

**MONTE CARLO SIMULATION OF MONETARY  
TRANSACTIONS:  
A SIMPLE MODEL FOR WEALTH DISTRIBUTION**

FYS3150: COMPUTATIONAL PHYSICS

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ABSTRACT. Lorem ipsum dolor sit amet, consectetur adipiscing elit. In id neque elementum, accumsan ligula at, lobortis tortor. Duis elementum pellen-tesque purus, sit amet euismod diam facilisis volutpat. Nulla facilisi. Mauris quis felis ante. Aliquam ac velit sit amet velit porta condimentum iaculis ut quam. Phasellus pretium libero nec lectus placerat, ac consectetur sem faucibus. Maecenas dictum porta finibus.

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## 1. INTRODUCTION

## 2. THEORETICAL BACKGROUND

**2.1. The simplest model for an economy.** Arguably, the most famous equation in macroeconomics is the “autarky identity”<sup>1</sup>

$$(1) \quad Y = C + G + I,$$

where  $Y$  is income,  $C$  is consumption,  $G$  is government spending and  $I$  is investment. The best-know, but not necessarily the best, measure of income  $Y$  is the Gross Domestic Product (GDP)<sup>2</sup>. Consumption  $C$  is the monetary value of all goods and services purchased in the private sector, while government spending  $G$  is the consumption of the government. Finally, investment  $I$  is the sum of private and public saving.

Equation 1 is autarkic because all interactions with other economies are excluded from the expression. There are no terms representing exports and captial inflow, for instance. We are dealing with a *closed* economy, alternatively the entire world as a whole. Moreover, let’s assume that the economy we are studying is a peaceful anarchy, without a governing authority of any sort, in effect setting  $G = 0$ . To begin with, we will also forgo the agents the ability to save such that the worth of every individual, or agent, in the economy must be spent at once. Equation 1 is reduced to  $Y = C$ , everything one agent spends is the income of another.

**2.2. Monetary transactions.** To simulate monetary transactions in our model economy we expand employ the framework introduced in Patriarca et al.[1]. We assume there are  $N$  agents that exchange money in pairs  $(i, j)$ . We assume also that all agents start with the same amount of money  $m_0 > 0$ . For every period an arbitrary pair of agents are picked at randomly and let them conduct business, id est a transaction takes place between them. Money is conserved during the transaction such that

$$(2) \quad m_i + m_j = m'_i + m'_j,$$

where the right-hand side is the agent  $i$  and  $j$ ’s updated wealth and the left-hand side represents the amount of money agents  $i$  and  $j$  had before the transaction. The exchange is done via a random reassignment factor  $\epsilon$ , such that

$$(3) \quad m'_i = \epsilon(m_i + m_j)$$

$$(4) \quad m'_j = (1 - \epsilon)(m_i + m_j)$$

No agent will ever have negative wealth, that is  $m \geq 0$ . Moreover, because of the conservation law in equation 2, the system eventually reaches an equilibrium state given by a Gibbs distribution

$$(5) \quad w_m = \beta e^{-\beta m}, \quad \beta = \frac{1}{\langle m \rangle},$$

where  $\langle m \rangle$  is the expected wealth, for which the arithmetic mean is an unbiased estimator. This implies that after an equilibrium has been reached the majority of agents is left with lower wealth than they had initially and the number of rich agents exponentially decrease.

It is easy to see that the logarithm of the Gibbs measure in equation 5 yields a linear equation

$$(6) \quad \ln(w_m) = \ln(\beta) - \beta m$$

<sup>1</sup>Any introductory text on macroeconomics will give a thorough elaboration, e.g. Gärtner.

<sup>2</sup>GDP = GNP (Gross National Product) in autarky.

**2.3. Pareto distributions.** How the income and wealth of a population has been studied for a long time. Already in 1897, Vilfredo Pareto, sought to describe both wealth and income distribution by power law probability distributions[2]. Pareto showed that the higher end of the distribution of money follows a distribution

$$(7) \quad w_m \propto m^{-1-\alpha}, \quad \alpha \in [1, 2].$$

Pareto's "80-20 rule", that 20% of the population controls 80% of the wealth, corresponds to a certain value of  $\alpha$ . Today, a whole class of distributions similar to the one in equation 7 are named after Pareto[3].

