

TAKE-HOME EXAM

FYS4170: QUANTUM FIELD THEORY

CANDIDATE NUMBER: 15020

1. TRACE OF DIRAC MATRIX PRODUCTS

The Dirac gamma matrices are in the chiral representation given in 2×2 block form as,

$$(1) \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli sigma matrices,

$$(2) \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The gamma matrices satisfy the anti-commutation relations

$$(3) \quad \{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \times \mathbb{1},$$

The fifth gamma matrix is defined by

$$(4) \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

and has the following properties

$$(5) \quad (\gamma^5)^\dagger = \gamma^5,$$

$$(6) \quad (\gamma^5)^2 = 1,$$

$$(7) \quad \{\gamma^5, \gamma^\mu\} = 0.$$

a. The trace of the product of two gamma matrices can be evaluated using the anti-commutator relation and the cyclic property of the trace of a matrix product,

$$(8) \quad \text{Tr}[\gamma^\mu \gamma^\nu] = \text{Tr}[2g^{\mu\nu} \times \mathbb{1} - \gamma^\nu \gamma^\mu]$$

$$(9) \quad = 2g^{\mu\nu} \text{Tr} \mathbb{1} - \text{Tr}[\gamma^\nu \gamma^\mu]$$

$$(10) \quad = 8g^{\mu\nu} - \text{Tr}[\gamma^\mu \gamma^\nu].$$

In this particular case $g^{\mu\nu}$ is a matrix element, not the metric, and can therefore be moved outside the trace operator as in 9. In 10 the cyclic property of the trace is employed. This yields

$$(11) \quad \text{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu}.$$

b. The trace of the product of two gamma matrices with the fifth gamma matrix is,

$$(12) \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^5] = \text{Tr}[\gamma^0 \gamma^0 \gamma^\mu \gamma^\nu \gamma^5]$$

$$(13) \quad = -\text{Tr}[\gamma^0 \gamma^5 \gamma^0 \gamma^\mu \gamma^\nu]$$

$$(14) \quad = -\text{Tr}[\gamma^\mu \gamma^\nu \gamma^0 \gamma^5 \gamma^0]$$

$$(15) \quad = -\text{Tr}[\gamma^\mu \gamma^\nu (\gamma^5)^\dagger]$$

$$(16) \quad = -\text{Tr}[\gamma^\mu \gamma^\nu \gamma^5].$$

Here we have first inserted $\mathbb{1} = \gamma^0 \gamma^0$, then the anticommutator relation from Equation 7 is employed three times. The cyclic property of the trace is employed and finally $\gamma^0 \gamma^\mu \gamma^0 = \gamma^\dagger$ and the fact that γ^5 is Hermitian (Equation 5). Adding $\text{Tr}[\gamma^\mu \gamma^\nu \gamma^5]$ to both sides gives $2 \text{Tr}[\gamma^\mu \gamma^\nu \gamma^5] = 0$ and we have that

$$(17) \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^5] = 0.$$

c. The trace of an odd number of gamma matrices is always zeros. Here is a proof for three gamma matrices

$$(18) \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho] = \text{Tr}[\gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho]$$

$$(19) \quad = -\text{Tr}[\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^5]$$

$$(20) \quad = -\text{Tr}[\gamma^5 \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho]$$

$$(21) \quad = -\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho],$$

where first the property in Equation 6 is employed, then three anticommutations from Equation 7 and the cyclic property of the trace. Adding $\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho]$ similar to before gives $2 \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho] = 0$ and we end up with,

$$(22) \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho] = 0.$$

This proof holds for any odd number of gamma matrices because the number of commutation relation “switches” needed will then also be odd, yielding the desired minus sign.

d. The trace of the product of four gamma matrices and the special γ^5 is zero in *nearly* all cases. In fact, this is the first non-vanishing trace involving γ^5 . Let us first try the same kind of trick as before,

$$(23) \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = \text{Tr}[\gamma^0 \gamma^0 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5]$$

$$(24) \quad = -\text{Tr}[\gamma^0 \gamma^5 \gamma^0 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma]$$

$$(25) \quad = -\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^0 \gamma^5 \gamma^0]$$

$$(26) \quad = -\text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5],$$

which in the same way as in the previous cases yields

$$(27) \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = 0.$$

However, the result is something else if all of Dirac's gamma matrices are represented in the trace. Take for instance $(\mu\nu\rho\sigma) = (0123)$,

$$(28) \quad \text{Tr}[\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^5] = i \text{Tr}[\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^1 \gamma^2 \gamma^3]$$

