

THE SOLAR SYSTEM: AN EXERCISE IN NUMERICAL INTEGRATION

_____ FYS3150: COMPUTATIONAL PHYSICS _____

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ABSTRACT. Two, three, multi. With Euler-Cromer and then Velocity Verlet

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1. INTRODUCTION

2. THEORY

2.1. Newtonian gravitation. When one studies the solar system and the way it moves, one inevitably needs to consider the gravity of the situation. The classical law of gravitation is given by Newton's law

$$(1) \quad F_G = \frac{GM_1M_2}{r^2},$$

where F_G is the gravitational force between two bodies, G ¹ is the gravitational constant, M_1 and M_2 are the masses of the two bodies and r is the distance between them.

In three dimensions, by employing Newton's second law of motion, one will get the following componental equations for the acceleration due to gravitational pull on a particular body i .

$$(2) \quad \frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_i}, \quad \frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_i}, \quad \frac{d^2z}{dt^2} = \frac{F_{G,z}}{M_i},$$

each of which can be integrated in order to find the position of the bodies at any given time and for a particular initial velocity and position. Moreover, a neat vectorial expression for the Newtonian gravity (equation 1) in three dimensions is

$$(3) \quad \mathbf{F}_G = \frac{GM_1M_2}{r^3} \mathbf{r}$$

where \mathbf{r} gives the position of the body in a cartesian coordinate system.

2.2. Relativity. The theory of general relativity (GR), put forth by Albert Einstein in 1915, is the current description of gravity in modern physics. On curiosity of the solar system that cannot be explained is the perihelion precession of Mercury's orbit. For every Mercury year the perihelion of the planets orbit moves slightly. The observed value of this effect is 43 arc-seconds per century. Einstein showed himself that GR could explain this anomaly.

Closed elliptical orbits are a special feature of the factor $(1/r^2)$ in Newton's law of gravitation. Any change to this factor will result in a change in the orbit. For a small such correction, each orbit will be almost the same, but as time progresses the orientation of the elliptical orbit will itself rotate. In this study we will not compute a space-time manifold in order to make this correction, but rather add a general relativistic error term to the Newtonian gravitational force (equation 1).

$$(4) \quad F_G = \frac{GM_\odot M_{Mercury}}{r^2} \left[1 + \frac{3l^2}{r^2 c^2} \right],$$

where $M_{MERCURY}$ is the mass of Mercury, r is the distance between Mercury and the Sun, $l = |\mathbf{r} \times \mathbf{v}|$ is the orbital angular momentum per unit mass, and c is the speed of light in vacuum.

The precession of Mercury's orbit is not easy to spot simply by plotting the planets orbit, but the angle of the perihelion² can be calculated easily enough by the following equation

$$(5) \quad \theta_p = \arctan \left(\frac{y_p}{x_p} \right)$$

¹ $G = 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$

²The point in the orbit closest to the sun.

3. UNITS

In order to make the system easy to work with and the calculations easier as well, a change of units is warranted. In this study we will therefore express the mass of any body as a fraction of solar masses, that is $M_{\odot} = 2 \times 10^{30} kg$. This means that the Earth³, for instance will have a mass of $M_{Earth} = 3 \times 10^{-6}$.

Additionally, the units for distance will be astronomical units AU . This is the mean distance between the earth and the sun. For a simple system and a particular coordinate system with the sun at origin, the initial position for the Earth could simply be $\mathbf{r}_{Earth} = (1, 0, 0)$.

The unit for time will be Earth years, yr . This means that velocity will be in units AU/yr .

Changing the variables will have consequences for the gravitationan constant G and velocities v as well. This can be deduced easily enough by picking a sample system consisting of the earth and the sun. Inserting in equation 1 gives

$$(6) \quad F_G = \frac{GM_{\odot}M_{Earth}}{r^2}$$

Furthermore, assuming the orbit of the earth is perfectly circular, will give the following relation

$$(7) \quad F_G = \frac{M_{Earth}v^2}{r}$$

Combining equations 6 and 7 yields the following

$$(8) \quad v^2 r = gM_{\odot} = 4\pi^2 AU^3 yr^{-2}$$

Because the mass of the sun as a fraction of solar masses is $M_{\odot} = 1$ and the distance between the earth and the sun is $r = 1AU$ we land at

