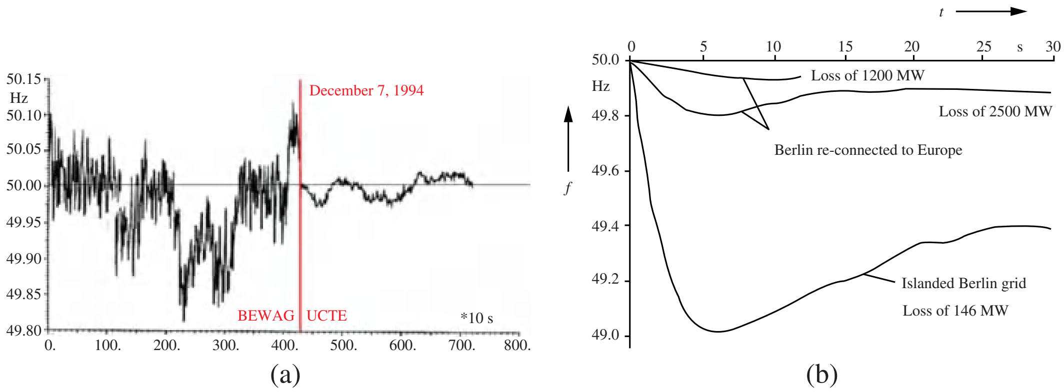


Figure 11.10 Frequency volatility (a) and nadir for loss-of-generation events (b) in West Berlin, before and after interconnection with Western Europe. Source: unknown.

11.7 Consider a system with two generators characterized by the following cost functions, where we will ignore generator limits:

$$C_1(P_1) = 400 + 5P_1 + 0.02P_1^2 \text{ \$/h}$$

$$C_2(P_2) = 800 + 5P_2 + 0.01P_2^2 \text{ \$/h}$$

(a) Find the least-cost dispatch to meet a system demand of $P_D = P_1^* + P_2^* = 800$ MW.
(b) Find the incremental cost λ in \$/MWh for each generator at this operating condition.
(c) Find the average cost in \$/MWh for energy provided by each generator. Do these values have to match? Explain.
(d) Suppose the system load increases to 1000 MW. Can you tell by inspection which generator would be recruited to provide the additional 200 MW?
(e) Suppose that Generator 2 in the previous problem is far from loads and entails a uniform 10% transmission losses, such that $P_L = 0.1P_2$. What is its penalty factor L_2, and what is the least-cost dispatch for a system load of 1000 MW when taking losses into account?

12

Power Flow

12.1 Introduction

Power flow analysis is concerned with describing the operating state of an entire power system, by which we mean a network of generators, transmission lines, and loads that could represent an area as small as a municipality or as large as several states. Given certain known quantities—typically, the amount of power generated and consumed at different locations—power flow analysis allows one to determine other quantities. The most important of these quantities are the voltages at locations throughout the transmission system, which, for alternating current (a.c.), consist of both a magnitude and a time element or phase angle. Once the voltages are known, the currents flowing through every transmission link can be easily calculated. Thus, the name *power flow*, or *load flow*, as it is often called in the industry: given the amount of power delivered and where it comes from, power flow analysis tells us how it flows to its destination.

Owing to the peculiarities of a.c., but also to the sheer size and complexity of a real power system—its elaborate topology with many nodes and links, and the large number of generators and loads—it is no mean feat to deduce what is happening in one part of the system from what is happening elsewhere, despite the fact that these events are intimately related through well-understood, deterministic laws of physics. Although we can readily calculate voltages and currents for direct current (d.c.) circuits in terms of each other (as seen in Chapter 2), even a small network of a handful of a.c. power sources and loads defies our ability to write down easy formulas for the relationships among all the variables.

A key problem is that equations for power are *nonlinear*. Given voltages and currents, it is straightforward to find power, but given power generated and consumed at various locations, it is difficult to determine what the voltages and currents throughout a network must be in order to give rise to those power flows. Mathematicians would say that the system cannot be solved analytically; there is no *closed-form solution*. We can only get a numerical answer through a process of successive approximation or *iteration*. In order to determine its operating state, we must in effect simulate the entire system.

Historically, such simulations were accomplished through an actual miniature d.c. model of the power system in use. Generators were represented by small d.c. power supplies, loads by resistors, and transmission lines by appropriately sized wires. The voltages and currents could be found empirically by direct measurement. To find out how much the current on line A would increase, for example, due to Generator X taking over power production from Generator Y, one would simply adjust the values on X and Y and go read the ammeter on line A. The d.c. model does not exactly

Electric Power Systems: A Conceptual Introduction, Second Edition. Alexandra von Meier.

Companion website: www.wiley.com/go/vonmeier/electricpowersystems2

match the behavior of the a.c. system, but it gives an approximation that is close enough for most practical purposes.

In the age of computers, we no longer need to physically build such models but can create them mathematically. With plenty of computational power, we can not only represent a d.c. system but also the a.c. system itself in a way that accounts for the subtleties of a.c. Such a simulation constitutes *power flow analysis*.

Power flow answers the question: What is the present operating state of the system, given certain known quantities? To do this, it uses a mathematical algorithm of successive approximation by iteration, or the repeated application of calculation steps. These steps represent a process of trial and error that starts with assuming one array of numbers for the entire system, comparing the relationships among the numbers to the laws of physics, and then repeatedly adjusting the numbers until the entire array is consistent with both physical law and the conditions stipulated by the user. In practice, this looks like a computer program to which the operator gives certain input information about the power system, and which then provides output that completes the picture of what is happening in the system—that is, how the power is flowing.

There are variations on what types of information are chosen as input and output, and there are also different computational techniques used by different programs to produce the output. Beyond the straightforward power flow program that simply calculates the variables pertaining to a single, existing system condition, there are more involved programs that analyze a multitude of hypothetical situations or system conditions and rank them according to some desired criteria; such programs are known as *optimal power flow* (OPF), discussed in Section 12.5.1.

This chapter is intended to provide the reader with a general sense of what power flow analysis is, how it is useful, and what it can and cannot do. Section 12.2 introduces the problem of power flow, showing how the power system is abstracted for the purpose of this analysis and how the known and unknown variables are defined. Section 12.3 discusses the interpretation of power flow results based on a sample case and points out some of the general features of power flow in large a.c. systems. Section 12.4 explicitly states the equations used in power flow analysis and outlines a basic mathematical algorithm used to solve the problem, including some simplifications or shortcuts. Section 12.5 addresses key applications of power flow analysis, and Section 12.6 considers the case of radial distribution systems.

12.2 The Power Flow Problem

12.2.1 Network Representation

In order to analyze any circuit, we use as a reference those points that are electrically distinct: that is, there is some impedance between them, which can sustain a potential difference. These reference points are called *nodes*. When representing a power system on a large scale, the nodes are called *buses*, since they represent an actual physical *busbar* where different components of the system are connected. A bus is electrically equivalent to a single point on a circuit, and it marks the location of one of two things: a generator that injects power, or a load that consumes power. At the degree of resolution generally desired on the larger scale of analysis, the load buses represent aggregations of loads (or very large individual industrial loads) at the location where they connect to the high-voltage transmission system. Such an aggregation may in reality be a transformer connection to a subtransmission system, which in turn branches out to a number of distribution substations; or it may be a single distribution substation from which originate a set of distribution feeders (see Figure 7.4).[1] In any case, whatever lies behind the bus is taken as a single load for purposes of the power flow analysis.

The buses in the system are connected by transmission lines. At this scale, one does not generally distinguish among the three phases of an a.c. transmission line (Section 4.1). Based on the assumption that, to a good approximation, the same thing is happening on each phase, the three are condensed by the model into a single line, making a so-called *one-line diagram*. Indeed, a single line between two buses in the model may represent more than one three-phase circuit. Still, for this analysis, all the important characteristics of these conductors can be condensed into a single quantity, the *impedance* of the one line (see Section 3.3). Since the impedance is essentially determined by the physical characteristics of the conductors (such as their material composition, diameter, and length), it is taken to be constant.[2] Note that this obviates the need for geographical accuracy, since the distance between buses is already accounted for within the line impedance, and the lines are drawn in whatever way they best fit on the page or the screen.

Thus, the model so far represents the existing hardware of the power system, drawn as a network of buses connected by single lines. An example of such a one-line diagram is shown in Figure 12.1, to illustrate the level of abstraction. This topology or characteristic connection of the network can be changed by switching operations, whereby, for example, an individual transmission line can be taken out of service. Such changes, of course, must be reflected by redrawing the one-line diagram, where now some lines may be omitted or assigned a new impedance value. For a given analysis run, though, the network topology is taken to be fixed.

The mathematical representation of a transmission network in the *bus admittance matrix* is found in Section 12.4.2.

12.2.2 Choice of Variables

From the analysis of simple d.c. circuits in Chapter 2, we are familiar with the notion of organizing the descriptive variables of the circuit into categories of "knowns" and "unknowns," whose relationships can subsequently be expressed in terms of multiple equations. Given sufficient information, these equations can then be manipulated with various techniques so as to yield numerical results for the hitherto unknowns. As readers may recall from high school algebra, the conditions under which such a system of equations is solvable (meaning that it can yield unambiguous numerical answers) are straightforward: there must be exactly one equation for each unknown quantity. Each equation represents one statement relating one unknown variable to some set of known quantities. This set of equations must not be redundant: if any one equation duplicates information implied by the others, it does not tell us anything new and therefore does not count toward making the whole system solvable. If there are fewer equations than unknowns, we do not have enough information to decide which values the unknowns must take (in other words, the information given does not rule out a multiplicity of possibilities); if there are more equations than unknowns, the system is overspecified, meaning that some equations are either redundant or

1 Of course, it is possible to run power flow analysis at different scales, including a smaller scale that explicitly incorporates more distribution system elements. In the present discussion, we emphasize the largest transmission scale because of its general and economic importance. Also note that distribution systems usually submit to simpler methods of analysis because of their radial structure, implying that power flows in only one direction. Power flow analysis is indispensable, however, for the meshed networks that characterize the transmission system.

2 Ambient conditions such as conductor temperature are hard to know exactly, but have a small enough effect on line impedance that they are usually neglected.

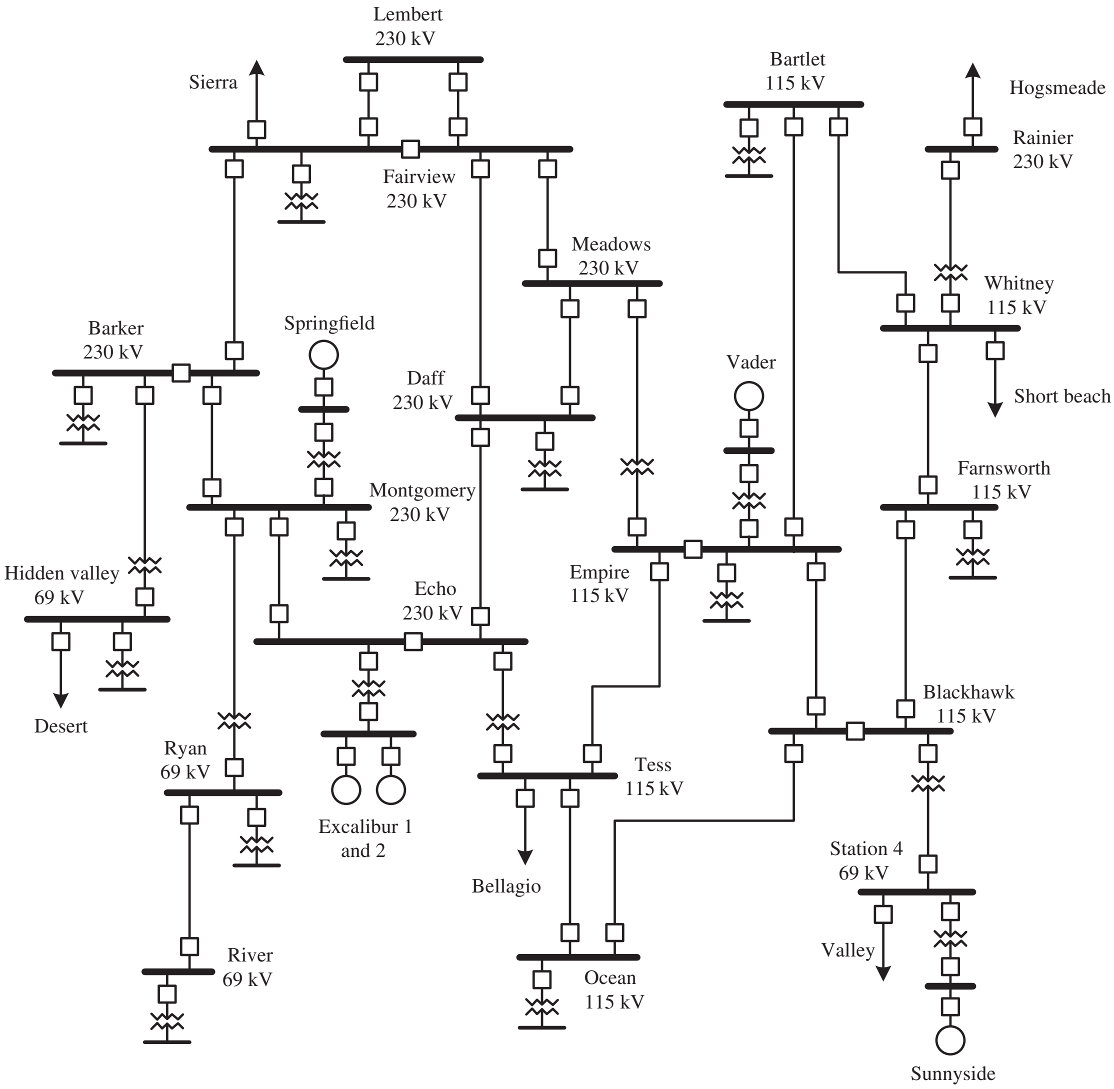


Figure 12.1 One-line diagram for a fictitious power system.

mutually contradictory. In order to determine whether an unambiguous, unique solution to a system of equations such as those describing an electric power system can be found, one must begin by taking an inventory of variables and information that translates into equations for those variables.

As introduced in Chapter 1, there are two basic quantities that describe the flow of electricity: voltage and current. Recognizing these quantities in simple d.c. circuits in Chapter 2, we saw that both voltage and current will vary from one location to another in a circuit, but they are everywhere related: the current through each circuit branch corresponds to the voltage or potential difference between the two nodes at either end, divided by the impedance of this branch. It is generally assumed that the impedances throughout the circuit are known, since these are more or less permanent properties of the hardware. Thus, if we are told the voltages at every node in the circuit, we can deduce from them the currents flowing through all the branches, and everything that is happening in the circuit is completely described. If one or more pieces of voltage information were missing, but we were given appropriate information about the current instead, we could still work backwards and solve the problem. In this sense, the number of variables in a circuit corresponds to the number of electrically distinct points in it: assuming we already know all the properties of the hardware, we need to be told one piece of information per node in order to figure out everything that's going on in a d.c. circuit.

For a.c. circuits, the situation is more complicated because we have introduced the dimension of time to capture an ongoing oscillation or movement. Thus each of the two main variables, voltage and current, has *two* numerical components: a magnitude component and a time component. To fully describe the voltage at any given node in an a.c. circuit, we must therefore specify two numbers: a voltage magnitude and a voltage angle (Section 3.1). Given the complex impedances (Section 3.3) of all the network branches, which are also number pairs, we can solve for the current magnitude and angle in each branch. The product of voltage and current gives the amount of power transferred at any point, which is again a pair of numbers with a real and a reactive component (Section 3.4.2). Thus, an a.c. circuit requires exactly two pieces of information per node for its operating state to be completely determined. More than two, and they are either redundant or contradictory; fewer than two, and possibilities are left open so that the system cannot be solved.

A word of caution is in order: Owing to the *nonlinear* nature of the power flow problem, it may be impossible to find one unique solution even with the proper number of equations, as more than one answer fits the given configuration. Nonlinearity is further discussed in Section 12.2.3. In most situations, it is straightforward to identify the "correct" solution among the mathematical possibilities based on physical plausibility and common sense. Conversely, there may be no solution at all, if the given information does not correspond to an actual physical situation.

Having discussed voltage and current, each with magnitude and angle, as the basic electrical quantities, which are known and which are unknown? In practice, current is not known; the currents through the various circuit branches turn out to be the last thing that we calculate once we have completed the power flow analysis. Voltage, as we will see, is known explicitly for some buses but not for others. More typically, what is known is the amount of *power* going into or out of a bus. Power flow analysis takes all the known real and reactive power flows at each bus, and those voltage magnitudes that are explicitly known, and from this information calculates the remaining voltage magnitudes and all the voltage angles. This is the hard part. The easy part, finally, is to calculate the current magnitudes and angles from the voltages.

From Section 3.4, we know how to calculate real and reactive power from voltage and current: power is basically the product of voltage and current, and the relative phase angle between voltage and current determines the respective contributions of real and reactive power. Conversely, one can deduce voltage or current magnitude and angle if real and reactive power are given, but it is far more difficult to work out mathematically in this direction. This is because each value of real and reactive power would be consistent with many different possible combinations of voltages and currents. In order to choose the correct ones, we have to check each node in relation to its neighboring nodes in the circuit and find a set of voltages and currents that are consistent all the way around the system. This is what power flow analysis does.

12.2.3 Nonlinearity

A linear function is one in which the output is proportional to the inputs. One consequence is that the principle of *superposition* holds: when more input is added, the new output is the sum of the outputs due to the original plus the newly added input. That is, for the linear function $f(x) = c\,x$ it is always true that

$$f(x_1 + x_2) = c(x_1 + x_2) = c\,x_1 + c\,x_2 = f(x_1) + f(x_2)$$

The same is not true for the nonlinear function $g(x) = c\,x^2$ where

$$g(x_1 + x_2) = c\,(x_1 + x_2)^2 \;\neq\; c\,x_1^2 + c\,x_2^2$$

We have already encountered superposition in Section 2.4, where we describe linear aspects of circuit behavior—specifically, Ohm's law. For linear circuit elements (including resistors, inductors, and capacitors), voltage is a linear function of current ($V = IZ$) and *vice versa* ($I = YV$), with the impedance Z or admittance Y playing the role of the proportionality constant. As a result, the voltage resulting from an added current source in a circuit can be added to the original voltage; or the current due to an added voltage source can be added to the original current.

For example, in Figure 12.2 (very similar to Figure 2.6), the current in the center branch of the circuit is the sum of currents due to the two voltage sources on either side. The reason we may simply add currents is because the behavior of the center branch is characterized by a linear equation: $V = IR$, where the resistance is a known constant.

Now suppose the center branch is characterized not by a constant impedance, but by a certain amount of power demand (i.e., it is modeled as a *constant power load*). Ohm's law still applies, but the impedance in it is no longer some known constant value. Instead, the impedance may shift around depending on the voltage conditions, so as to keep the product of voltage and current for that load constant. We can write an equation for power, $S = I^*V$, which at first glance might appear linear.[3] However, this equation is not in fact linear because V itself is a function of I, or I of V. If we substitute Ohm's law $V = IZ$ into the equation for power, we get

$$S = I^*IZ \quad \text{or} \quad S = \frac{VV^*}{Z^*}$$

which is quadratic in either I or V. Consequently, it is not at all obvious how to solve for the voltage across or current through the center circuit branch in Figure 12.3.

Kirchhoff's laws still apply, but don't entirely solve our problem. Although we can still write $I_3 = I_1 + I_2$, the actual values of I_1 and I_2 cannot be determined independently of the voltage V_3 at the central circuit node. Likewise, V_3 can be written as the difference between the voltage source and the voltage drop across the resistance on either side of the circuit, but those voltage drops cannot be determined independently of the current.