$$(29) \quad = -i \text{Tr}[\gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^1 \gamma^2 \gamma^3]$$

$$(30) \quad = -i \text{Tr}[\gamma^1 \gamma^1 \gamma^2 \gamma^3 \gamma^2 \gamma^3]$$

$$(31) \quad = i \text{Tr}[\gamma^2 \gamma^3 \gamma^2 \gamma^3]$$

$$(32) \quad = -i \text{Tr}[\gamma^2 \gamma^2 \gamma^3 \gamma^3]$$

$$(33) \quad = -i \text{Tr}[\mathbb{1}] = -i4.$$

A computation with the indices of two adjacent gamma matrices interchanged.

$$(34) \quad \text{Tr}[\gamma^0 \gamma^1 \gamma^3 \gamma^2 \gamma^5] = i \text{Tr}[\gamma^0 \gamma^1 \gamma^3 \gamma^2 \gamma^0 \gamma^1 \gamma^2 \gamma^3]$$

$$(35) \quad = -i \text{Tr}[\gamma^0 \gamma^0 \gamma^1 \gamma^3 \gamma^2 \gamma^1 \gamma^2 \gamma^3]$$

$$(36) \quad = -i \text{Tr}[\gamma^1 \gamma^1 \gamma^3 \gamma^2 \gamma^2 \gamma^3]$$

$$(37) \quad = i \text{Tr}[\gamma^3 \gamma^2 \gamma^2 \gamma^3]$$

$$(38) \quad = -i \text{Tr}[\gamma^3 \gamma^3]$$

$$(39) \quad = i \text{Tr}[\mathbb{1}] = i4.$$

The initial ordering of the for gamma matrices will change the sign compared to the initial case, because the number of commutation relation switches needed to complete the computation will change as well. This means that if two adjacent indices are interchanged, the sign will change. If two indices with another index between them are changed the sign stays the same - equivalent to two adjacent index exchanges. In other words, an even number of permutations will leave the sign unchanged, while an odd number of permutations will not. For all other cases, where two or more of the indices are equal the answer is zero. The result must therefore be proportional to the four-dimensional Levi-Civita symbol, as well as $-i4$ from the trial computation. In conclusion,

$$(40) \quad \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] = -i4\epsilon^{\mu\nu\rho\sigma}.$$

2. SCATTERING CROSS SECTION

A central result in quantum field theory is

$$(41) \quad d\sigma = \frac{1}{4vE_A E_B} |\mathcal{M}|^2 d\Pi_n,$$

a. In Equation 41, $d\sigma$ is the infinitesimal cross section, whereas the cross section is defined by

$$(42) \quad \sigma \equiv \frac{\text{Number of scattering events}}{\rho_A \ell_A \rho_B \ell_B A},$$

for two colliding bunches of particles A and B . ρ_A and ρ_B are the densities of the particle bunches, ℓ_A and ℓ_B are the lengths of the bunches of particles, and A is the cross-sectional area common to the two bunches. The cross-section can be thought of as the probability of scattering events. It has area as unit, as it gives a measure of an area that two particles need to be within to interact. When colliding two beams of particles with well-defined momenta, one can express the likelihood of any particular final state in terms of the cross section. The relative velocity of the two particle bunches is given by $v = |v_A - v_B|$ in the lab frame, while E_A and E_B are the energies of the particle beams.

\mathcal{M} is the matrix element of the interesting part of the S (cattering) matrix ($S = \mathbb{1} + iT$) after the four-momentum conservation term is factored out,

$$(43) \quad \langle \{\mathbf{p}_f\} | iT | \{\mathbf{k}_i\} \rangle \equiv {}_{\text{out}} \langle \{\mathbf{p}_f\} | \{\mathbf{k}_i\} \rangle_{\text{in}} = i\mathcal{M}(2\pi)^4 \delta^{(4)}(\Sigma k_i - \Sigma p_f),$$

where $\{\mathbf{p}_i$ and $\{\mathbf{p}_f\}$ are the sets of *in* and *out* states, respectively. \mathcal{M} is interesting because it describes the actual scattering. If the particles in question does not interact at all, S is simply the identity operator - the final state will be equal to the initial state.

Lastly, the infinitesimal relativistic invariant n -body final-state momenta phase space is (a tongue twister) given by

$$(44) \quad d\Pi_n = \left(\frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}(\Sigma p_i - \Sigma p_f).$$

It contains both the four-momentum conservation factor and a product of infinitesimal 3-momenta of the outgoing states.

b.