**2.4. Transactions and savings.** We can expand upon the model by introducing a savings rate  $\lambda$ . The savings rate is defined as a fraction of an agent's wealth that does not partake in a transaction for every period. One can gather from the macroeconomic identity in equation 1 that income must still be the same and the transaction law in equation 2 still holds. The updated wealth of agents  $i$  and  $j$  after a transaction becomes

$$(8) \quad m'_i = \lambda m_i + \epsilon(1 - \lambda)(m_i + m_j)$$

$$(9) \quad m'_j = \lambda m_j + (1 - \epsilon)(1 - \lambda)(m_i + m_j)$$

one can rewrite these expressions to

$$m'_i = m_i + \delta m$$

$$m'_j = m_j - \delta m$$

where

$$(10) \quad \delta m = (1 - \lambda)(\epsilon m_j - (1 - \epsilon)m_i)$$

**2.5. Economic inequality and social friction.** "The first historical series of income distribution statistics became available with the publication in 1953 of Kuznet's monumental Shares of Upper Income Groups in Income and Savings. Kuznet's series dealt with only one country (the United States) over a period of thirty-five years (1913-1948)" (Piketty & Goldhammer, 2014, Kindle locations 276-280[4]). Ever since that time there has been wealth and income inequality to a greater or lesser degree. According to the PolitiFact the top 400 richest Americans in 2011 "[had] more wealth than half of all Americans combined"[5].

One of the possible reasons for continued wealth and income inequality that has risen in popularity the last few years is the process of wealth concentration. This is a process by which, under certain conditions, newly created wealth concentrates in the possession of already-wealthy individuals or entities. Those who already have wealth have the means to invest in newly created sources and structures of wealth. Piketty argues that the fundamental force for wealth divergence is the usually greater return of capital than economic growth (Piketty & Ganser, 2014 p. 284 Table 12.2[4]).

To incorporate the social rigidities described above, we will make two further additions to the model. These additions build mostly on the work of Goswami and Sen[6]. Firstly, we will make it more likely for two agents to conduct a transaction if they have similar wealth. Second, we will make it more likely for two agents to make a transaction if they have made a transaction before. We define a probability

$$(11) \quad p_{ij} \propto |m_i - m_j|^{-\alpha} (c_{ij} + 1)^\gamma,$$

where the first factor  $|m_i - m_j|^{-\alpha}$  relates to wealth similarity and the second factor  $(c_{ij} + 1)^\gamma$  relates to previous transactions. A relatively large difference between  $m_i$  and  $m_j$  should translate to a lower probability. The strength of this effect is governed by the exponent  $\alpha$ , a larger value decreases the probability of a trade

further. The variable  $c_{ij}$  is simply the number of times agent  $i$  and  $j$  have conducted transactions in the past. The number 1 is added in order to ensure that if they have not interacted earlier they can still interact. The strength of this effect is governed by the exponent  $\gamma$ .

### 3. ALGORITHM & IMPLEMENTATION

All programming code used in this project is written in C++ and is publicly available at github by following the link on the first page. The program consists mainly of two files, `main.cpp` and `functions.cpp`. `main.cpp` contains the set-up of the program and is where the number of agents, initial wealth, number of transactions, number of simulations and all parameters ( $\lambda$ ,  $\alpha$  and  $\gamma$ ) are specified. The main part of the program however, is in `functions.cpp` which in addition to the `trade(...)`<sup>3</sup> function contains the function `output(...)`<sup>4</sup> which writes data to file.

