FYS4170, autumn 2017

Mid term exam: Solutions

Problem 1.
Unless otherwise stated, we use throughout only the following identities: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $\{\gamma^\mu, \gamma^5\} = 0$, $\left(\gamma^5\right)^2 = 1$, $\mathrm{Tr}\,[\mathbf{1}] = 4$ and $\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB]$ for *any* matrices A, B, C.

a)

$$\begin{aligned} \mathrm{Tr}\,[\gamma^\mu\gamma^\nu] &= \mathrm{Tr}\,[2g^{\mu\nu}\mathbf{1} - \gamma^\nu\gamma^\mu] = 8g^{\mu\nu} - \mathrm{Tr}\,[\gamma^\mu\gamma^\nu] & (1) \\ \rightsquigarrow \mathrm{Tr}\,[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu}\,. & (2) \end{aligned}$$

b)

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = (-1)^\alpha\frac{1}{4}\mathrm{Tr}\left[\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^5\right] \tag{3}$$

where we inserted the identity $\gamma^\alpha\gamma^\alpha = (-1)^\alpha \times \mathbf{1}$ (no summation over α!), with

$$(-1)^\alpha = \begin{cases} 1 & \text{if } \alpha = 0 \\ -1 & \text{if } \alpha \neq 0 \end{cases} \tag{4}$$

For given values of μ, ν we can always insert $\gamma^\alpha\gamma^\alpha$ with a value of α such that $\alpha \neq \mu,\ \nu$. This means that γ^α will anti-commute with γ^μ, γ^ν, and γ^5:

$$\begin{aligned} \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] &= -(-1)^\alpha\frac{1}{4}\mathrm{Tr}\left[\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^5\gamma^\alpha\right] & (5) \\ &= -(-1)^\alpha\frac{1}{4}\mathrm{Tr}\left[\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^5\right] & (6) \\ \rightsquigarrow \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] &= 0\,. & (7) \end{aligned}$$

c)

$$\begin{aligned} \mathrm{Tr}\,[\gamma^\mu\gamma^\nu\gamma^\rho] &= \frac{1}{4}\mathrm{Tr}\left[\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\right] & (8) \\ &= -\frac{1}{4}\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5\right] = -\frac{1}{4}\mathrm{Tr}\left[\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\right] = 0 & (9) \end{aligned}$$

d) Consider first the case where any of the four indices (μ, ν, ρ, σ) are equal. We can then, if necessary, anti-commute the matrices such that two equal matrices are located next to each other and use the identity $\gamma^\alpha\gamma^\alpha = (-1)^\alpha \times \mathbf{1}$, thus reducing the trace to that in problem **b)**. All indices must thus be different. Furthermore, the result must be anti-symmetric with respect to exchange of any two indices, and hence be proportional to the Levi-Civita symbol $\varepsilon^{\mu\nu\rho\sigma}$. We can find this proportionality constant by using the definition of $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$:

$$\begin{aligned} \mathrm{Tr}\left[\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\right] &= \mathrm{Tr}\left[-i\gamma^5\gamma^5\right] = -4i & (10) \\ \rightsquigarrow \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right] &= -4i\varepsilon^{\mu\nu\rho\sigma} & (11) \end{aligned}$$

Problem 2.
a)

$$d\sigma = \frac{1}{4vE_AE_B}\left|\mathcal{M}\right|^2 d\Pi_n \tag{12}$$

1. **cross section** σ
   Gives a measure of the probability, with dimensions of an area, of two particles to interact For a colliding beam of particles of species A and B, e.g., it is defined as $\sigma \equiv AN/(N_AN_B)$, where A is the overlap area between the two beams, N_A and N_B the number of particles of each species that encounter each other in a given time interval, and N the actual number of collisions that take place during this time. It can also be seen as the 'effective area' of a particle A, as 'seen' by an incoming flux of particle B; in this case a useful (equivalent!) definition is $\sigma \equiv (dN/dt)/(vn_B)$, where dN/dt denotes the number of collisions per unit time, v the beam velocity, and n_B the number density of particles B.