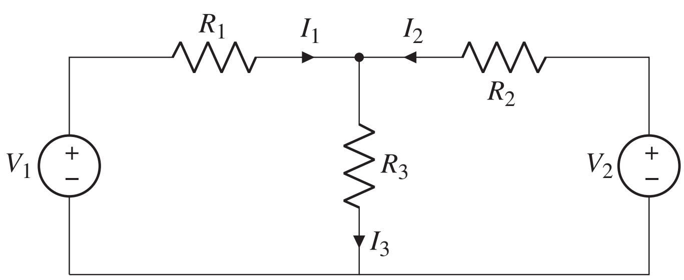


Figure 12.2 Current I_3 is found by adding currents $I_1 + I_2$, superimposing the two partial circuits on the center branch. This works because all circuit elements have linear characterizations.

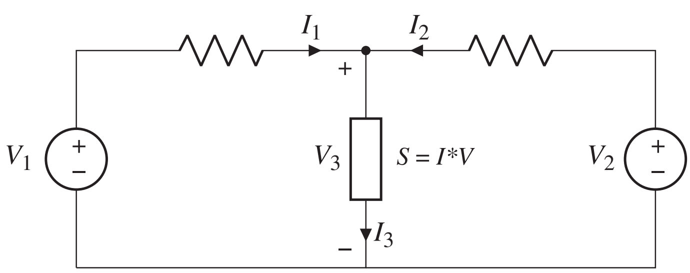


Figure 12.3 In a nonlinear system, we cannot directly solve for V_3 or I_3.

3 The asterisk denotes the complex conjugate; see Section 3.5.7.

Power flow iteration

What to do? We could make a guess at the current I_3, see what voltage V_3 it implies, and then check how consistent this is with V_1 and V_2. Or we could guess at V_3 and check the corresponding currents; if they don't add up, we can adjust our guess for V_3. With repeated guesses, we can try to get the discrepancy to shrink until the solution is both internally consistent and matches all given information. Such a brute-force, trial-and-error approach may seem desperate, but it is in fact how the power flow problem is handled professionally—by the fancier name of *iterative solution methods*.

12.2.4 Types of Buses

Let us now articulate which variables will be given for each bus as inputs to the analysis. Here we must distinguish between different types of buses based on their actual, practical operating constraints. The two main types are *generator buses* and *load buses*. At a load bus, we assume that the power consumption is given—determined by the consumer—and we specify two numbers, real and reactive power, for each load bus.[4] Referring to the symbols P and Q for real and reactive power, load buses are referred to as *P,Q buses* in power flow analysis.

At the generator buses we could in principle also specify P and Q. Here we run into two issues: one has to do with balancing the real power needs of the system, and the other with managing reactive power. To address real power balance, we will specify P for all but one generator, the slack bus. To represent the actual operational control for most generators, we typically use the generator bus voltage V instead of the reactive power Q as the second variable. Most generator buses are therefore called *P,V buses*.

12.2.5 Variables for Balancing Real Power

Balancing the system means that all the generators in the system collectively must supply power in exactly the amount demanded by the load, plus the amount lost on transmission lines. This applies to both real and reactive power, but let us consider only real power first. If we tried to specify a system in which the sum of P generated did not match the P consumed, our analysis would yield no solution, reflecting the fact that in real life the system would lose synchronicity and crash. Therefore, for all situations corresponding to a stable operation of the system, and thus a viable solution of the power flow problem, we must require that real power generated and consumed matches up. Of course, we can vary the contributions from individual generators—that is, we can choose a different *dispatch*—so long as the *sum* of their P's matches the amount demanded by the system. As mentioned earlier, this total P must not only match the load demand, it must actually *exceed* that amount in order to make up for the transmission losses, which are the resistive I^2R energy losses (Section 1.4).

Now we have a problem: How are we supposed to know ahead of time what the transmission losses are going to be? Once we have completed the power flow analysis, we will know what the current flows through all the transmission lines are going to be, and combining this information

4 We assume that the load's power demand is independent of the voltage at that bus. This may seem to fly in the face of everything said in Section 6.3.1. However, we are talking here about the voltage magnitude at a bus as modeled in the transmission system, which would typically represent a substation, which is not the actual service voltage where customer loads are connected. Owing to voltage regulating equipment in the distribution system that aims to keep the service voltage constant (Section 7.4), it is fair to assume that the actual service voltage to customers is independent of the transmission bus voltage, and that the power drawn at the bus should remain independent of bus voltage magnitude.

with the known line impedances will give us the losses. But we cannot tell *a priori* the amount of losses. The exact amount will vary depending on the dispatch, or amount of power coming from each generator, because a different dispatch will result in a different distribution of current over the various transmission paths, and not all transmission lines are the same. Therefore, if we were given a total P demanded at the load buses and attempted now to set the correct sum of P for all the generators, we could not do it.

The way to deal with this situation mathematically resembles the way it would be handled in actual operation. Knowing the total P demanded by the load, we could begin by assuming a typical percentage of losses (say, 5%). We now dispatch all the generators in the system so that the sum of their output approximately matches what we expect the total real power demand (load plus losses) to be (in this case, 105% of load demand). But since we do not yet know the exact value of the line losses for this particular dispatch (seeing that we have barely begun our power flow calculation), we will probably be off by a small amount. A different dispatch might, for example, result in 4.7% or 5.3% instead of 5% losses overall. We now make the plausible assumption that this uncertainty in the losses constitutes a sufficiently small amount of power that a single generator could readily provide it. So we choose one generator whose output we allow to adjust, depending on the system's needs: we allow it to "take up the slack" and generate more power if system losses are greater than expected, or less if they are smaller. In power flow analysis, this one generator bus is appropriately labeled the *slack bus*, or sometimes *swing bus*.

Thus, as the input information to our power flow analysis, we specify P for *one less* than the total number of buses. What takes the place of this piece of information for the last bus is the requirement that the system remain balanced. This requirement will be built into the equations used to solve the power flow and will ultimately determine what the as yet unknown P of the slack bus has got to be. The blank space in the initial specifications for the slack bus, where P is not given, will be filled by the voltage angle, to be discussed in Section 12.2.7.

12.2.6 Variables for Balancing Reactive Power

Analogous to real power, the total amount of reactive power generated throughout the system must match the amount of reactive power consumed by the loads.[5] Whereas in the case of a mismatch of real power, the system loses synchronicity, a mismatch of reactive power would lead to voltage collapse. Also analogous to real power transmission losses, there are *reactive power losses*. Reactive losses are defined simply as the difference between reactive power generated and reactive power consumed by the metered load.

Physically, these losses in Q reflect the fact that transmission lines have some reactance (Section 3.3) and thus tend to "consume" reactive power; in analogy to I^2R, we could call them I^2X losses. The term "consumption," however, like the reactive power "consumption" by a load, does not directly imply an energy consumption in the sense of energy being withdrawn from the system. To be precise, the presence of reactive power does necessitate the shuttling around of additional current, which in turn is associated with some real I^2R losses "in transit" of a much smaller magnitude. But these second-order I^2R losses (the side effect of a side effect) are already

5 Recall that in Section 3.4, we put quotation marks around the terms "supplied" and "consumed" as they apply to reactive power, since this is a somewhat arbitrary nomenclature. No net energy is produced or consumed by either generator or load due to reactive power exchange. However, instantaneous power must be balanced throughout the system at all times during each cycle. This is accomplished by balancing both P and Q. Physically, if instantaneous power is imbalanced, the difference is made up from the potential energy stored in electric and magnetic fields throughout the system. If this finite capacity were exhausted, voltage would collapse.

captured in the analysis of real power for the system. The term "reactive losses" thus does not refer to any physical measure of something lost, but rather should be thought of as an accounting device. While real power losses represent physical heat lost to the environment and therefore always have to be positive,[6] reactive losses on a given transmission link can be positive or negative, depending on whether inductive or capacitive reactance plays a dominant role.

In any case, what matters for both operation and power flow analysis is that Q, just like P, needs to be balanced at all times. Thus, just as for real power, all the generators in the system must generate enough reactive power to satisfy the load demand *plus* the amount that vanishes into the transmission lines.

This leaves us with the analogous problem of figuring out how much total Q our generators should produce, not knowing ahead of time what the total reactive losses for the system will turn out to be: as with real losses, the exact amount of reactive losses will depend on the dispatch. Operationally, though, the problem of balancing reactive power is considered in very different terms. When an individual generator is instructed to provide its share of reactive power, the control objective is usually expressed in terms of maintain a certain voltage magnitude at the generator bus, rather than injecting a certain number of MVAR. An automatic voltage regulator (AVR) continually and automatically adjusts through the generator's field current (Section 10.4.2) and thereby alters the reactive power output.

Generator bus voltage magnitude is a relatively straightforward variable to control. It is also an indicator of whether the correct amount of reactive power is being generated throughout the system. When the combined generation of reactive power by all the generators matches the amount consumed, their bus voltages hold steady. Conversely, if there is a need to increase or decrease reactive power generation, adjusting the field current at one or more generators so as to return to the voltage set point will automatically accomplish this objective.[7] The new value of MVAR produced by each generator can then be read off the dial for accounting purposes.

Conveniently for power flow analysis, then, there is no need to know explicitly the total amount of Q required for the system. Specifying the voltage magnitude is essentially equivalent to requiring a balanced Q. In principle, we could specify P and Q for each generator bus, except for one slack bus assigned the voltage regulation (and thus the onus of taking up the slack of reactive power). For this "reactive slack" bus we would need to specify voltage magnitude V instead of Q, with the understanding that this generator would adjust its Q output as necessary to accommodate variations in reactive line losses. In practice, however, since voltage is already the explicit operational control variable, it is customary to specify V instead of Q for all generator buses, which are therefore called *P,V buses*.[8] In a sense, this assignment implies that all generators share the "reactive slack," in contrast to the real slack that is taken up by only a single generator.

12.2.7 The Slack Bus

We have now, for our power flow analysis, three categories of buses: *P,Q buses*, which are generally load buses, but could in principle also be generator buses; *P,V buses*, which are necessarily generator buses (since loads have no means of voltage control); and then there is the slack bus, for which we cannot specify P, only V. What takes the place of P for the slack bus?

---

6 Lest we violate the second law of thermodynamics, which forbids heat from flowing spontaneously from the air into the wire and making electricity.

7 See Section 10.4.4 for more about how generators share MVAR load.

8 It is important to remember that the V of the P, V bus represents only voltage magnitude, not angle. To avoid any confusion, the careful notation P, $|V|$ is sometimes used, where the vertical lines indicate magnitude.

Frequency change due to power flow

As introduced in Section 10.4.1, *real* power balance manifests operationally as a steady frequency such as 60 Hz. A constant frequency is indicated by an unchanging voltage angle, which for this reason is also known as the *power angle*, at each generator. When more power is consumed than generated, the generators' rotation slows down: their electrical frequency drops, and their voltage angles fall farther and farther behind. Conversely, if excess power is generated, frequency increases and the voltage angles move forward. While generators are explicitly dispatched to produce a certain number of megawatts, the necessary small adjustments to balance real power in real time are made (by at least one or more *load-following* generators) through holding the generator frequency steady at a specified value. Not allowing the frequency to depart from this reference value is equivalent to not letting the voltage angle (relative to the rotating reference frame) to increase or decrease over time.

In power flow analysis, the slack bus is the one mathematically assigned to do the load following. Its instructions, as it were, are to do whatever is necessary to maintain real power balance in the system. Physically, this would mean holding the voltage angle constant. The place of P will therefore be taken by the *voltage angle*, which is the variable that in effect represents real power balance. We can think of the voltage angle here as analogous to the voltage magnitude in the context of reactive power. Specifying that bus voltage magnitude should be kept constant effectively amounts to saying that whatever is necessary should be done to keep the system reactive power balanced. Likewise, specifying a constant voltage angle at the generator bus amounts to saying that this generator should do whatever it takes to keep real power balanced.

We thus assign to the slack bus a voltage angle, which, in keeping with the conventional notation for the context of power flow analysis, we will call θ (lowercase Greek theta). This θ can be interpreted as the relative position of the slack bus voltage at time zero. It is the same quantity that is elsewhere called the *power angle* and labeled as δ.

What is important to understand here is that the actual numerical value of this individual voltage phase angle has physical meaning only in relation to a reference. It is the *difference* between the voltage angle at one bus and another, as well as its rate of change, that matters. Physically, the angle difference between voltage curves at two locations corresponds to the difference in the precise timing of the zero crossings (or voltage maxima), which in turn is related to the power transfer between those two locations in the network.[9] Also, a fixed value of θ has physical meaning in that it implies this angle will not change as the system operates. The choice of a particular numerical value for the first θ in a network amounts to a choice of coordinate system (i.e., what do we call Time Zero). Given this reference, the voltage angles for each of the other buses throughout the system will take on a different values depending on their contribution to real power. But as long as the entire system is in a state of equilibrium, where generation equals load, these angles will hold steady.

We now conveniently take advantage of the slack bus to establish a systemwide reference for timing, and we might as well make things simple and call the reference point "zero." This could be interpreted to mean that the alternating voltage at the slack bus has its maximum at the precise instant that we depress the "start" button of an imaginary stopwatch, which starts counting the milliseconds (in units of degrees within a complete cycle of 1/60th second) from time zero. In principle, we could pick any number between 0° and 360° as the voltage angle for the slack bus, but 0° is the simple and conventional choice.

9 Since the voltage continually alternates, it would be of little use to say that the voltage maximum at Bus A occurs at exactly 3:00:00 P.M. Rather, we would want to know that the voltage maximum at Bus A always occurs one-tenth of a cycle later than at Bus B.

Table 12.1 Variables in power flow analysis.

Type of Bus	Variables Given (Knowns)	Variables Found (Unknowns)
Generator	Real power (P)	Voltage angle (θ)
	Voltage magnitude (V)	Reactive power (Q)
Load or generator	Real power (P)	Voltage angle (θ)
	Reactive power (Q)	Voltage magnitude (V)
Slack	Voltage angle (θ)	Real power (P)
	Voltage magnitude (V)	Reactive power (Q)

12.2.8 Summary of Variables

To summarize, our three types of buses in power flow analysis are P, Q (load bus), P, V (generator bus), and θ, V (slack bus). Given these two input variables per bus, and knowing all the fixed properties of the system (i.e., the impedances of all the transmission links, as well as the a.c. frequency), we now have all the information required to completely and unambiguously determine the operating state of the system. This means that we can find values for all the variables that were not originally specified for each bus: θ and V for all the P, Q buses; θ and Q for the P, V buses; and P and Q for the slack bus. The known and unknown variables for each type of bus are listed in Table 12.1 for easy reference.

Once we know θ and V, the voltage angle and magnitude, at every bus, we can very easily find the current through every transmission link; it becomes a simple matter of applying Ohm's law to each individual link. (In fact, these currents have already been determined implicitly, so that by the time the program announces θ's and V's, all the hard work is done.) Depending on how the output of a power flow program is formatted, it may state only the basic output variables, as in Table 12.1, it may explicitly state the currents for all transmission links in amperes; or it may express the flow on each transmission link in terms of an amount of real and reactive power flowing, in megawatts (MW) and MVAR.

12.3 Example with Interpretation of Results

12.3.1 Six-bus Example

Consider the six-bus example illustrated in Figure 12.4.[10]

This example is simple enough for us to observe in detail, yet too complex to predict its behavior without numerical power flow analysis.

Each of the six buses has a load, and four of the buses also have generators. Bus 1, keeping with convention, is the slack bus. Buses 2–4, which have both generation and loads, are modeled as P, V buses; the local load is simply subtracted from the real and reactive generation at each. Buses 5 and 6, which have only loads, are modeled as P, Q buses.

10 This example is taken from PowerWorld™, a power flow software application available from www.powerworld.com for both commercial and educational use; a short version is available as a free download. The case illustrated here is from the menu of the standard demonstration cases in PowerWorld, labeled "Contour 6-Bus" (retrieved October 2004).

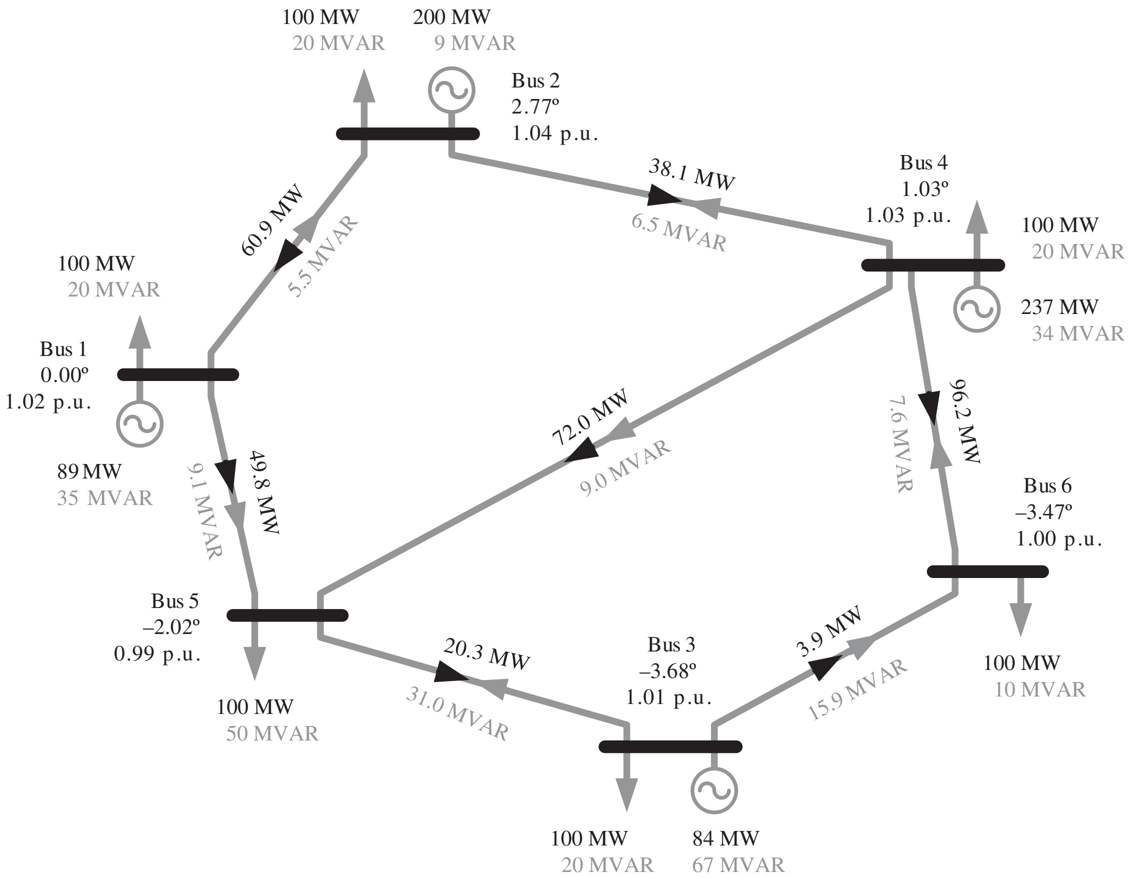


Figure 12.4 Six-bus power flow example.

The distribution of loads and the generation dispatch, for both real and reactive power, are completely determined somewhere outside the power flow program, whether in the real world or the program user's fantasy. The one exception is the generator at the slack bus, whose real power output varies so as to accommodate systemwide losses. In addition to the MW and MVAR loads and the MW generation levels for every generator (except the slack), the user specifies the voltage magnitudes to be maintained at each generator bus; the program then computes the MVAR generation necessary to maintain this voltage at each bus. (It is also possible to specify MVAR generation and allow the program to determine the voltage magnitudes, but, as mentioned earlier, the former method better resembles real-life operations.)

