SOLVING THE POISSON-EQUATION IN ONE DIMENSION

FYS3150: COMPUTATIONAL PHYSICS

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Abstract. Lorem ipsum dolor sit amet, consectur adipiscing lit. Nullam ut lacus eget lorem...

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1. Introduction

2. Theory

2.1. **The Poisson Equation.** The Poisson equation is a classical equation from electromagnetism. The electrostatic potential Φ is generated by a localized charge distribution $\rho(\mathbf{r})$. In three dimensions the equation reads

$$\nabla^2\Phi = -4\pi\rho(\mathbf{r}) \tag{1}$$

where ∇^2 is the Laplace operator. In three dimensions the Laplace operator can be expressed using spherical coordinates, but in this study I am assuming that Φ and ρ are spherically symmetric, thus reducing the equation to a one-dimensional problem. only dependent on radius r.

$$\nabla^2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) \tag{2}$$

By substituting $\Phi(r) = \phi(r)/r$ the Poisson equation is reduced to

$$\frac{d^2\phi}{dr^2} = -4\pi r\rho(r) \tag{3}$$

and by letting $\phi \to u$ and $r \to x$ one is left with the very simple equation

$$-u''(x) = f(x) \tag{4}$$

The inhomogenous term f, or source term, is given by the charge distribution ρ multiplied by r and the constant -4π. In this study, however, the source term will be $f(x) = 100e^{-10x}$ and the results can be compared to the analytical solution $u(x) = 1 - (1 - e^{-10})x - e^{-10x}$.

2.2. **Approximation of the Second Derivative.** In this study the one-dimensional Poisson equation will be solved with Dirichlet boundary conditions by rewriting it as a set of linear equations. The discretized approximation of u is defined as v_i with grid points $x_i = ih$, step size of $h = \frac{1}{n+1}$, in the interval $x_0 = 0$ to $x_{n+1} = 1$ and with boundary conditions $v_0 = v_n + 1 = 0$. The interior solution $v_i \forall i \in 1, ..., n$ is to be found. The second order derivative is approximated with the three point formula such that equation 4 becomes

$$-\frac{v_{i+1} - 2v_i + vi - 1}{h} = f_i \tag{5}$$

By defining $\tilde{\mathbf{b}} = h^2 f_i$ one can rewrite equation 5 as $-v_{i+1} - 2v_i + vi - 1 = h^2 f_i$. If we ignore the end points, $i = 0$ and $i = n+1$, this equation can be represented as a matrix equation.

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ & & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \tag{6}$$