**3.1. The `trade(...)` function.** This function runs a specified number of Monte Carlo cycles, where the transactions take place. For each iteration two random agents are picked, then some logic follows which will determine if the two agents are allowed to go into business. This is a very important part of the program and is therefore listed here:

```
while ((pow(fabs(m_i - m_j), -alpha)*
        pow(previous_transactions+1, gamma) < random_number)
        || (agent_i == agent_j)) {

    // Pick new agents ...
    agent_i = (int) rand() % N;
    agent_j = (int) rand() % N;

    // ... find wealth of these ...
    m_i = agents(agent_i);
    m_j = agents(agent_j);

    // ... update previous transactions ...
    previous_transactions = C(agent_i, agent_j);

    // ... and update the random comparison number
    random_number = (double) rand() / RAND_MAX;
```

One can see that the the argument of the while loop in the program listing above contains equation 11 which is compared against a random number and an equality test that will return `True` if agents  $i$  and  $j$  are the same agent. This part of the code makes sure that a new pair of randomly picked agents until we are satisfied that the agents are not the same and that they are close enough in wealth and in previous relations to go forward with a transaction. After everything is all-right and we have a good pair of agents a transaction takes place:

```
// Random value of transaction b/w 0 and 1
double epsilon = (double) rand() / RAND_MAX;

// Transaction takes place, but some is saved (lambda)
```

<sup>3</sup>The functions have several arguments: `trade(int N, int no_of_transactions, arma::vec (&agents), double lambda, double alpha, double gamma)`. To include all the arguments in the text would have decreased readability and they are replaced with dots.

<sup>4</sup>`output(...) = output(int N, arma::vec agents, std::string filename)`.

```

double delta_m = (1 - lambda) *
    (epsilon*m_j - (1 - epsilon)*m_i);
agents(agent_i) += delta_m;
agents(agent_j) -= delta_m;

// Register that a transaction has taken place
C(agent_i, agent_j) += 1;
C(agent_j, agent_i) += 1;

```

The program listing above should be straight-forward. Notice that the transaction is registered in matrix **C** by incrementing element  $i, j$  and  $j, i$  by 1.

The last part of the `trade(...)` function is not particularly important for the quality of the data that a simulation will provide, but vastly increased the speed of the program. This last functionality breaks the for loop within the `trade(...)` function when a steady state is reached. Listed in its entirety here:

```

double var = arma::var(agents);
cumVarBlock += var;
cumVar2Block += var*var;

// Enter here at the correct place: end of block
if (i % blockSize == 0) {
    // Variance of a block "averaged" over the block
    double avgVar = cumVarBlock / blockSize;
    double avgVar2 = cumVar2Block / blockSize;

    // Variance of a block of variance
    double varVarBlock = avgVar2 - avgVar*avgVar;

    // Check if variance is low enough
    if ((fabs(avgVarBlockOld - avgVar) / fabs(avgVarBlockOld))
        < 0.2) &&
        (fabs(varVarBlock - varVarBlockOld) / fabs(
            varVarBlockOld) < 0.5)) {

        // Telling a user that an equilibrium has been reached
        std::cout << "Equilibrium reached at transaction no. "
            << i << std::endl;

        // Done! For now.
        break;

    } else {
        avgVarBlockOld = avgVar;
        varVarBlockOld = varVarBlock;
    }
    // Reset cumulative variance
    cumVarBlock = 0;
    cumVar2Block = 0;
}

```

This code section does some calculations on the results after a certain amount of transactions have taken place, an amount predetermined by `blockSize`. The average variance of the block is computed as well as the average square variance of

the block size. Then the "variance of the variance" is computed. These measures are compared against the measures from the previous block. The relative size of the average variance must be less than 20% and the relative size of the variance of the variance must be less than 50% of the measures from the previous block. If both these logical statements evaluate to `True`, the for loop of the `trade(...)` function is broken, terminating the current simulation. We found that for most simulations an equilibrium is reached quite quickly, usually between 40000 and 120000 transactions have taken place between  $N = 1000$  agents. This speeds up the time for a simulation to finish tremendously, compared with the initially, and rather naively, picked  $10^7$  iterations. In order to find correct values `blockSize` as well as when the variance measures are "low enough" the behaviour of the variance measures were meticulously studied through several trial runs of the function.