2. **matrix element, or invariant amplitude $\mathcal{M}$**
   Defined as the 'non-trivial' part of the $S = 1 + iT$-matrix – which is essentially the time evolution operator that takes a state $|\mathbf{k}_A, \mathbf{k}_B\rangle$ of two localized wave packages in the remote past to a state consisting of arbitrary many 'particles' in the distant future – i.e. the part that describes actual scattering, with 4-momentum conservation factored out. For states defined at a common point in time, this reads formally

$$\langle \mathbf{p}_1, \mathbf{p}_2, ...|iT|\mathbf{k}_A, \mathbf{k}_B\rangle \ \equiv i\mathcal{M}(2\pi)^4\, \delta^{(4)}\big(k_A + k_B - \sum_f p_f\big) \tag{13}$$

3. **relativistically invariant phase space** $d\Pi_n \equiv \left(\Pi_{f=1}^n \frac{d^3pf}{(2\pi)^3 2E_f}\right)(2\pi)^4\delta^{(4)}$
   The differential provides a cell in phase space that corresponds to a specific momentum configuration of all final state particles. In its integrated version, the phase space thus counts the number of all distinct kinematical final state configurations that are possible

4. **Further kinematic quantities**
   E_A and E_B provide the energies of the two initial particles, and v their relative velocity . The product of (the inverse of) these three quantities transforms under Lorentz transformations as an area oriented perpendicular to the collision axis. In all reference frames that are boosted along this axes, the numerical value of σ is thus the same

b)

1. The amplitude for the transition of an 'in state', consisting of two particle states $|\phi_A\rangle$ and $|\phi_B\rangle$ prepared in the remote past, into an 'out state, consisting of an arbitrary number of states $|\phi_f\rangle$ as measured in the distant future, is given by ${}_{\rm out}\langle\phi_1\phi_2...|\phi_A\phi_B\rangle_{\rm in}$. The S matrixgives the same amplitude, but in terms of states constructed at a *common* reference time (which one can think of as 'the' time at which the interaction happens):

$$\begin{aligned} {}_{\rm out}\langle\phi_1\phi_2...|\phi_A\phi_B\rangle_{\rm in} &= \lim_{T\to\infty} \langle\phi_1\phi_2...|e^{-iH(2T)}|\phi_A\phi_B\rangle \qquad (14)\\ &\equiv \langle\phi_1\phi_2...|S|\phi_A\phi_B\rangle \qquad (15)\end{aligned}$$

2. We then chose to consider a specific realisation of external states, namely plane waves with fixed momentum $\mathbf{p}_f$ for the final states, $|\phi_f\rangle = |\mathbf{p}_f\rangle$, and two localized wave packages with a spatial displacement $\mathbf{b}$ for the initial state,

$$|\phi_A\phi_B\rangle = |\widetilde{\mathbf{k}_A\mathbf{k}_B}\rangle \equiv \int \frac{d^3k_A}{(2\pi)^3}\frac{d^3k_B}{(2\pi)^3}\frac{\varphi_A(\mathbf{k}_A)\varphi_B(\mathbf{k}_B)e^{i\mathbf{k}_B\cdot\mathbf{b}}}{2\sqrt{E_aE_B}}|\mathbf{k}_A\mathbf{k}_B\rangle\,. \tag{16}$$

   Here, φ_A and φ_B are any smooth functions that are sharply peaked at $\mathbf{k}_A$ and $\mathbf{k}_B$, respectively. $|\mathbf{k}_A\rangle$ and $|\mathbf{k}_B\rangle$ denote 1-particle plane wave solutions as for the final state.

3. The probability P of a scattering event to happen is thus given by

$$P \ = \ |\langle\phi_1\phi_2...|S|\phi_A\phi_B\rangle|^2\ , \tag{17}$$

   and the number N of scattering events resulting from N_B particles of type B impinging on one particle of type A therefore

$$N = \frac{N_B}{A}\int d^2bP\,. \tag{18}$$

   Here, the integration over the impact parameter $\mathbf{b}$ is perpendicular to the collision axis, and A is the total area that is integrated over.

4. We used the definition of the cross section from **a)** and adapted it to our specific example (with $N_A = 1$)

$$d\sigma = \frac{A}{N_B} dN \,. \tag{19}$$

At tis point, one just needs the definition of the matrix element, see **a)**, and plug in all the individual expressions introduced above, to arrive at the final expression for the cross section. In particular, eight delta functions from the definition of $\mathcal{M}$ plus two from $\int d^2b \, \exp[i\mathbf{k}_B \cdot \mathbf{b}]$ reduce the initial $3(2+n)$ momentum integrals to the $3n-4$ that appear in the definition of Π_n.