$$(9) \quad G = 4\pi^2 AU^3 yr^{-2} \quad v_{Earth} = 2\pi AU yr^{-1}$$

4. ALGORITHMS AND IMPLEMENTATION

4.1. Object orientation. Object orientation allows for more general code to be written and for easy reuse. Moreover, object orientation makes sense for humans, who tends to classify objects we see and interact with in everyday life. In this study we make use of all the advantages of object oriented programming by implementing several classes. A class diagram illustration of the program is included in appendix A. Bear in mind that this diagram does not represent the functionality of the program to a full extent, as many methods and a few classes are left out. Rather, it helps one to understand how the class hierarchy is built up and how the program as whole is supposed to function.

The **System** class contains all the information about the initial conditions of the system, which integration method should be used and also some helpful methods, like one that writes data to file.

The **InitialCondition** superclass contains a method for setting up particles. Depending on what system one wants to model, we have implemented three subclasses of **InitialCondition**: **TwoBody**, **ThreeBody** and **MultiBody**. The name of these classes speak mostly for themselves.

The interesting objects, contained within the **InitialCondition** subclasses are instances of the **Particle** class. The class could have another name, like "planet" or "body" because instances of it will represent the largest bodies in the solar system; the sun and (eventually) all nine planets⁴. Every particle instance takes a position

³In SI-units, $M_{Earth} \approx 6 \times 10^{24} kg$

⁴Including Pluto for historical reasons.

vector, a velocity vector and a mass as inputs and has only these attributes⁵. The extra abstraction to "particle" is therefore fitting, and the class could be employed elsewhere, for instance in a molecular dynamics simulation.

The `Integrator` superclass has two subclasses `EulerCromer` and `VelocityVerlet` each of which must overwrite the `integrateOneStep` method. These are the most important part of the program, because it is where the essential part of the two algorithms are implemented. The `integrateOneStep` method is called iteratively for a specified number of steps from a class instance of the `system` class, hence the name of the method. Within the `integrateOneStep` method, the integration is performed for every particle in the system. A further description of the integrator algorithms follows.

4.2. Euler-Cromer. Analytically the Euler-Cromer method, or the semi-implicit Euler method, can be expressed in one dimension by the following recursive relations

$$(10) \quad v_{n+1} = v_n + a_n dt$$

$$(11) \quad x_{n+1} = x_n + v_{n+1} dt,$$

where a_n is the acceleration, v_n is the velocity and x_n is the position, after a certain number of steps n . dt is the time step and $t_n = t_0 + ndt$ is the time after n steps. One sees that the algorithm is relatively cheap, with $4n$ FLOPS. In C++ code the `integrateOneStep` method of the `EulerCromer` class looks like this

```
void EulerCromer::integrateOneStep(std::vector<Particle*>
    particles) {

    m_system->computeForces();

    for (int i=0; i<particles.size(); i++) {
        Particle *p = particles.at(i);

        // Acceleration vector
        vec3 a = (p->getForce()) / (p->getMass());

        // Velocity update
        p->getVelocity() += m_dt*a;

        // Position update
        p->getPosition() += m_dt*p->getVelocity();
    }
}
```

Notice that the forces used to compute the acceleration are computed within the `System` class and that the method uses three-dimensional vectors for more realistic results.

The Euler Cromer method is a first-order integrator which means that it commits a global error of the order of dt . This means that by decreasing the time step one will get a more accurate result.

4.3. Velocity Verlet. The Verlet method is an integration method designed to integrate Newton's equation of motion. The method provides good numerical stability and more precise results compared with a regular method like the Euler-Cromer

⁵The position and velocity vectors are defined by its own class `Vec3`.

method. The reason for this is that it is symplectic, which means that it conserves state-space volume⁶.