3. CALCULATING \mathcal{M} , FEYNMAN RULES

a. The invariant amplitude \mathcal{M} is given by the formula

$$(45) \quad \langle \mathbf{p}_1 \mathbf{p}_2 \dots | iT | \mathbf{k}_B \mathbf{k}_A \rangle = i\mathcal{M}(2\pi)^4 \delta^{(4)}(\Sigma k_i - \Sigma k_f) = \langle f | T \left\{ \exp -i \int dt H_I \right\} | i \rangle,$$

where T is the time-ordering operator and H_I is the interaction Hamiltonian.

The general algorithm to calculate the quantity \mathcal{M} from a given interaction Hamiltonian in momentum space is as follows. First, one expands the exponential term between the bra and ket in a series of the interaction factor in the interaction Hamiltonian. The expressions of higher orders will represent interactions that are of lower probability, and we usually disregard

Fifth, integrate over undetermined momentum (if in momentum space) and divide by symmetry factor. The symmetry factor arises from the fact that there may be several ways to draw Feynman diagrams, analogous to several ways to do the same contractions.

¹At least in this course.

Above is just one of several possible Feynman diagrams describing the interactions. A vertex with six external legs as this can be drawn in $2!4!$ different ways. There are $4!$ possible ways of pairing ϕ_b with \mathbf{p}_f and $2!$ possible ways of pairing ϕ_a with \mathbf{k}_i . This gives rise to the Feynman rule,

$$(49) \quad -i48\kappa.$$

4. CURRENT CONSERVATION AND CHARGE OF A DIRAC FIELD

The axial vector current is defined by

$$(50) \quad j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

The current is conserved if the divergence is zero.

$$(51) \quad \partial_\mu j_A^\mu = (\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi)$$

$$(52) \quad = (\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu(\partial_\mu\psi).$$

It is not immediately clear when this expression is zero. A step on the way is to state the Lagrangian and find the Euler-Lagrange equations. The Lagrangian for the Lagrangian is

$$(53) \quad \mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}m\psi.$$

The Euler-Lagrange equations for $\bar{\psi}$ is

$$(54) \quad \partial_\mu \left(\frac{\mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$$

$$(55) \quad 0 - (i\gamma^\mu\partial_\mu\psi - m\psi) = 0$$

$$(56) \quad \gamma^\mu\partial_\mu\psi = -im\psi,$$

and for ψ ,

$$(57) \quad \partial_\mu \left(\frac{\mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

$$(58) \quad \partial_\mu i\bar{\psi}\gamma^\mu - (-m\bar{\psi}) = 0$$

$$(59) \quad \partial_\mu \bar{\psi}\gamma^\mu = im\bar{\psi}.$$

Inserting Equation 56 and Equation 59 into Equation 52 yields

$$(60) \quad \partial_\mu j_A^\mu = im\bar{\psi}\gamma^5\psi + \bar{\psi}\gamma^5im\psi = 2im\bar{\psi}\gamma^5\psi,$$

and we see that if $m = 0$ that the axial vector current must be conserved. In order to calculate the charge, we split the axial vector current up into a left-handed and right-handed part by a linear combination.

$$(61) \quad j_A^\mu = j_R^\mu - j_L^\mu = \bar{\psi}\gamma^\mu \left(\frac{1+\gamma^5}{2} \right) \psi - \bar{\psi}\gamma^\mu \left(\frac{1-\gamma^5}{2} \right) \psi = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

These left- and right-handed current are individually conserved. The Dirac equation in terms of left-handed and right-handed fields (ψ_L and ψ_R) is

$$(62) \quad (i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \sigma \cdot \nabla) \\ i(\partial_0 - \sigma \cdot \nabla) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

We see that when $m = 0$, we get a set of decoupled equations

$$(63) \quad i(\partial_0 - \sigma \cdot \nabla)\psi_L = 0$$

$$(64) \quad i(\partial_0 + \sigma \cdot \nabla)\psi_R = 0.$$

It will therefore be alright to compute the charge from the right- and left-handed current from Equation 61 separately. To be precise, one can look at one component of the Weyl spinor representation of ψ at a time and employ the following eigenvalue relation

$$(65) \quad \gamma^5 \psi_{L/R} = \mp \psi_{L/R}.$$

We get

$$(66) \quad j_L^\mu = (\bar{\psi}_L \quad \bar{\psi}_R) \gamma^\mu \left(\frac{1 - \gamma^5}{2} \right) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \bar{\psi}_L \gamma^\mu \psi_L$$

$$(67) \quad j_R^\mu = (\bar{\psi}_L \quad \bar{\psi}_R) \gamma^\mu \left(\frac{1 + \gamma^5}{2} \right) \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \bar{\psi}_R \gamma^\mu \psi_R.$$

