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On characteristic polynomials for a generalized chiral random matrix ensemble with a source

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Abstract

We evaluate averages involving characteristic polynomials, inverse characteristic polynomials and ratios of characteristic polynomials for a $N \times N$ random matrix taken from a L-deformed chiral Gaussian Unitary Ensemble with an external source Ω. Relation to a recently studied statistics of bi-orthogonal eigenvectors in the complex Ginibre ensemble, see Fyodorov (2017 arXiv:1710.04699), is briefly discussed as a motivation to study asymptotics of these objects in the case of external source proportional to the identity matrix. In particular, for an associated *complex bulk/chiral edge* scaling regime we retrieve the kernel related to Bessel/Macdonald functions.

Keywords: random characteristic polynomials, chiral unitary ensemble, Ginibre ensemble

(Some figures may appear in colour only in the online journal)

1. Introduction

Random characteristic polynomials are fascinating objects admitting rich mathematical structures, e.g. explicit expressions for expected moments, products and ratios [1, 2, 4, 8–10, 12, 13, 15, 24, 30, 32, 33, 39], duality formulas and integrability [14, 27, 34] as well as relations with the kernel of the underlying determinantal processes [6, 11]. Study of random characteristic polynomials is also strongly motivated by applications in number theory [13, 21, 28, 29] as well as in many areas of theoretical physics ranging from quantum chaos and quantum chaotic scattering [3, 22, 23], Anderson localization [35] and theory of glassy systems [19] to QCD [7, 20, 38] and string theory [31]. In a recent work [18] yet another application along these lines was revealed by the first author of the present paper. Namely, the distribution of diagonal entries of the 'nonorthogonality overlap' matrix between left and right eigenvectors of non-selfadjoint matrices from complex Ginibre ensemble (see (3.28) below) was expressed in terms of the average of the inverse characteristic polynomial of complex self-adjoint matrices from a particular deformed version of the chiral Gaussian Unitary Ensemble. In this paper we investigate the mathematical structure of the latter ensemble in more detail.

Our main object of interest here is an L-deformed chiral Gaussian Unitary Ensemble with an external source specified via the following joint probability density function defined on the space of complex matrices X of size $N \times N$:

$$\mathcal{P}^{(L)}(X)\mathrm{d}X = c \cdot \mathrm{e}^{-\operatorname{Tr}(XX^*)+\operatorname{Tr}(\Omega X)+\operatorname{Tr}(X^*\Omega^*)}\left[\det\left(X^*X\right)\right]^L \prod_{i,j=1}^{N} \mathrm{d}X^R_{i,j}\mathrm{d}X^I_{i,j}, \tag{1.1}$$

where Ω is a (in general, complex-valued) fixed non-random source matrix of size $N \times N$, $X_{i,j} = X^R_{i,j} + iX^I_{i,j}$ denote the sums of the real and imaginary parts of the matrix entries $X_{i,j}$, and c is an appropriate normalization constant. For $L = 0, \Omega = 0$, equation (1.1) defines the standard chiral Gaussian Unitary Ensemble (or, in a different interpretation, the Wishart–Laguerre ensemble) for which the general expectation values of product and ratios of characteristic polynomials were considered in detail in [20, 25]. For $L = 0, \Omega \neq 0$ the ensemble coincides with one studied in [16]. Note also that the ensemble defined via (1.1) (but with a Hermitian source $\Omega = \Omega^*$) is actually a special limiting case of a version of the QCD-motivated ensemble considered in [37]. The latter paper addressed general $n-$point correlation functions of eigenvalue densities for such type of a chiral ensemble, but their knowledge is not sufficient for our goals of studying the averages involving (inverse) characteristic polynomials associated with equation (1.1).

The outline of the paper is as follows. In section 2 we derive the joint pdf of the singular values for the matrix model given by equation (1.1). Sections 3–6 develop compact integral representations of:

- averaged inverse characteristic polynomials in theorem 3.4,
- averaged characteristic polynomials in theorem 4.1,
- averaged ratio of characteristic polynomials in theorem 5.1,
- the kernel in theorem 6.1,

starting from the formulae obtained in Desrosiers and Forrester (proposition 2 of [16]), and in Forrester and Liu (proposition 2.5 of [17]). Then, motivated by the applications to the problem considered in [18], see section 3.4 for more detail, we perform asymptotic analysis of the aforementioned objects in the *complex bulk/chiral edge scaling regime* (this nomenclature is explained in section 3.4) for the special degenerate case where the source matrix Ω is proportional to the unit matrix, see section 7. The main results of that section are summarized by propositions 7.1–7.3. Lastly, the appendices contain proofs of auxiliary propositions.

2. The joint density for singular values

Let X be a matrix taken from the ensemble defined by probability measure (1.1). Let $\omega_1, \ldots, \omega_N$ denote the squared singular values of Ω, and by $x_1, \ldots, x_N$ the squared singular values of X.

Proposition 2.1. *Assume that the parameters* $\delta_1, \ldots, \delta_N$ *are pairwise distinct. The joint probability density of* $(x_1, \ldots, x_N)$ *can be written as*

$$P^{(L)}(x_1, \ldots, x_N) = \frac{1}{\mathcal{N}_L} \cdot \triangle(x) \det\left({}_0F_1(1; \omega_k x_j)\right)_{k,j=1}^N \prod_{i=1}^N x_i^L \mathrm{e}^{-x_i}, \tag{2.1}$$

where $\mathcal{N}_L$ *is a normalizing constant and* $\triangle(x)$ *is the Vandermonde determinant and* ${}_0F_1(1; xy)$ *is a confluent hypergeometric function closely related to the Bessel function, see (2.3).*

Proof. The singular value decomposition for X is

$$X = UX_D V,$$

where U and V are unitary matrices of size $N \times N$ and $X_D = \mathrm{diag}(\sqrt{x_1}, \sqrt{x_2}, \cdots, \sqrt{x_N})$. We know that $\mathrm{d}X \sim \triangle(x)^2 \mathrm{d}\mu(U)\mathrm{d}\mu(V)$, with $\mathrm{d}\mu(U)$ being the invariant Haar measure on the unitary group. In order to obtain the formula in the statement of the proposition we use the Berezin–Karpelevich integral formula introduced in [5, 36]

$$\int_{U(N)} \mathrm{d}\mu(U) \int_{U(N)} \mathrm{d}\mu(V) \mathrm{e}^{\mathrm{Tr}(\Omega U X_D V) + \mathrm{Tr}(\Omega^* V^* X_D U^*)} = \frac{\mathrm{const}}{\triangle(x)} \frac{\det\left[I_0\left(2\sqrt{\omega_k x_j}\right)\right]_{k,j=1}^N}{\triangle(\delta)}, \tag{2.2}$$

and the well-known relation to the Bessel function of imaginary argument:

$${}_0F_1(1; xy) = I_0\left(2\sqrt{xy}\right). \tag{2.3}$$

□

3. Averages of inverse characteristic polynomials

From section 2 we conclude that the joint probability density $P^{(L)}(x_1, \ldots, x_N)$ can be written as

$$P^{(L)}(x_1, \ldots, x_N) = \frac{1}{\mathcal{N}_L} \det\left(\eta_i(x_j)\right)_{i,j=1}^N \det\left(\zeta_i(x_j)\right)_{i,j=1}^N, \tag{3.1}$$

where

$$\eta_i(x) = x^{i-1}, \quad \zeta_i(x) = {}_0F_1(1; \omega_i x)\, x^L \mathrm{e}^{-x}, \tag{3.2}$$

and the normalization reads

$$\mathcal{N}_L = N! \det G, \tag{3.3}$$

with elements of the matrix $G = (g_{i,j})_{i,j=1}^N$ given by the integral $g_{i,j} = \int_0^\infty \eta_i(x)\zeta_j(x)\mathrm{d}x$. We will assume that the parameters $\omega_1, \ldots, \omega_N$ are pairwise distinct. The following proposition was proved in [17] (see proposition 2.5 of [17]) for the averages of inverse characteristic polynomials.

Proposition 3.1. *Consider the ensemble defined by equation (3.1), and assume that* $\eta_i(x) = x^{i-1}$. *Set* $G = (g_{i,j})_{i,j=1}^N$, *where the matrix entries are defined by*

$$g_{i,j} = \int_0^\infty \eta_i(x)\zeta_j(x)\mathrm{d}x, \quad 1 \leqslant i, j \leqslant N. \tag{3.4}$$

Let C be the inverse of G, and let $c_{i,j}$ be the matrix entries of C^T, i.e. $C^T = (c_{i,j})_{i,j=1}^N$. Then the following formula holds true

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right] = \int_0^\infty \mathrm{d}u \frac{1}{y-u}\sum_{j=1}^{N} c_{N,j}\zeta_j(u) \tag{3.5}$$

where here and henceforth $\mathbb{E}(...)$ denotes averaging over jpdf given by equation (3.1).

3.1. Characterization of the matrix entries of C

We will use the same method as in [17] to compute the averages of inverse characteristic polynomials explicitly. First, we wish to obtain a formula characterizing the entries of C. For this purpose let use compute the entries of the matrix G explicitly. We have

$$g_{i,j} = \int_0^\infty x^{i+L-1}\mathrm{e}^{-x}{}_0F_1(1;\omega_j x)\mathrm{d}x, \quad i \leqslant i,j \leqslant N. \tag{3.6}$$

In order to compute this integral we will use the well-known formula, see e.g. [26],

$$\int_0^\infty x^{n+\frac{\nu}{2}}\mathrm{e}^{-\alpha x} I_\nu\left(2\beta\sqrt{x}\right)\mathrm{d}x = n!\beta^\nu \mathrm{e}^{\beta^2/\alpha}\alpha^{-n-\nu-1}\mathbf{L}_n^\nu\left(-\frac{\beta^2}{\alpha}\right), \tag{3.7}$$

where $\mathbf{L}_n^\nu(x)$ are generalized Laguerre polynomials, and find

$$g_{i,j} = (i+L-1)!\mathrm{e}^{\delta_j}\mathbf{L}_{i+L-1}\left(-\omega_j\right), \quad 1 \leqslant i,j \leqslant N. \tag{3.8}$$

Taking into account the definition of the matrix entries $c_{i,j}$ (recall that $c_{i,j}$ are the matrix entries of C^T, where $C = G^{-1}$, so $\sum_{j=1}^N c_{j,i}g_{j,k} = \delta_{i,k}$), we obtain

$$\sum_{j=1}^{N}(j+L-1)!\mathrm{e}^{\omega_k}\mathbf{L}_{j+L-1}\left(-\omega_k\right)c_{j,i} = \delta_{i,k}, \tag{3.9}$$

where $1 \leqslant i,k \leqslant N$, and $L \in \{0,1,\ldots\}$.

3.2. The case $L = 0$

When $L = 0$, the following equation between two polynomials in u holds true

$$\sum_{j=1}^{N}(j-1)!\mathbf{L}_{j-1}\left(u\right)c_{j,i} = \mathrm{e}^{-\omega_i}\prod_{\substack{\tau=1\\ \tau\neq i}}^{N}\frac{-u-\omega_\tau}{\omega_i-\omega_\tau}. \tag{3.10}$$

Indeed, the expressions on the left hand side and on the right hand side of equation (3.10) are both polynomials in u of degree $N-1$. In addition, we note that

$$\prod_{\substack{\tau=1\\ \tau\neq i}}^{N}\frac{\omega_k-\omega_\tau}{\omega_i-\omega_\tau} = \delta_{i,k}.$$

Therefore, equation (3.9) implies that equation (3.10) is satisfied at points $\omega_1, \ldots, \omega_N$.

We know that the leading term of the Laguerre polynomial is

$$n!\mathbf{L}_n^\nu(x) = (-x)^n + \ldots.$$

Therefore, taking $u \to \infty$ in equation (3.10) we should have

$$c_{N,i} = \frac{\mathrm{e}^{-\omega_i}}{\prod_{\substack{\tau=1\\ \tau\neq i}}^{N}(\omega_i - \omega_\tau)}, \quad 1 \leqslant i \leqslant N. \tag{3.11}$$

Remark 3.2. Taking into account the explicit expression for the matrix entries $g_{i,j}$ and formula above we see that the condition $\sum_{k=1}^{N} c_{N,k} g_{i,k} = \delta_{N,i}$ is equivalent to

$$\sum_{k=1}^{N} \frac{1}{\prod\limits_{\substack{\tau=1\\ \tau\neq k}}^{N}(\omega_k - \omega_\tau)} (i-1)!\mathbf{L}_{i-1}(-\omega_k) = \delta_{N,i}, \quad 1 \leqslant i \leqslant N. \tag{3.12}$$

The equation just written above follows from the formulae for Lagrange interpolation with the monic Laguerre polynomials.

Now we are ready to apply proposition 3.1. We have

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right]_{L=0} = \int_0^\infty \mathrm{d}u \frac{1}{y-u} \sum_{j=1}^{N} \frac{\mathrm{e}^{-\omega_j}}{\prod_{\substack{\tau=1\\ \tau\neq j}}^{N}(\omega_j - \omega_\tau)} {}_0F_1(1;\omega_j u)\mathrm{e}^{-u} \tag{3.13}$$

or

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right]_{L=0} = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{d}u \frac{\mathrm{e}^{-u}}{y-u} \int_C \frac{\mathrm{e}^{-v}{}_0F_1(1;vu)\,\mathrm{d}v}{\prod_{k=1}^{N}(v-\omega_k)}, \tag{3.14}$$

where the counterclockwise contour C encircles $\omega_1, \ldots, \omega_N$.