The velocity Verlet method can also be represented by a (one-dimensional) recursive relation

$$(12) \quad x_{n+1} = x_n + v_n dt + \frac{1}{2} a_n dt^2$$

$$(13) \quad v_{n+1} = v_n + \frac{a_n + a_{n+1}}{2} dt.$$

These equations tell us that the algorithm is slightly more expensive compared to the Euler-Cromer scheme at $11n$. Here follows an implementation of the algorithm in C++.

```
void VelocityVerlet::integrateOneStep(std::vector<Particle
*> particles) {
    m_system->computeForces();

    for (int i=0; i<particles.size(); i++) {
        Particle *p = particles.at(i);

        // Acceleration vector
        vec3 a = (p->getForce()) / (p->getMass());

        // Position update
        p->getPosition() += p->getVelocity()*m_dt + 0.5*a*m_dt*
            m_dt;

        m_system->computeForces();

        // New acceleration vector
        vec3 anew = (p->getForce()) / (p->getMass());

        // Velocity update
        p->getVelocity() += (a + anew)*m_dt/2;
    }
}
```

One sees that it is necessary to compute forces and update acceleration once more in order to update the velocity, which means more operations must be conducted. Time will show if the extra cost of this algorithm is worth the increase in precision it promises.

5. DATA

As we wish to determine the orbits of our solar system as accurately as possible, we use data from HORIZON Web-Interface, provided by the Jet Propulsion Laboratory (NASA) at the California institute of technology: <http://ssd.jpl.nasa.gov/horizons.cgi>. Here of data from several celestial bodies can be downloaded. It is simple to find both position and velocity for all particles in our bodies at high precision. Units of length can be set to *AU* and velocities to *AU* per day. The velocities must be converted for our simulation. We have faith that these data are accurate.

⁶One would probably read an entire book to understand this, so we will leave it at that.

Comparison of Euler-Cromer and Velocity Verlet

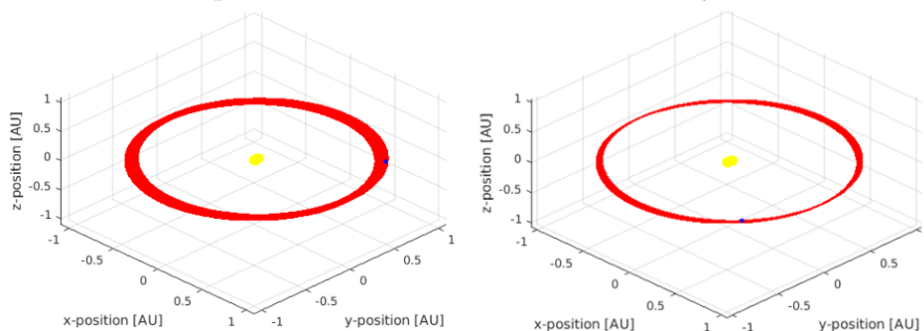


FIGURE 1. Figure showing the difference in precision between the Euler-Cromer method (left) and the velocity Verlet method (right). The time step is $dt = 10^{-2}$ and the number of points is $N = 10^6$

TABLE 1. Time comparison of Euler-Cromer and velocity Verlet applied to a two-body system.

N	Euler-Cromer [t]	Velocity Verlet [t]
10^3	0.023	0.036
10^4	0.20	0.17
10^5	1.92	1.83
10^6	18.42	17.88

6. RESULTS

The object-oriented model for the solar system that has been built is very flexible, as the reader will soon come to know. We start by constructing some simple models with two and three bodies, then the entire solar system, and lastly, simulate the perihelion precession of Mercury, as discussed in the theory section.

6.1. Two-Body system: Earth and Sun. A two-body system is a meager representation of the solar system, but is the first important stepping stone towards a more interesting and realistic model. The two-body system has two instances of the `Particle` class, representing the sun and the Earth. The orbits are found by solutions from both the Euler-Cromer method and the velocity Verlet method.

In figure 1 shows a plot of a solution by Euler-Cromer to the left and velocity Verlet to the right. The same time step $dt = 10^{-2}$ and the number of points $N = 10^6$ is employed in both simulations. The sun is the yellow dot in the middle of the plot and the Earth is the blue dot orbiting along the red path. The reason the past is wide is because the simulation has run for such a long time that the Earth has travelled several orbits⁷. The difference between the two methods is quite clear. The velocity Verlet method has a much narrower “orbit band” and is therefore more precise.

The model is easily adjustable and one can modify it to show many interesting phenomena. In figure 2 is a plot showing the Earth escaping its orbit around the sun. The escape velocity was found by trial and error; $v = \sqrt{8\pi AU/yr}$.

Table 1 shows a comparison of how much time the two algorithm requires to handle a problem of increasing size. The two algorithms were applied to the same two-body problem. As one can see, it appears that the velocity Verlet method is

⁷One hundred points per revolution and 100000 points should give 1000 revolutions in total.

