

TEK9010 - Evolutionary Dynamics

Sebastian G. Winther-Larsen

November 10, 2020

Quasispecies Theory

You need to incorporate three basic principles in order to model evolution; *reproduction*, *selection* and *mutation*. In the world of mathematical biology, we normally assume some sort of asexual reproduction.

$$\dot{x} = \frac{\partial x}{\partial t} = rx$$

The solution of this equation is exponential growth,

$$x(t) = x_0 = e^{rt},$$

but may be checked by resource limitation. Selection arises when different types of individuals reproduce at different rates.

$$\dot{x} = x(a - \phi),$$

$$\dot{y} = y(b - \phi),$$

such that $a \neq b$. The term ϕ ensures that $x + y = 1$, which is only possible if $\phi = ax + by$. We may exchange the values a and b with fitness values f_i , in the general case of n types of individuals,

$$\dot{x}_i = x_i(f_i - \phi).$$

Mutation arises when reproduction is not perfectly accurate, such that an individual of type i can transition to an individual of type j in reproduction. This is modelled by the mutation matrix $Q = [q_{ij}]$, which is a stochastic $n \times n$ matrix.

The quasispecies equation incorporates all of these concepts of reproduction, selection and mutation,

$$\dot{x}_i = \sum_{j=1}^n x_j f_j Q_{ji} - \phi(\mathbf{x}) x_i.$$

This equation gives the rate of change over time \dot{x}_i , i.e. the time derivative of the frequency of an individual of species i . The first term on the right-hand side contains the sum of product of all other individual frequencies x_j , the fitness f_j and the mutation rate Q_{ji} from species j to i . The fitness is also called the reproductive rate of the organism, which is determined by phenotype of the organism. The average fitness is given by $\phi(\mathbf{x}) = \sum_i f_i x_i$.

The quasispecies equation describes deterministic evolutionary dynamics in terms of mutation and constant selection acting on an infinitely large population. Generally, the quasispecies equation has one global equilibrium, consisting of a distribution of genomes in a mutation-selection balance.

Evolutionary Game Dynamics

In game theory, games can be formulated in terms of a payoff matrix, which specifies the payoff for one strategy when interacting with another.

Table 1: Pay-off matrix with interacting strategies A and B , with different payoffs a , b , c and d , dependent on choice.

	A	B
A	a	b
B	c	d

In evolutionary games we interpret the payoffs as fitness - a better strategy would lead to faster reproduction.

In game theory it is absolutely necessary to define the Nash equilibrium. If each player has a chosen strategy, and no player can increase its own expected payoff by changing its strategy while other players keep their unchanged, then the current set of strategy choices constitutes a Nash equilibrium. The Nash equilibrium is related to the evolutionary stable strategy (ESS). In general, for games with more than two strategies, we can define the two concepts in the following way. If $E(S_i, S_j)$ is the expected payoff for strategy S_i versus S_j , then;

- Strategy S_k is a strict Nash equilibrium if $E(S_k, S_k) > E(S_i, S_k)$ for all $i \neq k$,
- Strategy S_k is a (non-strict) Nash equilibrium if $E(S_k, S_k) \geq E(S_i, S_k)$ for all i ,
- Strategy S_k is ESS, if for all $i \neq k$ we have either $E(S_k, S_k) > E(S_i, S_k)$ or $E(S_k, S_k) = E(S_i, S_k)$ and $E(S_k, S_i) > E(S_i, S_i)$.
- Strategy S_k is stable against invasion by selection ("weak ESS") if for all $i \neq k$ we have either $E(S_k, S_k) > E(S_i, S_k)$ or $E(S_k, S_k) = E(S_i, S_k)$ and $E(S_k, S_i) \geq E(S_i, S_i)$.

Note; strict Nash implies ESS implies weak ESS implies Nash.

The replicator equation is the cornerstone of evolutionary game dynamics,

$$\dot{x}_i = x_i[f_i(\mathbf{x}) - \phi(\mathbf{x})].$$

It describes deterministic evolutionary game dynamics. For $n = 2$ strategies, there can be dominance, coexistence, bistability or neutrality. For $n \geq 3$ strategies, there can be heteroclinic cycles. For $n \geq 4$, there can be limit cycles and chaos. The replicator equation with n strategies

$$\dot{x}_i = x_i \left[\sum_{j=1}^n a_{ij} x_j - \phi(\mathbf{x}) \right].$$

is equivalent to the Lotka-Volterra equation from ecology,

$$\dot{y}_i = y_i \left(r_i + \sum_{j=1}^{n-1} b_{ij} y_j \right),$$

with the parameters $r_i = a_{in} - a_{nn}$ and $b_{ij} = a_{ij} - a_{nj}$.