By convention, the voltage angle at the slack bus is set to 0.00°. The power flow program computes the voltage angle at each of the other five buses in relation to the slack bus. We may now begin to observe the relationship between real power and voltage angle: a more positive voltage angle generally corresponds to an injection of power into the system and a more negative voltage angle to a consumption of real power. Buses 2 and 4, which both have generation exceeding local load, have positive voltage angles of 2.77° and 1.03°, respectively. Bus 3, though it has a generator, is still a net consumer of real power, with 100 MW load and only 84 MW generated; its voltage angle is −3.68°. Buses 5 and 6 have loads only and voltage angles of −2.02° and −3.47°, respectively.

Note, however, that the voltage angles are not in hierarchical order depending on the amount of power injected or withdrawn at each individual bus. This is because we also must consider the location of each bus relative to the others in the system and the direction of power flow between them. For example, consider Buses 2 and 4. Net generation at Bus 4 is greater than at Bus 2 (137 MW compared to 100 MW), yet the voltage angle at Bus 2 is more positive. We can see that this is due to the location of these buses in the system: real power is generally flowing from north to south, that is, from Bus 2 to the neighborhood of Buses 5, 3, and 6 where there is more load and less generation. As indicated by the black arrow on the transmission link, real power is flowing from Bus 2 to Bus 4. As a rule, real power flows from a greater to a smaller voltage angle. This rule holds true for six of the seven links in this sample case; the exception is Link 3–6, where both the power flow and the difference in voltage angle are very small. The reader can verify that throughout this case, while power flow and voltage angle are not exactly proportional, a greater flow along a transmission link is associated with a greater angle difference.

~~We~~ now turn to the relationship between reactive power and voltage magnitude, which is similar to that between real power and voltage angle. The nominal voltage of this hypothetical transmission system is 138 kV. However, just as the timing or angle of the voltage differs by a small fraction of a cycle at different locations in the grid, the magnitude, too, has a profile across the system with different areas a few percent higher or lower than the nominal value. Because it is this percentage, not the absolute value in volts, that is most telling about the relationship among different places in the grid, it is conventional to express voltage magnitude in *per-unit* terms (see Section 8.7). Per-unit (p.u.) notation simply indicates the local value as a multiple of the nominal value; in this case, 138 kV equals 1.00 p.u. The voltage magnitude at Bus 1 is given as 1.02 p.u., which translates into 141 kV; at Bus 5, the voltage magnitude of 0.99 p.u. means 137 kV.

As a rule, reactive power tends to flow in the direction from greater to smaller voltage magnitude. In our example, this rule holds true only for the larger flows of MVAR, along Links 1–5, 3–5, 4–5, and 3–6. The reactive power flows along Links 1–2, 4–2, and 6–4 do not follow the rule, but they are comparatively small.

Note that real and reactive power do not necessarily flow in the same direction on a given link. This should not be surprising, because the "direction" of reactive power flow is based on an arbitrary definition of the generation or consumption of VARs; there is in fact no net transfer of energy in the direction of the gray arrow for Q. Also, note that having Q flow opposite P does not imply any "relief" or reduction in current. For example, on Link 3–5, the real power flow P is 20.3 MW and reactive flow Q is 31.0 MVAR. In combination, this gives apparent power S of 37.1 MVA (using $S^2 = P^2 + Q^2$), regardless of the direction of Q. (Recall that MVA are the relevant units for thermal line loading limits, since total current depends on apparent power.)

From Figure 12.4, it is possible to evaluate the total real and reactive system losses, simply by observing the difference between total generation and total load. The four generators are supplying 89, 200, 84, and 237 MW, respectively, for a total of 610 MW of real power generated. Subtracting the six loads of 100 MW each, the total real power losses throughout the transmission system for this particular scenario are therefore 10 MW. On the reactive side, total generation is 145 MVAR, while total reactive load is 140 MVAR, and system reactive losses amount to roughly 5 MVAR.

The discerning reader may have noticed, however, that the stated line flows in Figure 12.4, which are average values for each link, cannot all be reconciled with the power balance at each bus. To account for losses in a consistent fashion, we must record both the power (real or reactive) entering and exiting each link. In Figure 12.5, these data are given for real power (MW) in black and reactive power (MVAR) in gray. The numbers in parentheses represent the losses, which are the difference between power flows at either end. Bus power, line flows, and losses are rounded to different decimal places, but the numbers do add up correctly for each bus and each link.

The most significant losses tend to occur on links with the greatest power flow. In this case, Link 4–6 has the greatest power flow with 96.2 MW real and 7.6 MVAR reactive, yielding 96.5 MVA apparent, and the greatest losses. While the real line losses are all positive, as they should be, the negative signs on some of the reactive losses indicate negative losses; we might consider them

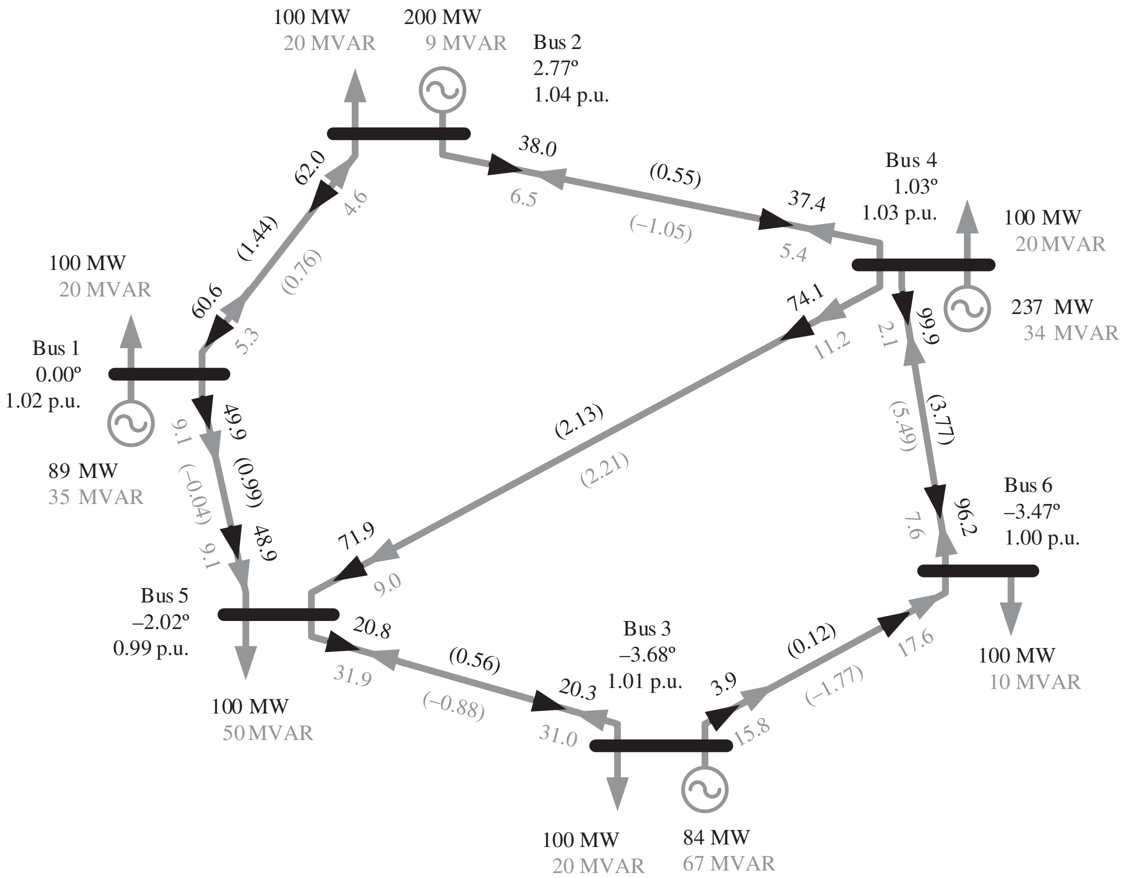


Figure 12.5 Six-bus power flow example with losses.

"gains," although nothing is actually gained. Reactive losses depend on operating conditions and impedance, where the model of a transmission link may incorporate reactive compensation such as capacitors. It is typical for system reactive losses to be positive overall, as they are in this example. Like real losses, reactive losses are related to the current and therefore apparent power flow. Thus, we also observe the greatest reactive losses in our example on Link 4–6. The real and reactive losses for every link can be totaled to confirm the estimated system losses given earlier.

12.3.2 Tweaking the Case

To gain a better sense of a power system's behavior and the information provided by power flow modeling, let us now make a small change to the operating state in the six-bus example and observe how the model responds. We simply increase the load at Bus 5 by 20% while maintaining the same power factor, thus changing it from 100 MW real and 50 MVAR reactive to 120 MW real and 60 MVAR reactive. This change is small enough for the generator at the slack bus to absorb, so we need not specify increased generation elsewhere. Indeed, generation at Bus 1 increases from 89 to 110 MW. Note that the difference amounts to 21, not 20 MW, as the increased load also entails some additional losses in the system. The new scenario is illustrated in Figure 12.6.

As we would expect, the line flows to Bus 5 increase by a total of 20 MW. The bulk (about 14.5 MW) of this additional power comes from Bus 1, about 4 MW from Bus 4, and the balance appears as a reduction of about 1.5 MW in the flow to Bus 3.

The changes do not stop here, however; they have repercussions for the remainder of the system. Three of the other buses are defined as P, V buses, and therefore have fixed voltage magnitudes.

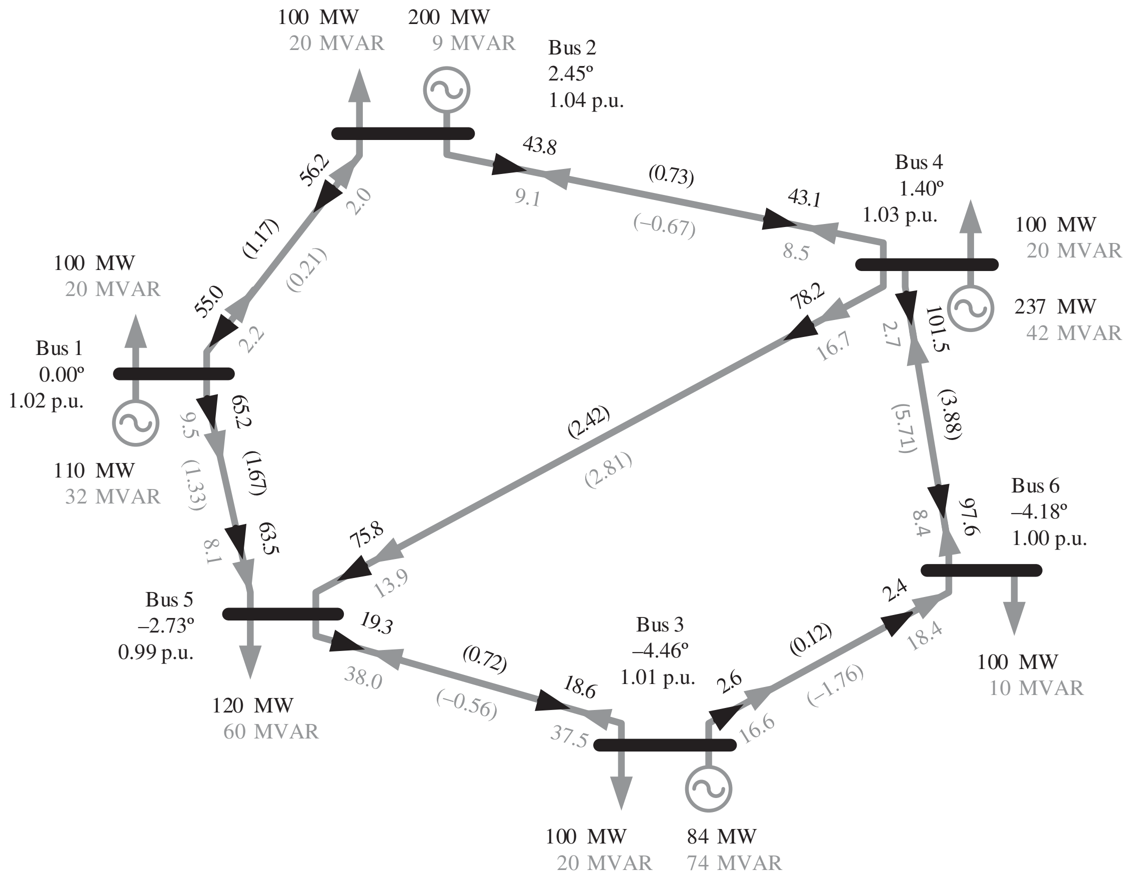


Figure 12.6 Modified six-bus power flow example.

The voltage magnitude at Bus 6 (a P, Q bus) is affected slightly (from 0.9951 to 0.9950 p.u.), although the change does not show up after rounding. In order to maintain the preset voltages at the P, V buses, reactive power generation increases at Buses 4 and 6, as it does at the slack bus. Indeed, system reactive generation now totals 157 MVAR, which are needed to accommodate the additional reactive load introduced at Bus 5 as well as a substantial increase in system reactive losses of 2.33 MVAR (up almost 50% from 4.74 to 7.07 MVAR). While real power is fixed at all buses other than 1 and 5, their voltage angles change as a result of the changed power flow pattern.

The most serious repercussion in this example occurs on Link 4–6, which was fully loaded before the change to Bus 5 was made. Owing to the vagaries of network flow, the change at Bus 5 results in a slight increase in real power flow from Bus 4 to Bus 6. Link 4–6 now carries slightly more current and apparent power, going from 100 to 101.5 MVA. This is significant because, in this hypothetical scenario, each transmission link has a thermal limit of 100 MVA. The power flow program thus shows the line becoming overloaded as a result of the change at Bus 5, even though Buses 4 and 6 are located on the opposite geographic end of the system, and neither generation nor load levels there were affected. In reality, this violation would mean that the proposed change is inadmissible and other options would have to be pursued—specifically, a generator other than Bus 1 would be required to increase generation—in order to meet the additional load without violating any transmission constraints.

The reader is encouraged to further tinker with power flow scenarios, for which the PowerWorld application is an invaluable tool. As this chapter illustrates, it is very difficult to produce an intuitive comprehension of power flow from the formal analytic description of the problem. Actual system operators, whose success hinges on just such an intuitive comprehension, develop it over the course of time by empirical observation of countless scenarios (see Section 16.3.1). Experimentation with a small network model like the one just discussed offers students of power systems an opportunity both to create some contextual meaning for the abstract mathematical power flow relationships, and to appreciate the complexity and challenge of the task faced by system operators and engineers.

12.3.3 Conceptualizing Power Flow

Time importance and rubber band analogy

Perhaps the most difficult conceptual aspect of a.c. power systems has to do with time. The synchronous oscillation along with its profile of voltage angles across the network provides a consistent temporal reference frame for an entire synchronous grid, which might span half a continent. The 60-cycle voltage oscillation is like a pulse that pervades this grid. Indeed, we can imagine a universal clock for the entire system, whose ticking marks the synchronized rhythm of all the connected generators. Yet, as seen in the power flow analysis, a time difference emerges and the oscillations do not coincide perfectly among the various buses. After arbitrarily assigning one bus as the reference for the system clock (a voltage angle of 0.00°), some buses will be slightly anticipating and others slightly dragging behind this pulse.[11] Converted into seconds, the time difference is minuscule: 1° is 1/360th of a cycle, which is 1/60th of a second, making 1° equal to about 46 microseconds (μs) or 0.0000463 seconds.

It could be tempting to attribute this time differential to the propagation time of a signal, but that would be incorrect. As mentioned in Chapter 1, transmission lines can easily be long enough so that the time it would take for an electrical signal to propagate at the speed of light from one end to the other is not negligible. Thus we might wonder about a delay between the voltage maximum (which could be considered a signal) occurring at different buses. Are the power-producing buses senders and the power-consuming buses receivers of a signal that is delayed by the distance between them?

Not exactly. The steady-state behavior we are concerned with does not correspond to the situation where a message originates from one distinct location to another at nearly the speed of light. A signal in the sense of "traveling message" in an a.c. grid would actually consist of a disturbance or departure from the 60-cycle oscillation in the background. Such disturbances (caused by faults, for example) do in fact propagate from one distinct location to another at the speed of light in a conductor, and when studying them, their "travel time" on transmission and distribution lines is meaningful (and can in fact be used to locate the disturbance source). These phenomena are distinct from power flow analysis, which deals with a steady operating state.

Initially establishing a steady state requires some pulse to make its rounds through the system as a disturbance signal—for example, while a section of the grid is energized by the first generator online, or as switches are closed between sections. But once these disturbances have echoed back and the transient effects have decayed, what is left is an ambient condition of being "energized" that resides everywhere in the system at once.

In fact, the time differentials expressed in the voltage-angle profile arise not from long-distance communication, but from the nature of a.c. power transfer as a strain across the transmission line. It is helpful to think of the angle difference like a *twist* in a rotating shaft that delivers mechanical power. Consider a *line shaft* as shown in Figure 12.7 that links numerous drive belts, pulleys or gears, commonly used in factories a century ago. A prime mover (such as a steam engine or water wheel) is driving the shaft forward, supplying mechanical power. Various machines are taking power from the shaft, by letting it drive their belts and thereby acting to hold back its rotation.

11 By musical analogy, this is like a band playing syncopated notes that don't coincide on the downbeat, but each player remains consistent from one measure to the next.

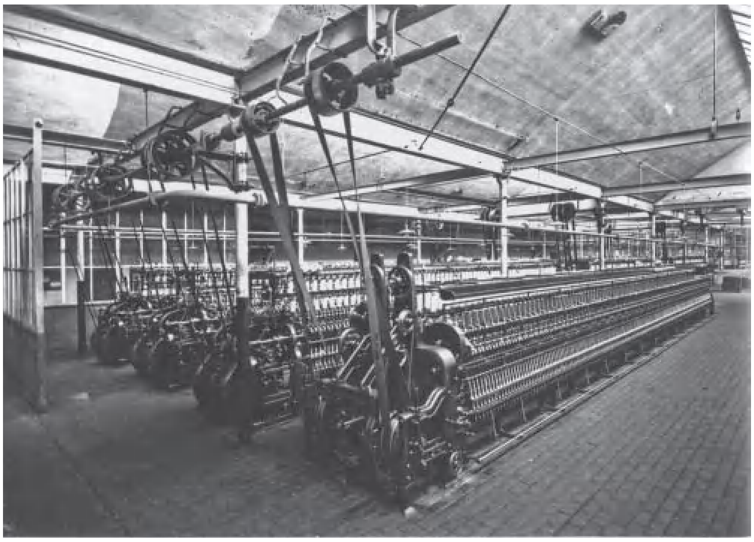


Figure 12.7 Common rotating line shaft in a yarn spinning factory (Leipzig, ca. 1925) as a mechanical analog to a.c. power transfer. Source: Atelier Hermann Walter/Wikimedia Commons/Public Domain.

Clearly, maintaining a constant rotational speed of the shaft hinges on the balance of torque from the belt or gears driving it forward, and those being driven. This is analogous to maintaining a constant a.c. frequency, except that in the case of the electric grid, there are multiple drivers of the shaft interspersed with the loads.

Now, a rigid steel shaft will not exhibit much deformation in the process. But imagine a line shaft made of rubber. Then it becomes intuitive that in those places where the shaft is being driven forward, it will twist in the forward direction relative to the places where belts are holding back the rotation. This mechanical twist angle is an excellent analogy to the voltage phase angle.

If we had marked the rubber shaft in advance with a straight line as a reference (or perhaps stuck a row of pins into it along its length), we could then observe the twist at any location along the rotating shaft by noting that the line (or the local pin) crosses a reference position (say, vertical) at a slightly different time than its neighbors elsewhere along the shaft. Note that nothing material is traveling from one place to another, nor does it take time to do so. The twist is just a characterization of the deformation of the shaft in its steady state of rotation. As such, it is perfectly analogous to the local voltage phase angle.