Problem 3

a) According to the LSZ reduction formula, the amplitude $\mathcal{M}$ for a $2 \to n$ process is given by

$$\langle \mathbf{p}_1, \mathbf{p}_2, ...| iT | \mathbf{k}_A, \mathbf{k}_B \rangle \equiv i\mathcal{M}(2\pi)^4 \, \delta^{(4)}\big(k_A + k_B - \sum_f p_f\big) \tag{20}$$

$$= \lim_{T\to\infty(1-i\epsilon)} {}_0\langle \mathbf{p}_1, \mathbf{p}_2, ...| T\Big\{ \exp\big[-i\int_{-T}^{T} dt H_I(t)\big]\Big\} |\mathbf{k}_A, \mathbf{k}_B\rangle_0 \Big|_{\text{no amp}} \tag{21}$$

This implies the following algorithm to calculate $\mathcal{M}$:

1. Expand the exponent to the order in the small coupling constant(s) for which we want to compute the amplitude. This results in $T\{\exp[...]\} \to T\{\text{some power of field operators}\}$.

2. Use Wick's theorem to expand $T\{...\} = N\{\text{sum of all contractions}\}$.

3. contract every external (initial or final) state with one of the field operators from the expansion. (if not possible, the corresponding contribution to $\mathcal{M}$ is zero.)

4. Contract all *remaining* operators with each other. (If any remain that cannot be contracted, the corresponding contribution to $\mathcal{M}$ is zero.)

5. Disregard any amplitudes constructed this way where there exists an internal propagator that is 'on shell', i.e. where the propagator mass and momentum satisfy $q^2 = m^2$.

b) The Lagrangian has to have mass dimension $[\mathcal{L}] = \text{mass}^4$, so we must have $[\kappa] = \text{mass}^{-2}$. There is only one (6-point) vertex in this theory, and the simplest way to calculate the corresponding Feynman rule is by considering the process $\phi_a\phi_a \to \phi_b\phi_b\phi_b\phi_b$. Following the above algorithm, with the interaction Hamiltonian given by $H_I = \kappa \int d^3x \, \phi_a^2 \phi_b^4$, we thus obtain

1.
$$\exp\left[-i\int dt H_I(t)\right] = \exp\left[-i\kappa \int d^4x \phi_a^2(x)\phi_b^4(x)\right] \tag{22}$$

$$= 1 - i\kappa \int d^4x \phi_a^2(x)\phi_b^4(x) + \mathcal{O}(\kappa^2) \,. \tag{23}$$

2.
$$T\left\{1 - i\kappa \int d^4x \phi_a^2(x)\phi_b^4(x)\right\} = 1 - i\kappa \int d^4x \, N\{\, \phi_a^2(x)\phi_b^4(x) + \text{contr.}\} \,. \tag{24}$$

3. For the process chosen here, we need at least two internal operators ϕ_a and four ϕ_b, so we can replace

$$T\left\{1 - i\kappa \int d^4x \phi_a^2(x)\phi_b^4(x)\right\} \to i\kappa \int d^4x N\{\phi_a(x)\phi_a(x)\phi_b(x)\phi_b(x)\phi_b(x)\phi_b(x)\} \,. \tag{25}$$

Contracting this with the external states gives for example

$$-i\kappa\int d^4x\;\;{}_0\langle \mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4|N\{\phi_b(x)\phi_b(x)\phi_b(x)\phi_b(x)\phi_a(x)\phi_a(x)\}|\mathbf{k}_A,\mathbf{k}_B\rangle_0 \tag{26}$$

A contraction with an initial (final) state state gives $\exp[-ik_i]$ ($\exp[+ip_f]$). Furthermore, there are 2! (4!) equivalent ways of contracting the initial (final) states. Therefore,

$$\begin{aligned} i\mathcal{M}(2\pi)^4\,\delta^{(4)}\big(k_A+k_B-\sum_f p_f\big) &= -i\kappa(2!)(4!)\int d^4x\exp\Big[-k_A-k_B+\sum_f p_f\Big] && (27)\\ &= -i\kappa(2!)(4!)\,(2\pi)^4\,\delta^{(4)}\big(k_A+k_B-\sum_f p_f\big)\,. && (28)\end{aligned}$$

From this, we can read off the Feynman rule for the 6-point vertex as $i\mathcal{M}=-48i\,\kappa$.