Remark 3.3. Assume that $\Omega = z\mathbf{1}_N$, $z \in \mathbb{C}$. Then $\Omega^*\Omega = |z|^2\mathbf{1}_N$, and $\omega_1 = \ldots = \omega_N = |z|^2$. In this degenerate case (denoted with a subscript 'deg') we obtain the formula

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right]_{\substack{\mathrm{deg}\\ L=0}} = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \mathrm{d}u \frac{\mathrm{e}^{-u}}{y-u} \int_C \frac{\mathrm{e}^{-v}{}_0F_1(1;vu)\,\mathrm{d}v}{(v-|z|^2)^N}. \tag{3.15}$$

It is not hard to check by direct calculations that this formula holds true at $N = 1$.

We conclude that equation (3.14) solves the problem of finding the mean inverse characteristic polynomial for the case $L = 0$ for a general source matrix, and equation (3.15) solves the corresponding degenerate problem considered in [18].

3.3. The case of a general L

In this section we prove theorem 3.4 which gives an explicit expression for the averages of inverse characteristic polynomial in the case of a general L.

Theorem 3.4. *Consider the probability distribution on the space of complex matrices of size $N \times N$ defined by equation (1.1).*

(A) *Let $\omega_1, \ldots, \omega_N$ be the squared singular values of Ω, and assume that $L \geqslant 1$. We have*

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right]=\frac{1}{\tilde{\mathcal{N}}_L}\frac{1}{2\pi \mathrm{i}}\int_0^\infty \mathrm{d}u\frac{\mathrm{e}^{-u}u^L}{y-u}\int_C\frac{\mathrm{e}^{-v}{}_0F_1\left(1;vu\right)\mathrm{d}v}{\prod\limits_{k=1}^{N}\left(v-\omega_k\right)}$$
$$\times\int_0^\infty\ldots\int_0^\infty\frac{\prod\limits_{i=1}^{L}\prod\limits_{j=1}^{N}\left(t_i+\omega_j\right)}{\prod\limits_{i=1}^{L}\left(t_i+v\right)}\triangle^2\left(t\right)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i, \tag{3.16}$$

where the counter-clockwise contour C encircles $\omega_1, \ldots, \omega_N$. The normalization constant is equal to

$$\tilde{\mathcal{N}}_L=\int_0^\infty\ldots\int_0^\infty\prod_{i=1}^{L}\prod_{j=1}^{N}\left(t_i+\omega_j\right)\triangle^2(t)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i \tag{3.17}$$

and is related to the normalization $\mathcal{N}_L$ of the density (2.1) as

$$\mathcal{N}_L=\frac{(-1)^{N(N-1)}N!\triangle(\omega)\prod_{i=1}^{N}\mathrm{e}^{\omega_i}}{L!\prod_{j=1}^{L-1}(j)!^2}\tilde{\mathcal{N}}_L. \tag{3.18}$$

(B) *Assume that $\Omega=z\mathbf{1}_N$, $z\in\mathbb{C}$. Then $\Omega^*\Omega=|z|^2\mathbf{1}_N$, and $\omega_1=\ldots=\omega_N=|z|^2$. In this case we obtain the formula*

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right]_{\deg}=\frac{1}{\tilde{\mathcal{N}}_L}\frac{1}{2\pi\mathrm{i}}\int_0^\infty\mathrm{d}u\frac{\mathrm{e}^{-u}u^L}{y-u}\int_C\frac{\mathrm{e}^{-v}{}_0F_1\left(1;vu\right)\mathrm{d}v}{\left(v-|z|^2\right)^N}$$
$$\times\int_0^\infty\ldots\int_0^\infty\frac{\prod_{i=1}^{L}\left(t_i+|z|^2\right)^N}{\prod_{i=1}^{L}(t_i+v)}\triangle^2\left(t\right)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i, \tag{3.19}$$

the counter-clockwise contour C encircles $|z|^2$ and the corresponding normalization is $\tilde{\mathcal{N}}_L=\int_0^\infty\ldots\int_0^\infty\prod_{i=1}^{L}\left(t_i+|z|^2\right)^N\triangle^2(t)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i$.

Proof. *It is convenient to assume that the parameters $\omega_1, \ldots, \omega_N$ are pairwise distinct. We begin from the observation that proposition 3.1 can be stated in an equivalent form.*

Proposition 3.5. *With the same notation as in the statement of proposition 3.1 the formula for the average of an inverse characteristic polynomial can be written as ratio of determinants, namely*

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right]=\frac{1}{\det G}\begin{vmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,N}\\ \vdots & \vdots & \ddots & \vdots\\ g_{N-1,1} & g_{N-1,2} & \cdots & g_{N-1,N}\\ \int_0^\infty\mathrm{d}u\frac{\zeta_1(u)}{y-u} & \int_0^\infty\mathrm{d}u\frac{\zeta_2(u)}{y-u} & \cdots & \int_0^\infty\mathrm{d}u\frac{\zeta_N(u)}{y-u}\end{vmatrix}. \tag{3.20}$$

Proof. See Desrosiers and Forrester [16] (proposition 2), Forrester and Liu [17] (proposition 2.5). □

By Laplace expansion, equation (3.20) can be rewritten as

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right] = \int_0^\infty \mathrm{d}u\frac{1}{y-u}\sum_{k=1}^{N}(-1)^{N-k}\zeta_k(u)\frac{\begin{vmatrix} g_{1,1} & \cdots & g_{1,k-1} & g_{1,k+1} & \cdots & g_{1,N}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ g_{N-1,1} & \cdots & g_{N-1,k-1} & g_{N-1,k+1} & \cdots & g_{N-1,N}\end{vmatrix}}{\begin{vmatrix} g_{1,1} & \cdots & g_{1,N}\\ \vdots & \ddots & \vdots\\ g_{N,1} & \cdots & g_{N,N}\end{vmatrix}}. \tag{3.21}$$

In our case the matrix entries $g_{i,j}$ given by equation (3.8) can be written in terms of the Laguerre monic polynomials $\{\pi_k(x)\}_{k=0}^{\infty}$ as

$$g_{i,j} = (-1)^{i+L-1}\pi_{i+L-1}(-\omega_j)\,\mathrm{e}^{\omega_j},\quad 1 \leqslant i,j \leqslant N, \tag{3.22}$$

where the Laguerre monic polynomials $\pi_k(x) = x^k + \ldots$ are defined in terms of the Laguerre polynomials as $\pi_k(x) = k!(-1)^k\mathbf{L}_k(x)$. So in our case formula (3.21) takes the form

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{y-x_i}\right] = (-1)^{L-1}\int_0^\infty \frac{\mathrm{d}u}{y-u}u^L\mathrm{e}^{-u}\sum_{k=1}^{N}(-1)^k\,{}_0F_1(1;\omega_k u)\,\mathrm{e}^{-\omega_k}$$
$$\times\frac{\begin{vmatrix} \pi_L(-\omega_1) & \cdots & \pi_L(-\omega_{k-1}) & \pi_L(-\omega_{k+1}) & \cdots & \pi_L(-\omega_N)\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \pi_{N+L-2}(-\omega_1) & \cdots & \pi_{N+L-2}(-\omega_{k-1}) & \pi_{N+L-2}(-\omega_{k+1}) & \cdots & \pi_{N+L-2}(-\omega_N)\end{vmatrix}}{\begin{vmatrix} \pi_L(-\omega_1) & \cdots & \pi_L(-\omega_1)\\ \vdots & \ddots & \vdots\\ \pi_{N+L-1}(-\omega_1) & \cdots & \pi_{N+L-1}(-\omega_N)\end{vmatrix}}. \tag{3.23}$$

Note the the ratio of determinants in formula (3.23) can be rewritten as

$$\frac{\begin{vmatrix} \pi_L(-\omega_1) & \cdots & \pi_{L+N-2}(-\omega_1)\\ \vdots & \ddots & \vdots\\ \pi_L(-\omega_{k-1}) & \cdots & \pi_{L+N-2}(-\omega_{k-1})\\ \pi_L(-\omega_{k+1}) & \cdots & \pi_{L+N-2}(-\omega_{k+1})\\ \vdots & \ddots & \vdots\\ \pi_L(-\omega_N) & \cdots & \pi_{L+N-2}(-\omega_N)\end{vmatrix}}{\begin{vmatrix} \pi_L(-\omega_1) & \cdots & \pi_{N+L-1}(-\omega_1)\\ \vdots & \ddots & \vdots\\ \pi_L(-\omega_N) & \cdots & \pi_{N+L-1}(-\omega_N)\end{vmatrix}}. \tag{3.24}$$

In order to evaluate the ratio of determinants just written above we will use the following result by Brezin and Hikami [9].

Proposition 3.6. *Let* $\mathrm{d}\alpha$ *be a measure with finite moments* $\int_{\mathbb{R}} |t|^k \mathrm{d}\alpha(t) < \infty, k = 0, 1, 2, \ldots$. *Let* $\pi_j(t) = t^j + \ldots$ *denote the jth monic orthogonal polynomial with respect to the measure* $\mathrm{d}\alpha$

$$\int_{\mathbb{R}} \pi_j(t)\pi_k(t)\mathrm{d}\alpha(t) = c_j c_k \delta_{j,k}, \quad j, k > 0.$$

Then

$$\begin{vmatrix} \pi_n(\mu_1) & \ldots & \pi_{n+m-1}(\mu_1) \\ \vdots & \ddots & \vdots \\ \pi_n(\mu_m) & \ldots & \pi_{n+m-1}(\mu_m) \end{vmatrix} = \triangle(\mu_1, \ldots, \mu_m) \frac{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^{n} \prod_{j=1}^{m} (\mu_j - t_i) \triangle^2(t_1, \ldots, t_n) \prod_{i=1}^{n} \mathrm{d}\alpha(t_i)}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \triangle^2(t_1, \ldots, t_n) \prod_{i=1}^{n} \mathrm{d}\alpha(t_i)}, \tag{3.25}$$

where

$$\triangle(\mu_1, \ldots, \mu_m) = \prod_{1 \leqslant i < j \leqslant m} (\mu_j - \mu_i).$$

Proof. See Brezin and Hikami [9]. □

We apply the formula stated in proposition 3.6 to both numerator (with $n = L$, $m = N - 1$, $\mu_1 = -\omega_1, \ldots, \mu_{k-1} = -\omega_{k-1}$, $\mu_k = -\omega_{k+1}, \ldots, \mu_{N-1} = -\omega_N$) and denominator (with $n = L$, $m = N$, $\mu_1 = -\omega_1, \ldots, \mu_N = -\omega_N$) of expression (3.24):

$$\begin{aligned} &\frac{\triangle(-\omega_1, \ldots, -\omega_{k-1}, -\omega_{k+1}, \ldots, -\omega_N)}{\triangle(-\omega_1, \ldots, -\omega_N)} \frac{\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{L} \prod_{\substack{j=1\\ j\neq k}}^{N} (-\omega_j - t_i) \triangle^2(t_1, \ldots, t_L) \prod_{i=1}^{L} \mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{L} \prod_{j=1}^{N} (-\omega_j - t_i) \triangle^2(t_1, \ldots, t_L) \prod_{i=1}^{L} \mathrm{e}^{-t_i}\mathrm{d}t_i} \\ &= (-1)^{1-N-L} \frac{\triangle(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_N)}{\triangle(\omega_1, \ldots, \omega_N)} \frac{\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{L} \prod_{\substack{j=1\\ j\neq k}}^{N} (\omega_j + t_i) \triangle^2(t_1, \ldots, t_L) \prod_{i=1}^{L} \mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{L} \prod_{j=1}^{N} (\omega_j + t_i) \triangle^2(t_1, \ldots, t_L) \prod_{i=1}^{L} \mathrm{e}^{-t_i}\mathrm{d}t_i}, \end{aligned}$$

where we used $\triangle(-\omega_1, \ldots, -\omega_N) = (-1)^{N(N-1)/2} \triangle(\omega_1, \ldots, \omega_N)$.

Taking into account that

$$\frac{\triangle(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_N)}{\triangle(\omega_1, \ldots, \omega_N)} = \frac{1}{\prod_{i=1}^{k-1} (\omega_k - \omega_i) \prod_{i=k+1}^{N} (\omega_i - \omega_k)} = \frac{(-1)^{N-k}}{\prod_{\substack{j=1\\ j\neq k}}^{N} (\omega_k - \omega_j)}, \tag{3.26}$$

we get a representation of the ratio of two determinants with orthogonal polynomial entries in terms of multiple integrals

$$\frac{\begin{vmatrix} \pi_L(-\omega_1) & \dots & \pi_{L+N-2}(-\omega_1) \\ \vdots & \ddots & \vdots \\ \pi_L(-\omega_{k-1}) & \dots & \pi_{L+N-2}(-\omega_{k-1}) \\ \pi_L(-\omega_{k+1}) & \dots & \pi_{L+N-2}(-\omega_{k+1}) \\ \vdots & \ddots & \vdots \\ \pi_L(-\omega_N) & \dots & \pi_{L+N-2}(-\omega_N) \end{vmatrix}}{\begin{vmatrix} \pi_L(-\omega_1) & \dots & \pi_{N+L-1}(-\omega_1) \\ \vdots & \ddots & \vdots \\ \pi_L(-\omega_N) & \dots & \pi_{N+L-1}(-\omega_N) \end{vmatrix}} = \frac{(-1)^{k+L-1}}{\prod_{\substack{j=1\\j\neq k}}^{N}(\omega_k-\omega_j)} \frac{\int_0^\infty \dots \int_0^\infty \prod_{i=1}^{L}\prod_{\substack{j=1\\j\neq k}}^{N}(\omega_j+t_i)\,\triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty \dots \int_0^\infty \prod_{i=1}^{L}\prod_{j=1}^{N}(\omega_j+t_i)\,\triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i}.$$

In order to get the formula in the statement of theorem 3.4, (A) insert the right hand side of equation (3.3) into the expression in the right hand side of formula (3.23), and use the basic residue theorem. By continuity, we can remove the condition that all parameters $\omega_1, \dots, \omega_N$ are pairwise distinct, and, in particular, to obtain the formula in the statement of theorem 3.4, (B).