**3.2. Example `main()`.** The parameters of the simulation is set up in the `main()` function. When all parameters are chosen the following for loop is almost everything that is needed

```
// Start simulations
for (int i = 0; i < simulations; i++) {

    // Print progress
    std::cout << "Simulation no. " << i << std::endl;

    // Assign initial wealth to all agents
    agents.fill(m0);

    // Trade!
    trade(N, transactions, agents, lambda, alpha, gamma);

    // Sort and add equilibrium wealth to total
    totagents += arma::sort(agents);
}
```

One can see that what must be specified is the number of simulations (`simulations`), initial wealth (`m0`), number of agents (`N`), number of transactions (`transactions`), an array containing the wealth of every agent (`agents`), savings rate (`lambda`), parameter governing weight of wealth similarity (`alpha`) and parameter governing weight of past transactions (`gamma`).

#### 4. RESULTS

Figure 1 shows plot of different wealth distributions after 1000 simulations for  $N = 500$  agents. We see that that without savings, the distribution shows a decreasing number of agents as wealth increases. As the savings rate  $\lambda$  is allowed to increase, the distribution moves to the right and eventually peaks at the initial wealth  $m_0 = 100$  when  $\lambda = 0.9$ . This plot, were it not for the colors, could be an exact copy of figure 1 in Patriarca et al.[1]. Without savings the distribution appears as a inverse exponential function ( $e^{-x}$ ) which is the same class of function as the Gibbs distribution (equation 5). However, these traits disappear from the wealth distributions once savings are allowed.

A further study of the wealth distributions from figure 1 is visualized in figure 2. Figure 2a shows the same graphs as in figure 1, but on with logarithmic scales on both axes. Figure 2b the wealth values evaluated by equation 5 and plotted with logarithmic  $y$ -axis. Observe that all all the wealth distributions give straight lines. This makes sense, as the logarithm of the Gibbs distribution yields a first order expression (equation 6).

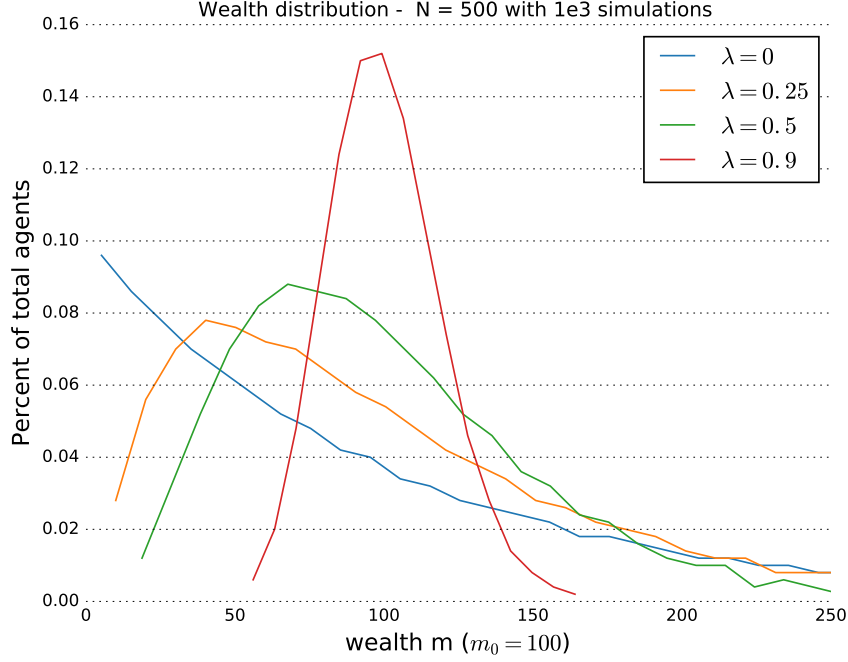


FIGURE 1. Wealth distribution for  $N = 500$  agents after 1000 simulations at different savings rates  $\lambda$