4. There are no internal fields left in our contraction, so points 4. and 5. from our algorithm are trivially performed.

Problem 4

A current is conserved if it is divergence-free, i.e. $\partial_\mu j^\mu=0$. Checking this condition for the axial vector current, we find that

$$\partial_\mu j^\mu_A=\partial_\mu\left(\bar\psi\right)\gamma^\mu\gamma^5\psi-\bar\psi\gamma^5\gamma^\mu\partial_\mu\psi=2im\bar\psi\gamma^5\psi, \tag{29}$$

where in the last equality we used the Dirac equation. This means that the axial vector current is only conserved if the particles are massless From Gauss' theorem, it follows that the corresponding conserved charge is given by $Q\equiv\int d^3x\,j^0$, i.e.

$$\begin{aligned} Q &= \int d^3x\,\psi^\dagger\gamma^0\gamma^0\gamma^5\psi && (30)\\ &= \int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}}\\ &\quad\times\sum_s\Big(a^s_{\mathbf{p}}u^s(p)e^{-ip\cdot x}+b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ip\cdot x}\Big)^\dagger\gamma^5\sum_r\Big(a^r_{\mathbf{p}'}u^r(p')e^{-ip'\cdot x}+b^{r\dagger}_{\mathbf{p}'}v^r(p')e^{ip'\cdot x}\Big) && (31)\\ &= \int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}}\sum_{s,r}\Big(a^{s\dagger}_{\mathbf{p}}a^r_{\mathbf{p}'}u^{s\dagger}(p)\gamma^5u^r(p')e^{ix\cdot(p-p')}+ && (32)\\ &\quad b^s_{\mathbf{p}}b^{r\dagger}_{\mathbf{p}'}v^{s\dagger}(p)\gamma^5v^r(p')e^{ix\cdot(p'-p)}+a^{s\dagger}_{\mathbf{p}}b^{r\dagger}_{\mathbf{p}'}u^{s\dagger}(p)\gamma^5v^r(p')e^{ix\cdot(p+p')}+b^s_{\mathbf{p}}a^r_{\mathbf{p}'}v^{s\dagger}(p)\gamma^5u^r(p')e^{-ix\cdot(p+p')}\Big)\\ &= \int\frac{d^3p}{(2\pi)^3}\int d^3p'\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}}\sum_{s,r}\Big(a^{s\dagger}_{\mathbf{p}}a^r_{\mathbf{p}'}u^{s\dagger}(p)\gamma^5u^r(p')e^{it(E_{\mathbf{p}}-E_{\mathbf{p}'})}\delta^3(\mathbf{p}-\mathbf{p}')+\\ &\quad b^s_{\mathbf{p}}b^{r\dagger}_{\mathbf{p}'}v^{s\dagger}(p)\gamma^5v^r(p')e^{it(E_{\mathbf{p}}-E_{\mathbf{p}'})}\delta^3(\mathbf{p}-\mathbf{p}')+a^{s\dagger}_{\mathbf{p}}b^{r\dagger}_{\mathbf{p}'}u^{s\dagger}(p)\gamma^5v^r(p')e^{it(E_{\mathbf{p}}+E_{\mathbf{p}'})}\delta^3(\mathbf{p}+\mathbf{p}')\\ &\quad +b^s_{\mathbf{p}}a^r_{\mathbf{p}'}v^{s\dagger}(p)\gamma^5u^r(p')e^{-it(E_{\mathbf{p}}+E_{\mathbf{p}'})}\delta^3(\mathbf{p}+\mathbf{p}')\Big) && (33)\\ &= \int\frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\sum_{s,r}\Big(a^{s\dagger}_{\mathbf{p}}a^r_{\mathbf{p}}u^{s\dagger}(p)\gamma^5u^r(p)+b^s_{\mathbf{p}}b^{r\dagger}_{\mathbf{p}}v^{s\dagger}(p)\gamma^5v^r(p)+a^{s\dagger}_{\mathbf{p}}b^{r\dagger}_{-\mathbf{p}'}u^{s\dagger}(\mathbf{p})\gamma^5v^r(-\mathbf{p})\\ &\quad +b^s_{\mathbf{p}}a^r_{-\mathbf{p}}v^{s\dagger}(\mathbf{p})\gamma^5u^r(-\mathbf{p})\Big) && (34)\end{aligned}$$