Computing the charge in parts, the left- and right-handed fields separately, gives the same result

$$\begin{aligned} Q_A = Q_R - Q_L &= \int d^3x j_R^0 - \int d^3x j_L^0 = \int d^3x \bar{\psi}_R \gamma^0 \psi_R - \int d^3x \bar{\psi}_L \gamma^0 \psi_L \\ &= \int d^3x \psi_R^\dagger \gamma^0 \psi_R - \int d^3x \psi_L^\dagger \gamma^0 \psi_L = \int d^3x \left[\psi^\dagger \left(\frac{1 + \gamma^5}{2} \right) \psi - \psi^\dagger \left(\frac{1 - \gamma^5}{2} \right) \psi \right] \\ &= \frac{1}{2} \int d^3x \left[\psi^\dagger 1 \psi - \psi^\dagger 1 \psi + 2\psi^\dagger \gamma^5 \psi \right] = \int d^3x \psi^\dagger \gamma^5 \psi. \end{aligned}$$

The axial charge becomes

$$(68)$$

$$Q_A = \int d^3x \psi^\dagger \gamma^5 \psi$$

$$(69)$$

$$\begin{aligned} &= \int d^3p \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^2} \frac{1}{2\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}}} \\ &\times \sum_s \left(a_{\mathbf{p}}^s u^s(p) e^{-ip \cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip \cdot x} \right)^\dagger \gamma^5 \sum_r \left(a_{\mathbf{q}}^r u^r(q) e^{-iq \cdot x} + b_{\mathbf{q}}^{r\dagger} v^r(q) e^{iq \cdot x} \right) \end{aligned}$$

$$(70)$$

$$\begin{aligned} &= \int d^3p \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^2} \frac{1}{2\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}}} \sum_{rs} \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{q}}^{s\dagger} u^{s\dagger}(p) \gamma^5 u^r(q) e^{ix \cdot (p-q)} \right. \\ &\quad \left. + b_{\mathbf{p}}^s b_{\mathbf{q}}^{r\dagger} v^{s\dagger}(p) \gamma^5 v^r(q) e^{ix \cdot (q-p)} + a_{\mathbf{p}}^{s\dagger} b_{\mathbf{q}}^{r\dagger} u^{s\dagger}(p) \gamma^5 v^r(q) e^{ix \cdot (p+q)} + b_{\mathbf{p}}^s a_{\mathbf{q}}^r v^{s\dagger}(p) \gamma^5 u^r(q) e^{ix \cdot (p+q)} \right) \end{aligned}$$

5. COMPLEX SCALAR FIELDS AND “PHOTONS”

The interaction of a complex scalar field ϕ is interacting with massless bosons (“photons”) B^γ . The interaction is described by the following interaction Lagrangian,

$$(71) \quad \mathcal{L}_{\text{Int}} = g^2 B_\mu B^\mu |\phi|^2 + ig B_\mu (\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi).$$

5.1. **a.** We wish to compute the amplitude of $\phi\phi^* \rightarrow \gamma\gamma$, two scalar particles annihilating into two photons, to lowest order in g .

$$(72) \quad i\mathcal{M} \hat{=} \langle \gamma\gamma | T \left\{ \exp \left(-i \int_{-T}^T dt H_I \right) \right\} | \phi\phi^* \rangle_{\text{connected, amputated}}$$

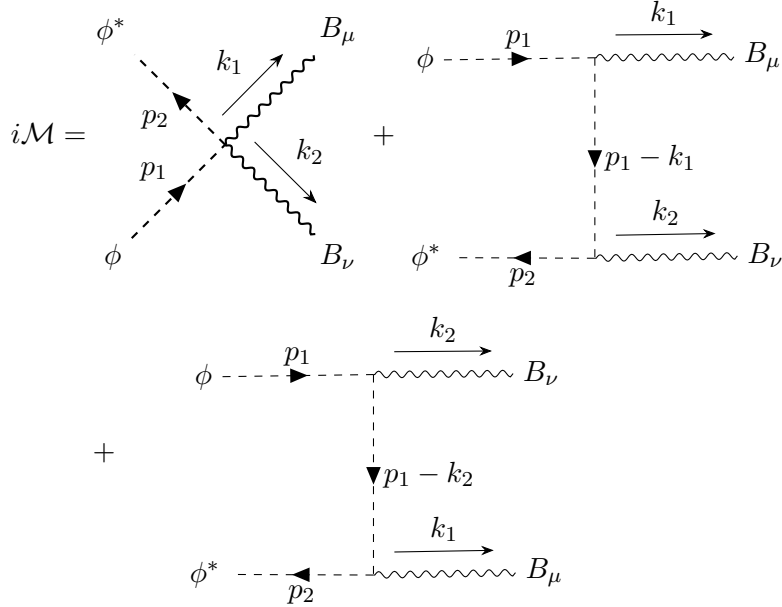
where $H_I = - \int d^x \mathcal{L}_{\text{Int}}$. So,