The normalization constant $\tilde{\mathcal{N}}_L$ is read off from the denominator of the above formula

$$\tilde{\mathcal{N}}_L = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^{L}\prod_{j=1}^{N}(\omega_j+t_i)\,\triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i, \tag{3.27}$$

and the relation to $\mathcal{N}_L$ given by expression (2.1) is found from the definition (3.3) and the formula (3.22)

$$\mathcal{N}_L = N!(-1)^{\frac{N(N+1)}{2}+N(L-1)}\prod_{i=1}^{N}\mathrm{e}^{\omega_i}\det\left(\pi_{i+L-1}(-\omega_j)\right)_{i,j=1}^{N}.$$

Similarly as before, the determinant is found by applying proposition 3.6

$$\det\left(\pi_{i+L-1}(-\omega_j)\right)_{i,j=1}^{N} = \frac{\triangle(-\omega_1,\dots,-\omega_N)\int_0^\infty \dots \int_0^\infty \prod_{i=1}^{L}\prod_{j=1}^{N}(-\omega_j-t_i)\,\triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty \dots \int_0^\infty \triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i},$$

where the numerator is proportional to $\tilde{\mathcal{N}}_L$ and denominator is a Mehta-type integral formula $\int_0^\infty \dots \int_0^\infty \triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i = L!\prod_{j=1}^{L-1}j!^2$. Combining above formulas gives the relation (3.18). □

3.4. Connection with the statistics of eigenvectors of a complex Ginibre matrix

One of the motivations in studying the deformed ensemble given by equation (1.1) is its connection to the statistics of eigenvectors in complex Ginibre ensemble as elucidated in [18]. To that end, consider a joint eigenvalue-eigenvector probability density function:

$$\mathcal{P}^{(2)}(t,z) = \mathbb{E}\left[\sum_{i=1}^{N}\delta(O_{ii}-1-t)\delta^{(2)}(z-\lambda_i)\right]_{\text{Ginibre}}, \tag{3.28}$$

where λ_i are the complex eigenvalues of the Ginibre matrix, and the diagonal entries O_{ii} of 'self-overlap' (also known as 'non-orthogonality') matrix O_{ij} are defined in terms of lengths

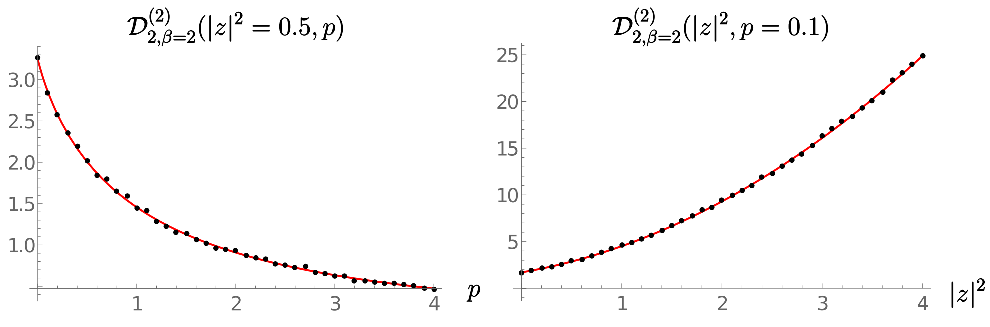

Figure 1. Plots of the function $\mathcal{D}^{(L=2)}_{N=2,\beta=2}(z,p)$ for fixed $|z|^2$ (left plot) or fixed p (right plot). Numerical results (black dots) computed with equation (3.30) are in a good agreement with analytical result equation (3.34) (red curves).

of the associated left $|L_i\rangle$ and right $|R_i\rangle$ eigenvectors as $O_{ii}=\langle L_i|L_i\rangle\,\langle R_i|R_i\rangle$. As was shown in [18] such a density is expressible as a Laplace transform

$$\mathcal{P}^{(2)}(t,z)\sim\int_0^\infty \mathrm{d}p\mathrm{e}^{-pt}\mathcal{D}^{(L=2)}_{N-1,\beta=2}(z,p), \tag{3.29}$$

of the following object:

$$\mathcal{D}^{(L)}_{N,\beta=2}(z,p)=\frac{1}{c}\int \mathrm{d}X\mathrm{e}^{-\operatorname{Tr}X^\dagger X}\frac{\det\left[(\bar{z}\mathbf{1}_N-X^\dagger)(z\mathbf{1}_N-X)\right]^L}{\det\left[p\mathbf{1}_N+(\bar{z}\mathbf{1}_N-X^\dagger)(z\mathbf{1}_N-X)\right]}, \tag{3.30}$$

with normalization $c=\int \mathrm{d}X\mathrm{e}^{-\operatorname{Tr}X^\dagger X}$ and a real parameter $p>0$. By setting $X\to X+z\mathbf{1}_N$, this correlation function reduces to an inverse characteristic polynomial averaged over the jpdf introduced in equation (1.1) for a degenerate source term Ω corresponding to $\omega_1=...=\omega_N=|z|^2$:

$$\mathcal{D}^{(L)}_{N,\beta=2}(z,p)=\frac{\mathcal{N}_L}{\mathcal{N}_0}\int \mathrm{d}xP^{(L)}(x)\prod_{i=1}^N\frac{1}{p+x_i}=\frac{\mathcal{N}_L}{\mathcal{N}_0}\mathbb{E}\left[\prod_{i=1}^N\frac{1}{p+x_i}\right]_{\mathrm{deg}}, \tag{3.31}$$

where $P^{(L)}(X)$ is the jpdf given by formula (2.1) and $\mathcal{N}_0$ is the special case of normalization given in equation (3.3). The renormalizing factor is taking into account the switch of conventions from viewing $\mathcal{D}^{(L)}$ as the ratio of characteristic polynomials $\mathbb{E}\left[\frac{\det^L(X^\dagger X)}{\det(p\mathbf{1}_N+X^\dagger X)}\right]_{\mathrm{Ginibre}}$ averaged over the (shifted by z) complex Ginibre ensemble to the inverse characteristic polynomial $\mathbb{E}\left[\frac{1}{\det(p\mathbf{1}_N+X^\dagger X)}\right]_{P^{(L)}}$ averaged over the deformed ensemble $P^{(L)}$ defined by formula (1.1). The ratio $\mathcal{D}^{(L)}$ is plotted in figure 1 as a function of both p and $|z|^2$ with both numerical and analytic methods.

We also comment on how the discussed connection to the complex Ginibre ensemble can lead to a confusion in the standard RMT 'bulk' versus 'edge' nomenclature in the asymptotic analysis performed in section 7. The problem arises since now the standard 'bulk' and 'edge' scaling regimes have an ambiguous interpretation, and depend on the applications in mind. Namely, the averages that we evaluate depend on parameters which for our choice are tuned to probe, simultaneously, (i) inside of the eigenvalue support ('bulk scaling') for complex eigenvalues of the non-Hermitian matrix X (Ginibre interpretation) or (ii) the region close to the edge ('edge scaling') of the underlying spectral density of real eigenvalues for the Hermitian matrix $X^\dagger X$ or its chiral counterpart (Wishart–Laguerre/chiral GUE intepretation). Indeed, in section 7 we inspect the bulk of the complex Ginibre ensemble by scaling $|z|^2 = NR$ with a fixed $0 \leqslant R < 1$ along with the edge regime of the characteristic polynomials arguments by scaling the parameter $p \sim N^{-1}$. For this reason we will call it the *complex bulk/chiral edge scaling regime*, for the lack of better terminology.

Explicit expressions for the object defined in the equation (3.31) in simplest cases $L = 0, 1, 2$ were found in [18] in the framework of a supersymmetry approach. In particular,

$$\mathcal{D}^{(0)}_{N,2}(z,p) = \frac{1}{(N-1)!}\int_0^\infty \frac{\mathrm{e}^{-pt-\frac{t|z|^2}{1+t}}}{t+1}\left(\frac{t}{1+t}\right)^{N-1}, \tag{3.32}$$

$$\mathcal{D}^{(1)}_{N,2}(z,p) = \frac{1}{(N-1)!}\mathrm{e}^{|z|^2}\int_0^\infty \frac{\mathrm{e}^{-pt}}{t(1+t)}\mathrm{e}^{-\frac{t|z|^2}{1+t}}\left(\frac{t}{1+t}\right)^{N}\left(\Gamma(N+1,|z|^2) - \frac{t}{1+t}|z|^2\Gamma(N,|z|^2)\right). \tag{3.33}$$

To see how these expressions follow in the present approach we use the following

Remark 3.7. For $L \geqslant 0$ the object defined in [18] and related to the averaged inverse characteristic polynomial by formula (3.31) is equal to

$$\mathcal{D}^{(L)}_{N,\beta=2}(z,p) = \frac{(-1)^{N-1}}{L!\prod_{j=1}^{L-1} j!^2}\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}u\frac{\mathrm{e}^{-u}u^L}{p+u}\int_C \frac{\mathrm{e}^{-v}{}_0F_1\left(1;vu\right)\mathrm{d}v}{\left(v-|z|^2\right)^N}$$
$$\times \int_0^\infty \ldots \int_0^\infty \frac{\prod_{i=1}^{L}\left(t_i+|z|^2\right)^N}{\prod_{i=1}^{L}(t_i+v)}\triangle^2(t)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i. \tag{3.34}$$

Proof. This follows directly from the relation (3.31) and the correction factor is found using the formula (3.18)

$$\frac{\mathcal{N}_L}{\mathcal{N}_0} = \frac{\tilde{\mathcal{N}}_L}{L!\prod_{j=1}^{L-1} j!^2}, \tag{3.35}$$

where $\tilde{\mathcal{N}}_0 = 1$. □

In the $L = 0$ case the formula (3.32) is related to remark 3.7 by the following proposition.

Proposition 3.8. *For $p > 0$ the following formula holds true*

$$\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}u\frac{\mathrm{e}^{-u}}{p+u}\int_C \frac{\mathrm{e}^{-v}{}_0F_1\left(1;vu\right)\mathrm{d}v}{\left(v-|z|^2\right)^N} = \frac{(-1)^{N-1}}{(N-1)!}\int_0^\infty \frac{\mathrm{d}t}{1+t}\mathrm{e}^{-tp}\left(\frac{t}{1+t}\right)^{N-1}\mathrm{e}^{-|z|^2\frac{t}{1+t}}. \tag{3.36}$$

Proof. We use $\frac{1}{p+u} = \int_0^\infty \mathrm{d}t\mathrm{e}^{-t(p+u)}$ to arrive at

$$\text{l.h.s.} = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}t\mathrm{e}^{-tp}\int_C \frac{\mathrm{d}v}{(v-|z|^2)^N}\mathrm{e}^{-v}\int_0^\infty \mathrm{d}u\mathrm{e}^{-u(1+t)}{}_0F_1\left(1;vu\right).$$

To evaluate the last integral we use again the identity (3.7) for $\nu = 0,\ n = 0,\ \beta = \sqrt{v},\ \alpha = 1+t$ and the integral in question is equal to $\frac{1}{1+t}\mathrm{e}^{\frac{v}{1+t}}$. Lastly, the contour integral is evaluated:

$$\text{l.h.s.} = \int_0^\infty \frac{\mathrm{d}t}{1+t}\mathrm{e}^{-tp}\frac{1}{2\pi \mathrm{i}}\int_C \frac{\mathrm{d}v}{(v-|z|^2)^N}\mathrm{e}^{-v\frac{t}{1+t}} = \frac{(-1)^{N-1}}{(N-1)!}\int_0^\infty \frac{\mathrm{d}t}{1+t}\mathrm{e}^{-tp}\left(\frac{t}{1+t}\right)^{N-1}\mathrm{e}^{-|z|^2\frac{t}{1+t}}.$$

□

Following the same lines and exploiting the identity (3.7) one can prove for general integer $L \geqslant 0$ the following equivalent representations (see (7.2) later on):

Proposition 3.9.

$$\mathcal{D}^{(L)}_{N,\beta=2}(z,p) = \int_0^\infty \frac{\mathrm{d}t}{(1+t)^{L+1}}\mathrm{e}^{-tp}\,\mathcal{G}^{(L)}_N\left(|z|^2, \frac{t}{1+t}\right) \tag{3.37}$$

where we defined the following function of $\rho = |z|^2$ *and* $\tau = t/(1+t)$*:*

$$\begin{aligned}\mathcal{G}^{(L)}_N(\rho,\tau) = (-1)^N L! \int_0^\infty \mathrm{d}t_1 \dots \int_0^\infty \mathrm{d}t_L \Delta^2(t_1,\dots,t_L)\prod_{k=1}^L (t_k+\rho)^N\,\mathrm{e}^{-t_k} \\ \times \frac{1}{2\pi\mathrm{i}}\oint_{Re(v)>0}\frac{\mathrm{d}v\mathrm{e}^{-v\tau}}{(v-\rho)^N}\mathbf{L}_L\left(-v(1-\tau)\right)\frac{1}{\prod_{k=1}^L(t_k+v)}\end{aligned} \tag{3.38}$$

which also can be presented in an explicitly real form:

$$\begin{aligned}\mathcal{G}^{(L)}_N(\rho,\tau) = \frac{(-1)^N L!}{(N-1)!}\int_0^\infty \mathrm{d}t_1 \dots \int_0^\infty \mathrm{d}t_L \Delta^2(t_1,\dots,t_L)\prod_{k=1}^L (t_k+\rho)^N\,\mathrm{e}^{-t_k} \\ \times \frac{\mathrm{d}^{N-1}}{d^{N-1}\rho}\left[\mathrm{e}^{-\rho\tau}\,\mathbf{L}_L\left(-\rho(1-\tau)\right)\frac{1}{\prod_{k=1}^L(t_k+\rho)}\right]\end{aligned} \tag{3.39}$$

where $\mathbf{L}_k(x)$ *are Laguerre polynomials.*

The above can be used for a verification of the $L=1$ case, which is a similar but substantially longer calculation than for $L=0$ and for this sake is relegated to the appendix A. The complex bulk/chiral edge scaling regime limit of $\mathcal{G}^{(L)}_N(\rho,\tau)$ relevant for the context of [18] is evaluated in the section 7 of the present paper, see equation (7.16).