At this point we need to determine the 2-spinor bilinears that appear in the above expression. Using the explicit form of $u^s(p)$ from the lectures, e.g., we immediately obtain

$$u^{s\dagger}(p)\gamma^5u^r(p)=\left(\xi^{s\dagger},\xi^{s\dagger}\right)\begin{pmatrix}-E+\mathbf{p}\cdot\sigma & 0\\ 0 & E+\mathbf{p}\cdot\sigma\end{pmatrix}\begin{pmatrix}\xi^r\\ \xi^r\end{pmatrix}\,. \tag{35}$$

We are free to choose a basis for the ξ^r, and the simplest is the one that corresponds to the spins being aligned and anti-aligned with $\mathbf{p}$, respectively. In this case, as discussed in detail in homework problem 12, the spinors are helicity eigenstates, with $\frac{1}{2}\mathbf{p}\cdot\sigma\,\xi^r = |\mathbf{p}|\,h_u^r\,\xi^r$. Here, the helicity (eigenvalue) is given by $h_u^r = +\frac{1}{2}$ ($h_u^r = -\frac{1}{2}$) for the spin aligned (anti-aligned) with the particle momentum (for anti-particles, it is the other way around). Using furthermore $E = |\mathbf{p}|$ for massless particles, we can simplify the above expression to

$$u^{s\dagger}(p)\gamma^5 u^r(p) = 4Eh_u^r \xi^{s\dagger}\xi^r = 4Eh_u^r\delta^{rs} \tag{36}$$

In the same way, we find

$$\begin{aligned} v^{s\dagger}(p)\gamma^5 v^r(p) &= -4Eh_v^r\delta^{rs} &(37)\\ v^{s\dagger}(p)\gamma^5 u^r(p) &= u^{s\dagger}(p)\gamma^5 v^r(p) = 0\,. &(38)\end{aligned}$$

Together, after anti-commuting b and $b^\dagger$, this gives

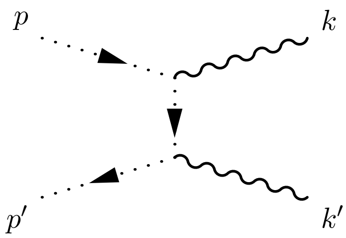
$$Q \propto \int \frac{d^3p}{(2\pi)^3}\sum_r \left(h_u^r a_{\mathbf{p}}^{r\dagger}a_{\mathbf{p}}^r + h_v^r b_{\mathbf{p}}^{r\dagger}b_{\mathbf{p}}^r\right), \tag{39}$$

i.e. the charge is simply the (total) *helicity* of our ensemble of particles. Indeed, we learned before that helicity is conserved – and only for massless particles – but now we have identified the corresponding Noether current.

Problem 5

a) Assigning momenta p (p') for the incoming ϕ (ϕ^*), and momenta k and k' for the outgoing photons, the amplitude takes the form $\mathcal{M} = \mathcal{M}_{\mu\nu}\epsilon^{*\mu}(k)\epsilon^{*\nu}(k')$. At lowest order three Feynman diagrams contribute, $\mathcal{M} = \mathcal{M}^t + \mathcal{M}^u + \mathcal{M}^{4p}$, with

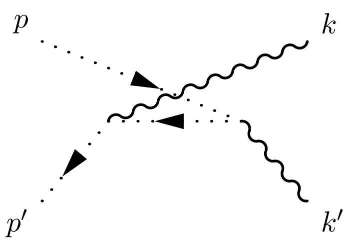
1. t-channel:



$$\begin{aligned} i\mathcal{M}^t_{\mu\nu} &= \left(-ig[-p' + (k'-p')]_\nu\right)\frac{i}{(p-k)^2 - m_\phi^2 + i\epsilon}\left(-ig[p + (p-k)]_\mu\right) &(40)\\ &= \frac{ig^2}{t - m_\phi^2}\left(2p_\mu - k_\mu\right)\left(2p'_\nu - k'_\nu\right) \to \frac{4ig^2}{t-m_\phi^2}p_\mu p'_\nu &(41)\end{aligned}$$

In the last step, we used that $\mathcal{M}_{\mu\nu}$ is contracted with the photon polarization vectors. Since $k_\mu\epsilon^\mu(k) = k'_\mu\epsilon^\mu(k') = 0$, we can thus replace $k_\mu \to 0$ and $k'_\nu \to 0$ in any expression for $\mathcal{M}_{\mu\nu}$.