The line shaft analogy also illustrates a key property of electric power that often challenges economists: namely, that there is no such thing as a package of goods being physically delivered from source to load. Rather, power is transferred into or out of the system locally by each generator or load as they interact with the energized medium through electromagnetic forces, where the power injected or withdrawn by a rotating machine can be expressed as torque times rpm. The transfer of power through the medium could be imagined in terms of a series of microscopic distortions along the shaft, but there is no unique packet of energy traveling. Rather, because the shaft is in a global state of rotation, it can be transacted with locally.

Likewise, the key property of the energized grid is that an alternating voltage is already present everywhere, like an ambient vibration. Therefore, it is more appropriate to think of power transfer as being perfectly instantaneous at every location. It would be incorrect physically to associate some specific power injected in one place with some specific power consumed elsewhere—despite the fact that electricity contracts are expressed as if this were the case. In this sense, our whole notion of "power flow" is always on the verge of being wrong, if we attempt to identify particular quantities of power as though they were unique entities.

The elasticity of the rubber shaft is analogous to the impedance of a transmission line. This is consistent with the idea that a more elastic shaft will twist farther for a given amount of torque. Likewise, the voltage phase angle difference associated with some amount power transfer will be greater if there is a higher impedance, as seen in Eq. (7.1).

Further, we can think about dynamic behaviors. Imagine if a big engine driving our rubber shaft suddenly stops working. From the shock, the shaft will reverberate in a twist, with some oscillations as it slows down. This is analogous to the stability problem (see Section 13.4.4). Clearly, the bouncing behavior will get worse with a longer and more flexible shaft under high load—just like a long transmission line with high impedance.

The spinning rubber shaft is a great visualization aid for power transfer along a single transmission link. But now, what about an entire network? A different mechanical analogy imagines transmission lines as rubber bands tied together into a grid. Each bus is a place where the rubber bands are either suspended by hooks from the ceiling (generation) or have weights hanging from them (loads). The real power in megawatts injected or consumed at each node corresponds to the weight or amount of force pulling the node up or down. The voltage angle roughly corresponds to the elevation of each point in the rubber-band grid. The requirement that power injected equals power consumed corresponds to the requirement that the rubber-band grid be in balance, that is, neither fall down nor snap to the ceiling.

The rubber-band model visualizes dynamic stability in terms of what happens when a weight suddenly falls off (a load is lost) or a hook pops out of the ceiling (a generator goes offline). Even if we assume that the remaining hooks can accommodate the weight (i.e., generators compensate for the change in system load), the dynamic problem is that the network of rubber bands will bounce up and down following the sudden change. Thus, dynamic stability addresses the question of how much bouncing the hooks will tolerate before the whole web of rubber bands comes falling off.

Clearly, any given rubber band can only be stretched so far before it breaks. This translates into the observation that any given transmission link can only sustain a limited difference in voltage angle between its two ends (*steady-state stability*; see Section 13.4.3). Once this limit is exceeded and synchronicity is lost, the link no longer transmits power, just as the broken rubber band no longer transmits a force from weight to ceiling hook.

This analogy becomes a bit awkward if we try to stretch it further (so to speak) and bring line impedance into it. Rubber bands come in different elasticities and strengths, referring to the amount by which they stretch under a given tension, and their ability to withstand tension without breaking. A transmission line with a high impedance would correspond to a band that stretches farther under a given tension, either because it is longer or because it is more elastic. (We have a visualization problem here in that the dimension of linear distance in the rubber band model relates to voltage angle, not geographical distance.) To make the analogy work, we must require that all rubber bands "break" (i.e., violate a stability limit) when elongated by a certain number of inches. It would then hold true that the stability limit is increasingly important for longer lines and those with higher impedance. The thermal limit, by contrast, would be related to the amount of tension or force that can be sustained by each band, regardless of stretch.

A superconducting DC transmission line (Section 7.2.5) would translate into a perfectly strong and firm cord with no stretch at all. In the rotating shaft analogy, it would be perfectly rigid with no twist. If our electric grid were connected by such ideal links, nothing would bounce or stretch, and the subject of power flow analysis would become rather uninteresting.

This is about as far as the rotating shaft and rubber-band analogies go; trying to incorporate reactive power and voltage magnitude into these model is too contrived to be useful. Perhaps the most appropriate conclusion is that a.c. power systems have a certain complexity which, in its defiance of human intuition, is unmatched by any mechanical system.

12.4 Power Flow Equations and Solution Methods

12.4.1 Derivation of Power Flow Equations

In Section 12.2.2, we stated the known and unknown variables for each of the different types of buses in power flow analysis. The *power flow equations* show explicitly how these variables are related to each other.

The complete set of power flow equations for a network consists of one equation for each node or branch point in this network, referred to as a bus, stating that the complex power injected or consumed at this bus is the product of the voltage at this bus and the current flowing into or out of the bus. Because each bus can have several transmission links connecting it to other buses, we must consider the sum of power entering or leaving by all possible routes. To help with the accounting, we will use a summation index i to keep track of the bus for which we are writing down the power equation, and a second index k to keep track of all the buses connected to i.

We express power in complex notation, which takes into account the two-dimensionality—magnitude and time—of current and voltage in an a.c. system. As shown in Section 3.5.2, complex power S can be written[12] in shorthand notation as

$$S = VI^*$$

where all variables are complex quantities and the asterisk denotes the complex conjugate of the current.[13]

Recall that S represents the complex sum of real power P and reactive power Q, where P is the real and Q the imaginary component. At different times it may be convenient to either refer to P and Q separately or simply to S as the combination.

In the most concise notation, the power flow equations can be stated as

$$S_i = V_i I_i^*$$

where the index i indicates the node of the network for which we are writing the equation. Thus, the full set of equations for a network with n buses would look like

$$S_1 = V_1 I_1^*$$

$$S_2 = V_2 I_2^*$$

$$\dots$$

$$S_n = V_n I_n^*$$

12 In this chapter, we reserve boldface notation for matrices.
13 The complex conjugate of a complex number has the same real part but the opposite (negative) imaginary part; see Section 3.3.5. It is used here to produce the correct relationship between voltage and current angle—their difference, not their sum—for purposes of computing power.

We can choose to define power as positive either going into or coming out of that node, as long as we are consistent. Thus, if the power at load buses is positive, that at generator buses is negative.

So far, these equations are not very helpful, since we have no idea what the I_i are. In order to mold the power flow equations into something we can actually work with, we must make use of the information we presumably have about the network itself. Specifically, we want to write down the impedances of all the transmission links between nodes. Then we can use Ohm's law to substitute known variables (voltages and impedances) for the unknowns (currents).

Written in the conventional form, Ohm's law is $V = IZ$ (where Z is the complex impedance). However, when solving for the current I, it is easier to use the admittance Y (where $Y = 1/Z$), so that Ohm's law becomes $I = VY$. This allows us to indicate the absence of a transmission link with a *zero* (for zero admittance), creating a *sparse* matrix for large systems that greatly facilitates computation.

The relationship $I = VY$ is what we wish to write down and substitute for every I_i that appears in the power flow equations. But now we face the next complication: the total current I_i coming out of any given node or bus is in fact the sum of many different currents going between bus i and all other buses that physically connect to i. We will indicate these connected buses with the index k, where k could include all buses in the network from 1 to n. In practice, fortunately, only a few of these will actually have links connecting to bus i. For any bus k that is not connected to i, $Y_{ik} = 0$.

For the current from node i to node k, we would generally write

$$I_{ik} = (V_k - V_i)Y_{ik}$$

where Y_{ik} is the admittance between i and k.

Suppose we are analyzing Bus 1, which is connected to Buses 2 and 3. (In this case $i = 1$, and $k = 1, 2,$ and 3.) By Kirchhoff's current law, the net current injection at Bus 1 from generation and/or load must equal the net outflow of current into the network, thus:

$$I_1 = I_{12} + I_{13}$$

Writing this in terms of bus voltages and admittances, and rearranging terms, we get

$$\begin{aligned} I_1 &= (V_1 - V_2)Y_{12} + (V_1 - V_3)Y_{13} \\ &= V_1(Y_{12} + Y_{13}) - V_2Y_{12} - V_3Y_{13} \end{aligned}$$

Here it becomes convenient to change the sign of the branch admittance, which allows us to write the current as a sum of positive terms. By letting $y_{12} = -Y_{12}$, we get:

$$I_1 = V_1(Y_{12} + Y_{13}) + V_2y_{12} + V_3y_{13}$$

This will extend very nicely when we sum over all nodes k that could possibly be connected to node i, in tidy summation notation.[14] This summation over the index k means accounting for all the current that is entering or leaving this one particular node, i, by way of the various links it has to nodes k. (For the complete system of power flow equations, we will consider every value of the index i so as to consider power flow for every bus.)

Thus,

$$S_i = V_iI_i^* = V_i\left(\sum_{k=1}^{n} y_{ik}V_k\right)^*$$

It is important to note that the $k = i$ term is included in the summation. The admittance y_{ii} is called the *self-admittance*. This self-admittance is defined as the (positive) sum of all the

14 The notation with the capital Greek sigma (for "sum") indicates the sum of indexed terms, with the index running from the value below the sigma ($k = 1$) to the value above it ($k = n$).

admittances connected to the ith bus. In the above example, $y_{11} = Y_{12} + Y_{13}$. By including the term for $k = i$ and making the branch admittances negative, we avoid the need to write out voltage differences between buses.

Recall from Section 3.3.7, the complex admittance $Y = G + jB$, whose real part G is the *conductance* and whose imaginary part B is called *susceptance*. The admittances of all the links in the network can be summarized by way of an *admittance matrix* $\mathbf{Y}$, where the lowercase $y_{ik} = g_{ik} + jb_{ik}$ indicates the matrix element that associates nodes i with k. More about the admittance matrix in Section 12.4.2.

To complete our expression for complex power at the ith bus, we expand the y's into g's and b's (noting that the complex conjugate gives a minus sign in front of the jb):

$$S_i = V_i \sum_{k=1}^{n} (g_{ik} - jb_{ik}) V_k^*$$

After rearranging terms to look more organized, we write the voltage phasors out in longhand, first as exponentials and then broken up into sines and cosines:

$$\begin{aligned} S_i &= \sum_{k=1}^{n} |V_i||V_k| e^{j(\theta_i - \theta_k)} (g_{ik} - jb_{ik}) \\ &= \sum_{k=1}^{n} |V_i||V_k| [\cos(\theta_i - \theta_k) + j\sin(\theta_i - \theta_k)](g_{ik} - jb_{ik}) \end{aligned}$$

The term $(\theta_i - \theta_k)$ in this equation denote the *difference* in voltage phase angle between nodes i and k, where the minus sign came from having used the complex conjugate of I initially. It is often convenient to abbreviate it $(\theta_i - \theta_k) = \theta_{ik}$.[15]

The equation we now have for S_i entails the product of two complex quantities, written out in terms of their real and imaginary components. By cross-multiplying all the real and imaginary terms, we can separate the real and imaginary parts of the result S, which will be the familiar P and Q. Taking care with the sign of j^2, we obtain:

$$\begin{aligned} P_i &= \sum_{k=1}^{n} |V_i||V_k| [g_{ik} \cos(\theta_i - \theta_k) + b_{ik} \sin(\theta_i - \theta_k)] \\ Q_i &= \sum_{k=1}^{n} |V_i||V_k| [g_{ik} \sin(\theta_i - \theta_k) - b_{ik} \cos(\theta_i - \theta_k)] \end{aligned} \tag{12.1}$$

The complete set of power flow equations for a network of n buses contains n such equations for S_i, or pairs of equations for P_i and Q_i. This complete set will account for every node and its interaction with every other node in the network.

12.4.2 The Bus Admittance Matrix

There are many possible ways of organizing information about the electrical connectivity and impedance characteristics of a network.

One basic choice is whether to describe the network in terms of impedances or admittances. As mentioned earlier, admittance has the advantage of having many smaller or zero values, since the vast majority of nodes in a network are not directly connected to each other. This creates a more sparse matrix that is easier and faster to manipulate, while the information it contains about the

15 This voltage phase angle difference is called δ_{12} elsewhere, but θ is more commonly used in the context power flow analysis.

network is the same. There are some calculations for which an impedance matrix is better suited than an admittance matrix; one important example is fault analysis. These techniques are beyond the scope of this text.[16] We will limit ourselves to the admittance matrix here, as it is far more commonly used for power flow analysis.

Another choice is whether to describe network *branches* or *nodes*. In the branch description, voltages are understood as being across and currents through an individual branch, so that Ohm's law can be written for each branch individually. The impedances or admittances of individual branches are sometimes called *primitive*. They can be collected in a primitive impedance or admittance matrix.[17] Such a compilation of branch impedances or admittances is distinct from the information about their connectivity, that is, which network branches actually meet at a node. Our choice here will be to summarize information by node.

The bus admittance matrix $\mathbf{Y}_{\text{bus}}$, also called *nodal admittance matrix*, contains information about both the connectivity of the network and the numerical values of the admittances connecting each pair of nodes. It is the inverse of the *bus impedance matrix* $\mathbf{Z}_{\text{bus}}$:

$$\mathbf{Y}_{\text{bus}} = \mathbf{Z}_{\text{bus}}^{-1}$$

Both of these matrices are symmetrical (i.e., values can be flipped about the main diagonal). The entries along the main diagonal characterize an individual node, and the off-diagonal elements characterize the connections between a respective pair of buses without any implied sense of directionality: $y_{ik} = y_{ki}$ and $z_{ik} = z_{ki}$. The diagonal elements of $\mathbf{Y}_{\text{bus}}$ are called *self-admittances* and the off-diagonal elements are the *negative branch admittances*. These definitions, including the negative signs, account for the connectivity. They let us obtain currents throughout the network just by multiplying the nodal admittance matrix by all the nodal bus voltages, without having to take any explicit voltage differences:

$$\mathbf{I} = \mathbf{Y}_{\text{bus}}\mathbf{V}$$

The way to construct an admittance matrix for any arbitrary network is first to convert that network into a *Norton equivalent* circuit (see Section 2.5.2). This means expressing all elements as some combination of current sources and parallel admittances.[18] If branches were characterized in terms of impedances (in ohms), these must be converted to admittances (in siemens). The generators and loads in the power system are represented by current sources in the Norton equivalent. These do not affect the admittances. When creating the admittance matrix for a circuit diagram, the current sources are ignored.

Note that in a conventional circuit diagram, we include a "zero" node corresponding to ground. This ground or reference node is not counted in the bus admittance matrix. In the standard one-line

16 An excellent reference for constructing and using the impedance matrix remains the classic text by J.J. Grainger and W.D. Stevenson, *Power System Analysis* (McGraw Hill, 1994).

17 The *primitive* or *branch admittance matrix* is commonly denoted by $[y]$. It captures only the admittance of each individual branch, and needs to be combined with an *adjacency* or *bus incidence matrix* A that contains information about the connectivity of the network along with reference directions for currents and voltages. For connoisseurs of linear algebra, $\mathbf{Y}_{\text{bus}} = A^{\text{T}}[y]A$. The reference directions are assigned in order to relate branch voltages to nodal voltages that are measured with respect to one common reference node, and branch currents to nodal current injections (i.e., defining current positive into or out of each branch at a given node). In graph theory, one would say that the network is represented by a *directed graph*. The reference direction is captured by assigning a +1 or −1 to each connection, depending on whether it is "entering" or "leaving" the node. Either convention works as long as it is applied consistently; in power systems, we conventionally assign +1 to a branch leaving a node. The adjacency matrix has zeroes along the main diagonal.

18 The corresponding procedure for an impedance matrix draws a Thévenin equivalent circuit, with a combination of voltage sources and series impedances.

diagram of a power system network, we don't explicitly show the ground; generators and loads are depicted as just dangling off their buses. The way to reconcile these representations is to imagine a ground plane behind the bus network diagram, to which all these dangling devices connect: this is simply stretching the circuit diagram into three dimensions, with the reference node out of view. Thus, when we speak of a "current injection" at Bus i, we are talking about current going from the ground plane into the network. Applying Kirchhoff's current law at every bus ensures that the overall net sum of currents to ground is zero. If we characterized the ground node or plane as its own circuit node, it would contain only redundant information. Therefore we simply omit it from the drawing of buses, and it does not get its own row or column in the admittance matrix.

To construct $\mathbf{Y}_{bus}$, we inventory the self-admittances for each node, by adding all admittances that are connected to that node. This includes the branch admittances to all adjacent nodes, as well as any shunt admittances associated with the individual node.[19] The self-admittances become the diagonal matrix elements y_{ii}. The off-diagonal matrix elements $y_{ik} = y_{ki}$ will be the negative branch admittances.[20]

In the bus network diagram as used in power flow analysis, the current injections are implicit in loads and generation, which are specified in terms of power. These power injections also implicitly capture the admittances through loads and generators to ground, so we don't need to include them in the network description. This representation is an alternative to the more detailed description from the Norton equivalent circuit, where anything connected between a node and ground appears as a combination of current source and impedance.

Note that in all of these cases, it is possible to specify impedances and admittances in physical units of ohms and siemens, respectively, but especially for large networks it is most common to use per-unit quantities.

Example

Consider the network diagram in Figure 12.8. Nothing will happen on this circuit because it has no source—but we can still write an admittance matrix for it. In this example, we are given impedances, so these need to be converted. We first write the primitive admittances Y_{ik} for each

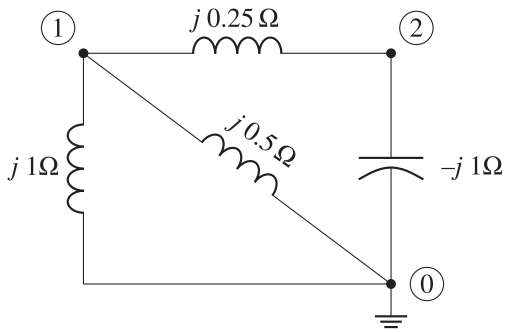


Figure 12.8 Simple network to illustrate obtaining branch admittances.

19 For example, in the medium transmission line model (Section 9.3.5), capacitances are divided in half, and each half is associated with one line end. These capacitances count toward the self-admittance, but not the branch admittance

20 That negative sign comes out of the conversion from branch to nodal admittances, as detailed previously. It ensures that all voltage differences between nodes are accounted for correctly when we multiply the bus admittance matrix by nodal voltages to obtain nodal currents.

branch, and then the matrix elements y_{ik}[21]:

Pq não pega todos os caminhos do ponto 1

$$Y_{10} = \frac{1}{j1} + \frac{1}{j0.5} = -j2$$

$$Y_{12} = \frac{1}{j0.25} = -j4$$

$$Y_{20} = \frac{1}{-j1} = j1$$

$$y_{11} = Y_{10} + Y_{12} = -j2 - j4 = -j6$$

$$y_{22} = Y_{12} + Y_{20} = -j4 + j1 = -j3$$

$$y_{12} = -Y_{12} = j4$$

The bus admittance matrix looks like this:

$$\mathbf{Y}_{\text{BUS}} = j\begin{bmatrix} -6 & 4 \\ 4 & -3 \end{bmatrix}$$

Note that the matrix has dimension 2×2 because one of the circuit nodes in the diagram is the ground reference.

Example

Consider the diagram in Figure 12.9, where two buses are connected by a transmission line modeled as a medium-length line, with series impedance $Z = j0.2$ p.u. and shunt admittance (due to capacitance) $Y = j0.2$ p.u. The capacitance is split in half and allocated to each end. The admittance matrix elements are as follows:

$$y_{11} = y_{22} = \underbrace{\frac{1}{Z}}_{Y_{12}} + \underbrace{\frac{Y}{2}}_{Y_{10} = Y_{20}} = \frac{1}{j0.2} + j0.1 = -j5 + j0.1 = -j4.9$$

$$y_{12} = y_{21} = -\frac{1}{Z} = j5$$

$y_{11} = Y_{10} + Y_{12}$

$y_{22} = Y_{12} + Y_{20}$

The matrix looks like this:

$$\mathbf{Y}_{\text{BUS}} = j\begin{bmatrix} -4.9 & 5 \\ 5 & -4.9 \end{bmatrix}$$

When transmission line data are tabulated for input into power flow solvers, the shunt admittance is conventionally listed as B (indicating it is a susceptance only), in a separate column from the series components R and X. Very occasionally, there may be nonzero entries for shunt conductance G, which would similarly be split in half.