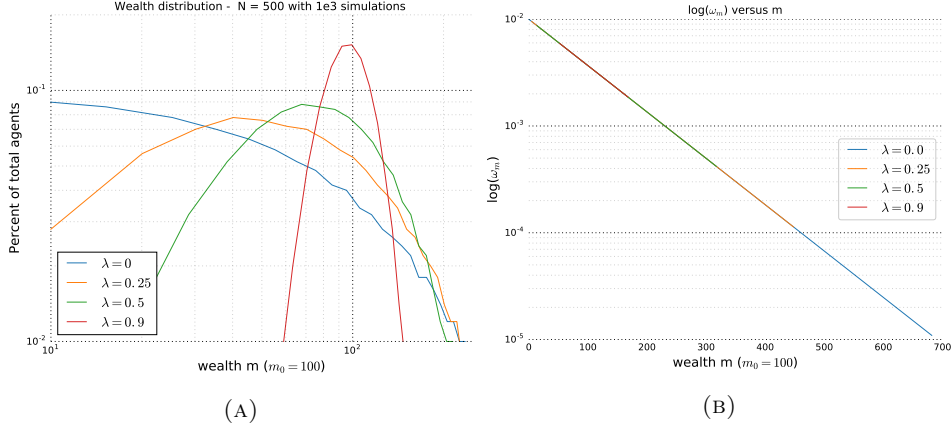


FIGURE 2. Logarithmic plot of wealth distribution from figure 1 2a), and the logarithm of the Gibbs measure (equation 5) of every data point (2b).

Figure 3a shows two different Pareto distributions with  $\alpha = 1.8$  and  $\alpha = 2$  (equation 7) plotted against a distribution at  $\lambda = 0$ . We see that they fit adequately with the higher end of the distribution, but not perfectly. The simple Pareto distribution becomes a straight line because of the logarithmic axes in the plot. The same figure also contains a parametrization of the wealth <sup>5</sup>. Figure 3b also shows such a parametrization, but for a higher savings rate  $\lambda = 0.9$ . The parametrizations are indicated by a label  $P(x)$ .

<sup>5</sup>This is done in the way Partriarca et al. (2004)[1] does. A recipe is shown in appendix A

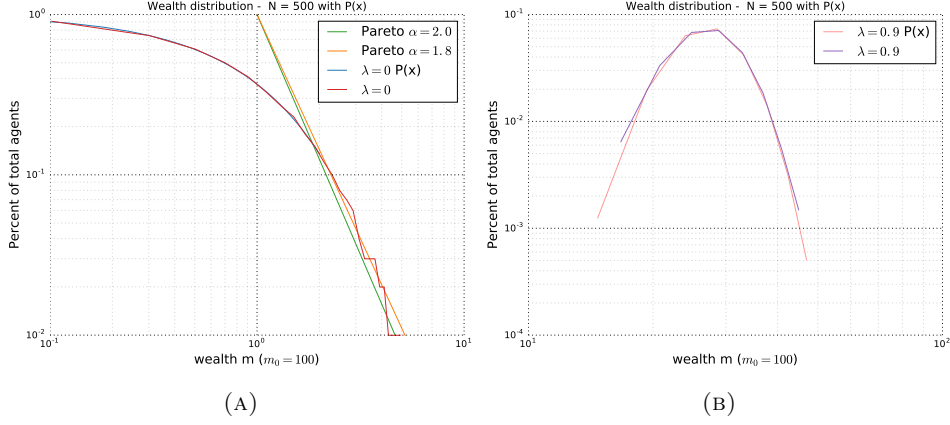


FIGURE 3. Parametrization of two different wealth distributions,  $\lambda = 0$  and  $\lambda = 0.9$ . Method for parametrization is found in appendix A. For  $\lambda = 0$  two Pareto power law distributions are also shown.