2. u-channel:



$$
\begin{aligned}
i\mathcal{M}^{u}_{\mu\nu} &= \left(-ig[-p' + (k-p')]_\mu\right) \frac{i}{(p-k')^2 - m_\phi^2 + i\epsilon} \left(-ig[p + (p-k')]_\nu\right) && (42)\\
&= \frac{ig^2}{u - m_\phi^2} \left(2{p'}_\mu - k_\mu\right)\left(2p_\nu - {k'}_\nu\right) \to \frac{4ig^2}{u-m_\phi^2} {p'}_\mu p_\nu && (43)
\end{aligned}
$$

3. 4-point-vertex (same Feynman diagram as vertex rule in problem!):

$$
i\mathcal{M}^{4p}_{\mu\nu} = 2ig^2 g_{\mu\nu} \qquad (44)
$$

For the Ward identity to be satisfied, we have to show that $\mathcal{M}_{\mu\nu}k^\mu = \mathcal{M}_{\mu\nu}k'^\nu = 0$. So let's check that the first expression indeed vanishes (note that $k^2 = k'^2 = 0$):

$$
\begin{aligned}
\mathcal{M}_{\mu\nu}k^\mu &= (\mathcal{M}^t_{\mu\nu} + \mathcal{M}^u_{\mu\nu} + \mathcal{M}^{4p}_{\mu\nu})k^\mu && (45)\\
&= g^2\left[\frac{2p\cdot k}{t-m_\phi^2}\left(2p'_\nu - k'_\nu\right) + \frac{2p'\cdot k}{u - m_\phi^2}\left(2p_\nu - k'_\nu\right) + 2k_\nu\right] && (46)\\
&= g^2\left[-\left(2p'_\nu - k'_\nu\right) - \left(2p_\nu - k'_\nu\right) + 2k_\nu\right] && (47)\\
&= 2g^2\left[k_\nu + k'_\nu - p_\nu - p'_\nu\right] = 0 && (48)
\end{aligned}
$$

because of 4-momentum conservation (in the third step, we have used that $t = (p-k)^2 = m_\phi^2 - 2p\cdot k$ and $u = (p-k')^2 = m_\phi^2 - 2p\cdot k'$). As it should be the case, we would have arrived at the same conclusion when using the simplified expressions for the amplitude (after replacing $k_\mu \to 0$ and $k'_\nu \to 0$).
The second case, where we contract instead with k'^ν, follows in full analogy because the amplitude is symmetric under $(k \leftrightarrow k', \mu \leftrightarrow \nu)$. This implies that we can replace

$$
\sum_{\text{pol.}} \epsilon^{*\mu}(k)\epsilon^{\tilde{\mu}}(k) \to -g^{\mu\tilde{\mu}} \qquad (49)
$$

in the expression for the *full* amplitude squared, when summing over all photon polarizations (and correspondingly for k').

b) First, we need to square the amplitude, using the simplification allowed by the Ward identity (note that there are no spins associated to the initial state!):

$$
\begin{aligned}
\overline{|\mathcal{M}|}^2 &= \frac{1}{1\cdot 1}\sum_{\text{pol.}} \epsilon^{*\mu}(k)\epsilon^{\tilde{\mu}}(k)\epsilon^{*\nu}(k')\epsilon^{\tilde{\nu}}(k')\mathcal{M}_{\mu\nu}\mathcal{M}^*_{\tilde{\mu}\tilde{\nu}} && (50)\\
&= \mathcal{M}_{\mu\nu}\mathcal{M}^{*\mu\nu} && (51)\\
&= \mathcal{M}^t_{\mu\nu}\mathcal{M}^{t\mu\nu} + \mathcal{M}^u_{\mu\nu}\mathcal{M}^{u\mu\nu} + \mathcal{M}^{4p}_{\mu\nu}\mathcal{M}^{4p\mu\nu} + 2\mathcal{M}^t_{\mu\nu}\mathcal{M}^{u\mu\nu}\\
&\quad +2\mathcal{M}^t_{\mu\nu}\mathcal{M}^{4p\mu\nu} + 2\mathcal{M}^u_{\mu\nu}\mathcal{M}^{4p\mu\nu}\,. && (52)
\end{aligned}
$$