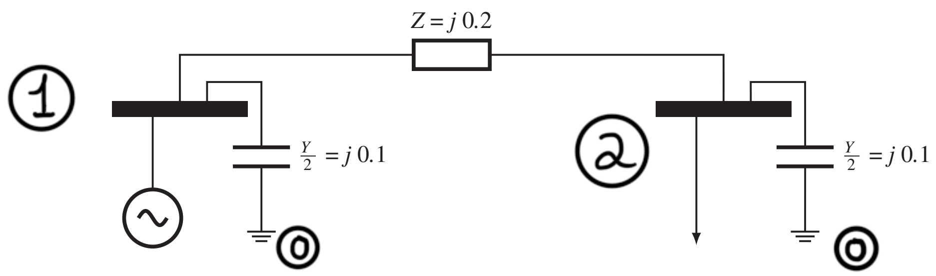


Figure 12.9 Simple two-bus network to illustrate allocation of transmission line capacitance.

21 Here we use lowercase letters for the elements of the bus admittance matrix and capital letters for the primitive values; some authors may use a reverse convention.

12.4.3 Solution Methods

There is no analytical, closed-form solution for the set of power flow equations given in Section 12.4.1. In order to solve the system of equations, we must proceed by a numerical approximation that is essentially a sophisticated form of trial and error.

To begin with, we assume certain values for the unknown variables. For clarity, let us suppose that these unknowns are the voltage angles and magnitudes at every bus except the slack, making them all P, Q buses (it turns out that having some P, V buses eases the computational volume in practice, but it does not help the theoretical presentation). In the absence of any better information, we would probably choose a *flat start*, where we assume the initial values of all voltage angles to be zero (the same as the slack bus) and the voltage magnitudes to be 100% of the nominal value, or 1 p.u. In other words, for lack of a better guess, we suppose that the voltage magnitude and angle profile across the system is completely flat.

We then plug these values into the power flow equations. Of course, we know they do not describe the actual state of the system, which was supposed to be consistent with the known variables (P's and Q's). Essentially, this will produce a contradiction: based on the starting values, the power flow equations will predict a different set of P's and Q's than we stipulated at the beginning. Our objective is to make this contradiction go away by repeatedly inserting a better set of voltage magnitudes and angles: as our voltage profile matches reality more and more closely, the discrepancy between the Ps and Qs, known as the *mismatch*, will shrink. Depending on our patience and the degree of precision we require, we can continue this process to reach some arbitrarily close approximation. This type of process is known as an *iterative* solution method, where "to iterate" means to repeatedly perform the same manipulation.

The heart of the iterative method is to know how to modify each guess in the right direction and by the right amount with each round of computation (iteration), so as to arrive at the correct solution as quickly as possible. Specifically, we wish to glean information from our equations that tells us which value was too high, which was too low, and approximately by how much, so that we can prepare a well-informed next guess, rather than blindly groping around in the dark for a better set of numbers.

There are several standard techniques for doing this. The ones most commonly used in power flow analysis are the Newton–Raphson, the Gauss, and the Gauss–Seidel iterations. We introduce the basic idea of Newton's method in Section 12.4.4, and work through a step-by-step example with the Newton–Raphson algorithm. Once this process is understood conceptually, readers should be able to interpret alternative techniques presented in other references. In practice, the choice of algorithm for a particular situation involves a trade-off among the number of iterations required to arrive at the solution, the amount of computation required for each iteration, and the degree of certainty with which the solution is found.

Regardless of which method is used, we will need to press our power flow equations for the crucial information about the error in each iteration, to determine the next one. Some readers may recognize this as a kind of *sensitivity analysis*, which asks how much one variable is affected by changes in another. We obtain this information by writing down the *partial derivatives* of the power flow equations, or their rates of change with respect to individual variables. Specifically, we need to know the rate of change of real and reactive power, each with respect to voltage angle or magnitude. This yields four possible combinations of partial derivatives.[22] For example, $\partial P/\partial\theta$ (read: "partial P partial theta") is the partial derivative of real power with respect to voltage angle, and similarly there

---

22 The partial derivative means the rate of change of a function with respect to only one of several variables, and is conventionally indicated by the curly ∂ instead of plain d for "differential element."

are $\partial P/\partial V$, $\partial Q/\partial \theta$, and $\partial Q/\partial V$.[23] Each of these combination is in fact a matrix (known as the *Jacobian* matrix) that, in turn, includes every bus combined with every other bus. In expanded form, with three buses, $\partial P/\partial \theta$ will look like this:

$$\frac{\partial P}{\partial \theta} = \left\{ \begin{matrix} \frac{\partial P_1}{\partial \theta_1} & \frac{\partial P_1}{\partial \theta_2} & \frac{\partial P_1}{\partial \theta_3} \\ \frac{\partial P_2}{\partial \theta_1} & \frac{\partial P_2}{\partial \theta_2} & \frac{\partial P_2}{\partial \theta_3} \\ \frac{\partial P_3}{\partial \theta_1} & \frac{\partial P_3}{\partial \theta_2} & \frac{\partial P_3}{\partial \theta_3} \end{matrix} \right.$$

Each of these four types of partial derivatives ($\partial P/\partial \theta$, $\partial P/\partial V$, $\partial Q/\partial \theta$, and $\partial Q/\partial V$) constitutes one partition of the big Jacobian matrix **J**:

$$\mathbf{J} = \left[\begin{array}{c|c} \frac{\partial P}{\partial \theta} & \frac{\partial P}{\partial V} \\ \hline \frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial V} \end{array} \right]$$

Within each partition there are rows for the θ or V and columns for the P or Q belonging to every bus (except the slack bus, so that the dimensionality of each partition is one less than the number of buses in the system).

We must now combine the system of equations and its partial derivatives with our guess for the unknown variables in such a way that it suggests to us a helpful modification of the unknowns, which will become our improved guess in the next iteration.

12.4.4 Iterative Computation

Our task can be stated as trying to find an unknown value of a function whose explicit form we do not know, based on information from elsewhere along the function. This can be done with a *Taylor series expansion*, which may be familiar to readers who have studied calculus. The idea is that we can express the unknown value of the function $f(x)$ at some particular x in terms of two pieces of information: the function's value at a different, nearby x; and the rate of change of the function with respect to x—its slope—at the same nearby x. Suppose we already know the value f at location x, and we also know how steep the function is there. Now we want to learn the value of f at the nearby location that we call $x + \Delta x$, so that Δx represents the difference between the two x's. If the function $f(x)$ is a straight line, we can write

$$f(x + \Delta x) = f(x) + f'(x)\Delta x$$

where $f'(x)$ is the *first derivative* or slope of the line at location x.

In the more general case, where $f(x)$ is not a straight line, but some type of curve, this equation is incomplete; we would have to include additional higher-order terms that correct for the curvature. Specifically, we would include the second derivative (the rate of change of the rate of change, which describes the upward or downward curvature) to correct the straight-line approximation, multiplied once again by the increment Δx. We may also need to include the third derivative or more, depending on how curvy the function is. Each nth term gets successively scaled down by a factor of $n!$ (n factorial). Note that this procedure applies only to functions that are infinitely differentiable; in other words, they cannot have corners, spikes or discontinuities.

23 Note that when we write V we mean the magnitude of V, which would be more properly designated by $|V|$ except that the notation is already cumbersome enough without the absolute value signs.

Written out, the Taylor series looks like

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \frac{f'''(x)}{3!}\Delta x^3 + \cdots$$

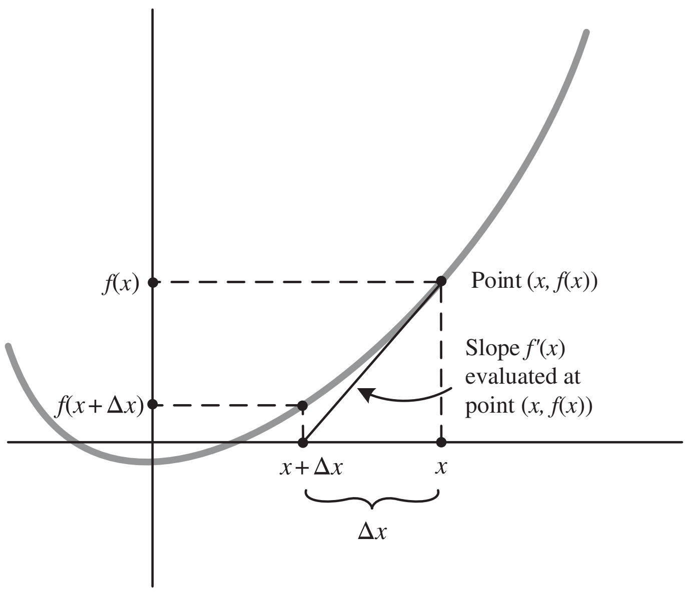
up to the point where the function's higher-order derivatives are zero.

Fortunately, we are in the business of making approximations rather than finding exact values. If the Δx is not too big and the function not too radically curved, then the higher-order terms ought to be quite small compared to the first derivative term. Thus, we can use the linear version for a general function, with the understanding that it will not take us exactly to the new value $f(x+\Delta x)$, but a good bit of the way there and almost certainly closer than we were before.[24]

The reader may wonder why we don't simply plug the new x-value, $x+\Delta x$, into the function $f(x)$. Usually, this is because we do not know how to write down $f(x)$ or $f'(x)$ in algebraic form, even though we have numerical values for f and f' at some particular x. In our application, we happen to know what $f(x)$ looks like algebraically—the power flow equations—but we cannot solve it in the way that we would like. Specifically, we would like to go backwards to find the x that will yield a particular value of f. Even though we know how to get f from any given x, we can't simply solve for x given the f because the equation does not allow itself to be turned inside out. Specifically, given the θ's and V's in the power flow equations, we can readily solve for the P's and Q's, yet we cannot go backwards from the P's and Q's to explicitly solve for the θ's and V's. Thus, we are forced to try out different sets of θ's and V's (represented by the x's) until we hit the right P's and Q's (represented by the f).

The standard way to proceed is to rearrange the equations as necessary so that the target value is $f(x) = 0$. The problem can then be stated in the tidy format, "Find the x that makes $f(x) = 0$ a true statement." It is illustrated in Figure 12.10, where we start with a certain x and a known value $f(x)$ that is not zero, but wish to find the Δx for which the value of the function $f(x+\Delta x)$ is zero. Note that in this diagram Δx happens to be negative (i.e., measuring to the left), but it could go either way. After writing down the first terms of the Taylor series and declaring that $f(x+\Delta x) = 0$, it takes

Figure 12.10 Newton's method.



24 This is true unless the function is very badly behaved or we started in an awkward spot.

only a minor manipulation to solve explicitly for the Δx that makes this true:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x$$
$$0 = f(x) + f'(x)\Delta x$$
$$\Delta x = -f(x)/f'(x)$$

We have then found the Δx that must be added to the original x to obtain the new x-value, for which the function is zero.

But because the function is curved, not straight, our answer will not be exactly right. We have evaluated the derivative $f'(x)$ at the location of our original x, meaning that we have used the slope at that location to extrapolate where the function is going. But in reality, the function's slope may change along the way. The higher-order terms of the Taylor series would address this problem, but they contain awkward squares and cubes. Instead of dealing with such terms, we simply repeat the linear process: we use the new location, $x + \Delta x$, as our starting point for another iteration. Since $x + \Delta x$ is presumably much closer to our target than the original x—which we can verify by checking that $f(x + \Delta x)$ is closer to zero than $f(x)$—the next time it should be easier to get even closer, with a smaller Δx. Depending on the precision we desire in getting f to zero, we can repeat the process again with more iterations after that, or call it a day. This, in essence, is Newton's method for finding the zero-crossing of a function.

In any case, we will probably have many x's and Δx's around and ought to keep track of which iteration they belong to. One way to label them is with a superscript like x^ν, where ν (Greek lowercase nu) stands for the number of iterations ($x^1, x^2, \ldots$ and $\Delta x^1, \Delta x^2, \ldots$) and is not to be mistaken for an exponent. The process of approaching a value of x for which $f(x) \approx 0$ is illustrated in Figure 12.11 for two iteration steps. Clearly, the more the slope changes between x^0 and the solution (i.e., the more the function is curved), the more steps will be required to get close. Based on the diagram, the

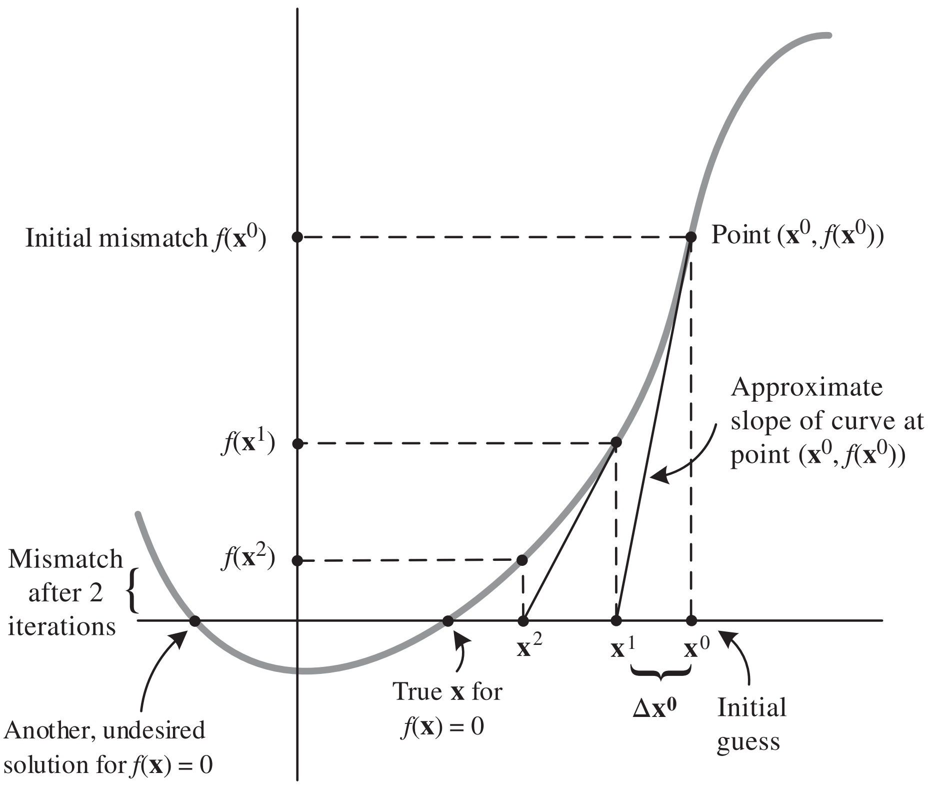


Figure 12.11 Iterative process of approximating $f(x) = 0$.

reader can also visualize how this approximation process can still succeed even if the slope of the line drawn to choose the next x^{ν} is not precisely equal (but bears a reasonable resemblance) to the actual slope of the curve; this property is used in shortcuts such as the dishonest Newton-Raphson method discussed later in this chapter that avoid some of the tedious computation of the exact derivatives.

The Jacobian matrix is essentially a large version of the derivative $f'(x)$ used in Newton's method, with multiple f's and x's tidily summarized into the single, bulky object labeled $\mathbf{J}$. The function $f(x)$ itself captures the power flow equations $P(\theta, V)$ and $Q(\theta, V)$. We write $\mathbf{f}(x)$ in boldface to indicate that it is a vector containing an organized set of several numbers (one P and one Q for each bus except the slack). However, in order to keep with the format of searching for $f(x) = 0$, we define $\mathbf{f}(x)$ as the *difference* between the $P(\theta, V)$, $Q(\theta, V)$ computed from the power flow equations as a function of x, and the P and Q injections at each of the buses that are given at the outset. In other words, $\mathbf{f}(x)$ represent the *mismatch*, which we want to get as close to zero as reasonably possible.

In principle, we are now able to combine all the information at our disposal into improving our guess for the θ's and V's (the x's). Let us label our initial guess for θ and V as $\mathbf{x}^0$, where the boldface $\mathbf{x}$ indicates the vector composed of one θ and one V for each bus (except the slack) and the superscript 0 indicates the zeroth iteration. Now we might simply adapt the expression for Δx from Newton's method (or some variation that corresponds to other solution methods, though the basic idea is always the same),

$$\Delta x = -f(x)/f'(x)$$

and substitute our matrix and vector quantities

$$\Delta \mathbf{x} = -\mathbf{f}(\mathbf{x})/\mathbf{J}$$

But stop! We've just made every mathematician cringe, because dividing by a matrix is not something one does. We need to use the proper *inverse* of the matrix, $\mathbf{J}^{-1}$, and write

$$\Delta \mathbf{x} = -\mathbf{J}^{-1}\mathbf{f}(\mathbf{x})$$

The inverse of a matrix is obtained by a tedious but tractable procedure in linear algebra, which quickly grows more cumbersome with increasing size of the matrix.[25] In the days of paper and pencil, half a dozen rows and columns would have easily defeated the most diligent scribe, and iterative power flow solution for large a.c. networks was simply not an option. Modern computing allows us to solve systems with hundreds and even thousands of buses, where inverting the Jacobian matrix is the critical part of the computational effort. Reasonably sized matrices are easily inverted today with calculators, Matlab or online tools—a capability still best appreciated through the character-building experience of solving a small system by hand.

Having obtained a correction $\Delta \mathbf{x}$ by hook or by crook, we add it to the old $\mathbf{x}$ in order to proceed to the next iteration. Specifically, we write:

$$\mathbf{x}^{(\nu+1)} = \mathbf{x}^{(\nu)} + \Delta \mathbf{x} = \mathbf{x}^{(\nu)} - \mathbf{J}(\mathbf{x}^{(\nu)})^{-1}\mathbf{f}(\mathbf{x}^{(\nu)})$$

where the superscript ν indicates the iteration number, with optional parentheses to prevent mistaking it for an exponent.[26]

25 Analogous to the inverse of a scalar, which gives 1 when multiplied by the original number, the inverse of a matrix produces the *identity matrix* when multiplied with the original matrix. Note that rearranging the equation to read $\mathbf{J}\Delta \mathbf{x} = -\mathbf{f}(\mathbf{x})$ does not remove the awkwardness because we must still solve for $\Delta \mathbf{x}$.

26 We will sometimes drop the parentheses for convenience. Other notation options are subscripts or simply $x(\nu)$ to denote iteration count.

With successive iterations, the value of $\mathbf{f}(\mathbf{x}^{(v)})$ should get smaller. When it reaches zero—or close enough, according to a chosen convergence threshold—it means that there is no more mismatch. The $\mathbf{x}^{(v)}$ at that point (i.e., the θ's and V's from the vth iteration) give us the operating state of the power system that is consistent with the P's and Q's we specified initially. We may need to verify, though, that $\mathbf{x}^{(v)}$ is a realistic and true solution for our power system, as opposed to some mathematical fluke, which can occur when a function has more than one zero-crossing.

We have now found θ and V for each P, Q bus. For any P, V buses, we would have found θ and Q as part of our computational process. There are several steps left to produce the complete output of the power flow analysis. First, by writing the power flow equation for the slack bus, we determine the amount of real power generated there. This tells us how many MW of losses there are in the system, as we can now compare the total MW generated to the total MW of load demand. Also, by using Ohm's law for every transmission link, we solve explicitly for each line flow, in amperes or MVA. Through the θ's and V's at each bus, we have information about the real and reactive power both going into and coming out of each link, and by subtracting we can specify the real and reactive losses on each link. Finally, we format the output and compare it to external constraints such as line flow limits so as to flag any violations. We have now completely described the system's operating state based on a given generation dispatch and combination of loads.