Prisoner's Dilemma and Cooperation

Table 2: A payoff matrix depicts a prisoner's dilemma game if $T > R > P > S$.

	C	D
C	R	S
D	T	P

In a prisoner's dilemma game you can either cooperate (C) or defect (D). Defection is "rational," because it maximises the payoff. But, if my opponent analyses the game the same way that I do, then we both choose defection which leads to a suboptimal payoff. The social optimum does not provide the highest payoff for the individual. Thus, the prisoner's dilemma captures the essence of cooperation and how defection can dominate.

The repeated Prisoner’s dilemma is a tool for studying direct reciprocity, which represents a mechanics for the evolution of cooperation. In a series of “tournaments,” Robert Axelrod invited participants to submit strategies for a repeated game. The clear winner in these tournaments was the simple Tit-for-Tat (TFT) statetegy. TFT starts with a cooperation, then does whatever the oponent did the previous round. TFT has a couple of weaknesses.

Table 3: Tit-for-Tat (TFT) payoff against Always Defect (ALLD).
The expected number of rounds played is given by \bar{m} .

	TFT	ALLD
TFT	$\bar{m}R$	$S + (\bar{m} - 1)P$
ALLD	$T + (\bar{m} - 1)P$	$\bar{m}P$

TFT cannot prevent netutral drift leading to Always Cooperate (ALLC) and it cannot correct mistakes.

An improvement on TFT is the Generous TFT (GTFT). This strategy cooperate whenever the oponent has cooperated and sometimes even cooperates when the oponent has defected. It is therefore able to correct mistakes.

In a repeadtet PD game where one allow for evolution of reactive strategies, it is revealed that TFT is a catalyst for cooperation, but will be replaced by GTFT. Both of these strategies are outcompeted by Win-stay, lose-shift (WSLS), which can correct mistakes and is stable against neutral drift to ALLC.

Stochastic Description of Finite Populations

An imporant concept in stochastic theory describing populations of finite size, is the Moran process. The Moran process is a birth-death process: Pick one individual for reproduction and one for death. The offspring of the first individual replaces the second. The individual can be the same. We typically have two types of individuals, A and B.

An interesting process to model is the one where the initial state is one A individual and $N - 1$ B individuals. The probability that A takes over the whole population is called the fixation probability. This is interesting because it says something about the likelihood that a mutation can take over the whole population.

In a population of size N , a neutral mutant will reach fixation with probability $1/N$. A mutant with realtive fitness r , will reach fixation with probability $\rho = (1 - 1/N)/(1 - 1/r^N)$. Introducing fitnes to this kind of model makes things more interesting, as we could model a situation where the mutation is favoured. If the mutation rate is constant, then neutral mutations accumulate at a constant rate, in effect introducing a “molecular clock.”

Games in Finite Populations

Extension of the Moran process provides a framework for studying evolutionary game dynamics for populations of a finite size. Computing fixation probabilities can determine if a selection one strategy over another. The payoff’s of the game makes a contribution to the fitness and therefore the selection. We differentiate between strong- and weak selection regimes.

Consider a standard payoff matrix in table 1. Natural selection would favour A replacing B if $b > c$ in a sufficiently small population. Moreover, natural selection favours A replacing B in a sufficiently large population, and *weak* selection, provided $a + 2b > c + 2d$.

Intensity of selection is implemented by introducing a parameter $w \in [0, 1]$. The fitness of A and B is then given by

$$f_i = 1 - w + wF_i,$$

and

$$g_i = 1 - w + wG_i,$$

with F_i and G_i representing functions for the expected payoffs for A and B , respectively.

There is a surprising $1/3$ law for evolutionary games in finite populations. If A and B are best counters to themselves, i.e. $a > c$ and $b < d$, The unstable equilibrium of the replicator equation is

$$x^* = \frac{d - b}{a - b - c + d}.$$

If $x^* < 1/3$ then the fixation probability is $\rho > 1/N$. In other words, a strategy has a fixation probability higher than $1/N$, if it has a higher fitness at frequency $1/3$. This holds only for weak selection and a large population size.

The traditional ESS and Nash equilibria are neither necessary nor sufficient to imply protection by selection in finite populations. There are special ESS that hold for finite population size N . Strategy B is ESS_N if

1. Selection opposes A invading B , $b(N-1) < c + d(N-2)$,
2. Selection opposes A replacing B , $a(N-2) + b(2N-1) < c(N+1) + d(2N-4)$

Game dynamics change for finite populations. Natural selection can favor the replacement of ALLD by a cooperative strategy such as TFT, when starting from a single individual using that strategy.