Making use of the symmetries between the u- and t-channel, we find

$$
\begin{aligned}
\mathcal{M}^t_{\mu\nu}\mathcal{M}^{t\mu\nu} &= \frac{16g^4}{(t-m_\phi^2)^2} p^\mu p_\mu\, {p'}^\nu {p'}_\nu = \frac{16 g^4 m_\phi^4}{(t-m_\phi^2)^2} && (53)\\
\rightsquigarrow \mathcal{M}^u_{\mu\nu}\mathcal{M}^{u\mu\nu} &= \mathcal{M}^t_{\mu\nu}\mathcal{M}^{t\mu\nu}\,(p\leftrightarrow p') = \frac{16g^4 m_\phi^4}{(u-m_\phi^2)^2} && (54)\\
2\mathcal{M}^t_{\mu\nu}\mathcal{M}^{4p\mu\nu} &= \frac{16g^4}{t-m_\phi^2}p^\mu p'_\mu = \frac{8g^4}{t-m_\phi^2}(s-2m_\phi^2) && (55)\\
\rightsquigarrow 2\mathcal{M}^u_{\mu\nu}\mathcal{M}^{4p\mu\nu} &= \mathcal{M}^t_{\mu\nu}\mathcal{M}^{4p\mu\nu}\,(p\leftrightarrow p') = \frac{8g^4}{u-m_\phi^2}(s-2m_\phi^2)\,. && (56)
\end{aligned}
$$

With

$$\frac{1}{t-m_\phi^2}+\frac{1}{u-m_\phi^2}=\frac{t+u-2m_\phi^2}{(t-m_\phi^2)(u-m_\phi^2)}=\frac{-s}{(t-m_\phi^2)(u-m_\phi^2)}\,, \tag{57}$$

this can be combined to

$$\mathcal{M}^t_{\mu\nu}\mathcal{M}^{t\mu\nu}+\mathcal{M}^u_{\mu\nu}\mathcal{M}^{u\mu\nu} = \frac{16g^4m_\phi^4s^2}{(t-m_\phi^2)^2(u-m_\phi^2)^2}-\frac{32g^4m_\phi^4}{(t-m_\phi^2)(u-m_\phi^2)}\,, \tag{58}$$

$$2\mathcal{M}^t_{\mu\nu}\mathcal{M}^{4p\mu\nu}+2\mathcal{M}^u_{\mu\nu}\mathcal{M}^{4p\mu\nu} = -\frac{8g^4s(s-2m_\phi^2)}{(t-m_\phi^2)(u-m_\phi^2)}\,. \tag{59}$$

For the remaining two terms, we find:

$$\mathcal{M}^{4p}_{\mu\nu}\mathcal{M}^{4p\mu\nu} = 4g^4g_{\mu\nu}g^{\mu\nu}=16g^4 \tag{60}$$

$$2\mathcal{M}^t_{\mu\nu}\mathcal{M}^{u\mu\nu} = \frac{32g^4}{(t-m_\phi^2)(u-m_\phi^2)}p^\mu p'_\mu p'^\nu p_\nu = 8g^4\frac{\left(2m_\phi^2-s\right)^2}{(t-m_\phi^2)(u-m_\phi^2)}. \tag{61}$$

Using the expressions derived above, we finally find for the polarization-summed amplitude

$$\overline{|\mathcal{M}|}^2 = 16g^4\left[1-\frac{m_\phi^2s}{(t-m_\phi^2)(u-m_\phi^2)}+\frac{m_\phi^4s^2}{(t-m_\phi^2)^2(u-m_\phi^2)^2}\right]. \tag{62}$$

As shown in problem 17 of the weekly exercises, the differential cross section in the CM frame can then be written as

$$\frac{d\sigma}{dt}=\frac{1}{64\pi s}\frac{\overline{|\mathcal{M}|}^2}{\mathbf{p}_{1\,\mathrm{cms}}^2}=\frac{\overline{|\mathcal{M}|}^2}{16\pi s(s-4m_\phi^2)}\,, \tag{63}$$

where it is understood that $u=u(t,s)=2m_\phi^2-t-s$ in the expression for $\overline{|\mathcal{M}|}^2$.