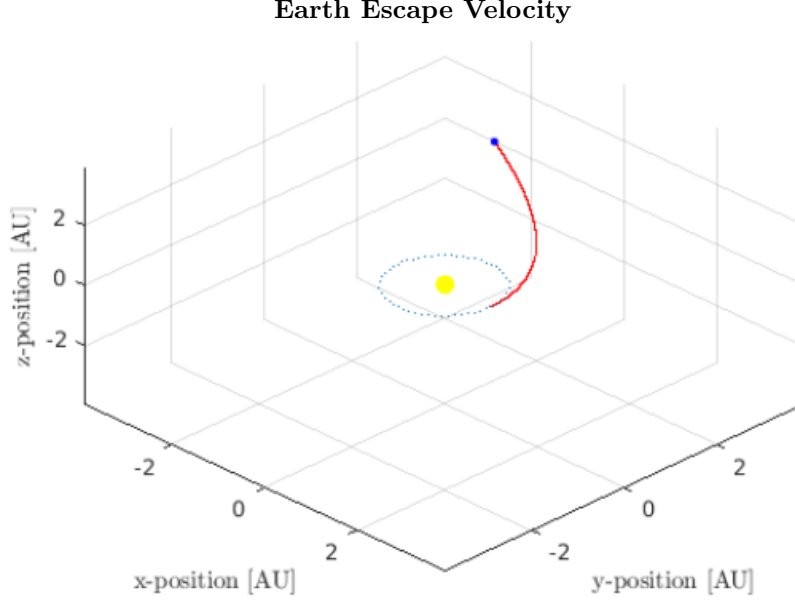


FIGURE 2. Illustration of Earth escaping orbit at $v = \sqrt{8\pi}AU/yr$

a bit quicker, but these results are probably not statistically significant. More on this in the discussion below.

6.2. Three-Body system: Earth, Sun and Jupiter. A natural progression from a two-body system is a three-body system. We have implemented a model consisting of particles representing Jupiter, Earth, and the Sun. The top left subfigure of figure 3 in two dimensions. The velocity Verlet method is employed here.

The rest of the subfigures in figure 3 show what would happen if Jupiter had larger mass than it has. The top right subplot shows this scenario if Jupiter's mass was $10\times$ larger than normal. One can see that this would drag the Earth along into the Sun, which stays mostly put, and likely skew the entire solar system.

The two bottom figures show what would happen if Jupiter would have had $1000\times$ normal mass. The plot speaks for itself; the entire system is pulled apart. This situation is shown in 2D to the left and 3D to the right.

6.3. Multi-Body system: all planets. Finally for the pinnacle of this study, a model of the entire solar system!⁸ Figure 4 shows a multi-body simulation representing the solar system, integrated using the velocity Verlet algorithm. One can see that the algorithm handles the problem quite well.

6.4. The perihelion precession of Mercury. Figure 5 shows the angle of the perihelion of Mercury calculated by equation 5 in the theory section. This plot is over a period of 30 Earth years and with a time step of $dt = 10^{-6}yr.s$. One can see a clear trend in the angle of the perihelion as predicted.

7. DISCUSSION

7.1. Integration Algorithm. Velocity Verlet for the win!! :D

8. CONCLUSION

⁸Excluding moons and other satellites, but including Pluto.