12.4.5 Power Flow Example

For illustration, let us work through the smallest possible example, a two-bus power system, with the Newton–Raphson method.[27] In this tiny system, there is only one bus to be solved for by iteration, and all the arithmetic is tractable by hand.

Figure 12.12 illustrates the case. We have Bus 1 as the slack bus, whose voltage is set at $V_1 = 1.00\angle 0°$ p.u. We don't know yet how much power the generator at Bus 1 needs to inject. What we do know is that the load at Bus 2 is demanding exactly 2.0 p.u. of real and 1.0 p.u. of reactive power (after choosing $S_{BASE} = 100$ MVA); that there is an impedance of $Z = j0.1$ p.u. between Bus 1 and 2; and that there will be some reactive but no resistive losses (since the line impedance is purely imaginary). We don't know what the voltage at Bus 2 will have to be in order to deliver the demanded power, or how much current has to flow.

Constructing a bus admittance matrix $\mathbf{Y}_{bus}$ seems overkill for this example, since it is built on just a single value, the impedance $Z = j0.1$ between Bus 1 and 2. But creating the proper matrix with self- and mutual admittance terms is a useful exercise and reminder of where the minus signs go. Per the procedure introduced in Section 12.4.2, it is given by:

$$\mathbf{Y}_{bus} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} -j10 & j10 \\ j10 & -j10 \end{bmatrix}$$

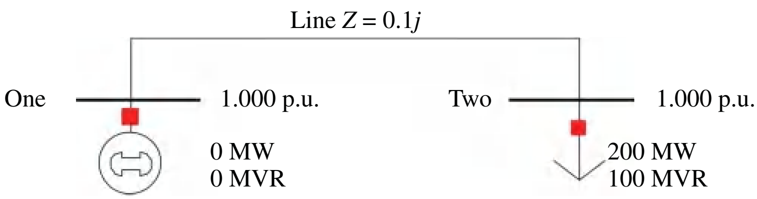


Figure 12.12 Two-bus power flow example visualized in *PowerWorld*. Source: U.S.-Canada Power System Outage Task Force, 2004/United States Department of Energy/Public Domain.

27 This teaching example was created by Tom Overbye, author of *PowerWorld*™. A free educational version of *PowerWorld*™ is downloadable from www.powerworld.com.

Using the general power flow equations (12.1) with $i = 2$, the net real and reactive power injections at Bus 2 are given by

$$P_2 = \sum_{k=1}^{2} |V_2||V_k|[g_{2k}\cos(\theta_2 - \theta_k) + b_{2k}\sin(\theta_2 - \theta_k)]$$

$$Q_2 = \sum_{k=1}^{2} |V_2||V_k|[g_{2k}\sin(\theta_2 - \theta_k) - b_{2k}\cos(\theta_2 - \theta_k)]$$

where the summation index k will take on the values 1 and 2.

We will now write equations for net power at each bus to be solved for (only Bus 2 in this example), which are the entries of the mismatch function $\mathbf{f}(\mathbf{x})$. When we have found the correct $\mathbf{x}$, this mismatch will approach zero, stating that the power injected at each bus by generation or load (as given in the problem statement) equals the power injected from that bus into the network by way of the power flow equations (as calculated from $\mathbf{x}$). Note that a positive power injection corresponds to generation, and power flow *away from* that bus on the transmission lines. Loads represent a negative injection. Since we subtract the injections from the power flows, loads appear as positive terms in the mismatch.

Thus, using the given values of $|V_1| = 1.0$, $g_{ik} = 0$ (since the line is purely inductive), $b_{21} = y_{21} = j10$, $P_2 = -2.0$ and $Q_2 = -1.0$, and setting net power equal to zero, we get:

$$\begin{aligned} P_{2\text{net}} &= |V_2|(10\sin\theta_2) + 2.0 = 0 \\ Q_{2\text{net}} &= |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 = 0 \end{aligned} \tag{12.2}$$

where all the terms with g_{ik} have dropped out. Also, in the expression for P_2, the $k = 2$ term disappears because $\sin(\theta_2 - \theta_2) = 0$. To set up a power flow solution by Newton–Raphson for the two bus example, we define a solution vector $\mathbf{x}$ for the voltage at Bus 2:

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ |V_2| \end{bmatrix}$$

It is standard practice to always express θ in units of radians within the power flow calculation.

Before we choose a starting value and begin to perform any iterations, let us write out the Jacobian matrix whose elements are the partial derivatives of the power flow equations (12.2) with respect to the two components of $\mathbf{x}$:

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial P_2}{\partial \theta_2} & \frac{\partial P_2}{\partial |V|_2} \\ \frac{\partial Q_2}{\partial \theta_2} & \frac{\partial Q_2}{\partial |V|_2} \end{bmatrix} = \begin{bmatrix} 10|V_2|\cos\theta_2 & 10\sin\theta_2 \\ 10|V_2|\sin\theta_2 & -10\cos\theta_2 + 20|V_2| \end{bmatrix}$$

We will evaluate the four elements of this Jacobian for each successive iteration of $\mathbf{x}$. For our zeroth iteration, we choose the standard *flat start*, zero angle and 1 p.u. magnitude:

$$\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

As our first calculation step, we plug these values for θ and $|V|$ into the power flow equations and into the Jacobian matrix:

$$\mathbf{f}(\mathbf{x}^{(0)}) = \begin{bmatrix} P_{2\text{net}} \\ Q_{2\text{net}} \end{bmatrix} = \begin{bmatrix} |V_2|(10\sin\theta_2) + 2.0 \\ |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix}$$

These numbers look just like the load P_L and Q_L, because the flat start implies that no power is transferred at all (since the voltage magnitude and angle are identical between Bus 1 and 2). Since we want P_{2net} and Q_{2net} to converge to zero, we have a long way to go. Evaluating the Jacobian for $\theta_2 = 0$ and $|V_2| = 1.0$, we get:

$$\mathbf{J}(\mathbf{x}^{(0)}) = \begin{bmatrix} 10|V_2|(\cos\theta_2) & 10\sin\theta_2 \\ 10|V_2|(\sin\theta_2) & -10\cos\theta_2 + 20|V_2| \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Now we are ready to put it all together and compute the next iteration $\mathbf{x}^{(1)}$. Recall that we must invert the Jacobian to compute the correction term for pointing us in the best direction toward the new $\mathbf{x}$ that will shrink $\mathbf{f}(\mathbf{x})$.

$$\mathbf{x}^{(\nu+1)} = \mathbf{x}^{(\nu)} - \mathbf{J}(\mathbf{x}^{(\nu)})^{-1}\mathbf{f}(\mathbf{x}^{(\nu)})$$

Plugging in the values from the $\nu = 0$ iteration, we get:

$$\mathbf{x}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.9 \end{bmatrix}$$

A quick sanity check suggests that a small negative voltage angle θ_2 and a voltage magnitude $|V_2|$ a bit less than 1.0 both make sense, because real and reactive power are flowing from Bus 1 to Bus 2. With this encouraging news, we repeat the procedure of evaluating $\mathbf{f}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$, now at $\mathbf{x}^{(1)}$:

$$\mathbf{f}(\mathbf{x}^{(1)}) = \begin{bmatrix} 0.9(10\sin(-0.2)) + 2.0 \\ 0.9(-10\cos(-0.2)) + 0.9^2(10) + 1.0 \end{bmatrix} = \begin{bmatrix} 0.212 \\ 0.279 \end{bmatrix}$$

We're pleased to note that $\mathbf{f}(\mathbf{x})$ is shrinking. For the Jacobian evaluated at the new $\mathbf{x}$, we have:

$$\mathbf{J}(\mathbf{x}^{(1)}) = \begin{bmatrix} 9(\cos(-0.2)) & 10\sin(-0.2) \\ 9(\sin(-0.2)) & -10\cos(-0.2) + 18 \end{bmatrix} = \begin{bmatrix} 8.82 & -1.986 \\ -1.788 & 8.199 \end{bmatrix}$$

That produces the following for the next iteration:

$$\mathbf{x}^{(2)} = \begin{bmatrix} -0.2 \\ 0.9 \end{bmatrix} - \begin{bmatrix} 8.82 & -1.986 \\ -1.788 & 8.199 \end{bmatrix}^{-1} \begin{bmatrix} 0.212 \\ 0.279 \end{bmatrix} = \begin{bmatrix} -0.233 \\ 0.8586 \end{bmatrix}$$

This time, the corrections to $\mathbf{x}$ were more modest, and evaluating $\mathbf{f}(\mathbf{x})$ already gets us fairly close to zero:

$$\mathbf{f}(\mathbf{x}^{(2)}) = \begin{bmatrix} 0.0145 \\ 0.0190 \end{bmatrix}$$

To take it one more step, the reader is invited to evaluate the Jacobian at $\mathbf{x}^{(2)}$ and confirm that the next iteration comes to

$$\mathbf{x}^{(3)} = \begin{bmatrix} -0.236 \\ 0.8554 \end{bmatrix}$$

which yields

$$\mathbf{f}(\mathbf{x}^{(3)}) = \begin{bmatrix} 0.0000906 \\ 0.0001175 \end{bmatrix}$$

This tiny mismatch should fall below a reasonable error threshold, which means that we can consider $\mathbf{x}^{(3)}$ our solution. Converting θ_2 from radians into degrees, we have solved for

$$V_2 = 0.8554\angle - 13.52^\circ \text{ p.u.} = 0.832 + j0.200 \text{ p.u.}$$

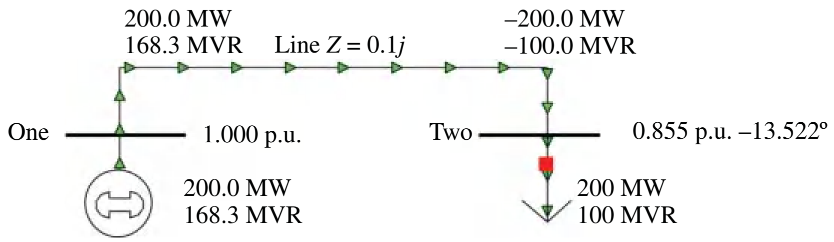


Figure 12.13 Two-bus power flow example solved in *PowerWorld*™.

This solution for the complex bus voltage fully and unambiguously specifies the operating state of the system, since we already know V_1. With this information, it is straightforward to calculate all the other system values.

From Ohm's law, the current on the transmission line is

$$I = \frac{(V_1 - V_2)}{Z} = \frac{0.168 - j0.2}{j0.1} = 2 - j1.68 = 2.61\angle - 40.0^\circ$$

and therefore the power injection at the slack bus is

$$S = I^* V = 2.61\angle 40.0^\circ \ 1.0\angle 0^\circ = 2.61\angle 40.0 = 2.00 + j1.68 \text{ p.u.}$$

With $S_{\text{BASE}} = 100\,\text{MVA}$, this gives

$$P_1 = 200\,\text{MW} \quad \text{and} \quad Q_1 = 168\,\text{MVAR}$$

for the slack bus real and reactive power injections, as illustrated in Figure 12.13. As expected, there are no real losses on this purely inductive transmission line, but the reactive losses are significant. The generator at the slack bus must provide the additional 68 MVAR of reactive power, or else the voltage at Bus 2 could not be maintained.

12.4.5.1 Low-voltage Solution

Although the numerical example above is about as simple as could possibly be contrived, it illustrates the diabolical nonlinearity of the power flow problem: specifying the complex power at Bus 2 does not unambiguously specify the state of the system! In fact, there are two mathematical solutions for V_2 that are equally consistent with the same power values at Bus 2. In the Newton–Raphson method, our choice of starting value $\mathbf{x}^{(0)}$ determines which solution the algorithm will converge to.

The flat start with $|V| = 1.0$ p.u. and $\theta = 0^\circ$ is generally a safe bet, because any solution in that neighborhood is likely to be physically reasonable and operationally stable. It is a conservative assumption that amounts to saying, let's suppose the power flows in the system are quite small, and the voltage differences between buses are not too dramatic.

In the above example, a starting guess with a small voltage magnitude (less than about 0.5) or a large angle (in the tens of degrees) will make the Newton–Raphson algorithm converge to the low-voltage solution $V_2 = 0.26\angle - 49.9^\circ$ p.u. As illustrated in Figure 12.14, this solution involves ridiculously high line losses—in fact, the transmission line becomes the major (inductive) load and sustains the majority of the voltage drop. In mathematical terms, the stipulated power condition at Bus 2 can be equally well met with low voltage and high current, but it would be unrealistic to actually operate the system in this condition in practice (see Section 13.4.5).

To further illustrate the complexity hiding within this toy example, we can map the fate of various starting guesses for $\mathbf{x}$. Figure 12.15 shows the *convergence regions*, where starting values in the dark gray (red in original) region will converge to the high-voltage solution, and the light gray (yellow)

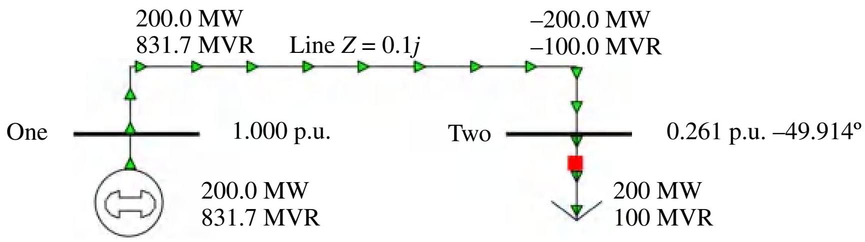


Figure 12.14 The unrealistic low-voltage solution to the two-bus power flow example.

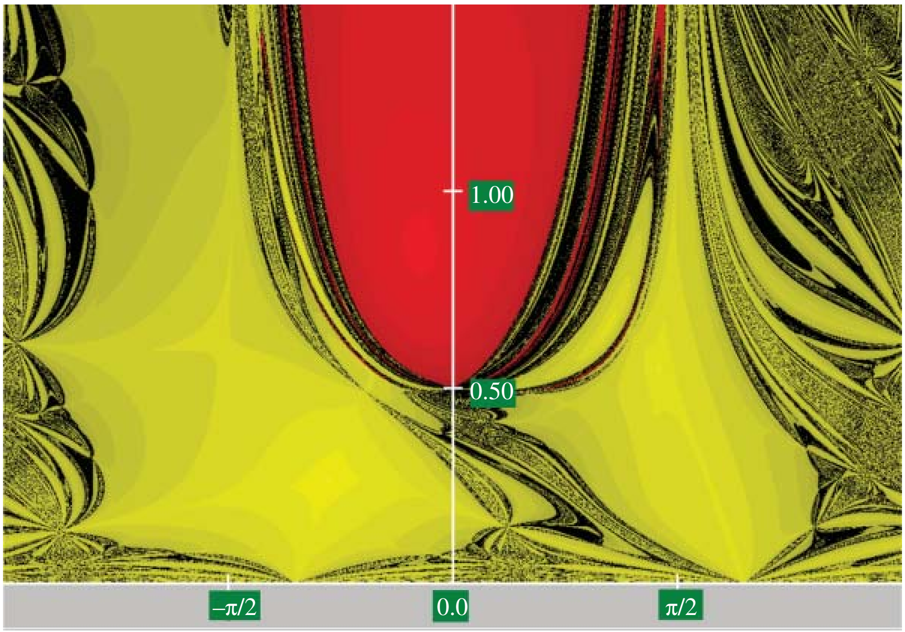


Figure 12.15 Convergence regions for the two-bus power flow example. Source: Courtesy of Tom Overbye.

region to the low-voltage solution. The shading within each region indicates how many iteration steps it takes to converge, with the lightest areas centering on the actual solutions. The dark bands are starting values for which the algorithm will be particularly challenged and tend to bounce back and forth through many iterations. The fractal boundary between the red and yellow regions is especially fascinating: even this simple example exhibits the *chaotic* property, where an arbitrarily small difference in the initial condition can have a dramatic effect on the outcome.[28] And especially in larger, more realistic systems, the solution algorithm may just fail to converge altogether.

Different iterative solution methods have different strengths and weaknesses. The Newton–Raphson method is most popular because it tends to converge relatively fast, and the number of iterations is independent of the number of buses in the network. It also does not matter which bus is selected as the slack bus. However, as we just saw, the results can be very sensitive to the initial guess.

The Gauss–Seidel method, by contrast, iterates on variables sequentially rather than all at once. It is based on Gaussian elimination and substitution, a routine but tedious procedure in linear algebra. There is no Jacobian matrix for the entire system to invert, so each iteration is computationally fast, but many iterations may be needed, and that number increases with the number of buses and branches in the network. The Gauss–Seidel method is less sensitive to the

28 For further reading and inspiration, the classic and gorgeously illustrated hardcover *The Beauty of Fractals* by Heinz-Otto Peitgen and Peter Richter (Springer, 1986) is highly recommended.

initial guess for bus voltages, but more sensitive to the choice of slack bus. Gauss–Seidel sometimes converges where Newton–Raphson does not, and *vice versa*.

12.4.6 Shortcuts

12.4.6.1 Dishonest Newton–Raphson

As noted earlier, inverting the Jacobian matrix, which is used in the iterative procedure to find a better $x + \Delta x$, is the most computation intensive aspect of the Newton–Raphson method. One way to considerably reduce the computation volume and thereby speed up the process is simply not to update the Jacobian with every iteration step. This is known as the "dishonest" or "lazy" Newton–Raphson method. Of course, without using the correct derivatives, we may not be pointed toward the best Δx in each iteration. This brings some risk of making a very bad choice for the next x, but usually it just means it will take more iteration steps to converge.

We can visualize the dishonest Newton–Raphson method in Figure 12.11 as using the slope of $f(x)$ evaluated at x^0 to find not only the next adjustment Δx^0, but reusing the same slope to choose Δx^1 and even subsequent iterations. For the curve in this figure, this would produce an x^2 not quite as close to the true x, but still progressing in the right direction. The computational savings from not repeatedly inverting a matrix are often worth the additional iteration steps.

Note that the accuracy of the solution is not affected by this shortcut, as we would choose the same error threshold for the mismatch; the method will either converge or it won't. Using an outdated version of the Jacobian carries some risk, though, of derailing the solution progress and causing it to fail to converge, or sending it so far astray that it converges to the wrong solution.

Although this section is titled "shortcuts," "clever detours" might be a better analogy. Using an inaccurate Jacobian is a bit like giving driving directions to the same destination along a detour route with fewer stop signs: we expect that even though it is less direct, it will be faster. Also, we hope that the driver won't get lost.

12.4.6.2 Decoupled Power Flow

Instead of neglecting to update the Jacobian altogether, another way to save computation time is to use a simpler, approximate version of it that is more sparse (i.e., it contains many zero entries) and therefore faster to invert. Specifically, we may choose to decouple real power from voltage magnitude and reactive power for voltage angle, based on the common rule that real power flow relates mainly to differences in voltage angle, and reactive power mainly to differences in voltage magnitude. This rule derives from the key assumption that the reactive properties of transmission lines tend to outweigh their resistance, and also from the assumption that angle differences tend to be small.

Mathematically, we are claiming that two of the four distinct partitions of the Jacobian matrix, which contain the partial derivatives of P or Q with respect to θ and V, can be safely neglected. Without inserting any numerical values, we can examine the mathematical form of these partial derivatives and conclude which terms ought to be large and which ought to be small, based on the two assumptions just cited. What we will find is that the dependence of real power on voltage angle, $\partial P/\partial\theta$, ought to be substantial, while the dependence of real power on voltage magnitude, $\partial P/\partial V$, ought to be much smaller by comparison. For reactive power, we will find the opposite: $\partial Q/\partial\theta$ ought to be small, but $\partial Q/\partial V$ should be substantial.

Let's walk through the process of examining the derivatives for a sample bus pair, 2 and 3 (to be general, we would write i and k). Since we are interested in power flow from one bus to another, we will consider only the derivatives with unequal indices (such as $\partial P_2/\partial\theta_3$, as opposed to $\partial P_2/\partial\theta_2$).