Problem 6

a) In the lab frame, the initial electron is at rest and we can choose a coordinate system such that the momentum of the initial photon is aligned with the z-axis. Hence,

$$p^\mu=(m_e,0,0,0) \qquad k^\mu=\omega(1,0,0,1)\,, \tag{64}$$

where $k^0=|\mathbf{k}|=k^3$ because the photon is massless. We still have the freedom to rotate our coordinate system around the z-axis, such that we can always choose the scattering to take place in the (x,z)-plane. With this choice of coordinates, the final state momenta take the form

$$p'^\mu=(E',p_x,0,p_z) \qquad k'^\mu=\omega'(1,\sin\theta,0,\cos\theta)\,. \tag{65}$$

For the contractions that we encounter in the expression for $|\mathcal{M}|^2$, we thus simply find

$$p\cdot k = m_e\,\omega\,, \tag{66}$$

$$p\cdot k' = m_e\,\omega'\,. \tag{67}$$

b) The simplest way to find this relation is to use 4-momentum conservation:

$$m_e^2=p'^2 = (p+k-k')^2 \tag{68}$$

$$= \underbrace{p^2}_{m_e^2}+2\underbrace{p\cdot(k-k')}_{m_e(\omega-\omega')}+\underbrace{(k-k')^2}_{-2k\cdot k'=-2\omega\omega'(1-\cos\theta)}\,, \tag{69}$$

where for the last term we use that photons are massless, as well as the explicit representations of k and k' from problem **a)**. Solving for ω', this gives

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m_e}(1 - \cos\theta)} \,. \tag{70}$$

c)

$$d\Pi_2 \equiv \frac{d^3k'}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2\omega' 2E'}(2\pi)^4\,\delta^{(4)}(k + p - k' - p') \tag{71}$$

$$= \frac{\omega'^2 d\Omega d\omega'}{(2\pi)^3}\frac{1}{4\omega' E'}(2\pi)\,\delta(\omega + E - \omega' - E')\Big|_{E'=\sqrt{m_e^2+(\mathbf{k}-\mathbf{k}'+\mathbf{p})^2}} \,. \tag{72}$$

In the lab frame, we have $\mathbf{p} = \mathbf{0}$; inserting the explicit form of $\mathbf{k}$ and $\mathbf{k}'$, we find $E'(\omega') = \sqrt{m_e^2 + \omega^2 + \omega'^2 - 2\omega\omega'(1 - \cos\theta)}$. For a real function $g(x)$ with zero x_0, we furthermore have $\delta\left(g(x)\right) = |g'(x_0)|^{-1}\,\delta(x - x_0)$. Applied to the last expression for Π_2, this implies

$$d\Pi_2 = \frac{\omega'^2 d\Omega d\omega'}{(2\pi)^2}\frac{1}{4\omega' E'}\left|-1 - \frac{\omega' - \omega\cos\theta}{E'}\right|^{-1} \tag{73}$$

$$= \frac{d\cos\theta}{8\pi}\frac{\omega'}{E' + \omega' - \omega\cos\theta} \qquad \Big| E' + \omega' = E + \omega \tag{74}$$

$$= \frac{d\cos\theta}{8\pi}\frac{\omega'}{m_e + \omega(1 - \cos\theta)} \tag{75}$$

$$= \frac{d\cos\theta}{8\pi}\frac{\omega'^2}{m_e\omega}\,, \tag{76}$$

where in the last step we have used the result from b).

d) We just have to put together the results from **a)** - **c)** in the expression given in problem 2:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{4\,c\,m_e\,\omega}\overline{|\mathcal{M}|}^2\frac{1}{8\pi}\frac{\omega'^2}{m_e\omega} \tag{77}$$

$$= \underbrace{\frac{e^4}{16\pi m_e^2}}_{\pi\alpha^2/m_e^2}\left(\frac{\omega'}{\omega}\right)^2\left[\underbrace{\frac{p\cdot k'}{p\cdot k}}_{\omega'/\omega} + \underbrace{\frac{p\cdot k}{p\cdot k'}}_{\omega'/\omega} + \Big(1 + \underbrace{\frac{m_e^2}{p\cdot k}}_{m_e/\omega} - \underbrace{\frac{m_e^2}{p\cdot k'}}_{m_e/\omega'=m_e/\omega+1-\cos\theta}\Big)^2 - 1\right] \tag{78}$$

$$= \frac{\pi\alpha^2}{m_e^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right) \tag{79}$$

This is the famous *Klein-Nishina* formula. Taking the non-relativistic limit, $E' - m_e \ll m_e \rightsquigarrow \omega \ll m_e \rightsquigarrow \omega' \approx \omega$, we find the *Thompson* cross section:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m_e^2}\left(1 + \cos^2\theta\right) \quad\rightsquigarrow\quad \sigma = \frac{8\pi\alpha^2}{3m_e^2}\,. \tag{80}$$