Earth, Sun and Jupiter

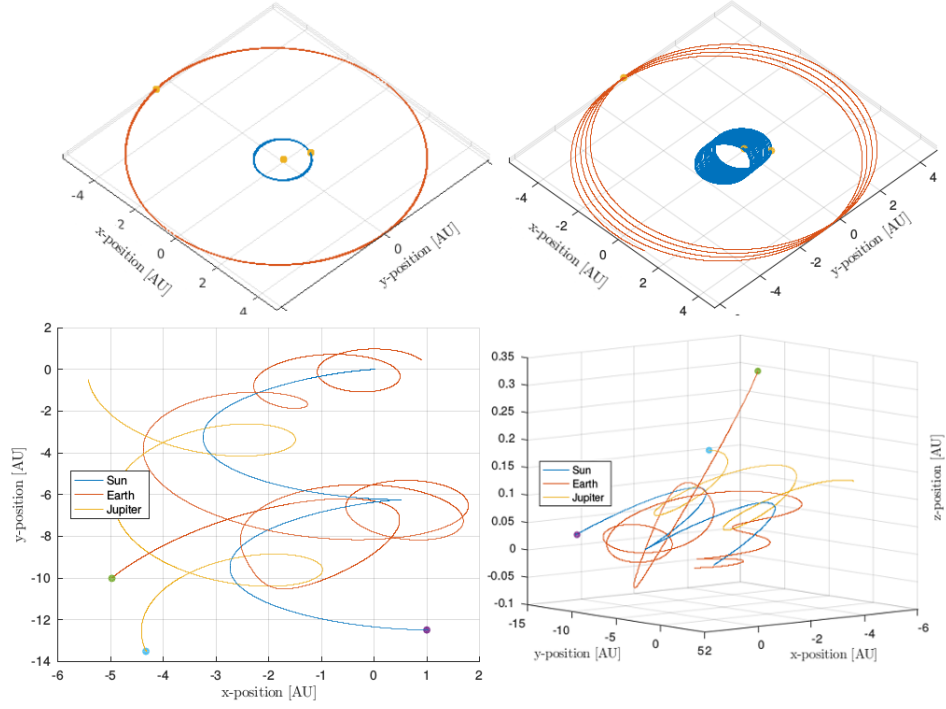


FIGURE 3. Model outcome for a Jupiter with different masses. Top left: Normal mass. Top right: Jupiter mass $10\times$ normal. Bottom: Jupiter mass $1000\times$ normal, in 2D (left) and 3D (right)

Multi-body model of the Solar System

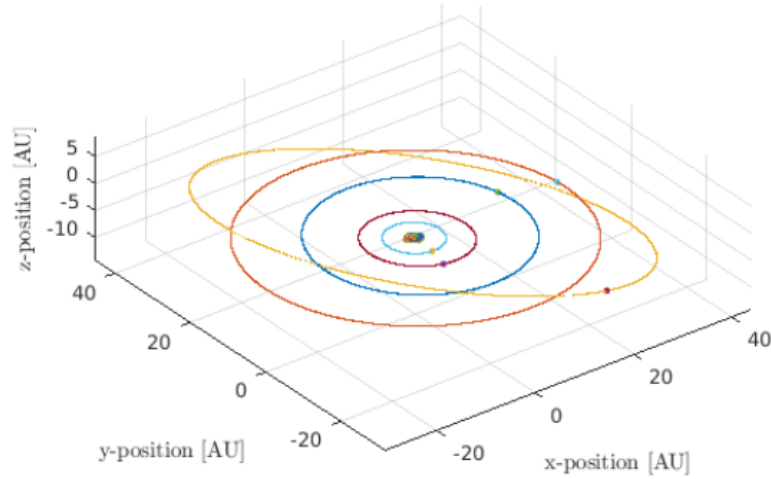


FIGURE 4. All planets of our solar system (including Pluto)

APPENDIX A. CLASS DIAGRAM

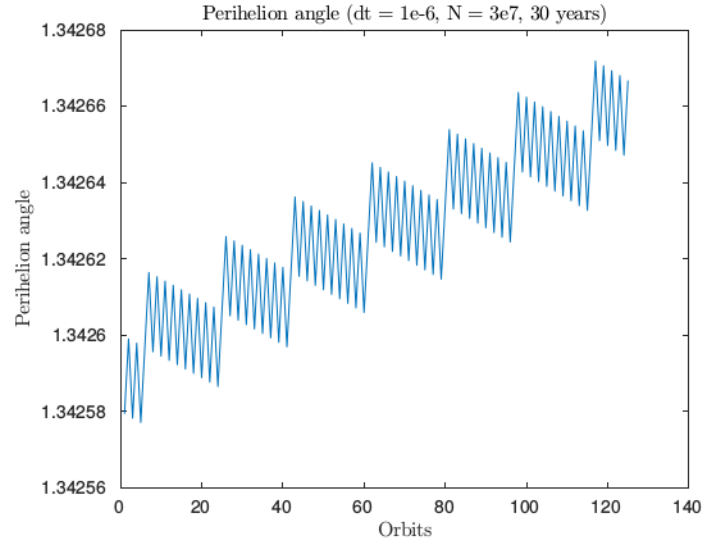


FIGURE 5. The precession of Mercury's orbit

