$$

$$= \frac{4\cancel{(x_\nu x^\nu)} - 2\cancel{(x^\mu x_\mu)}}{(x_\nu x^\nu)^{\cancel{2}}} = \underline{\underline{\frac{2}{x_\nu x^\nu}}}$$

③ A cathode corpuscle with charge e, moves in constant electric field $\vec{E}$. The motion is determined by the relativistic Newton equation

$$\frac{d}{dt}\vec{p} = e\vec{E} \qquad (*)$$

wher $\vec{p}$ is relativistic momentum $\vec{p} = m_e \gamma \vec{v}$.

a) Show: if $v=0$, $t=0$ then γ depends on t as $\gamma = \sqrt{1+K^2t^2}$.

$$(*) \rightarrow \quad d\vec{p} = e\vec{E}dt$$

$$\int d\vec{p} = \int e\vec{E}dt$$

$$\vec{p} = e\vec{E}t + p_0.$$

since $v=0$ at $t=0 \rightarrow p_0 = 0$.

$$E = \sqrt{p^2c^2 + m_e^2c^4} = \sqrt{(e\vec{E}t)^2c^2 + m_ec^4} = \gamma m_ec^2$$

$$\rightarrow \gamma = \frac{1}{m_ec^2}\sqrt{(e\vec{E}t)^2c^2 + m_ec^4}$$

$$= \underline{\sqrt{1+\left(\frac{c\vec{E}}{m_ec}\right)^2 t^2}}, \quad K = \frac{e\vec{E}}{m_ec}$$

b) $\boxed{\frac{dt}{d\tau} = \gamma}$, if $\gamma = \cosh \mathcal{K}\tau \rightarrow$ satisfied.

$$\gamma = \sqrt{1 + \mathcal{K}^2 t^2} \rightarrow \gamma^2 - 1 = \mathcal{K}^2 t^2$$

$$\rightarrow t = \frac{1}{\mathcal{K}}\sqrt{\gamma^2 - 1}.$$

$$t = \frac{1}{\mathcal{K}}\sqrt{\cosh^2 \mathcal{K}\tau - 1} \quad \leftarrow \cosh^2 x - \sinh^2 x = 1$$

$$= \frac{1}{\mathcal{K}} \sinh \mathcal{K}\tau$$

$$\frac{dt}{d\tau} = \cosh \mathcal{K}\tau = \gamma.$$

everything is as it should be... $\boxed{\Xi}$

c) For linear motion: $a_0 = \gamma^3 a$. Use to show that electron has constant proper acceleration $\vec{a}_0 = e\vec{E}/m_e$

$$\vec{a} = \frac{d\vec{v}}{dt} \qquad \vec{p} = \gamma m_e \vec{v}$$

the vel. is here

$$\frac{d\vec{p}}{dt} = m_e\left(\gamma \frac{d\vec{v}}{dt} + \vec{v}\frac{d\gamma}{dt}\right) = m_e\gamma\vec{a} + m_e\vec{v}\frac{d\gamma}{dt}$$

from lecture notes 7.15

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{c^2}{v^2}\right)^{-\frac{1}{2}} = \ldots = \gamma^3 \frac{1}{c^2} \vec{v}\cdot\vec{a}$$

then:

$$\frac{d}{dt}\vec{p} = m_e\left[\gamma\vec{a} + \gamma^3\frac{1}{c^2}\vec{v}(\vec{v}\cdot\vec{a})\right]$$

linear motion: $\vec{v}\parallel\vec{a} \rightarrow \vec{v}(\vec{v}\cdot\vec{a}) = v^2\vec{a}$

$$\frac{d\vec{p}}{dt} = m_e\left[\gamma\vec{a} + \gamma^3\frac{v^2}{c^2}\vec{a}\right] = m_e\gamma\vec{a}\left(1+\gamma^2\frac{v^2}{c^2}\right)$$

$$= m_e\gamma\vec{a}\left(1+\frac{v^2/c^2}{1-v^2/c^2}\right) = m_e\gamma\vec{a}\left(\frac{1-\cancel{v^2/c^2}+\cancel{v^2/c^2}}{1-v^2/c^2}\right)$$

$$= \underline{m\gamma^3\vec{a}}$$

$$e\vec{E} = \frac{d\vec{p}}{dt}$$

$$\cancel{m_e}\vec{a}_0 = \cancel{m_e}\gamma^3\vec{a}$$

$$\vec{a}_0 = \gamma^3\vec{a} \qquad \text{Q.E.D.}$$