First, we write out the derivatives $\partial P_2/\partial V_3$ and $\partial Q_2/\partial \theta_3$, which we will show to be small:

$$\frac{\partial P_2}{\partial V_3} = |V_2|[g_{23}\cos(\theta_2 - \theta_3) + b_{23}\sin(\theta_2 - \theta_3)]$$

$$\frac{\partial Q_2}{\partial \theta_3} = |V_2||V_3|[g_{23}\cos(\theta_2 - \theta_3) + b_{23}\sin(\theta_2 - \theta_3)]$$

We now observe the implications of our two assumptions. If a transmission link's reactive effects substantially outweigh its resistive effects, this means its conductance g_{23} is a much smaller number than its susceptance b_{23}.[29] This makes the cosine terms small, as they are multiplied by the g's. The sine terms are multiplied by the b's, so they could be substantial based on that consideration. However, the sine terms are also small, for a different reason: if the voltage angle difference $\theta_2 - \theta_3$ between buses is small, then the sine of $(\theta_2 - \theta_3)$ is small. Thus, each of the preceding derivatives consists of the sum of two small terms, and we might deem them small enough to be negligible.

By contrast, consider the derivatives $\partial P_2/\partial \theta_3$ and $\partial Q_2/\partial V_3$. Here, the g's multiply the sine terms, so these terms vanish on both accounts. But this leaves us with the cosine terms multiplied by b's, neither of which are small (since the cosine of a small angle is nearly 1).

$$\frac{\partial P_2}{\partial \theta_3} = |V_2||V_3|[g_{23}\sin(\theta_2 - \theta_3) - b_{23}\cos(\theta_2 - \theta_3)]$$

$$\frac{\partial Q_2}{\partial V_3} = |V_2|[g_{23}\sin(\theta_2 - \theta_3) - b_{23}\cos(\theta_2 - \theta_3)]$$

Indeed, if we consider the sine terms negligible and the cosine roughly equal to 1, we obtain the following approximations:

$$\frac{\partial P_2}{\partial \theta_3} \approx -|V_2||V_3|b_{23}$$

$$\frac{\partial Q_2}{\partial V_3} \approx -|V_2|b_{23}$$

Thus, the partial derivatives $\partial P_2/\partial \theta_3$ and $\partial Q_2/\partial V_3$ make up the "meat" of the Jacobian matrix. By assuming the small derivatives to all be negligible, we set two of the four partitions of the Jacobian matrix to zero. This sparsity makes it much faster to invert.

As in the dishonest Newton–Raphson method, using a slightly inaccurate Jacobian means we should expect to need more iterations, and we are trading some risk for speed. What is different in the decoupled power flow method is that the simplifications have a physical rationale. If the decoupling was a reasonably good assumption for the network, then we should still be headed in the right direction at each iteration, and should be quite confident to converge to the solution.

12.4.6.3 Fast-Decoupled Power Flow

An even more radical simplification of the Jacobian matrix is possible, called *fast-decoupled power flow*. Here we make a third assumption: that the voltage magnitude profile throughout the system is flat, meaning that all buses are very near the same voltage magnitude (i.e., the nominal system voltage, 1.0 p.u.). We then observe the effect of this assumption, combined with the previous two assumptions about transmission lines and voltage angles, on the Jacobian matrix. By a process of approximation and cancellation of terms, the assumption of a flat voltage profile leads to a much handier version of the Jacobian, including a portion that stays the same during each iteration

29 Recall that $G = R/Z^2$, so that when R approaches zero and there is only reactance ($Z \approx X$), $G \approx 0$ as well.

and therefore saves even more computational effort.[30] Again, this should affect only the process of converging on the correct solution, not the solution itself. If the simplifying assumptions were reasonable—in other words, if the simplified derivatives did not lead us in a grossly wrong direction—the computation is vastly sped up.

Note that the power flow solution obtained by the fast-decoupled algorithm will expressly produce a certain profile of voltage angle and magnitude throughout the system that contradicts our literal assumption that these profiles would be flat. Thus, we should think of the flat profiles as merely a procedural crutch along the way to discovering what the true profiles are. The reason we can get away with this apparent conflict is that the iteration process is self-correcting in nature. We can thus incorporate a statement that we know to be false when taken literally (i.e., the voltage profiles are exactly flat) into the directional guidance for our next iteration (the derivatives in the Jacobian matrix), without contradicting the solution at which we ultimately arrive.

Likewise, note the apparent contradiction between the existence of line losses, which can result only from line resistance, and the approximation that the conductances are negligible. Again, the simplifying assumption of ignoring the g's is only a crutch for the process of approaching the correct power flow solution, and the solution itself will be consistent with the actual, nonzero values of conductance and resistance. This solution combined with the explicit resistance values—which, for this purpose, are anything but negligible—then yields the losses for each transmission link.

12.4.6.4 DC Power Flow

The ultimate simplification of a.c. power flow analysis has the misleading name of "DC power flow." Here we not only assume decoupling and set voltage magnitudes to 1.0 p.u., but we completely ignore reactive power and voltage magnitude to focus exclusively on real power, calculating a solution only in terms of voltage phase angle. The reason it is called DC is that we are dealing with only a single state variable at each node instead of two, as would be the case in a direct-current network, even though the variables still describe an alternating voltage and current.

DC power flow differs from the other shortcuts in that it produces a set of linear equations that can be solved without iteration (just like an actual direct-current network), so it very reliably produces a solution—but not the correct one. The DC power flow solution can only be approximate because the method never computes the actual voltage magnitudes, which are not exactly 1.0 p.u. However, it is by far the fastest way to get a rough handle on a large network. For systems with a reasonable voltage magnitude profile, DC power flow offers an adequate quick overview. This is especially useful if many repetitions of the power flow solution process are required.

One important such application is *contingency analysis* (Section 13.3), where power flow is solved for many different scenarios such as the sudden loss of a transmission line or a generator. This type of analysis is not concerned with getting a highly accurate power flow solution, but with the

30 In the preceding discussion of partial derivatives we have only considered pairs of variables from neighboring buses, that is, the rate of change of real or reactive power at bus i (in our example, 2) with respect to voltage angle or magnitude at one neighboring bus k (in our example, 3). Having chosen that neighboring bus $k = 3$ and taking the derivative, we were able to cheerfully drop the summation sign with all its various k's, since our rate of change is independent of what happens at all these other buses. However, the Jacobian matrix also contains partial derivatives of power with respect to voltage at the same bus; for example, $\partial P_2/\partial\theta_2$. These terms, which appear along the diagonal of the matrix, are the ones affected by the assumption of a flat voltage profile. These diagonal terms look different in that they retain the summation over all the other k's. They succumb, however, to approximation and cancellation of various b's, leading to vastly simplified expressions. A thorough discussion appears in Arthur Bergen, *Power Systems Analysis* (Englewood Cliffs, NJ: Prentice Hall, 1986).

question of whether a given contingency might result in an egregious violation. If DC power flow raises a flag, that particular scenario can then be studied more carefully.

DC power flow is a true shortcut, which leads not quite exactly to the destination. By analogy, this is like driving directions to a store in a shopping mall that leave you on the wrong side of a parking lot barrier: depending on the nature of your errands, sometimes that's okay.

Besides assuming that voltage magnitudes are 1.0 p.u., DC power flow also assumes that angle differences are small and that line resistances are negligible (i.e., the assumptions for decoupling).[31] These assumptions produce a linear simplification of the real power flow from Eq. (12.1),

$$P_{ik} \approx \frac{|V_i||V_k|}{x_{ik}} \sin(\theta_k - \theta_i) \approx \frac{1}{x_{ik}}(\theta_i - \theta_k) \tag{12.3}$$

where x_{ik} is the inductive reactance of the lossless line connecting buses i and k, and P_{ik} is positive flowing from bus i to k.

The inverse reactance x can also be written as the admittance or susceptance, b. In fact, the entire admittance matrix $\mathbf{Y}_{\text{bus}}$ simplifies into a susceptance matrix if all the entries are purely imaginary, with $y_{ik} = b_{ik}$.[32] However, $\mathbf{B}$ is defined as the imaginary components of $\mathbf{Y}_{\text{bus}}$ without the row and column corresponding to the slack bus, since we do not solve for the state variables at that bus.

With $\mathbf{B}$, we can write a concise linear equation for (approximate) real power flow across the entire network, where $\boldsymbol{\theta}$ includes all the bus voltage angles. A negative sign in front of $\mathbf{B}$ accounts for the ordering of the bus angles and admittances in the matrix operations.[33] As before, power is defined as positive when injected into the network (i.e., generated at the bus).

$$\mathbf{P} = -\mathbf{B}\,\boldsymbol{\theta} \quad \text{or} \quad \boldsymbol{\theta} = -\mathbf{B}^{-1}\,\mathbf{P} \tag{12.4}$$

Note that the $\mathbf{B}$ matrix only needs to be inverted once and that like the admittance matrix $\mathbf{Y}_{\text{bus}}$ it is sparse, so the computational effort should be minimal. The reason we can get away with linearizing the power flow problem in this way is that we are neglecting losses, since there is no resistance in the network as modeled. Consequently, the total amount of power to be generated throughout the system—just the sum of net demand—is known *a priori*, and no iteration is needed.

Out of curiosity, could we reduce the dimensionality of the a.c. power flow problem in the opposite way, by considering only reactive power and voltage magnitude? No, because the total amount of Q that needs to be generated throughout the system depends on reactive I^2X losses on the transmission lines. The analogous simplification would require us to ignore reactance and consider only resistance—but that assumption would seriously disagree with the physical reality of transmission lines, where it is generally true that $X \gg R$.

Example

Consider the three-bus system in Figure 12.16, with two generator buses and one load bus. Bus 1 (generator) is the slack bus. Bus 2 (generator, minus some load) and Bus 2 (load only) are both modeled as PQ buses.[34] Our objective is to estimate the power flow on each of the three lines.

Bus 3

31 Transformer tap settings are another detail that is ignored.
32 To review, each diagonal element b_{ii} equals the sum of line admittances connected to Bus i, and each off-diagonal element b_{ik} is the negative admittance between Bus i and Bus k.
33 In an alternative convention, the minus sign in Eq. (12.4) is subsumed within the definition of $\mathbf{B}$, in which case all the $b_{ik} = -y_{ik}$.
34 Bus 2 might instead be a PV bus, in which case we would be given the bus voltage magnitude instead of reactive power injection. For DC power flow, this wouldn't matter at all.

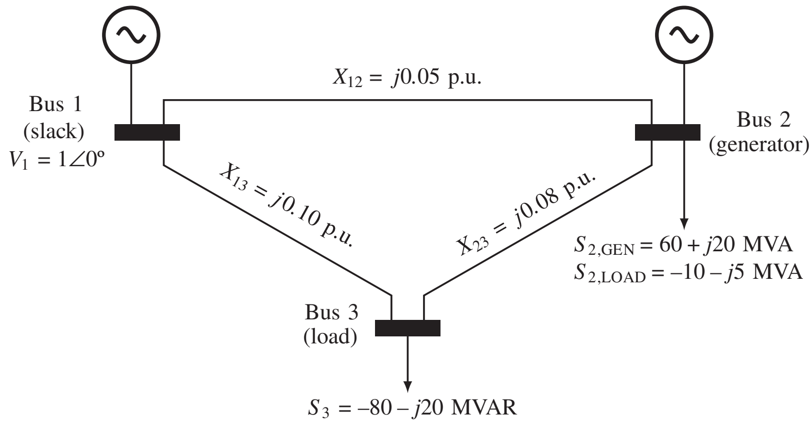


Figure 12.16 Three-bus network.

The power generation and demand are given as follows:

Bus Number	Bus Type	P_G	Q_G	P_D	Q_D
1	Slack	Unknown	Unknown	0	0
2	PQ	60 MW	20 MVAR	10 MW	5 MVAR
3	PQ	0	0	80 MW	20 MVAR

The branch admittances in per-unit are the inverse of the impedances shown in Figure 12.16: $y_{12} = -j20$, $y_{13} = -j10$, and $y_{23} = -j12.5$. The bus admittance matrix is

$$\mathbf{Y}_{\mathrm{bus}} = \begin{bmatrix} -30 & 20 & 10 \\ 20 & -32.5 & 12.5 \\ 10 & 12.5 & -22.5 \end{bmatrix}$$

Since Bus 1 is the slack bus, we have voltage angles at two buses to solve for, θ_2 and θ_3. Using a base power $S_{\mathrm{base}} = 100$ MVA, the net power injections at these buses are $P_2 = 0.5$ and $P_3 = -0.8$ p.u. The information about Q will be ignored entirely.

We are thus solving a linear system with two input and two output variables. Therefore, the **B** matrix has to be reduced to the same 2×2 size (in linear algebra terms, it must be of the appropriate *rank*). This is done by simply deleting the row and column corresponding to the slack bus. In our example, we delete the first row and first column of $\mathbf{Y}_{\mathrm{bus}}$.[35]

Equation (12.4) then becomes:

$$\begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = -\mathbf{B}^{-1}\,\mathbf{P} = \begin{bmatrix} 32.5 & -12.5 \\ -12.5 & 22.5 \end{bmatrix}^{-1} \begin{bmatrix} 0.50 \\ -0.80 \end{bmatrix} \rightarrow \frac{60-10}{100}$$

The easy-to-find, exact, and guaranteed unique solution (because **B** is invertible) to the wrong problem (because it does not quite represent the system under consideration) is

$$\begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0.002173 \\ -0.03435 \end{bmatrix} \text{rad} = \begin{bmatrix} 0.124° \\ -1.968° \end{bmatrix}$$

35 Note that the information about impedances connected to the slack bus is not lost, but is still captured in the self-admittances for Buses 2 and 3. Without deleting the slack bus row and column, the matrix would contain redundant information and could not be used to solve a 3×3 system of equations. We would discover this as soon as we tried to invert the matrix.

These results pass an initial sanity test in that the angle at Bus 3 should be moderately negative. The direction of power flow between Buses 1 and 2 is not obvious *a priori*, so a small angle difference is plausible.

From the estimated voltage angles, we can directly calculate power flows on each line, using Eq. (12.3) with angles in radians and $\theta_1 = 0$:

$$P_{12} \approx y_{12}(\theta_1 - \theta_2) = 20 \cdot (-0.00217) = -0.0435 \text{ p.u.}$$
$$P_{13} \approx y_{13}(\theta_1 - \theta_3) = 10 \cdot 0.03435 = 0.3435 \text{ p.u.}$$
$$P_{23} \approx y_{23}(\theta_2 - \theta_3) = 12.5 \cdot 0.03652 = 0.4565 \text{ p.u.}$$

We read this as 4.35 MW flowing from Bus 2 to Bus 1, 34.35 MW flowing from Bus 1 to Bus 3, and 45.65 MW flowing from Bus 2 to Bus 3. The numbers add up correctly for each bus, including the slack bus power injection that we already knew to be 30 MW (since the network is lossless).

12.5 Applications

Power flow analysis is a fundamental and essential tool for operating a power system, as it answers the basic question: What happens to the overall *state* of the system if generation and load change in certain ways, at certain locations? This question may be posed in the context of either day-to-day operations or longer-term planning.

In the short run, a key part of a system operator's responsibility is to approve generation schedules that have been prepared on the basis of some economic considerations—whether by central planning or by competitive bidding—and scrutinize them for technical feasibility. This assessment hinges on power flow studies to predict the system's operating state under a proposed dispatch scenario. If the analysis shows that important constraints such as line loading limits would be violated, the schedule is deemed infeasible and must be changed.

Even with feasible schedules in hand, reality does not always conform to plans, requiring operators to monitor any changes and, if necessary, make adjustments to the system in real time. Power flow analysis is the only comprehensive way to predict the consequences of changes such as increasing or decreasing generation levels, increasing or decreasing loads, or switching transmission links and assessing whether such changes are safe or desirable for the system. Specifically, operators need to know impacts of any actions on voltage levels (are they within proper range?), line flows (are any thermal or stability limits violated?), line losses (are they excessive?), and security (is the operating state too vulnerable to individual equipment failures?). Similarly, power flow analysis is a fundamental tool in the planning context to evaluate changes to generation capacity or the transmission and distribution infrastructure.

12.5.1 Optimal Power Flow

Sometimes it is necessary to compare several hypothetical operating scenarios for the power system to guide operating and planning decisions. Specifically, one often wishes to compare and evaluate different hypothetical dispatches of generation units that could meet a given loading condition. Such an evaluation is performed by an *optimal power flow* (OPF) program, whose objective is to identify the operating configuration or "solution" that best meets a particular set of criteria. These criteria may include the total cost of generation, transmission line losses, and various requirements concerning the system's security, or resilience with respect to disturbances.

An OPF algorithm consists of numerous power flow analysis runs, one for each hypothetical dispatch scenario that could meet the specified load demand without violating any constraints. Clearly, this makes OPF vastly more computation-intensive than just a basic power flow analysis. The output of each individual power flow run, which is a power flow solution in terms of bus voltage magnitudes and angles, is evaluated according to one or more criteria that can be wrapped into a single quantitative metric or *objective function*—for example, the sum of all line losses in megawatts, or the sum of all generating costs in dollars when line losses are included. The OPF program then devises another scenario with different real and reactive power contributions from the various generators and performs the power flow routine on it, then another, and so on until the scenarios do not get any better and one is identified as optimal with respect to the chosen metric. This winning configuration with real and reactive power dispatches constitutes the output of the OPF run. OPF solutions may then provide guidance for on-line operations as well as generation and transmission planning.

Especially for applications in a market environment, where planning and operating decisions may have sensitive economic or political implications for various parties, it is crucial to recognize the inherently subjective nature of OPF. Power flow analysis in and of itself answers a question of physics. By contrast, OPF answers a question about human preferences, coded in terms of quantitative measures. Thus, what is found to constitute an "optimal" operating configuration for the system depends on how the objective function is defined, which may include the assignment of prices, values, or trade-offs among different individual criteria. In short, "optimality" does not derive from a power system's intrinsic technical properties, but from external considerations.

It is also important to understand that the translation of an OPF solution into actual planning and operating decisions is not clear-cut and has always involved some level of human judgment. For example, the computer program may be too simplistic in its treatment of security constraints to allow for sensible risk trade-offs under dynamically changing conditions, which then calls for some engineering judgment to adapt the OPF recommendation in practice.

At the same time, the computational process is already complex enough that different OPF program packages may not offer identical solutions to the same problem. Therefore, the output of power flow analysis including OPF constitutes advisory information rather than deterministic prescriptions. Indeed, the complexity of the power flow problem underscores the difficulty of managing power systems through static formulas and procedures that would lend themselves to automation, especially if a system is expected to perform near its physical limits.

12.5.2 State Estimation

The set of voltage magnitudes and angles at every network node define the *state* of the system. Power flow analysis is about identifying that state based on a set of input variables, specifically power injections at the various nodes or buses. In practice, however, there may not be sufficiently recent or reliable data available for every bus to determine the system state in a time frame relevant for operational decisions (say, on the order of minutes).

State estimation is the process of combining available information to propose a solution for the system state that best fits the available data. These data typically include direct physical measurements, pseudo-measurements (i.e., values that are known even if not physically measured, such as zero power injection at a node with neither generation nor load connected to it), a network model with impedances and connectivity, Ohm's law and Kirchhoff's laws. By corroborating information across multiple sources, state estimation can make up for lacking data at a particular node. Moreover, it recognizes that any given data point could be erroneous, and it can assign different weights or credibility to different reported values. For example, a measurement may not be properly time-aligned, the instrument may not be calibrated accurately, or there could be noise, delay or intermittency in communication—and some reporting locations might be chronically more suspect than others.