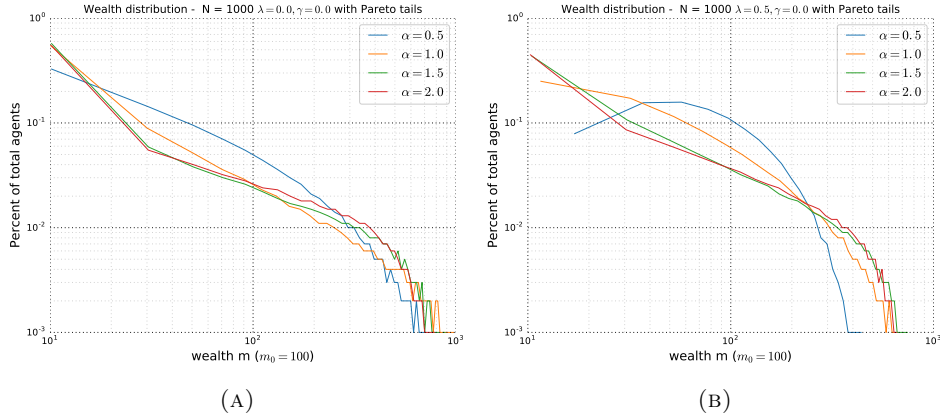


FIGURE 4. The effect of different weights of propensity to trade with nearest neighbours,  $\alpha$ . Figure 4a is excludes saving ( $\lambda = 0$ ), figure 4b includes saving ( $\lambda = 0.5$ ).

The effect of “nearest neighbour” interactions are summarized in figures 4 and 5. Figure 4 shows logarithmic plots of simulations for different values of  $\alpha$  from equation 11. The parameter for previous transactions *gamma* is set to zero for now. The only difference between the subfigures in figure 4 is that figure 4a is for  $\lambda = 0$  and figure 4b is for  $\lambda = 0.5$ , colloquially with and without saving respectively. Figure 4 shows similar results to that of Goswami and Sen (2014)[6].

Figure 5 shows the simulations for nearest neighbour interaction fitted to a Pareto distribution. The Pareto distribution should fit only for higher values of the wealth distribution, id est the tails of the distribution. Figure 5a show that the power law fittings (Pareto distributions) appear to have similar shapes to the results from the simulations. One can see, however, that the data from the simulations tend drop quicker than their Pareto power law counterparts. Figure 5b shows a logarithmic version of figure 5a. Here the Pareto fittings are straight lines, as one would assume, but the data from the simulations apparently inhibit more complexity than the a simple power law relationship between wealth and frequency.



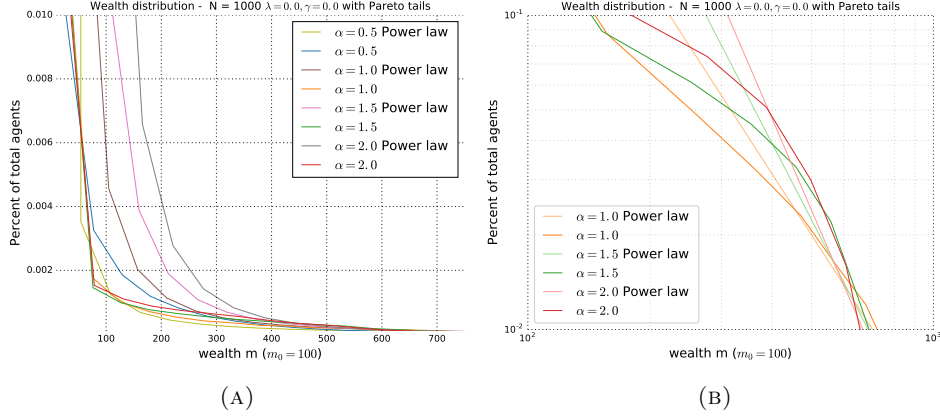


FIGURE 5. Power law fittings of neighbourhood effect wealth distributions. Figure 5a is a linear plot, while figure 5b is double-logarithmic.