$$\begin{aligned} i\mathcal{M} &= \langle \gamma\gamma | T \left\{ \exp \left(i \int d^4x g^2 B_\mu B^\mu \phi^* \phi + ig B_\mu (\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) \right) \right\} | \phi\phi^* \rangle \\ &= \langle \gamma\gamma | T \left\{ \text{ext} \left(i \int d^4x g^2 B_\mu B^\mu \phi\phi^* \right) \text{ext} \left(ig \int d^4x B_\mu \phi \partial^\mu \phi^* \right) \text{ext} \left(-ig \int d^4x B_\mu \phi^* \partial^\mu \phi \right) \right\} | \phi\phi^* \rangle. \end{aligned}$$

Performing an expansion to lowest order of the exponential factors in g yields,

$$\begin{aligned} i\mathcal{M} &= \langle \gamma\gamma | T \left\{ \left(\mathbb{1} + ig^2 \int d^4x B_\mu B^\mu \phi\phi^* + \dots \right) \left(\mathbb{1} + ig \int d^4x B_\mu \phi \partial^\mu \phi^* + \dots \right) \right. \\ &\quad \left. \times \left(\mathbb{1} - ig \int d^4x B_\mu \phi^* \partial^\mu \phi + \dots \right) \right\} | \phi\phi^* \rangle \\ &= \langle \gamma\gamma | \mathbb{1} | \psi\psi^* \rangle + \langle \gamma\gamma | ig^2 \int d^4x B_\mu B^\mu \phi^* \phi | \psi\psi^* \rangle \\ &\quad + \langle \gamma\gamma | g^2 \int d^4x B_\mu \phi \partial^\mu \phi^* B_\nu \phi^* \partial^\nu \phi | \psi\psi^* \rangle \\ &\quad + \langle \gamma\gamma | T \left\{ ig \int d^4x B_\mu \phi \partial^\mu \phi^* \right\} | \psi\psi^* \rangle \\ &\quad + \langle \gamma\gamma | T \left\{ -ig \int d^4x B_\mu \phi^* \partial^\mu \phi \right\} | \psi\psi^* \rangle + \dots \end{aligned}$$

The two last terms will not be fully contracted and is therefore ignored, terms that are in higher of second order in g are also ignored. The two contracted terms leads to the following three Feynman diagrams.



Evaluating the Feynman diagrams with the provided rules yields

(73)

$$\begin{aligned}
 i\mathcal{M} &= i\mathcal{M}_I + i\mathcal{M}_{II} + i\mathcal{M}_{III} \\
 &= 2ig^2 g^{\mu\nu} \epsilon_\mu^* \epsilon_\nu^* \\
 &\quad + (-ig(p_1 + p_1 - k_1))^\mu \frac{i}{(p_1 + k_1)^2 + m^2} (-ig(p_1 - k_1 - p_2)^\nu \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2)) \\
 &\quad + (-ig(p_1 + p_1 - k_2))^\mu \frac{i}{(p_1 + k_2)^2 + m^2} (-ig(p_1 - k_2 - p_2)^\nu \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1)).
 \end{aligned}$$

These equations can be simplified by employing conservation of momentum,

$$(74) \quad i\mathcal{M}_I = 2ig^2 g^{\mu\nu} \epsilon_\mu^*(k_1) \epsilon_\nu^*$$

$$(75) \quad i\mathcal{M}_{II} = i(-ig)^2 \frac{(2p_1 - k_1)^\mu (k_2 - 2p_2)^\nu}{(p_1 - k_1)^2 - m^2} \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2)$$

$$(76) \quad i\mathcal{M}_{III} = i(-ig)^2 \frac{(2p_1 - k_2)^\mu (k_1 - 2p_2)^\nu}{(p_1 - k_2)^2 - m^2} \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1).$$

