PROBLEM SET 1
FYS4170

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1. Tensor notation

1.1. Index form. Some regular operations on index form.

$$
\begin{aligned}
(\nabla S)_i &= \partial_i S \\
(\nabla \cdot \mathbf{A})_i &= \partial_i \mathbf{A} \\
(\nabla \times \mathbf{A})_i &= \epsilon_{ijk} \partial_j \mathbf{A} \\
\operatorname{tr} M &= M^i{}_j \\
M^T = (M_{ij})^T &= M_{ji}
\end{aligned}
$$

1.2. Proof of 3D identities.

$$\nabla \cdot (\nabla \times \mathbf{A}) = \partial^i \epsilon_{ijk} \partial^j A^k = \epsilon_{ijk} \partial^i \partial^j A^k = 0$$

The reasoning behind why this is zero is that when you change two indices in the Levi-Civita symbol, the sign changes. If any of the indices in the other symbols (in accordance with the Levi-Civita change), the sign does not change. One will end up with terms that cancel in pairs.

$$(\nabla \times (\nabla S))_i = \epsilon_{ijk} \partial^j \partial^k S = 0$$

The same reasoning works here.

1.3. Are the equalities valid? The following is true,

$$\partial_\mu x^\nu = \delta_\mu{}^\nu.$$

Where the delta is a rank two tensor ("matrix"), with ones along the diagonal.

The following is not true

$$\partial_\mu x^\mu = 1. \tag{1}$$

It is quite easy to see why

$$\partial_\mu x^\mu = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}) \cdot (x^0, x^1, x^2, x^3) = 1 + 1 + 1 + 1 = 4$$

The following is true

$$\partial^\mu x^\nu = g^{\mu\nu}.$$

Here is why

$$\partial^\mu x^\nu = g^{\mu\rho} \partial_\rho x^\nu = g^{\mu\rho} \delta_\rho{}^\nu = g^{\mu\nu}.$$

This is OK

$$T_\alpha{}^\beta{}_\gamma = g^{\beta\gamma} T_{\alpha\gamma\mu} = g^{\gamma\beta} T_{\alpha\gamma\mu}.$$

The metric raises γ and the metric is its own transpose (and inverse). No worries.

This is not OK.

$$T_{\alpha\ \beta}^{\ \beta} = g_{\alpha\mu} g^{\beta\alpha} T^{\mu}_{\ \alpha\beta}.$$

Some indices are reused and rules are violated. We need to change some letters

$$T_{\alpha\ \beta}^{\ \beta} = g_{\alpha\mu} g^{\beta\rho} T^{\mu}_{\ \rho\beta}.$$

1.4. **Constructing stuff.** All independent Lorentz scalar from two four-vectors A and B:

$$A^{\mu} B_{\mu} \quad A^{\mu} B^{\mu} \quad B^{\mu} B^{\mu}$$