In sum, a state estimation algorithm seeks the state of the system that is most consistent and plausible in view of all the available information. One common approach is the *least squares* or *weighted least squares* fit, used in many other applications of statistical analysis. In essence, it asserts that the solution most consistent with the data is the one that minimizes the collective discrepancies between the empirical measurements and the points proposed by the solution, where discrepancy is defined as the squared difference.

The mathematical problem statement distinguishes the true state vector $\mathbf{x}$ (the set of all bus voltage magnitudes and angles), the function $\mathbf{h}(\mathbf{x})$ of the true state vector that specifies what all the correct measurements should be, and the set of empirical measurements $\mathbf{z}$ that can be expressed as the correct measurements plus an error term $\boldsymbol{\varepsilon}$. Thus, for the ith node in the network, we would write

$$z_i = h_i(\mathbf{x}) + \varepsilon_i$$

Note that each h_i term is a function not only of the state variables x_i at the local bus, but the state $\mathbf{x}$ of the entire network. In this way, $\mathbf{h}(\mathbf{x})$ captures the laws of physics: for example, you cannot reasonably measure a voltage at one bus that is terribly different from its neighbor, or a current that doesn't obey Ohm's law.

If the errors are random and follow a Gaussian distribution,[36] the maximum likelihood estimate of the true state based on m measurements is the solution to the least squares problem: find $\mathbf{x}$ that satisfies

$$\min \sum_{i=1}^{m} \varepsilon_i^2 = \min\ [\mathbf{z} - \mathbf{h}(\mathbf{x})]^{\mathrm{T}}[\mathbf{z} - \mathbf{h}(\mathbf{x})]$$

(where multiplying by the transpose of the vector is simply the proper way to square it). The weighted least squares formulation includes an *a priori* emphasis (captured by the vector $\mathbf{W}$) on measurements known to be more accurate,

$$\min \sum_{i=1}^{m} \varepsilon_i^2 = \min\ [\mathbf{z} - \mathbf{h}(\mathbf{x})]^{\mathrm{T}}\mathbf{W}[\mathbf{z} - \mathbf{h}(\mathbf{x})]$$

The solution can be found by iterative computation, taking successive guesses at $\mathbf{x}$ and improving the guess with each iteration toward reducing the error.

One important observation is that state estimation requires some redundancy of measurement. If there are n nodes in the network, we need $m > n$ data points for the minimization problem to have a unique solution. A network for which sufficient information is available to produce a unique state estimate is called *observable*.

Many modern control rooms have state estimator applications that issue an updated report every few minutes. Besides providing grid operators the closest thing to a full view of the system in real-time, state estimation can be used to verify input information. Through its iteration process, the state estimation algorithm can identify if a particular data source makes especially egregious

36 In statistics, the bell-shaped Gaussian distribution describes a collection of observations that vary due to random chance.

error contributions. Such an observation can flag bad measurement data as well as bad assumptions about impedance parameters, topology, or other modeling errors. The state estimator may even fail to converge on a solution altogether, in a sign that something is very wrong.

For example, suppose a transmission line tripped offline and is no longer connected, but this event was not reported. Because the connectivity assumed by the state estimator no longer reflects the actual physical relationship among the measurement points, a large error between the empirical and the presumed correct measurements would result, causing a failure of the state estimator to converge. This would in turn trigger an alarm, prompting operators to check for the source of the problem or inconsistency.[37]

In commercial practice to date, state estimation has been used almost exclusively at the transmission level, although growth in active distributed resources (Section 15.2) motivates the development and adoption of distribution state estimation tools. Distribution systems pose several unique challenges in this context. They are more difficult both to model and to observe, simply because there are so many connecting points for loads, while measurements are scarce (especially in real-time). Also, distribution state estimation requires a three-phase line model that accounts explicitly for each phase and all the mutual impedances, since lines are untransposed (Section 7.2.2) and load imbalance is more pronounced. Yet distribution system models at this level of granularity are notoriously inaccurate, owing to the vast number of variables and the myriad ways in which the real-world condition of the network might change but go unreported. Finally, for the purpose of state estimation, the radial topology of distribution systems—which otherwise facilitates both operation and analysis—is a disadvantage, because it affords less redundancy from Kirchhoff's laws for mutual corroboration of measurements.

12.6 LinDistFlow

Some useful simplifications of power flow analysis can be made for the special case of radial networks, which includes most distribution systems. The *LinDistFlow* equations provide an approximate overview of the relationships between power flow and voltage drop along a radial line.[38]

The full power flow equations as presented earlier in this chapter specify the relationship between real and reactive power transfer, line impedance, and voltage magnitude and phase angle at either end. As the reader surely appreciates by now, these equations are nonlinear, cumbersome to manipulate, and not readily intuitive. LinDistFlow, by contrast, can help build intuition about how voltage phasors relate to power flows, with the aid of a few simplifications. The resulting approximate equations, even though they don't give exact solutions, provide a general sense of how these quantities can be expected to vary. In particular, the effect of capacitors or distributed generation on voltage drop can be helpfully approximated with LinDistFlow.

Again, this analysis applies only to radial systems, or radial regions within a network. Specifically, this means that any circuit branch to be analyzed is a single branch connecting two nodes (labeled

37 When the above scenario occurred in Ohio on August 14, 2003, this alarm happened to be disabled and operators remained unaware of a crucial transmission line trip in their area—one of several factors leading to the Northeast Blackout. See *Final Report on the August 14, 2003 Blackout in the United States and Canada: Causes and Recommendations*, U.S.-Canada Power System Outage Task Force, 2004.

38 The terminology and derivation originates from a seminal paper by Mesut Baran and Felix Wu, "Network Reconfiguration in Distribution Systems for Loss Reduction and Load Balancing," *IEEE Transactions on Power Delivery* 4(2), 1401–1407, 1989. The presentation in this section is based on work by Roel Dobbe, Michael Sankur, and Dan Arnold.

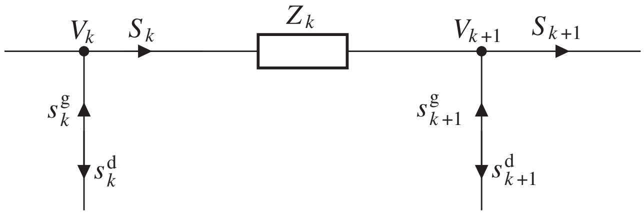


Figure 12.17 Radial distribution branch for the derivation of LinDistFlow equations.

in Figure 12.17 as k and $k+1$), with no other parallel circuit branch or "back way" connecting those two points. Our objective is to write an expression that relates complex power $\mathbf{S}_k$ flowing from node k to node $k+1$ across the line impedance $\mathbf{Z}_k = R + jX$, and the voltage phasors $\mathbf{V}_k$ and $\mathbf{V}_{k+1}$ at the two nodes.[39] For generality, we also include power demand and generation s^d and s^g that result in some net power injection at each node, but these will simply add to the power entering and leaving each node to other directions and be subsumed within $\mathbf{S}_k$ or $\mathbf{S}_{k+1}$.[40] What matters here is that we can isolate the power flow $\mathbf{S}_k$ on the single branch connecting any two nodes in question. We will simply let $k = 1$ for the following discussion.

Let us present the LinDistFlow equations first and appreciate the insights they convey, and then return to their derivation. The linearized approximation for voltage magnitudes[41] V_1 and V_2 at adjacent nodes in a radial system is

$$V_1^2 - V_2^2 \approx 2(RP + XQ) \tag{12.5}$$

Note that Eq. (12.5) describes a linear relationship, even though the voltage magnitudes appear squared, because the square terms are completely isolated and we can simply redefine a variable such as $E_1 = V_1^2$.[42] The main assumption for this approximation is that power losses are negligible. This will be closer to true when impedances and power flows are small.

A key point illustrated by Eq. (12.5) has to do with the relative importance of real and reactive power for voltage magnitude drop, depending on the relative dominance of resistance or reactance on the line. Recall the general rule that voltage drop is more sensitive to reactive power than real power transfer. In truth, this rule applies only to lines that are mostly inductive, and especially when $X \gg R$.[43] That condition happens to hold true for almost all transmission lines and, to a lesser degree, for many distribution lines.

Equation (12.5) also tells us, though, that in case a line's resistance is comparable to its reactance, then real power flow will contribute appreciably to the voltage drop. The relative impact of P and Q could even be reversed if, hypothetically, a line's resistance exceeded its reactance. In practice, the impact of distributed generation on feeder voltage depends significantly on the line's X/R ratio. In any case, because of the plus sign in Eq. (12.5), both real and reactive power flow from node 1 to 2 are associated with the voltage magnitude decreasing from 1 to 2.

---

39 It turns out that it doesn't matter if we label the power flow S_k or S_{k+1}. The reason is twofold: In the voltage magnitude equation, we will assume that there are no losses. Therefore, power out of node 1 equals power into node 2, and the subscript assignment is a matter of stylistic preference. In the angle equation, we have a difference of phase angles, which will be the same regardless of whether we take node 1 or 2 as the reference.

40 The generality is important in case we want to string together multiple branches and use consistent labels to account for all nodes in relation to each other. This will work as long as the system is radial.

41 In this section, we drop the absolute value signs and use boldface notation for phasors.

42 Solving for V^2 still leaves two mathematically possible solutions for V, but one is negative and therefore physically implausible.

43 Note that this condition and conclusion are perfectly consistent with our earlier analysis of the full nonlinear power flow equations and the Jacobian matrix. LinDistFlow just makes the relationship much easier to see.

The corresponding relationship for voltage phase angles can be shown at any of three successive levels of simplification, the last two of which qualify as linear equations:

$$\sin(\delta_1 - \delta_2) = \frac{1}{V_1 V_2}(XP - RQ) \tag{12.6}$$

$$\delta_1 - \delta_2 \approx \frac{1}{V_1 V_2}(XP - RQ) \tag{12.7}$$

$$\delta_1 - \delta_2 \approx (XP - RQ) \tag{12.8}$$

The first form, Eq. (12.6), is exact and requires no specific assumptions other than the radial topology. For the case where $X \gg R$, the RQ term becomes negligible, and the expression reduces to the common approximation for real power flow versus angle (as in Section 7.3.2).

Because voltage phase angle separations in distribution systems tend to be small, we can get the linearized equation (12.7) by making the small-angle approximation $\sin\theta \approx \theta$ (where θ is expressed in radians, not degrees), which is usually excellent.[44] Getting rid of the trigonometric function is a huge advantage, especially if we want to use these variables in some linear algebra formulation.

A further simplification, shown in Eq. (12.8), assumes that the voltage magnitudes are both equal to 1.0 p.u. (Section 8.7), and the fraction with V_1 and V_2 simply goes away. This allows us to make a statement about voltage phase angles without any information about the magnitudes, which can be very helpful.

This "well-behaved voltage" approximation will tend to introduce a greater numerical error than the small-angle approximation, but it won't qualitatively affect the relationship of variables to each other. Typical voltage drops in distribution systems are on the order of single-digit percent, as the general operational standard calls for maintaining voltage magnitudes in the range of 0.95 to 1.05 p.u. (95% to 105% of nominal). In the worst case that still meets this standard, we would have about a 10% error:

$$\frac{1}{V_1\ V_2} = \frac{1}{0.95 \cdot 0.95} = 1.108 \quad \text{or} \quad \frac{1}{V_1\ V_2} = \frac{1}{1.05 \cdot 1.05} = 0.907 \text{ p.u.}$$

The relationships described by the pair of LinDistFlow equations for voltage magnitude and angle are especially useful in the context of a modern system with distributed generation, where both real and reactive power might flow in either direction. For example, there may be very little difference in voltage magnitude due to opposing effects of P and Q flow, while the current is greater than expected. If current measurements are not available, complementing voltage magnitude information with voltage angle measurements would provide conclusive information about whether P and Q are both small or acting in opposition.

The associations of voltage magnitude and angle profiles with positive, zero, or negative real and reactive power flows, and the dependence of this association on the impedance of the line, are illustrated in Figure 12.18 for two special cases, with $X \gg R$ ($R \approx 0$) and $R \approx X$. Figure 12.18a illustrates the familiar result from the transmission context: namely, that real power flows from greater to smaller voltage phase angle, and reactive power flows from greater to smaller voltage magnitude. In distribution systems, however, line resistance may be much more significant, and $X \gg R$ may be a poor approximation. Figure 12.18b illustrates the special case where resistance and reactance are equal. The result is a clockwise rotation of the iso-V and iso-δ lines by 45°.

One intuitive insight conveyed by this qualitative analysis is the crucial importance of the X/R ratio for characterizing distribution lines, and for predicting the impacts of distributed generation

44 For example, with a phase angle difference of 1°, we have $\sin 1° = \sin\ 0.017453\,\text{rad} = 0.017452$ and the approximate equality is good to four significant figures.

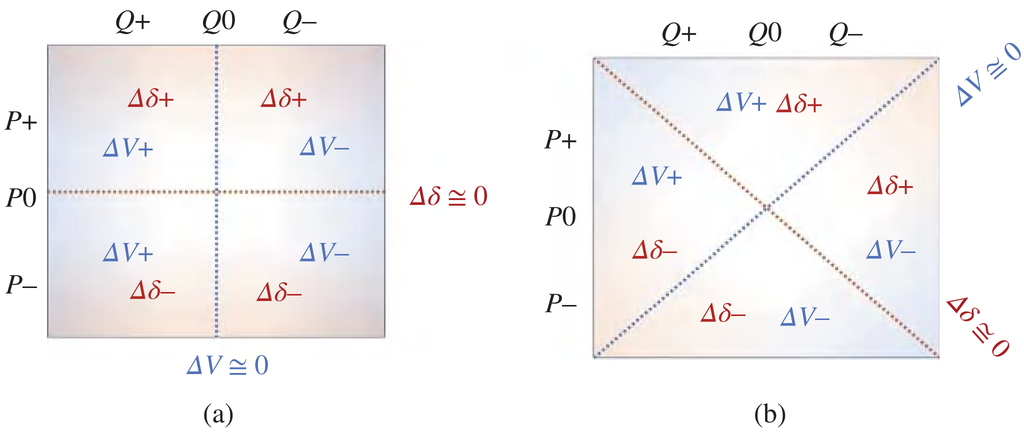


Figure 12.18 Qualitative association of voltage drop $\Delta V = V_1 - V_2$ and angle difference $\Delta\delta = \delta_1 - \delta_2$ with the direction of real and reactive power flow, based on Eqs. (12.5) and (12.8). $P+$ and $Q+$ indicate positive power flowing from Node 1 to 2, in the sense of S_k in Figure 12.17. $\Delta V+$ indicates that $|V_1| - |V_2| > 0$, and $\Delta\delta+$ that $\delta_1 - \delta_2 > 0$. (a) When $X \gg R$, the sign of $\Delta\delta$ is aligned with real and ΔV with reactive power flow. (b) When $X \approx R$, the relationship rotates by 45°.

(Section 15.2.2). Another observation is that for managing voltage on radial distribution feeders, real and reactive power control can play complementary roles. This motivates the dual option for both volt-VAR and watt-VAR droop curves for inverter control (Section 14.4.3).

12.6.1 Derivation

To derive LinDistFlow equations, we first write down Kirchhoff's voltage law (KVL), stating that one nodal voltage is given by the sum of the other, plus the voltage drop between them:

$$\mathbf{V_1} = \mathbf{V_2} + \mathbf{IZ}$$

It does not matter which node we label as 1 or 2, as long as the current is defined as positive in the direction consistent with KVL. We use boldface notation here to remember that these are all complex phasor quantities.

Our objective is to obtain separate, real expressions for voltage magnitudes and angles, respectively. The general trick is to multiply complex expressions by their complex conjugate.

First, for the voltage magnitude equation, we multiply both sides of the equation by $\mathbf{V_1^*}$:

$$\begin{aligned}\mathbf{V_1V_1^*} &= (\mathbf{V_2} + \mathbf{IZ})(\mathbf{V_2^*} + \mathbf{I^*Z^*})\\ &= \mathbf{V_2V_2^*} + \mathbf{V_2I^*Z^*} + \mathbf{IZV_2^*} + \mathbf{IZI^*Z^*}\end{aligned}$$

Initially, the result of cross-multiplying these terms looks confusing. But realizing that the product of a complex number and its complex conjugate is just the magnitude squared, and employing the identity $ab^* + a^*b = 2\,\mathrm{Re}\{ab^*\}$, we get

$$V_1^2 = V_2^2 + 2\,\mathrm{Re}\{\mathbf{V_2I^*Z^*}\} + I^2Z^2$$

The last term represents a measure of line losses multiplied (again) by impedance. We can reasonably expect this term to be small (since the line impedance should be a small number, especially when squared) and will choose to neglect it; this is the essence of the linearization. Consequently, the equality is only approximate from here on.

Rearranging terms and substituting complex power $\mathbf{S} = \mathbf{I^*V_2}$ we get

$$V_1^2 - V_2^2 \approx 2\,\mathrm{Re}\{\mathbf{SZ^*}\}$$

which expands into

$$V_1^2 - V_2^2 \approx 2\,\mathrm{Re}\{(P + jQ)(R - jX)\} = 2\,(RP + XQ)$$

where taking the real part retains half the pairings from the cross-multiplication to yield Eq. (12.5).

For the derivation of the angle equation, we again start with KVL, but now take the complex conjugate of both sides and multiply by $\mathbf{V_2}$, to obtain

$$\mathbf{V_1^*V_2} = \mathbf{V_2^*V_2} + \mathbf{I^*Z^*V_2}$$

Notice that the product $\mathbf{V_1^*V_2}$ will give us the difference between the two voltage phase angles.

We again substitute complex power $\mathbf{S} = \mathbf{I^*V_2}$ and this time take only the imaginary part of both sides of the equation, which produces the sine on the left hand side and retains the opposite pairings of cross-multiplied terms on the right,

$$\begin{aligned} V_1V_2\sin(\delta_1 - \delta_2) &= \mathrm{Im}\{\mathbf{SZ}^*\} \\ &= \mathrm{Im}\{(P + jQ)(R - jX)\} = RQ - XP \end{aligned}$$

yielding Eq. (12.6).

Problems and Questions

12.1 Repeat the iterative Newton–Raphson solution of the two-bus power flow example with a line impedance of $Z = j0.05$ p.u. How do you expect the solution for V_2 and the losses will be different?

12.2 Repeat the iterative Newton–Raphson solution of the two-bus power flow example with a (physically unrealistic!) purely resistive line with $Z = 0.1$ p.u. How do you expect the solution for V_2 and the losses will be different?

12.3 Repeat the iterative Newton–Raphson solution of the two-bus power flow example with a starting guess of $\mathbf{x} = [0, 0.25\text{ p.u.}]$, and comment.

12.4 For the three-bus example in Figure 12.19, use the DC power flow approximation to estimate real power flow on each of the three lines.

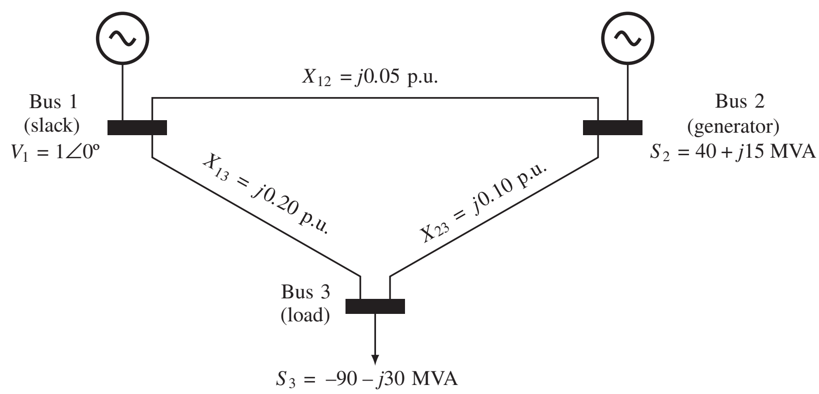


Figure 12.19 Three-bus example for Problems 12.4 and 12.5.