4. Average of characteristic polynomials

We turn our attention to the average characteristic polynomial evaluated for a general integer parameter $L \geqslant 1$.

Theorem 4.1. *Consider the probability distribution on the space of complex matrices of size* $N \times N$ *defined by equation (1.1).*

(A) *Let* $\omega_1, \dots, \omega_N$ *be the squared singular values of* Ω*, and assume that* $L \geqslant 1$*. We have*

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^N (z-x_i)\right] = \frac{(-1)^{N+L}}{\tilde{\mathcal{N}}_L}\frac{\mathrm{e}^z}{z^L}\int_0^\infty \mathrm{d}y\mathrm{e}^{-y}{}_0F_1(1;-zy)\prod_{j=1}^N(\omega_j+y) \\ \times \int_0^\infty \dots \int_0^\infty \prod_{i=1}^L\prod_{j=1}^N(\omega_j+t_i)\,\triangle^2(t)\prod_{i=1}^L(y-t_i)\mathrm{e}^{-t_i}\mathrm{d}t_i\end{aligned} \tag{4.1}$$

where the normalization coefficient $\tilde{\mathcal{N}}_L$ *is given by equation (3.17).*

(B) *Assume that* $\Omega = z\mathbf{1}_N$, $z \in \mathbb{C}$. *Then* $\Omega^*\Omega = |z|^2\mathbf{1}_N$, *and* $\omega_1 = \ldots = \omega_N = |z|^2$. *In this degenerate case we obtain the formula*

$$\mathbb{E}\left[\prod_{i=1}^{N}(z-x_i)\right]_{\mathrm{deg}} = \frac{(-1)^{N+L}}{\tilde{\mathcal{N}}_L}\frac{e^z}{z^L}\int_0^\infty \mathrm{d}y e^{-y}{}_0F_1(1;-zy)\left(|z|^2+y\right)^N \times \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{L}\left(|z|^2+t_i\right)^N \triangle^2(t)\prod_{i=1}^{L}(y-t_i)e^{-t_i}\mathrm{d}t_i \tag{4.2}$$

with the normalization factor $\tilde{\mathcal{N}}_L = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{L}\left(t_i+|z|^2\right)^N \triangle^2(t)\prod_{i=1}^{L}e^{-t_i}\mathrm{d}t_i$.

Proof. We assume that the parameters $\omega_1, \ldots, \omega_N$ are pairwise distinct. We begin from the observation that proposition 3.1 can be stated in an equivalent form.

Proposition 4.2. *We have*

$$\mathbb{E}\left[\prod_{i=1}^{N}(z-x_i)\right] = \frac{1}{\det G}\begin{vmatrix} g_{1,1} & \cdots & g_{1,N} & \eta_1(z) \\ g_{2,1} & \cdots & g_{2,N} & \eta_2(z) \\ \vdots & \ddots & \vdots & \vdots \\ g_{N+1,1} & \cdots & g_{N+1,N} & \eta_{N+1}(z) \end{vmatrix}, \tag{4.3}$$

where $\eta_i(x) = x^{i-1} + \ldots$ *is any system of monic polynomials and the matrix* $g_{i,j}$ *is given by equation (3.4).*

Proof. See Desrosiers and Forrester [16], proposition 2. □

Recall that in our case the matrix entries $g_{i,j}$ can be rewritten in terms of the monic Laguerre polynomials, see equation (3.22), so we can write

$$\mathbb{E}\left[\prod_{i=1}^{N}(z-x_i)\right] = \frac{\begin{vmatrix} (-1)^L\pi_L(-\omega_1)e^{\omega_1} & \cdots & (-1)^L\pi_L(-\omega_N)e^{\omega_N} & 1 \\ (-1)^{L+1}\pi_{L+1}(-\omega_1)e^{\omega_1} & \cdots & (-1)^{L+1}\pi_{L+1}(-\omega_N)e^{\omega_N} & z \\ \vdots & \ddots & \vdots & \vdots \\ (-1)^{L+N}\pi_{L+N}(-\omega_1)e^{\omega_1} & \cdots & (-1)^{L+N}\pi_{L+N}(-\omega_N)e^{\omega_N} & z^N \end{vmatrix}}{\begin{vmatrix} (-1)^L\pi_L(-\omega_1)e^{\omega_1} & \cdots & (-1)^L\pi_L(-\omega_N)e^{\omega_N} \\ (-1)^{L+1}\pi_{L+1}(-\omega_1)e^{\omega_1} & \cdots & (-1)^{L+1}\pi_{L+1}(-\omega_N)e^{\omega_N} \\ \vdots & \ddots & \vdots \\ (-1)^{L+N-1}\pi_{L+N-1}(-\omega_1)e^{\omega_1} & \cdots & (-1)^{L+N-1}\pi_{L+N-1}(-\omega_N)e^{\omega_N} \end{vmatrix}}, \tag{4.4}$$

or

$$\mathbb{E}\left[\prod_{i=1}^{N}(z-x_i)\right] = \frac{(-1)^L}{z^L}\frac{\begin{vmatrix} \pi_L(-\omega_1) & \cdots & \pi_L(-\omega_N) & (-z)^L \\ \pi_{L+1}(-\omega_1) & \cdots & \pi_{L+1}(-\omega_N) & (-z)^{L+1} \\ \vdots & \ddots & \vdots & \vdots \\ \pi_{L+N}(-\omega_1) & \cdots & \pi_{L+N}(-\omega_N) & (-z)^{L+N} \end{vmatrix}}{\begin{vmatrix} \pi_L(-\omega_1) & \cdots & \pi_L(-\omega_N) \\ \pi_{L+1}(-\omega_1) & \cdots & \pi_{L+1}(-\omega_N) \\ \vdots & \ddots & \vdots \\ \pi_{L+N-1}(-\omega_1) & \cdots & \pi_{L+N-1}(-\omega_N) \end{vmatrix}}. \tag{4.5}$$

We know that

$$z^n = n!\mathrm{e}^z \int_0^\infty \mathbf{L}_n(y)\mathrm{e}^{-y}{}_0F_1(1;-zy)\mathrm{d}y, \tag{4.6}$$

see, for example, equation (2.12) in Forrester and Liu [17]. For our purposes it is convenient to rewrite equation (4.6) as

$$(-z)^n = \mathrm{e}^z \int_0^\infty \pi_n(y)\mathrm{e}^{-y}{}_0F_1\left(1;-zy\right)\mathrm{d}y. \tag{4.7}$$

Inserting formula (4.7) into equation (4.5), we find

$$\mathbb{E}\left[\prod_{i=1}^N (z-x_i)\right] = \frac{(-1)^{L+N}\mathrm{e}^z}{z^L}\int_0^\infty \mathrm{e}^{-y}{}_0F_1(1;-zy)\frac{\begin{vmatrix} \pi_L(-\omega_1) & \dots & \pi_L(-\omega_N) & \pi_L(y) \\ \pi_{L+1}(-\omega_1) & \dots & \pi_{L+1}(-\omega_N) & \pi_{L+1}(y) \\ \vdots & \ddots & \vdots & \vdots \\ \pi_{L+N}(-\omega_1) & \dots & \pi_{L+N}(-\omega_N) & \pi_{L+N}(y) \end{vmatrix}}{\begin{vmatrix} \pi_L(-\omega_1) & \dots & \pi_L(-\omega_N) \\ \pi_{L+1}(-\omega_1) & \dots & \pi_{L+1}(-\omega_N) \\ \vdots & \ddots & \vdots \\ \pi_{L+N-1}(-\omega_1) & \dots & \pi_{L+N-1}(-\omega_N) \end{vmatrix}}\mathrm{d}y. \tag{4.8}$$

It remains to compute the ratio of determinants in the formula just written above in terms of multiple integrals. We use proposition 3.6 to rewrite both the numerator (with $n = L$, $m = N+1$, $\mu_1 = -\omega_1$, $\dots$, $\mu_N = -\omega_N$, $\mu_{N+1} = y$) and the denominator (with $n = L$, $m = N+1$, $\mu_1 = -\omega_1$, $\dots$, $\mu_N = -\omega_N$, $\mu_{N+1} = y$) as

$$\frac{\triangle\left(-\omega_1,\dots,-\omega_N,y\right)}{\triangle\left(-\omega_1,\dots,-\omega_N\right)}\frac{\int_0^\infty\dots\int_0^\infty\prod_{i=1}^L\prod_{j=1}^N\left(-\omega_j-t_i\right)\prod_{i=1}^L\left(y-t_i\right)\triangle^2\left(t_1,\dots,t_L\right)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty\dots\int_0^\infty\prod_{i=1}^L\prod_{j=1}^N\left(-\omega_j-t_i\right)\triangle^2\left(t_1,\dots,t_L\right)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}. \tag{4.9}$$

The ratio of the Vandermonde determinants can be written as

$$\frac{\triangle\left(-\omega_1,\dots,-\omega_N,y\right)}{\triangle\left(-\omega_1,\dots,-\omega_N\right)} = (-1)^N\prod_{j=1}^N\left(-\omega_j-y\right) = \prod_{j=1}^N\left(\omega_j+y\right), \tag{4.10}$$

and we finally arrive at the following formula for the averaged characteristic polynomial:

$$\mathbb{E}\left[\prod_{i=1}^N (z-x_i)\right] = (-1)^{N+L}\frac{\mathrm{e}^z}{z^L}\int_0^\infty \mathrm{d}y\mathrm{e}^{-y}{}_0F_1(1;-zy)\prod_{j=1}^N\left(\omega_j+y\right)$$
$$\times\frac{\int_0^\infty\dots\int_0^\infty\prod_{i=1}^L\prod_{j=1}^N\left(\omega_j+t_i\right)\prod_{i=1}^L\left(y-t_i\right)\triangle^2\left(t_1,\dots,t_L\right)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty\dots\int_0^\infty\prod_{i=1}^L\prod_{j=1}^N\left(\omega_j+t_i\right)\triangle^2\left(t_1,\dots,t_L\right)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}. \tag{4.11}$$

5. Average of ratios of characteristic polynomials

Theorem 5.1. *Consider the probability distribution on the space of complex matrices of size $N\times N$ defined by equation (1.1).*

(A) *Let $\omega_1, \ldots, \omega_N$ be the squared singular values of Ω, and assume that $L \geqslant 1$. We have*

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{v-x_i}{z-x_i}\right] = (-1)^{N+L+1}\frac{\mathrm{e}^{v}v^{-L}}{\tilde{\mathcal{N}}_L}\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}x\frac{v-x}{z-x}x^L\mathrm{e}^{-x}\int_C\frac{\mathrm{d}u\,{}_0F_1(1;ux)\mathrm{e}^{-u}}{\prod_{j=1}^{N}(u-\omega_j)}$$
$$\times\int_0^\infty\frac{\mathrm{d}s\mathrm{e}^{-s}{}_0F_1(1;-vs)}{s+u}\prod_{j=1}^{N}(s+\omega_j)\int_0^\infty\ldots\int_0^\infty\prod_{k=1}^{L}\prod_{j=1}^{N}(\omega_j+t_k)\,\triangle^2(t)\prod_{i=1}^{L}\frac{s-t_i}{u+t_i}\mathrm{e}^{-t_i}\mathrm{d}t_i, \tag{5.1}$$

where C is a counter-clockwise contour encircling the points $\omega_1, \ldots, \omega_N$, and leaving the real negative numbers $-s, -t_1, \ldots, -t_L$ outside. The normalization constant $\tilde{\mathcal{N}}_L$ is given by the formula (3.17).