By definition, the product of the photon polarization vector with its related photon momentum is zero, i.e. $\epsilon_\mu^*(k_2) k_2^\mu = 0$. This simplifies two of the above expressions

$$(77) \quad i\mathcal{M}_{II} = 4ig^2 \frac{p_1^\mu p_2^\nu}{(p_1 - k_1)^2 - m^2} \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2)$$

$$(78) \quad i\mathcal{M}_{III} = 4ig^2 \frac{p_1^\mu p_2^\nu}{(p_1 - k_2)^2 - m^2} \epsilon_\mu^*(k_2) \epsilon_\nu^*(k_1),$$

in turn concatenating Equation 73 to
(79)

$$i\mathcal{M} = 4ig^2 \left(\frac{1}{2}g^{\mu\nu} + \frac{p_1^\mu p_2^\nu}{(p_1 - k_1)^2 - m^2} + \frac{p_1^\nu p_2^\mu}{(p_1 - k_2)^2 - m^2} \right) \epsilon_\mu^*(k_1) \nu^*(k_2).$$

Notice the index change necessary to make this work.

The remaining goal of this venture is to use Ward's identity to show that we have found the correct expression for $i\mathcal{M}$. Ward's identity can be stated as

$$(80) \quad p_\mu \mathcal{M}^\mu = 0,$$

where p_μ is the momentum of an external photon in the interaction process, in a simple form stated as $\mathcal{M} = \epsilon_\mu \mathcal{M}^\mu$. This identity is essentially a statement of current conservation, which is a consequence of the gauge symmetry of QED². The amplitude vanishes when the polarization is replaced by the momentum.

Replacing $\epsilon_\mu^*(k_1)$ with k_1 , one "diagram" at a time gives

$$(81) \quad k_{1\mu} \mathcal{M}_I^\mu = 2g^2 k_1^\nu \epsilon_\nu^*(k_2)$$

$$(82) \quad k_{1\mu} \mathcal{M}_{II}^\mu = (ig)^2 \frac{(2p_1^\mu k_{1\mu} - k_1^2)(k_2^\nu \epsilon_\nu^*(k_2) - 2p_2^\nu \epsilon_\nu^*(k_2))}{k_1 - 2p_1 k_1} \\ = g^2(k_2 \epsilon^*(k_2) - 2p_2 \epsilon^*(k_2)),$$

wherein Equation 82 we done the following replacement, $(p_1 - k_1)^2 - m^2 = k_1^2 - 2k_1 p_1 + p_1 - m^2 = k_1^2 - 2k_1 p_1$. Employing a similar trick like this and making use of conservation of momentum to change variables, gives us an alternative expression for the last term,

$$(83) \quad i\mathcal{M}_{III} = i(ig)^2 \frac{(2p_2 - k_1)^\mu (k_2 - 2p_1)}{p_3^2 - 2p_3 p_2} \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2).$$

Exchanging the polarization vector with the momentum vector yields,

$$(84) \quad k_{1\mu} \mathcal{M}_{III} = (ig)^2 \frac{(2p_2^\mu k_{1\mu} - k_1^2)(k_2^\nu \epsilon_\nu^*(k_2) - 2p_1^\nu \epsilon_\nu^*(k_2))}{k_1^2 - 2k_1 p_2} \\ = g^2(k_2 \epsilon^*(k_2) - 2p_1 \epsilon^*(k_2)).$$

Finally, adding up all the terms,

$$(85) \quad k_{1\mu} \mathcal{M}^\mu = 2g^2 k_1 \epsilon^*(k_2) + g^2(k_2 \epsilon^*(k_2) - 2p_2 \epsilon^*(k_2)) + g^2(k_2 \epsilon^*(k_2) - 2p_1 \epsilon^*(k_2)) \\ (86) \quad = 2g^2 \epsilon^*(k_2)(k_1 + k_2 - p_1 - p_2) = 0,$$

we see that we indeed have Ward's identity!

6. COMPTON SCATTERING

The expression for the spin-averaged squared matrix element can be simplified to

$$(87) \quad |\bar{\mathcal{M}}|^2 = 2e^4 \left[\frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k}{p \cdot k'} + \left(1 + \frac{m_e^2}{p \cdot k} - \frac{m_e^2}{p \cdot k'} \right) - 1 \right]$$

² $\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$, $A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$.