Evolutionary Graph Theory

A graph can represent the spatial configuration of a population, the differentiation hierarchy of cells in a multicellular organism, or a social network. Individuals are placed on the vertices of the graph and the edges of a graph determine competitive interaction. All individuals of the population are labelled $i \in [0, N]$, at each time step, an individual is chosen for reproduction. The probability that the offspring of i replaces j is w_{ij} , i.e. the process is determined by an $N \times N$ matrix W , where all entries are probabilities. The Moran process is given by the *complete graph* with identical weights.

The temperature of a vertex is given by,

$$T_j = \sum_{i=1}^N w_{ij}.$$

If all vertices have the same temperature, then the fixation probability is equivalent to the Moran process. This is called the isothermal theorem.

The cycle and directed cycles are isothermal. All symmetric graphs $w_{ij} = w_{ji}$ are isothermal.

Special graphs can both amplify and reduce selection. Amplifiers increase the fixation probability of advantageous mutants and reduce the fixation probability of disadvantageous mutants. Suppressor graphs have selection mechanics that function in the opposite manner.

Games on graphs can be studied by assuming that individuals interact with their nearest neighbors and thereby accumulate payoff. Some games are the birth-death, death-birth and imitation process games. A simple rule for evolution of cooperation on graphs is $b/c > k$; selection favours cooperation if the benefit-to-cost ratio exceeds the number of neighbors.

Spatial Games

Evolutionary games can be studied in a spatial setting, where players interacting with their nearest neighbors. It is possible to formulate entirely deterministic spatial game dynamics. In spatial games, the theory of cellular automata meets game theory. Visualisation of spatial games make some pretty pictures. In some parameter regions, we discover spatial chaos, dynamic fractals and evolutionary kaleidoscopes.

An implementation of a spatial game is the spatial prisoner's dilemma, where the payoff matrix is a bit simplified (table 4).

Table 4: Spatial prisoner's dilemma payoff matrix.

	C	D
C	1 0	
D	b 0	

Each individual has eight surrounding neighbors, for which a payoff is computed. A cooperator surrounded by 8 cooperators receives payoff 8. A defector surrounded by 8 cooperators receives payoff $8b$. Martin Nowak has studied such games extensively (Nowak and May 1993).

I have written a class in Python that can simulate such games, included in an abridged form here:

```
class Board:

    def __init__(self, N=2, b=1.5):
        self.N = N
        self.b = b
        self.lattice = np.ones((N, N))
        self.payoff_lattice = np.zeros((N, N))

    def set_up_simple_square(self):
        self.lattice = np.zeros((self.N, self.N))
        self.lattice[self.N//2, self.N//2 - 1] = 1
        self.lattice[self.N//2 - 1, self.N//2] = 1
        self.lattice[self.N//2 - 1, self.N//2 - 1] = 1
        self.lattice[self.N//2, self.N//2] = 1

    def set_up_single_defector(self):
        self.lattice = np.ones((self.N, self.N))
        self.lattice[self.N//2, self.N//2] = 0

    def advance(self):
        old_strategies = self.lattice.copy()

        self.compute_payoffs()
        for i in range(self.N):
            for j in range(self.N):
                dir = self.dir_happiest_neighbor(i, j)
                coords = self.move(dir, i, j)
                # Change strategy if someone did better
                if self.payoff_lattice[coords] > self.payoff_lattice[i, j]:
                    self.lattice[i, j] = old_strategies[coords]

    def set_up_random(self):
        self.lattice = (np.random.rand(self.N, self.N) > 0.5).astype(int)

    def compute_site_payoff(self, i, j):

        # Am I cooperating?
        payoff_multiple = 1 if self.lattice[i, j] else self.b

        payoff = 0

        for dir in range(8):
            payoff += self.lattice[self.move(dir, i, j)]

        return payoff * payoff_multiple

    def compute_payoffs(self):
        for i in range(self.N):
            for j in range(self.N):
                self.payoff_lattice[i, j] = self.compute_site_payoff(i, j)

    def dir_happiest_neighbor(self, i, j):
        values = []

        for dir in range(8):
            values.append(self.payoff_lattice[self.move(dir, i, j)])

        best_dir = np.asarray(values).argmax()

        return best_dir
```

Nowak, M., and R. May. 1993. "The Spatial Dilemmas of Evolution." *World Scientific* 3 (01): 35–78.