(B) *Assume that $\Omega = z\mathbf{1}_N$, $z \in \mathbb{C}$. Then $\Omega^*\Omega = |z|^2\mathbf{1}_N$, and $\omega_1 = \ldots = \omega_N = |z|^2$. In this case we obtain the formula*

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{v-x_i}{z-x_i}\right]_{\deg} = (-1)^{N+L+1}\frac{\mathrm{e}^{v}v^{-L}}{\tilde{\mathcal{N}}_L}\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}x\frac{v-x}{z-x}x^L\mathrm{e}^{-x}\int_C\frac{\mathrm{d}u\,{}_0F_1(1;ux)\mathrm{e}^{-u}}{(u-|z|^2)^N}$$
$$\times\int_0^\infty\frac{\mathrm{d}s\mathrm{e}^{-s}{}_0F_1(1;-vs)}{s+u}\left(s+|z|^2\right)^N\int_0^\infty\ldots\int_0^\infty\prod_{k=1}^{L}\left(|z|^2+t_k\right)^N\triangle^2(t)\prod_{i=1}^{L}\frac{s-t_i}{u+t_i}\mathrm{e}^{-t_i}\mathrm{d}t_i, \tag{5.2}$$

where the normalization $\tilde{\mathcal{N}}_L = \int_0^\infty\ldots\int_0^\infty\prod_{i=1}^{L}\left(t_i+|z|^2\right)^N\triangle^2(t)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i$.

Proof. We begin from the following proposition

Proposition 5.2. *With the same notation as in the statement of proposition 3.1 the formula for the average of a ration of two characteristic polynomials can be written as*

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{v-x_i}{z-x_i}\right] = \int_0^\infty\mathrm{d}x\frac{v-x}{z-x}$$
$$\times\sum_{i=1}^{N}\zeta_i(x)\frac{\begin{vmatrix} g_{1,1} & \cdots & g_{1,i-1} & 1 & g_{1,i+1} & \cdots & g_{1,N}\\ g_{2,1} & \cdots & g_{2,i-1} & v & g_{2,i+1} & \cdots & g_{2,N}\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ g_{N,1} & \cdots & g_{N,i-1} & v^{N-1} & g_{N,i+1} & \cdots & g_{N,N}\end{vmatrix}}{\begin{vmatrix} g_{1,1} & \cdots & g_{1,N}\\ \vdots & \ddots & \vdots\\ g_{N,1} & \cdots & g_{N,N}\end{vmatrix}}. \tag{5.3}$$

Proof. This formula can be derived by the procedure very similar to that in Desrosiers and Forrester [16], see proposition 2. □

The ratio of determinants in the formula above can be rewritten as

$$\frac{\mathrm{e}^{-\omega_i}}{v^L}\frac{\begin{vmatrix}\pi_L(-\omega_1) & \dots & \pi_L(-\omega_{i-1}) & (-v)^L & \pi_L(-\omega_{i+1}) & \dots & \pi_L(-\omega_N)\\ \pi_{L+1}(-\omega_1) & \dots & \pi_{L+1}(-\omega_{i-1}) & (-v)^{L+1} & \pi_{L+1}(-\omega_{i+1}) & \dots & \pi_{L+1}(-\omega_N)\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \pi_{L+N-1}(-\omega_1) & \dots & \pi_{L+N-1}(-\omega_{i-1}) & (-v)^{L+N-1} & \pi_{L+N-1}(-\omega_{i+1}) & \dots & \pi_{L+N-1}(-\omega_N)\end{vmatrix}}{\begin{vmatrix}\pi_L(-\omega_1) & \dots & \pi_L(-\omega_N)\\ \pi_{L+1}(-\omega_1) & \dots & \pi_{L+1}(-\omega_N)\\ \vdots & \ddots & \vdots\\ \pi_{L+N-1}(-\omega_1) & \dots & \pi_{L+N-1}(-\omega_N)\end{vmatrix}}.$$

We use formula (4.7) to rewrite the equation just written above as

$$\frac{\mathrm{e}^{-\omega_i+v}}{v^L}\int_0^\infty \mathrm{d}s\mathrm{e}^{-s}{}_0F_1(1;-vs)$$
$$\times\frac{\begin{vmatrix}\pi_L(-\omega_1) & \dots & \pi_L(-\omega_{i-1}) & \pi_L(s) & \pi_L(-\omega_{i+1}) & \dots & \pi_L(-\omega_N)\\ \pi_{L+1}(-\omega_1) & \dots & \pi_{L+1}(-\omega_{i-1}) & \pi_{L+1}(s) & \pi_{L+1}(-\omega_{i+1}) & \dots & \pi_{L+1}(-\omega_N)\\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \pi_{L+N-1}(-\omega_1) & \dots & \pi_{L+N-1}(-\omega_{i-1}) & \pi_{L+N-1}(s) & \pi_{L+N-1}(-\omega_{i+1}) & \dots & \pi_{L+N-1}(-\omega_N)\end{vmatrix}}{\begin{vmatrix}\pi_L(-\omega_1) & \dots & \pi_L(-\omega_N)\\ \pi_{L+1}(-\omega_1) & \dots & \pi_{L+1}(-\omega_N)\\ \vdots & \ddots & \vdots\\ \pi_{L+N-1}(-\omega_1) & \dots & \pi_{L+N-1}(-\omega_N)\end{vmatrix}}.$$

Again, we represent the ratio of determinants with orthogonal polynomial entries in terms of multiple integrals using proposition 3.6 to both numerator and denominator:

$$\frac{\triangle(-\omega_1,...,-\omega_{i-1},s,-\omega_{i+1},...,-\omega_N)}{\triangle(-\omega_1,...,-\omega_N)}\frac{\int_0^\infty ...\int_0^\infty \prod_{k=1}^L\prod_{\substack{j=1\\ j\neq i}}^N(-\omega_j-t_k)\prod_{k=1}^L(s-t_k)\triangle^2(t_1,...,t_L)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty ...\int_0^\infty \prod_{k=1}^L\prod_{j=1}^N(-\omega_j-t_k)\triangle^2(t_1,...,t_L)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}. \tag{5.4}$$

The ratio of the Vandermonde determinants can be simplified as

$$\frac{\triangle(-\omega_1,\dots,-\omega_{i-1},s,-\omega_{i+1},\dots,-\omega_N)}{\triangle(-\omega_1,\dots,-\omega_N)}=\prod_{\substack{k=1\\ k\neq i}}^N\frac{\omega_k+s}{\omega_k-\omega_i}. \tag{5.5}$$

Therefore, the involved ratio of the determinants can be rewritten as

$$(-1)^L\prod_{\substack{k=1\\ k\neq i}}^N\frac{\omega_k+s}{\omega_k-\omega_i}\frac{\int_0^\infty ...\int_0^\infty \prod_{k=1}^L\prod_{\substack{j=1\\ j\neq i}}^N(\omega_j+t_k)\prod_{k=1}^L(s-t_k)\triangle^2(t_1,...,t_L)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty ...\int_0^\infty \prod_{k=1}^L\prod_{j=1}^N(\omega_j+t_k)\triangle^2(t_1,...,t_L)\prod_{i=1}^L\mathrm{e}^{-t_i}\mathrm{d}t_i}.$$

These calculations give us the following formula for the average of ratios of characteristic polynomials

$$\mathbb{E}\left[\prod_{i=1}^{N}\frac{v-x_i}{z-x_i}\right]=\int_0^\infty \mathrm{d}x\frac{v-x}{z-x}$$
$$\times\sum_{i=1}^{N}\zeta_i(x)\left(\frac{\mathrm{e}^{-\omega_i+v}}{v^L}\int_0^\infty \mathrm{d}s\mathrm{e}^{-s}{}_0F_1(1;-vs)(-1)^L\prod_{\substack{k=1\\k\neq i}}^{N}\frac{\omega_k+s}{\omega_k-\omega_i}\right)$$
$$\times\frac{\int_0^\infty\dots\int_0^\infty\prod_{k=1}^{L}\prod_{\substack{j=1\\j\neq i}}^{N}(\omega_j+t_k)\prod_{k=1}^{L}(s-t_k)\triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i}{\int_0^\infty\dots\int_0^\infty\prod_{k=1}^{L}\prod_{j=1}^{N}(\omega_j+t_k)\triangle^2(t_1,\dots,t_L)\prod_{i=1}^{L}\mathrm{e}^{-t_i}\mathrm{d}t_i}. \tag{5.6}$$

Taking into account the explicit form of the functions $\zeta_i(x)$ (see equation (3.2)), and applying the basic residue theorem, we obtain the formula in the statement of theorem 5.1 after some straightforward manipulations.

6. The formula for the correlation kernel

Theorem 6.1. *Consider the determinantal process formed by the squared singular values of a random matrix X whose probability distribution is defined by formula (1.1). The correlation kernel $K_N(x,y)$ for this determinantal point process can be written as*

$$K_N(x,y)=\frac{(-1)^{N+L+1}\mathrm{e}^{x-y}}{\tilde{\mathcal{N}}_L}\left(\frac{y}{x}\right)^L\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}u\ {}_0F_1(1;uy)\mathrm{e}^{-u}}{\prod_{j=1}^{N}(u-\omega_j)}\int_0^\infty\frac{\mathrm{d}s\mathrm{e}^{-s}{}_0F_1(1;-xs)}{s+u}\prod_{j=1}^{N}(s+\omega_j)$$
$$\times\int_0^\infty\dots\int_0^\infty\prod_{k=1}^{L}\prod_{j=1}^{N}(\omega_j+t_k)\triangle^2(t)\prod_{i=1}^{L}\frac{s-t_i}{u+t_i}\mathrm{e}^{-t_i}\mathrm{d}t_i, \tag{6.1}$$

where C is a counter-clockwise contour encircling the points $\omega_1,\dots,\omega_N$, and leaving the real negative numbers $-s,-t_1,\dots,-t_L$ outside. The normalization constant $\tilde{\mathcal{N}}_L$ is given by formula (3.17).

Proof. It is known that for polynomial ensembles the correlation kernel can be written in terms of averages of ratios of characteristic polynomials as follows

$$K_N(x,y)=\frac{1}{x-y}\operatorname*{Res}_{z=y}\left(\mathbb{E}\left[\prod_{i=1}^{N}\frac{x-x_i}{z-x_i}\right]\right), \tag{6.2}$$

see Desrosiers and Forrester [16], equation (11). Use the equation just written above, and theorem 5.1. □

7. Complex bulk/chiral edge scaling regime asymptotics in the degenerate case

We study the $\omega_1=\dots=\omega_N=|z|^2$ degenerate case of averages considered in theorems 3.4(B), 4.1(B) and 5.1(B) in a *complex bulk/chiral edge scaling* asymptotic regime where $|z|^2=NR$, $R\in[0,1)$ (bulk regime from the point of view of complex eigenvalues of the matrix X) and the argument of characteristic polynomials scales as N^{-1} (the 'edge' regime from the point of view of real eigenvalues of the matrix $X^\dagger X$ or its chiral counterpart). Such choice is motivated, in particular, by the applications [18] and a detailed discussion of this is given in section 3.4.

7.1. Asymptotics of the averaged inverse characteristic polynomial

Proposition 7.1. *In the degenerate case $\omega_1 = \ldots = \omega_N = |z|^2$, the averaged inverse characteristic polynomial $\mathbb{E}\left[\prod_{i=1}^{N}\frac{1}{p+x_i}\right]_{\deg} \equiv Q_N(p)$ given by equation (3.19) has the following complex bulk/chiral edge asymptotic behaviour for general $L \geqslant 0$*

$$\lim_{N\to\infty}\frac{N^{N+L-1/2}}{\mathrm{e}^{NR_*}}Q_N\left(p=\frac{\xi}{NR_*}\right)=\sqrt{\frac{2}{\pi}}\xi^{\frac{L}{2}}K_L\left(2\sqrt{\xi}\right), \tag{7.1}$$

where $R_* = 1 - R$ and the source is scaled with N as $|z|^2 = NR$ with fixed $R \in [0, 1)$.