6.1. **a.** In the lab frame, where the electron is initially at rest, we have the following momenta:

$$(88) \quad k = (\omega, 0, 0, \omega)$$

$$(89) \quad p = (m, 0, 0, 0)$$

$$(90) \quad k' = (\omega', \omega' \sin \theta, 0, \omega' \cos \theta)$$

$$(91) \quad p' = (E', \mathbf{p}')$$

where ω is the photon energy and θ is the scattering angle of the photon. In $|\mathcal{M}|^2$, one can contract the four-momenta to

$$(92) \quad p \cdot k = m\omega$$

$$(93) \quad p \cdot k' = m\omega'$$

6.2. **b.** We now want to express ω' in terms of ω and θ . Remembering the Lorentz invariance of the square of the four momentum ($p^2 = p'^2 = m^2$) and employing conservation of momentum we get

$$(94) \quad m^2 = (p')^2 = (p+k-k')^2 = p^2 + 2p(k-k') + (k-k')^2 = p^2 + 2p(k-k') + k^2 - 2k \cdot k' + k'^2$$

employing Equation 92 and Equation 93, computing $k \cdot k'$ and that $p^2 = m^2$ gives

$$\begin{aligned} m^2 &= m^2 + 2m(\omega - \omega') - 2\omega\omega'(1 - \cos \theta) \\ 0 &= m\omega - m\omega' - \omega\omega' + \omega\omega' \cos \theta \\ 0 &= m\omega - \omega'(m + \omega - \omega \cos \theta) \\ \omega' &= \frac{m\omega}{m + \omega - \omega \cos \theta} \\ (95) \quad \omega' &= \frac{\omega}{1 + \frac{m}{\omega}(1 - \cos \theta)} \end{aligned}$$

6.3. **c.** The n -body relativistically invariant phase space is given by Equation 44. The two-body integrated version of this is

$$(96) \quad \int d\Pi_2 = \int \frac{d^3p' d^3k'}{(2\pi)^6} \frac{1}{2E'2\omega'} (2\pi)^4 \delta^{(4)}(k' + p' - k - p).$$

First we want to make a variable change, $dk' = (\omega')^2 \sin \theta d\omega' d\theta d\varphi$, where we insert the infinitesimal solid angle $d\Omega = \sin \theta d\theta d\varphi$, which gives $d^3k' = (\omega')^2 d\omega' d\Omega$. Inserting this, and using conservation of momentum ($k' + p' = k + p \rightarrow -p' = k' - k - p$) in the spatial part of the Dirac delta and crossing out equal factors in the numerator and denominator yields

$$(97) \quad \int d\Pi_2 = \int \frac{d^3p'}{(2\pi)^3} \frac{(\omega')^2 d\omega' d\Omega}{4\omega' E'} \delta^{(3)}(2\mathbf{p}') \delta(\omega' + E' - \omega - m)$$

$$(98) \quad = \int \frac{(\omega')^2 d\omega' d\Omega}{(2\pi)^3} \frac{1}{4\omega' E'} \delta(\omega' + E' - \omega - m).$$

Now we want to do something with the solid angle. Consider $d\Omega = \sin \theta d\theta d\varphi$ and notice that there are no φ in our expression. We can therefore evaluate the integral, $\int d\phi = 2\pi$. Moreover, $\int_0^\pi \sin \theta d\theta = \int_{-1}^1 -d \cos \theta =$

$\int_1^{-1} d \cos \theta$ allowing the change $\sin \theta d\theta = d \cos \theta$. This should now leave us with

$$(99) \quad \int d\Pi_2 = \int \frac{d\omega' d \cos \theta}{2\pi} \frac{1}{4E'} \delta(\omega' + E' - \omega - m).$$

Now to get rid of the remaining Dirac delta. The energy-momentum relation gives

$$(100) \quad p'^2 = E'^2 - |\mathbf{p}|^2 = m^2 \rightarrow E' = \sqrt{|\mathbf{p}|^2 + m^2},$$

where

$$(101) \quad \mathbf{p}'^2 = \mathbf{p} + \mathbf{k} - \mathbf{k}' = \mathbf{0} + \omega \hat{z} - (\omega \sin \theta, 0, \omega \cos \theta)$$

$$(102) \quad |\mathbf{p}'|^2 = \omega' \sin^2 \theta + \omega^2 - 2\omega\omega' \cos \theta + \omega'^2 \cos^2 \theta$$

$$(103) \quad = \omega^2 + (\omega')^2 - 2\omega\omega' \cos \theta,$$

so that

$$(104) \quad E' = \sqrt{m^2 + \omega^2 + (\omega')^2 - 2\omega\omega' \cos \theta}.$$