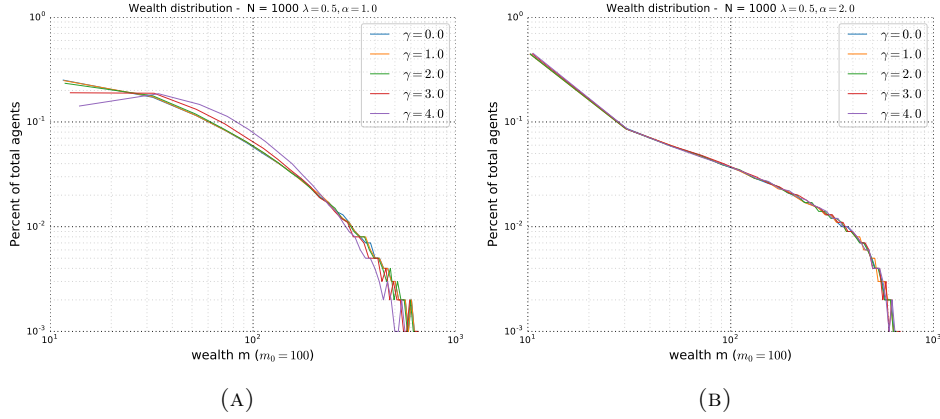


FIGURE 6. Wealth distributions when weighing of previous transactions, in addition, has been taken into account. Figure 6a weighs wealth similarity less strongly ( $\alpha = 1$ ) than figure 6b ( $\alpha = 2.0$ ).

The last effect to consider is the former transaction effect, in combination with savings and nearest neighbour interactions. This effect is illustrated in figure 6. This figure is an attempt to replicate figure 5 in Goswami and Sen (2014)[6]. Both subfigures show the wealth distribution for different values of  $\gamma$  (propensity to trade with agents you have traded with before), but figure 6a weighs neighbour interactions less strongly than figure 6b, with  $\alpha = 1.0$  and  $\alpha = 2.0$  respectively.

Lastly, figure 7 show a power law fitting of the wealth distributions first presented in figure 6. The figure both includes ( $\lambda = 0.5$ ) and excludes savings ( $\lambda = 0$ ). The plot is logarithmic and the Pareto power distribution is therefore presented as a straight line.

## 5. DISCUSSION

Most part of this study was a replications of the work done by Patriaraca et al. (2004)[1] and Goswami and Sen (2014)[6] and we have more than sufficiently managed to do so, but the path to that point has regrettably involved a bit of guesswork. We found that both articles were somewhat lacking in procedural and

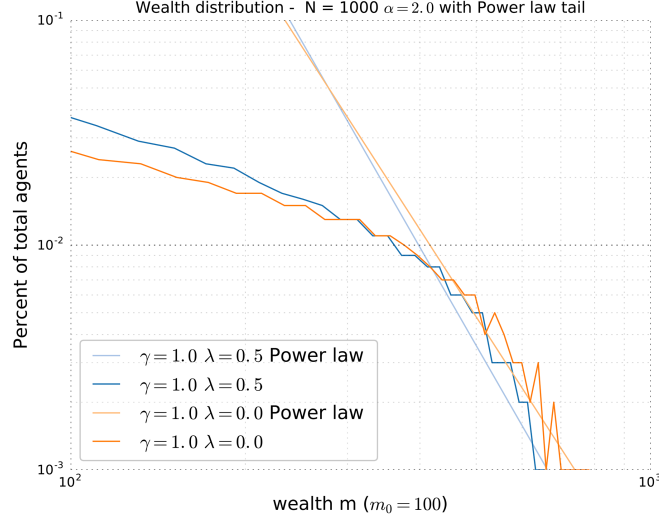


FIGURE 7. Power law fitting of wealth distribution including savings, nearest neighbour- and previous transaction effects. The plot is logarithim and both includes ( $\alpha = 0$ ) and excludes ( $\alpha = 0.5$ ) savings.

algorithmic descriptions, which we find not to be consistent with sound science and the scientific method<sup>6</sup>. “[Just] as other information should be available to those who want to learn and understand, program source code is the only means for programmers to learn the art from their predecessors. It would be unthinkable for playwrights not to allow other playwrights to read their plays (...)” (Erik Naggum)[7]. That being said, we have confidence that our own results are very much comparable to both of the studies that this study is based upon.