Proof. We proceed similarly to the proof of proposition 3.8. The formula (3.19) is rewritten by using the expression $\frac{1}{p+u} = \int_0^1 \frac{\mathrm{d}\tau}{(1-\tau)^2}\mathrm{e}^{-\frac{(p+u)\tau}{1-\tau}}$ and the identity (3.7) to evaluate the integral over u. The result reads (see equation (3.38))

$$Q_N(p)=\int_0^1 \mathrm{d}\tau(1-\tau)^{L-1}\mathrm{e}^{-\frac{\tau p}{1-\tau}}\,G_N^{(L)}\left(|z|^2,\tau\right), \tag{7.2}$$

where

$$G_N^{(L)}\left(|z|^2,\tau\right)=\frac{(-1)^{N-1}L!}{\tilde{\mathcal{N}}_L}\int_0^\infty\ldots\int_0^\infty \triangle^2(t)\prod_{k=1}^{L}(t_k+|z|^2)^N\,\mathrm{e}^{-t_k}\mathrm{d}t_k\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}v\mathrm{e}^{-v\tau}}{(v-|z|^2)^N}\frac{\mathbf{L}_L\left(v(\tau-1)\right)}{\prod_{k=1}^{L}(t_k+v)}.$$

where $\mathbf{L}_k(x)$ are Laguerre polynomials. To extract the scaling limit of $G_N^{(L)}\left(|z|^2,\tau\right)$, we rescale the parameters $|z|^2 = NR$ and $\tau = 1 - a/N$ and integration variables $v \to Nv$ and $t_k = Nq_k$. In this way we define the rescaled function $g(R,a)=\frac{G_N^{(L)}\left(NR,1-\frac{a}{N}\right)}{N^{(N+L)(L-1)+1}}$ being equal to

$$g(R,a)=\frac{(-1)^{N-1}L!}{\tilde{\mathcal{N}}_L}\int_0^\infty\ldots\int_0^\infty \triangle^2(q)\prod_{k=1}^{L}(q_k+R)^N\,\mathrm{e}^{-Nq_k}\mathrm{d}q_k\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}v\mathrm{e}^{-Nv}}{(v-R)^N}\frac{\mathrm{e}^{va}\mathbf{L}_L\left(-va\right)}{\prod_{k=1}^{L}(q_k+v)}. \tag{7.3}$$

We use an identity

$$\prod_{k=1}^{L}\frac{1}{q_k+v}=\sum_{k=1}^{L}\frac{1}{q_k+v}\prod_{\substack{m=1\\m\neq k}}^{L}\frac{1}{(q_m-q_k)} \tag{7.4}$$

which, when substituted back to equation (7.3), reduces the sum over k to its last term $\frac{1}{q_L+v}\prod_{m=1}^{L-1}\frac{1}{(q_m-q_L)}$ and multiply the result by L due to the symmetry of q_k integrals. The distinguished variable is renamed $q_L \to q$

$$\begin{aligned}g(R,a)=&\frac{(-1)^{N+L-2}LL!}{\tilde{\mathcal{N}}_L}\int_0^\infty\ldots\int_0^\infty \triangle^2(q_1,\ldots,q_{L-1})\prod_{k=1}^{L-1}(q_k+R)^N\,\mathrm{e}^{-Nq_k}\mathrm{d}q_k\\&\times\int_0^\infty\mathrm{d}q(q+R)^N\mathrm{e}^{-Nq}\prod_{i=1}^{L-1}(q-q_i)\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}v\mathrm{e}^{-Nv}}{(v-R)^N}\frac{\mathrm{e}^{va}\mathbf{L}_L\left(-va\right)}{q+v}.\end{aligned} \tag{7.5}$$

We consider the contour integral

$$I(q) = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}v \mathrm{e}^{-N\mathcal{L}_1(v)} \frac{\mathrm{e}^{va}\mathbf{L}_L(-va)}{q+v}, \tag{7.6}$$

with $\mathcal{L}_1(v) = v + \log(v - R)$ which we approximate by the method of steepest descent. Denoting $R_* = 1 - R$, the saddle point is found at $v_* = -R_*$ and located on the negative real axis as $R < 1$. Since the orginal contour C encircles the pole at $v = R$ and does not cross the negative real axis, it has to be deformed to pass through the saddle v_*. This adds a contribution to the integral coming from the pole at $v = -q$ as long as $-q > v_*$:

$$\int_C = \int_{C_{\mathrm{sp}}(R)} - \int_{C(-q)} \theta(-q - v_*),$$

where $C_{\mathrm{sp}}(R)$ is a contour encircling R and passing through the saddle point v_* whereas $C(-q)$ is a path encircling $-q$. We compute both contributions

$$I(q) = \frac{1}{2\pi \mathrm{i}} \int_{C_{\mathrm{sp}}(R)} \mathrm{d}v \mathrm{e}^{-N\mathcal{L}_1(v)} \frac{\mathrm{e}^{va}\mathbf{L}_L(-va)}{q+v} + \theta(R_* - q)(-1)^{N+1}\mathrm{e}^{N\mathcal{L}_2(q)-aq}\mathbf{L}_L(aq) \tag{7.7}$$

and substitute it back into the equation (7.5) to obtain:

$$g(R, a) \sim g_1 + g_2,$$

where

$$\begin{aligned} g_1 =& \frac{(-1)^{N+L}LL!}{\tilde{\mathcal{N}}_L} \int_0^\infty \dots \int_0^\infty \triangle^2(q_1, \dots, q_{L-1}) \prod_{k=1}^{L-1} \mathrm{e}^{-N\mathcal{L}_2(q_k)}\mathrm{d}q_k \\ &\times \int_0^\infty \mathrm{d}q \prod_{i=1}^{L-1}(q - q_i)\mathrm{e}^{-N\mathcal{L}_2(q)} \frac{1}{2\pi\mathrm{i}} \int_{C_{\mathrm{sp}}(R)} \mathrm{d}v \mathrm{e}^{-N\mathcal{L}_1(v)} \frac{\mathrm{e}^{va}\mathbf{L}_L(-va)}{q+v}, \\ g_2 =& \frac{(-1)^{L-1}LL!}{\tilde{\mathcal{N}}_L} \int_0^\infty \dots \int_0^\infty \triangle^2(q_1, \dots, q_{L-1}) \prod_{k=1}^{L-1} \mathrm{e}^{-N\mathcal{L}_2(q_k)}\mathrm{d}q_k \int_0^{R_*} \mathrm{d}q \prod_{i=1}^{L-1}(q - q_i)\mathrm{e}^{-aq}\mathbf{L}_L(aq) \end{aligned} \tag{7.8}$$

and $\mathcal{L}_2(q) = q - \ln(q + R)$. We first compute the g_1 term by approximating the q_k, q integrals around the saddle points $q_k = R_* + N^{-1/2}x_k$, $q = R_* + N^{-1/2}x$ and the contour integral v around $v = -R_* + \mathrm{i}N^{-1/2}y$:

$$\begin{aligned} g_1 \sim& \frac{(-1)^L LL!\mathrm{e}^{-N(L-1)R_*}}{N^{(L-1)^2/2}\tilde{\mathcal{N}}_L} \frac{\mathrm{e}^{-aR_*}\mathbf{L}_L(aR_*)}{2\pi N^{L/2}} \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \triangle^2(x_1, \dots, x_{L-1}) \prod_{k=1}^{L-1} \mathrm{e}^{-\frac{x_k^2}{2}}\mathrm{d}x_k \\ &\times \int_{-\infty}^\infty \mathrm{d}x\mathrm{d}y \frac{\mathrm{e}^{-\frac{x^2}{2}-\frac{y^2}{2}}}{x+iy} \prod_{i=1}^{L-1}(x - x_i). \end{aligned} \tag{7.9}$$

In the above formula, we introduce the following representation of the Hermite monic polynomials h_n, see e.g. [24]

$$\int_{-\infty}^\infty \prod_{i=1}^{L-1} \mathrm{e}^{-\frac{x_i^2}{2}}\mathrm{d}x_i \triangle^2(x) \prod_{i=1}^{L-1}(z - x_i) = h_{L-1}(z) \prod_{k=1}^{L-1}\left(k!\sqrt{2\pi}\right), \tag{7.10}$$

which results in the first contribution g_1 equal to

$$g_1(R,a) \sim \frac{(-1)^L LL! \mathrm{e}^{-N(L-1)R_*}}{N^{(L-1)^2/2}\tilde{\mathcal{N}}_L} \prod_{i=1}^{L-1}(i!\sqrt{2\pi}) \frac{\mathrm{e}^{-aR_*}\mathbf{L}_L(aR_*)}{2\pi N^{L/2}} \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y \frac{\mathrm{e}^{-\frac{x^2}{2}-\frac{y^2}{2}}}{x+iy} h_{L-1}(x), \tag{7.11}$$

where the remaining x, y integrals do not depend on N any longer. We now turn our attention to the second term g_2 given by equation (7.8). In this case, the q integral is computed exactly due to an exact cancellation of the N-dependent parts:

$$g_2(R,a) \sim \frac{(-1)^{L-1} LL! \mathrm{e}^{-N(L-1)R_*}}{N^{(L-1)^2/2}\tilde{\mathcal{N}}_L} \prod_{i=1}^{L-1}(i!\sqrt{2\pi}) \int_0^{R_*} \mathrm{d}q(q-R_*)^{L-1}\mathrm{e}^{-aq}\mathbf{L}_L(aq), \tag{7.12}$$

and using a Mehta-type integral formula $\int_{-\infty}^{\infty} \triangle^2(x_1,...,x_n)\prod_{k=1}^{n} \mathrm{e}^{-\frac{x_k^2}{2}}\mathrm{d}x_k = \prod_{i=1}^{n}(i!\sqrt{2\pi})$.

By comparing both contributions to g given by equations (7.11) and (7.12) we see that $g_2/g_1 \sim N^{L/2}$. Hence, the term g_2 is dominant in the large N limit and we can safely disregard the contribution g_1 in what follows. The remaining q integral in the contribution coming from formula (7.12) is found exactly by expressing Laguerre polynomial as $\mathrm{e}^{-x}\mathbf{L}_n(x) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^n)$ and integrating by parts:

$$\int_0^{R_*} \mathrm{d}q(q-R_*)^{L-1}\mathrm{e}^{-aq}\mathbf{L}_L(aq) = -\frac{1}{L}\mathrm{e}^{-aR_*}(-R_*)^L. \tag{7.13}$$

In a similar fashion we approximate the normalization constant $\tilde{\mathcal{N}}_L$ given by formula (3.17) as

$$\tilde{\mathcal{N}}_L \sim N^{NL+L^2/2}\mathrm{e}^{-NLR_*}\prod_{i=1}^{L}(i!\sqrt{2\pi}). \tag{7.14}$$

At last, by using equations (7.13) and (7.14) in equation (7.15) we arrive at

$$G_N^{(L)}\left(R, 1-\frac{a}{N}\right) \sim \frac{\mathrm{e}^{NR_*}}{N^{N-1/2}}\frac{\mathrm{e}^{-aR_*}R_*^L}{\sqrt{2\pi}}. \tag{7.15}$$

Such an expression and an integral representation of the Bessel–Macdonald function

$$\int_0^{\infty} \mathrm{d}a a^{L-1}\mathrm{e}^{-\frac{\xi}{a}-aR_*} = 2\left(\frac{\xi}{R_*}\right)^{\frac{L}{2}} K_L\left(2\sqrt{\xi R_*}\right)$$

can now be used in equation (7.2) to obtain our final asymptotic formula for the 'complex bulk/chiral edge asymptotics' of the mean of inverse characteristic polynomial:

$$Q_N\left(p=\frac{\xi}{NR_*}\right) \sim \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{NR_*}}{N^{N+L-1/2}}\xi^{\frac{L}{2}}K_L\left(2\sqrt{\xi}\right)$$

equivalent to the statement of the proposition. □

The above considerations can be trivially translated into the asymptotic behavior of the object $\mathcal{G}_N^{(L)}(\rho,\tau)$ defined in formulas (3.37) and (3.38) and featuring in applications in [18]. Namely, we have after rescaling $\rho = Nw^2$ and exploiting equation (7.15)

$$\lim_{N\to\infty} \frac{1}{N^{(N-1/2)(L-1)+1}} \mathcal{G}_N^{(L)}(Nw^2, \tau = 1 - a/N) = \left(\prod_{k=1}^{L} k!\right) (2\pi)^{(L-1)/2} \mathrm{e}^{-a(1-w^2)} (1 - w^2)^L. \tag{7.16}$$

By renaming $a \to 1/s$ and considering the special case $L = 2$ one can check that this formula is exactly equivalent, after appropriate interpretation, to the expression equation (2.24) in [18] derived there via a relatively tedious procedure.

7.2. Asymptotics of the averaged characteristic polynomial

Proposition 7.2. *In the degenerate case $\omega_1 = ... = \omega_N = |z|^2$, the averaged characteristic polynomial $\mathbb{E}\left[\prod_{i=1}^N (p + x_i)\right]_{\mathrm{deg}} \equiv P_N(p)$ given by equation (4.2) has the following complex bulk/chiral edge asymptotic behaviour for general $L \geqslant 0$*

$$\lim_{N\to\infty} \frac{\mathrm{e}^{NR_*}}{N^{L+N+1/2}} P_N\left(p = \frac{\xi}{NR_*}\right) = (-1)^L \sqrt{2\pi} \xi^{-\frac{L}{2}} I_L\left(2\sqrt{\xi}\right), \tag{7.17}$$

where $R_* = 1 - R$, the source scaled as $|z|^2 = NR$ with a fixed $R \in [0, 1)$ and $I_L(x)$ stands for the Bessel function of imaginary argument.

