TAKE-HOME EXAM

FYS4170: QUANTUM FIELD THEORY

CANDIDATE NUMBER: [?????]

1. Trace of Dirac Matrix Products

The Dirac gamma matrices are in the chiral representation given in 2×2 block form as,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma & 0 \end{pmatrix}, \tag{1}$$

where σ^i are the Pauli sigma matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The gamma matrices satisfy the anti-commutation relations

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu = 2g^{\mu\nu} \times \mathbb{1}, \tag{3}$$

The fifth gamma matrix is defined by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \tag{4}$$

and has the following properties

$$(\gamma^5)^\dagger = \gamma^5, \tag{5}$$

$$(\gamma^5)^2 = 1, \tag{6}$$

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{7}$$

a. The trace of the product of two gamma matrices can be evaluated using the anti-commutator relation and the cyclic property of the trace of a matrix product,

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = \mathrm{Tr}[2g^{\mu\nu} \times \mathbb{1} - \gamma^\nu\gamma^\mu] \tag{8}$$

$$= 2g^{\mu\nu}\,\mathrm{Tr}\,\mathbb{1} - \mathrm{Tr}[\gamma^\nu\gamma^\mu] \tag{9}$$

$$= 8g^{\mu\nu} - \mathrm{Tr}[\gamma^\mu\gamma^\nu]. \tag{10}$$

In this particular case $g^{\mu\nu}$ is a matrix element, not the metric, and can therefore be moved outside the trace operator as in 9. In 10 the cyclic property of the trace is employed. This yields

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$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}. \tag{11}$$

b. The trace of the product of two gamma matrices with the fifth gamma matrix is,

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = \mathrm{Tr}\left[\gamma^0\gamma^0\gamma^\mu\gamma^\nu\gamma^5\right] \tag{12}$$
$$= -\,\mathrm{Tr}\left[\gamma^0\gamma^5\gamma^0\gamma^\mu\gamma^\nu\right] \tag{13}$$
$$= -\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^0\gamma^5\gamma^0\right] \tag{14}$$
$$= -\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu(\gamma^5)^\dagger\right] \tag{15}$$
$$= -\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right]. \tag{16}$$

Here we have first inserted $\mathbb{1} = \gamma^0\gamma^0$, then the anticommutator relation from Equation 7 is employed three times. The cyclic property of the trace is employed and finally $\gamma^0\gamma^\mu\gamma^0 = \gamma^\dagger$ and the fact that γ^5 is Hermitian (Equation 5). Adding $\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right]$ to both sides gives $2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = 0$ and we have that

$$\mathrm{Tr}\left|\gamma^\mu\gamma^\nu\gamma^5\right] = 0. \tag{17}$$

c. The trace of an odd number of gamma matrices is always zeros. Here is a proof for three gamma matrices

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = \mathrm{Tr}\left[\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\right] \tag{18}$$
$$= -\,\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5\right] \tag{19}$$
$$= -\,\mathrm{Tr}\left[\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\right] \tag{20}$$
$$= -\,\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho], \tag{21}$$

where first the property in Equation 6 is employed, then three anticommutations from Equation 7 and the cyclic property of the trace. Adding $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho]$ similar to before gives $2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = 0$ and we end up with,

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho] = 0. \tag{22}$$

This proof holds for any odd number of gamma matrices because the number of commutation relation “switches” needed will den be odd, yielding the minus sign.