**5.1. The “savings rate” and other quirks.** In this study we call the factor  $\lambda \in [0, 0.9]$  the savings rate. It makes sense to employ it to the model described herein, as one would not assume that anyone would spend (potentially) their entire net worth on a single transaction. On the other hand, a savings rate would usually be defined as the fraction of income that an individual saves over a certain amount of time, e.g. a year. The term “savings rate” as it is used in this model can therefore lead to some confusion, mostly because there is no concept of how time progresses throughout the simulation. The only thing that happens is a series of transactions, and after a steady state is reached we have a snapshot of the monetary distribution, but no feel for the dynamics of the economy.

**5.2. Future developments.** We found, after delving into the data, that the Parato distributions than follow the higher tails of the simulation distribution data best are best described by the following rule

$$(12) \quad w_m \propto m^{-1-\lambda-\alpha-\gamma}$$

where, of course,  $\lambda$  is the savings rate,  $\alpha$  is the importance of wealth similarity and  $\gamma$  is the importance of previous transactions. This is a very neat result, but we have only confirmed that the Pareto distributions fits the most-wealth in a visual manner. It would be good to confirm the explanatory power by some sort of

<sup>6</sup>We went as far as to e-mail some of the co-authors of the articles, asking (very nicely!) for descriptions of algorithms, software and methods. At the time of writing we are still without answer.

regression analysis or a goodness-of-fit test. This was regrettably beyond the scope of our study.

A very important part of the social sciences of which economics is a part is that they have applications for the real world and help in a way to explain the behaviour of human beings. It would therefore be nice to compare the results of these simulations to real wealth distributions. As Piketty points out[4], data on wealth and income distributions have been available in the US since 1953. One study[8], in the Norwegian business magazine Kaptial uses the magazine’s study of the 400 richest Norwegians to show that their wealth are indeed Pareto distributed.

Norway is a special case because it is one of very few OECD<sup>7</sup> member countries that with a wealth tax. These include Switzerland, Hungary, Spain and France[9]. In Norway, the wealth tax has a tremendous impact on high net worth individuals, making up 57.1% of the total tax burden of the 400 richest individuals[10]. It would therefore be beneficial to include tax effects in the model in some way, as different tax regimes may have an impact of the monetary distribution.

We started the theory section by stating a macroeconomic identity (equation 1). In it we included government spending, which necessarily is very important to an economy. Depending on the relative size of the government, the spendings of the government could impact wealth distribution through welfare. Even though poor individuals may not have a high personal wealth it may not be necessary for them, because of the security that a welfare state provides. This would in turn affect savings rate in an economy. A huge “social security net” could lead to lower savings and vice versa.

In a modern, open economy one inevitably must trade with other economies. The macroeconomic identity in equation 1 can be expanded to net exports  $X - M$ , where  $X$  is exports and  $M$  is imports

$$(13) \quad Y = C + G + I + (X - M).$$

It would be interesting to see what opening the economy up for trade would do for wealth distribution. According to the Stolper-Samuelson theorem, opening an economy to trade should make unskilled workers producing traded goods in a high-skilled economy worse off as trade increases[11]. Advocates of protectionism usually use such theories to underline their argumentation. Building a model consisting of several “countries” with increasing amount of trade could be beneficial in this debate, especially as we have seen an increase in protectionist political rhetoric in recent times.

## 6. SUMMARY REMARKS

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<sup>7</sup>Organisation for Economic Co-operation and Development. Colloquially defined as a “club for rich countries”.

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## Appendices

### A. PARAMETRIZATION FROM PATRIARCA ET AL.

A corresponding exact solution of an income distributions for a generic value of  $\lambda$  with  $0 < \lambda < 1$  is provided in Patriarca et al.[1]. Fir one employs the reduced variable

$$(14) \quad x = \frac{m}{\langle m \rangle}$$

the agent money in units of average money  $\langle m \rangle$  and the parameter

$$(15) \quad n(\lambda) = 1 + \frac{3\lambda}{1 - \lambda}.$$

The money distributions, for arbitrary values of  $\lambda$ , are well fitted by the function

$$(16) \quad P_n(x) = a_n x^{n-1} e^{-nx}$$

where  $a_n$  is a normalization factor shown to be

$$(17) \quad a_n = \frac{n^n}{\Gamma(n)}$$