Proof. We start with the formula (4.2) in which we rescale the parameters accordingly $|z|^2 = NR$, $y \to Ny$, $p = \xi/N$, $t_i = Nq_i$ to find

$$\frac{P_N\left(p = \frac{\xi}{N}\right)}{N^{(N+L+1)(L+1)}} = \frac{1}{\tilde{\mathcal{N}}_L} \frac{\mathrm{e}^{-\xi/N}}{\xi^L} \int_0^\infty \mathrm{d}y \mathrm{e}^{-N\mathcal{L}_2(y)} I_0(2\sqrt{\xi y}) \int_0^\infty ... \int_0^\infty \triangle^2(q) \prod_{i=1}^{L} (y - q_i) \mathrm{e}^{-N\mathcal{L}_2(q_i)} \mathrm{d}q_i,$$

with $\mathcal{L}_2(y) = y - \ln(R + y)$. The integral is evaluated in the leading approximation via the saddle-point method by setting $q_i = R_* + N^{-1/2}x_i$, $y = R_* + N^{-1/2}z$ where $R_* = 1 - R$ and using an representation of Hermite monic polynomials introduced in formula (7.10). With the approximate normalization coefficient given by formula (7.14), the result reads

$$P_N\left(p = \frac{\xi}{N}\right) \sim \frac{N^{1/2+\frac{3}{2}L+N} \mathrm{e}^{-NR_*}}{\xi^L} \int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-\frac{z^2}{2}} I_0\left(2\sqrt{\xi R_* + N^{-1/2} z\xi}\right) h_L(z).$$

By using the Rodrigues identity $h_L(z) = (-1)^L \mathrm{e}^{z^2/2} \frac{\mathrm{d}^L}{\mathrm{d}z^L} \mathrm{e}^{-z^2/2}$, the last integral is evaluated as

$$\int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-\frac{z^2}{2}} I_0\left(2\sqrt{\xi R_* + N^{-1/2} z\xi}\right) h_L(z) = \int_{-\infty}^{\infty} \mathrm{d}z \mathrm{e}^{-z^2/2} \frac{\mathrm{d}^L}{\mathrm{d}z^L} I_0\left(2\sqrt{\xi R_* + N^{-1/2} z\xi}\right)$$
$$\sim (-1)^L \sqrt{2\pi} N^{-L/2} \frac{\mathrm{d}^L}{\mathrm{d}R^L} I_0\left(2\sqrt{\xi R_*}\right) = (-1)^L \sqrt{2\pi} N^{-L/2} \left(\frac{\xi}{R_*}\right)^{L/2} I_L\left(2\sqrt{\xi R_*}\right).$$

and plugged into previous expression to obtain

$$P_N\left(p=\frac{\xi}{N}\right)\sim(-1)^L\frac{N^{1/2+L+N}\sqrt{2\pi}}{\mathrm{e}^{NR_*}}\left(\frac{1}{\xi R_*}\right)^{L/2}I_L\left(2\sqrt{\xi R_*}\right),$$

giving the final formula by rescaling $\xi\to\xi/R_*$. □

7.3. Asymptotics of the kernel

Proposition 7.3. *In the degenerate case $\omega_1=...=\omega_N=|z|^2$, the kernel $K_N(x,y)$ given by equation (6.1) has the following complex bulk/chiral edge asymptotic behaviour for general $L\geqslant 0$*

$$\lim_{N\to\infty}\frac{1}{NR_*}K_N\left(x=\frac{\alpha}{NR_*},y=\frac{\beta}{NR_*}\right)=\left(\frac{\beta}{\alpha}\right)^{\frac{L}{2}}\int_0^1\mathrm{d}\tau J_L\left(2\sqrt{\alpha\tau}\right)J_L\left(2\sqrt{\beta\tau}\right),\tag{7.18}$$

where $R_*=1-R$, the source is scaled as $|z|^2=NR$ with a fixed $R\in[0,1)$ and $J_L(x)$ stands for the standard Bessel function.

Proof. The formula for the kernel in the degenerate case is given by expression (6.1) upon setting $\omega_1=...=\omega_N=|z|^2$. We first rescale variables in that equation $|z|^2=NR$, $t_i\to Nt_i$, $u\to Nu$, $s\to Ns$ and introduce a rescaled kernel

$$k(\xi,\eta)=\frac{K_N\left(x=\frac{\xi}{N},y=\frac{\eta}{N}\right)}{N^{L(N+L)+1}}.\tag{7.19}$$

Taking into account the explicit formula for the kernel $K_N(x,y)$ obtained in theorem 6.1 we can represent $k(\xi,\eta)$ as

$$k(\xi,\eta)=\frac{(-1)^{N+L+1}\mathrm{e}^{\frac{\xi-\eta}{N}}\left(\frac{\eta}{\xi}\right)^L}{\tilde{\mathcal{N}}_L}\frac{1}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}u\,{}_0F_1(1;u\eta)\mathrm{e}^{-Nu}}{(u-R)^N}$$
$$\times\int_0^\infty\frac{\mathrm{d}s\mathrm{e}^{-Ns}{}_0F_1(1;-\xi s)}{s+u}(s+R)^N\int_0^\infty...\int_0^\infty\prod_{k=1}^L(R+t_k)^N\triangle^2(t)\prod_{i=1}^L\frac{s-t_i}{u+t_i}\mathrm{e}^{-Nt_i}\mathrm{d}t_i.\tag{7.20}$$

Similarly as in the proof of proposition 7.1, the term $\prod_{i=1}^L\frac{1}{u+t_i}$ of the above formula is expressed by the identity (7.4), the last integration variable is singled out and renamed $t_L=t$

$$k(\xi,\eta)=\frac{L(-1)^N\mathrm{e}^{\frac{\xi-\eta}{N}}\left(\frac{\eta}{\xi}\right)^L}{\tilde{\mathcal{N}}_L}\int_0^\infty\mathrm{d}t_1...\int_0^\infty\mathrm{d}t_{L-1}\int_0^\infty\mathrm{d}s\int_0^\infty\mathrm{d}tA_2,\int_C\mathrm{d}uA_1\tag{7.21}$$

where the integrands are equal to

$$A_1=\frac{1}{2\pi\mathrm{i}}\mathrm{e}^{-N\mathcal{L}_1(u)}{}_0F_1(1;u\eta)\frac{s-t}{(u+t)(u+s)},$$
$$A_2=\mathrm{e}^{-N\mathcal{L}_2(t)-N\mathcal{L}_2(s)}\prod_{k=1}^{L-1}\left[\mathrm{e}^{-N\mathcal{L}_2(t_k)}(s-t_k)(t-t_k)\right]{}_0F_1(1;-\xi s)\triangle^2(t_1...t_{L-1}),\tag{7.22}$$

with notation $\mathcal{L}_1(x) = x + \log(x - R), \mathcal{L}_2(x) = x - \ln(R + x)$ introduced already in the proofs of propositions 7.1 and 7.2. Both the order of integrations and grouping of integrands are selected with the view to facilitate the subsequent asymptotic analysis. The first integral in equation (7.21)

$$\int_C \mathrm{d}u A_1 = I_C(t) - I_C(s) \tag{7.23}$$

is, by the formula $\frac{s-t}{(u+t)(u+s)} = \frac{1}{u+t} - \frac{1}{u+s}$, equal to the difference between the values of the same contour integral I_C evaluated at two different arguments. This fundamental object reads

$$I_C(x) = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}u \mathrm{e}^{-N\mathcal{L}_1(u)} \frac{{}_0F_1(1; u\eta)}{u + x}$$

and differs from the previously studied integral (7.6) only by the N-independent part. Then, following the same analysis as has been done in the proof of proposition 7.1, the saddle-point method identifies the saddle point at $u_* = -R_*$ where $R_* = 1 - R$ and the leading-order approximation comprising two contributions is given by

$$I_C(x) \sim \frac{1}{2\pi \mathrm{i}} \int_{C_{sp}(R)} \mathrm{d}u \mathrm{e}^{-N\mathcal{L}_1(u)} \frac{{}_0F_1(1; u\eta)}{u + x} + (-1)^{N+1} \frac{\mathrm{e}^{Nx} {}_0F_1(1; -x\eta)}{(x + R)^N} \theta(R_* - x).$$

One can again argue that the first term yields eventually a sub-leading contribution, whereas $C_{sp}(R)$ denotes a contour encircling the pole at R and passing through the saddle point u_*. Thus, the first integration yields

$$\int_C \mathrm{d}u A_1 \sim (-1)^{N+1} \mathrm{e}^{N\mathcal{L}_2(t)} {}_0F_1(1; -t\eta) \theta(R_* - t) - (-1)^{N+1} \mathrm{e}^{N\mathcal{L}_2(s)} {}_0F_1(1; -s\eta) \theta(R_* - s)$$

which in turn allows to represent the rescaled kernel (7.21) as a sum of two parts

$$k(\xi, \eta) \sim \frac{L}{\tilde{\mathcal{N}}_L} \left(\frac{\eta}{\xi} \right)^L (J_2 - J_1), \tag{7.24}$$

where we denoted

$$J_1 = \int_0^\infty \mathrm{d}t_1 ... \int_0^\infty \mathrm{d}t_{L-1} \int_0^\infty \mathrm{d}s \int_0^{1-R} \mathrm{d}t A_2^{(t)}, \qquad J_2 = \int_0^\infty \mathrm{d}t_1 ... \int_0^\infty \mathrm{d}t_{L-1} \int_0^{1-R} \mathrm{d}s \int_0^\infty \mathrm{d}t A_2^{(s)},$$

and rescaled the integrands as $A_2^{(t)} = A_2 \mathrm{e}^{N\mathcal{L}_2(t)} {}_0F_1(1; -t\eta)$, $A_2^{(s)} = A_2 \mathrm{e}^{N\mathcal{L}_2(s)} {}_0F_1(1; -s\eta)$.

7.3.1 Asymptotics of J_1. We evaluate J_1 integral by the saddle-point procedure applied first to the t_i's integrals by expanding $t_i = R_* + N^{-1/2} x_i$ around the corresponding saddle points $R_* = 1 - R$:

$$J_1 \sim \frac{\mathrm{e}^{-N(L-1)R_*}}{N^{\frac{L^2}{2} - \frac{1}{2}}} \int_0^{R_*} \mathrm{d}t \, {}_0F_1 \left(1; -\eta t \right) \int_0^\infty \mathrm{d}s \mathrm{e}^{-N\mathcal{L}_2(s)} {}_0F_1 \left(1, -\xi s \right)$$
$$\times \int_{-\infty}^\infty ... \int_{-\infty}^\infty \prod_{i=1}^{L-1} \mathrm{e}^{-\frac{x_i^2}{2}} \mathrm{d}x_i \triangle^2(x_1, ..., x_{L-1}) \prod_{k=1}^{L-1} \left(\sqrt{N}(s - R_*) - x_k \right) \left(\sqrt{N}(t - R_*) - x_k \right). \tag{7.25}$$

The $L-1$ integrals can be then re-expressed in terms of the Hermite monic polynomials h_n by the formula found in [24]:

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\prod_{i=1}^{L-1}\mathrm{e}^{-\frac{x_i^2}{2}}\mathrm{d}x_i\triangle^2(x_1\ldots x_{L-1})\prod_{i=1}^{L-1}(\lambda_1-x_i)(\lambda_2-x_i)=\frac{\prod_{j=1}^{L-1}(j!\sqrt{2\pi})}{\lambda_2-\lambda_1}\begin{vmatrix}h_{L-1}(\lambda_1) & h_{L-1}(\lambda_2)\\ h_L(\lambda_1) & h_L(\lambda_2)\end{vmatrix}. \tag{7.26}$$

The resulting expression for J_1 reads

$$J_1\sim\prod_{j=1}^{L-1}(j!\sqrt{2\pi})\frac{\mathrm{e}^{-N(L-1)R_*}}{N^{L^2/2}}\int_0^{R_*}\mathrm{d}t\,{}_0F_1\Big(1;-\eta t\Big)\int_0^{\infty}\mathrm{d}s\mathrm{e}^{-N\mathcal{L}_2(s)}{}_0F_1\Big(1,-\xi s\Big)$$
$$\times\frac{1}{t-s}\begin{vmatrix}h_{L-1}\Big(\sqrt{N}(s-R_*)\Big) & h_{L-1}\Big(\sqrt{N}(t-R_*)\Big)\\ h_L\Big(\sqrt{N}(s-R_*)\Big) & h_L\Big(\sqrt{N}(t-R_*)\Big)\end{vmatrix} \tag{7.27}$$

and contains two terms due to the determinant. The saddle-point approximation of the s integral alone is performed around the corresponding saddle point $s=R_*+N^{-1/2}\sigma$ and with the use of Rodrigues formula for the Hermite monic polynomials:

$$\int_0^{\infty}\mathrm{d}s\mathrm{e}^{-N\mathcal{L}_2(s)}\frac{h_n\left(\sqrt{N}(s-R_*)\right){}_0F_1\Big(1,-\xi s\Big)}{t-s}\sim\frac{\mathrm{e}^{-NR_*}}{\sqrt{N}}\int_{-\infty}^{\infty}\mathrm{d}\sigma\mathrm{e}^{-\frac{\sigma^2}{2}}h_n(\sigma)\frac{{}_0F_1\Big(1,-\xi\left(R_*+\frac{\sigma}{\sqrt{N}}\right)\Big)}{t-R_*-\frac{\sigma}{\sqrt{N}}}$$
$$=\frac{\mathrm{e}^{-NR_*}}{\sqrt{N}}\int_{-\infty}^{\infty}\mathrm{d}\sigma\mathrm{e}^{-\frac{\sigma^2}{2}}\frac{\mathrm{d}^n}{\mathrm{d}\sigma^n}\left[\frac{{}_0F_1\Big(1,-\xi\left(R_*+\frac{\sigma}{\sqrt{N}}\right)\Big)}{t-R_*-\frac{\sigma}{\sqrt{N}}}\right]\sim\frac{\sqrt{2\pi}(-1)^n\mathrm{e}^{-NR_*}}{\sqrt{N}^{n+1}}\frac{\mathrm{d}^n}{\mathrm{d}R^n}\left[\frac{{}_0F_1\Big(1,-\xi R_*\Big)}{t-R_*}\right].$$

To identify the leading contribution to J_1, it is safe to approximate the t-dependent Hermite polynomials by its highest power i.e. $h_n(\sqrt{N}(t-R_*))\sim\sqrt{N}^n(t-R_*)^n$. By counting the powers of N, leading order contribution to J_1 is proportional to $h_{L-1}\Big(\sqrt{N}(s-R_*)\Big)h_L\Big(\sqrt{N}(t-R_*)\Big)$ and the formula (7.27) reads

$$J_1\sim(-1)^{L-1}\prod_{j=0}^{L-1}j!\frac{\mathrm{e}^{-NLR_*}\sqrt{2\pi}^L}{N^{L^2/2}}\int_0^{R_*}\mathrm{d}t\,{}_0F_1\Big(1;-\eta t\Big)\frac{\mathrm{d}^{L-1}}{\mathrm{d}R^{L-1}}\left[\frac{{}_0F_1\Big(1,-\xi R_*\Big)}{t-R_*}\right](t-R_*)^L.$$