This leads to yet another new expression for the phase space

$$(105) \quad \int d\Pi_2 = \int \frac{d\omega' d \cos \theta}{2\pi} \frac{1}{4E'} \delta(\omega' + \sqrt{m^2 + \omega^2 + (\omega')^2 - 2\omega\omega' \cos \theta} - \omega - m).$$

In order to proceed we need to use this neat rule

$$(106) \quad \int dx f(x) \delta(g(x)) = \int dx \frac{f(x)}{|g'(x)|}.$$

Differentiating the expression inside the Dirac delta function gives us

$$(107) \quad \frac{\partial}{\partial \omega'} \left(\omega' + \sqrt{m^2 + \omega^2 + (\omega')^2 - 2\omega\omega' \cos \theta} - \omega - m \right) = 1 + \frac{\omega' - \omega \cos \theta}{E'}.$$

Inserting this now gives us

$$(108) \quad \int d\Pi_2 = \int \frac{d \cos \theta}{2\pi} \frac{\omega'}{4E'} \frac{1}{\left| 1 + \frac{\omega' - \omega \cos \theta}{E'} \right|}$$

$$(109) \quad = \frac{1}{8\pi} \int d \cos \theta \frac{\omega'}{E' + \omega' - \omega \cos \theta}$$

$$(110) \quad = \frac{1}{8\pi} \int d \cos \theta \frac{\omega'}{m + \omega - \omega' + \omega' - \omega \cos \theta}$$

$$(111) \quad = \frac{1}{8\pi} \int d \cos \theta \frac{\omega'}{m + \omega(1 - \cos \theta)}.$$

Finally, notice that we are not far from Equation 95 in the fraction

$$(112) \quad \int d\Pi_2 = \frac{1}{8\pi} \int d\cos\theta \frac{\omega'}{m \left(1 + \frac{\omega}{m}(1 - \cos\theta)\right)}$$

$$(113) \quad = \frac{1}{8\pi} \int d\cos\theta \frac{(\omega')^2}{\omega m}.$$

Removing the integral signs on both sides and we land on an expression for the two-particle final state phase space,

$$(114) \quad d\Pi_2 = \frac{1}{8\pi} d\cos\theta \frac{(\omega')^2}{\omega m}.$$

6.4. **d.** The infinitesimal cross section is given by Equation 41. Inserting $|\bar{\mathcal{M}}|^2$, $d\Pi_2$, $v = 1$, $E_A = \omega$, $E_B = m$ and dividing through by $d\cos\theta$ yields

$$(115) \quad \frac{d\sigma}{d\cos\theta} = \frac{1}{4\omega m} 2e^4 \left[\frac{m\omega'}{m\omega} + \frac{m\omega}{m\omega'} + \left(1 + \frac{m^2}{m\omega} + \frac{m^2}{m\omega'}\right)^2 - 1 \right] \frac{1}{8\pi} \frac{(\omega')^2}{\omega m}$$

$$(116) \quad = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + \left(1 + \frac{m}{\omega} + \frac{m}{\omega'}\right)^2 - 1 \right]$$

$$(117) \quad = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + \left(1 + m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right)\right)^2 - 1 \right]$$

$$(118) \quad = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + (1 - 1 + \cos\theta)^2 - 1 \right]$$

$$(119) \quad = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + \cos^2\theta - 1 \right]$$

$$(120) \quad = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right],$$

where the fine-structure constant $\alpha = \frac{e^2}{4\pi}$ has been inserted in Equation 116 and Compton's formula for shift in photon wavelength³ has been inserted in Equation 118. In the relativistic limit, where $\omega \rightarrow 0$, we see from Equation 95 that $\omega'/\omega \rightarrow 1$. Equation 120 becomes

$$(121) \quad \frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2} (1 + \cos^2\theta).$$

The full cross section is found by integration

$$(122) \quad \int d\sigma = \sigma = \int_{-1}^1 \frac{\pi\alpha^2}{m^2} (1 + \cos^2\theta) d\cos\theta = \int_{-1}^1 \frac{\pi\alpha^2}{m^2} (1 + u^2) du = \frac{\pi\alpha^2}{m^2} \left[u + \frac{1}{3}u^3 \right]_{-1}^1$$

$$(123) \quad = \frac{\pi\alpha^2}{m^2} \left(1 + \frac{1}{3} + 1 + \frac{1}{3} \right) \frac{8\pi\alpha^2}{3m^2},$$

³ $\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1 - \cos\theta)$.

which is the Thompson cross-section for scattering of classical electromagnetic radiation by a free electron.