The remaining integral can be represented as a sum

$$\int_0^{R_*}\mathrm{d}t\,{}_0F_1\Big(1;-\eta t\Big)\frac{\mathrm{d}^{L-1}}{\mathrm{d}R^{L-1}}\left[\frac{{}_0F_1\Big(1,-\xi R_*\Big)}{t-R_*}\right](t-R_*)^L$$
$$=(-1)^{L-1}(L-1)!\sum_{k=0}^{L-1}\frac{(-\xi)^k}{k!^2}{}_0F_1\Big(1+k;-\xi R_*\Big)\int_0^{R_*}\mathrm{d}t(t-R_*)^k{}_0F_1\Big(1;-\eta t\Big),$$

which can be verified by using the identities $\frac{\mathrm{d}^k}{\mathrm{d}R^k}{}_0F_1\Big(1,-\xi R_*\Big)=\frac{\xi^k}{k!}{}_0F_1\Big(1+k,-\xi R_*\Big)$ and $\frac{\mathrm{d}^{L-1-k}}{\mathrm{d}R^{L-1-k}}\frac{1}{t-R_*}=(-1)^{L-1-k}\frac{(L-1-k)!}{(t-R_*)^{L-k}}$. Finally the formula for J_1 takes the form

$$J_1\sim\prod_{j=0}^{L-1}j!\frac{\sqrt{2\pi}^L(L-1)!\mathrm{e}^{-NLR_*}}{N^{L^2/2}}\sum_{k=0}^{L-1}\frac{(-\xi)^k}{k!^2}{}_0F_1\Big(1+k;-\xi R_*\Big)\int_0^{R_*}\mathrm{d}t(t-1+R)^k{}_0F_1\Big(1;-\eta t\Big). \tag{7.28}$$

7.3.2 Asymptotics of J_2. The procedure of approximating the term J_2 in equation (7.24) is very similar to previously discussed. First, applying the saddle-point method to t_i integrals and using formula (7.26) for the Hermite monic polynomials gives

$$J_2 \sim \prod_{j=1}^{L-1}(j!\sqrt{2\pi})\frac{\mathrm{e}^{-N(L-1)R_*}}{N^{L^2/2}}\int_0^{R_*}\mathrm{d}s\,{}_0F_1(1;-\eta s){}_0F_1(1;-\xi s)\int_0^{\infty}\mathrm{d}t\mathrm{e}^{-N\mathcal{L}_2(t)}\frac{1}{t-s}$$
$$\times\left[h_{L-1}\left(\sqrt{N}(s-R_*)\right)h_L\left(\sqrt{N}(t-R_*)\right)-h_{L-1}\left(\sqrt{N}(t-R_*)\right)h_L\left(\sqrt{N}(s-R_*)\right)\right].$$

The t integral is in turn found through the saddle-point approximation by expanding $t=R_*+N^{-1/2}\xi$:

$$\int_0^{\infty}\mathrm{d}t\mathrm{e}^{-N\mathcal{L}_2(t)}\frac{1}{t-s}h_n\left(\sqrt{N}(t-1+R)\right)\sim\frac{(-1)^n\mathrm{e}^{-NR_*}\sqrt{2\pi}}{\sqrt{N}^{n+1}}\frac{n!}{(R_*-s)^{n+1}}.$$

Similarly as in the computation of J_1, the leading order term is proportional to $h_{L-1}\left(\sqrt{N}(t-R_*)\right)h_L\left(\sqrt{N}(s-R_*)\right)$ which gives

$$J_2\sim\prod_{j=0}^{L-1}j!\frac{\sqrt{2\pi}^L(L-1)!\mathrm{e}^{-NLR_*}}{N^{L^2/2}}\int_0^{R_*}\mathrm{d}s\,{}_0F_1(1;-\eta s){}_0F_1(1;-\xi s). \tag{7.29}$$

We substitute both formulas (7.28) and (7.29) for J_1 and J_2 respectively into the equation (7.24), and use the asymptotics of normalization constant $\tilde{\mathcal{N}}_L$ given by formula (7.14). After reintroducing the renormalized kernel defined in the equation (7.20) and applying straightforward manipulations, we arrive at

$$\frac{1}{N}K_N\left(x=\frac{\xi}{N},y=\frac{\eta}{N}\right)$$
$$\sim\left(\frac{\eta}{\xi}\right)^L\int_0^{R_*}\mathrm{d}s\,{}_0F_1(1;-\eta s)\left({}_0F_1(1;-\xi s)-\sum_{k=0}^{L-1}\frac{(-\xi)^k}{k!^2}{}_0F_1\left(1+k;-\xi R_*\right)(s-R_*)^k\right).$$

Next we define a 'macroscopic' kernel

$$\kappa_L(\alpha,\beta)\equiv\lim_{N\to\infty}\frac{1}{NR_*}K_N\left(x=\frac{\alpha}{NR_*},y=\frac{\beta}{NR_*}\right),$$

which is found by changing the variables as $\xi=\frac{\alpha}{R_*},\eta=\frac{\beta}{R_*}$, $s=R_*\tau$ and introducing Bessel functions via the identity ${}_0F_1(1+k;-\alpha)=\frac{k!}{\alpha^{k/2}}J_k(2\sqrt{\alpha})$:

$$\kappa_L(\alpha,\beta)=\left(\frac{\beta}{\alpha}\right)^L\int_0^1\mathrm{d}\tau\,J_0(2\sqrt{\beta\tau})\left(J_0(2\sqrt{\alpha\tau})-\sum_{k=0}^{L-1}(1-\tau)^k\frac{\alpha^{k/2}}{k!}J_k(2\sqrt{\alpha})\right).$$

One then arrives at the final expression (7.18) by using the following proposition for $a=2\sqrt{\alpha}$ and $b=2\sqrt{\beta}$. □

Proposition 7.4. *For $a, b > 0$ the following identity holds true*

$$\left(\frac{b}{a}\right)^{2L}\int_0^1 \mathrm{d}\tau\, J_0(b\sqrt{\tau})\left(J_0(a\sqrt{\tau}) - \sum_{k=0}^{L-1}(1-\tau)^k\frac{(a/2)^k}{k!}J_k(a)\right) = \left(\frac{b}{a}\right)^{L}\int_0^1 \mathrm{d}\tau J_L(a\sqrt{\tau})J_L(b\sqrt{\tau}). \tag{7.30}$$

Proof. Proof of this formula is provided in the appendix B. □

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Appendix A. Equivalence between equation (3.33) and remark 3.31

Equality of the formula (3.33) and the $L = 1$ case of remark 3.31 follows by the following proposition.

Proposition A.1. *For $p > 0$ the following relation holds true*

$$\frac{(-1)^{N-1}}{2\pi \mathrm{i}}\int_0^\infty \mathrm{d}u\frac{\mathrm{e}^{-u}u}{p+u}\int_C \frac{\mathrm{e}^{-v}{}_0F_1\,(1; vu)\,\mathrm{d}v}{(v-|z|^2)^N}\int_0^\infty \frac{\left(q+|z|^2\right)^N}{q+v}\mathrm{e}^{-q}\mathrm{d}q$$
$$= \frac{1}{(N-1)!}\mathrm{e}^{|z|^2}\int_0^\infty \frac{\mathrm{e}^{-pt}}{t(1+t)}\mathrm{e}^{-\frac{t|z|^2}{1+t}}\left(\frac{t}{1+t}\right)^N\left(\Gamma(N+1,|z|^2) - \frac{t}{1+t}|z|^2\Gamma(N,|z|^2)\right). \tag{A.1}$$

Proof. We follow the same approach as in the proof of proposition 3.8. We first bring the l.h.s. to the following form, see proposition 3.9,

$$\text{l.h.s.} = \int_0^\infty \frac{\mathrm{d}t}{(1+t)^2}\mathrm{e}^{-tp}\,\mathcal{G}_N^{(1)}\,(\rho,\tau)\,, \tag{A.2}$$

where $\tau = \frac{t}{1+t}$, $\rho = |z|^2$ and (see 3.39)

$$\mathcal{G}_N^{(1)}\,(\rho,\tau) = \frac{(-1)^{N-1}}{(N-1)!}\int_0^\infty \mathrm{d}q\,(q+\rho)^N\mathrm{e}^{-q}\frac{\mathrm{d}^{N-1}}{\mathrm{d}\rho^{N-1}}\left[\frac{\mathrm{e}^{-\rho\tau}(1+\rho(1-\tau))}{q+\rho}\right]. \tag{A.3}$$

Now we use the Leibniz formula:

$$\frac{\mathrm{d}^{N-1}}{\mathrm{d}\rho^{N-1}}\left[\mathrm{e}^{-\rho\tau}\frac{1+\rho(1-\tau)}{q+\rho}\right] = (N-1)!\sum_{l=0}^{N-1}\frac{(-1)^l}{(N-1-l)!}\frac{1}{(q+\rho)^{l+1}}\frac{\mathrm{d}^{N-1-l}}{\mathrm{d}\rho^{N-1-l}}\left[\mathrm{e}^{-\rho\tau}(1+\rho(1-\tau))\right], \tag{A.4}$$

which allows to perform the integration over q in formula (A.3) explicitly:

$$\int_0^\infty \mathrm{d}q\,(q+\rho)^{N-l-1}\mathrm{e}^{-q} = (N-l-1)!\sum_{m=0}^{N-l-1}\frac{\rho^m}{m!},$$

and get, after introducing $k = N - l - 1$, that

$$\mathcal{G}_N^{(1)}\,(\rho,\tau) = \sum_{k=0}^{N-1}(-1)^k\left(\sum_{m=0}^{k}\frac{\rho^m}{m!}\right)\frac{\mathrm{d}^k}{\mathrm{d}\rho^k}\left[\mathrm{e}^{-\rho\tau}(1+\rho(1-\tau))\right]. \tag{A.5}$$

By using the Leibniz formula again

$$\frac{\mathrm{d}^k}{\mathrm{d}\rho^k}\left(\mathrm{e}^{-\rho\tau}(1+\rho(1-\tau))\right)=(-1)^k\mathrm{e}^{-\rho\tau}\left[\tau^k(1+\rho(1-\tau))-k\tau^{k-1}(1-\tau)\right], \tag{A.6}$$

we find

$$\mathcal{G}_N^{(1)}(\rho,\tau)=\mathrm{e}^{-\rho\tau}\sum_{k=0}^{N-1}\left[-\tau^{k+1}\rho+\tau^k(k+1+\rho)-k\tau^{k-1}\right]\left(\sum_{m=0}^{k}\frac{\rho^m}{m!}\right), \tag{A.7}$$

which can be expanded in powers of τ as

$$\begin{aligned}\mathcal{G}_N^{(1)}(\rho,\tau)=\mathrm{e}^{-\rho\tau}&\left\{-\tau^N\rho\sum_{m=0}^{N-1}\frac{\rho^m}{m!}+\tau^{N-1}\left(N\sum_{m=0}^{N-1}\frac{\rho^m}{m!}+\frac{\rho^N}{(N-1)!}\right)\right.\\&\left.+\sum_{l=2}^{N}\tau^{N-l}\left[-(N-l+1)\sum_{m=0}^{N-l+1}\frac{\rho^m}{m!}+(N-l+1+\rho)\sum_{m=0}^{N-l}\frac{\rho^m}{m!}-\rho\sum_{m=0}^{N-l-1}\frac{\rho^m}{m!}\right]\right\}.\end{aligned} \tag{A.8}$$

It is easy to see that coefficients of τ^k for $k\leqslant N-2$ are zero, hence we arrive at

$$\mathcal{G}_N^{(1)}(\rho,\tau)=\mathrm{e}^{-\rho\tau}\tau^{N-1}\left[-\tau\rho\sum_{m=0}^{N-1}\frac{\rho^m}{m!}+N\sum_{m=0}^{N}\frac{\rho^m}{m!}\right]. \tag{A.9}$$

We recall the definition of the incomplete $\Gamma-$function $\Gamma(n,x)=(n-1)!\mathrm{e}^{-x}\sum_{m=0}^{n-1}\frac{x^m}{m!}$ and find

$$\mathcal{G}_N^{(1)}(\rho,\tau)=\frac{\mathrm{e}^{\rho(1-\tau)}\tau^{N-1}}{(N-1)!}\Big[-\tau\rho\Gamma(N,\rho)+\Gamma(N+1,\rho)\Big]. \tag{A.10}$$

Finally, by recalling that $\tau=\frac{t}{1+t}$ and $\rho=|z|^2$ we see that $\mathcal{G}_N^{(1)}\left(|z|^2,\frac{t}{1+t}\right)$ substituted to (A.2) coincides with (A.1). □

Appendix B. Proof of proposition 7.4

In this appendix we provide a proof of proposition 7.4.

Proof. The left- and right-hand side of the formula (7.30) is denoted by κ_L and κ'_L respectively:

$$\begin{aligned}\kappa_L&=\left(\frac{b}{a}\right)^{2L}\int_0^1\mathrm{d}\tau\,J_0(b\sqrt{\tau})\left(J_0(a\sqrt{\tau})-\sum_{k=0}^{L-1}(1-\tau)^k\frac{(a/2)^k}{k!}J_k(a)\right),\\\kappa'_L&=\left(\frac{b}{a}\right)^{L}\int_0^1\mathrm{d}\tau J_L(a\sqrt{\tau})J_L(b\sqrt{\tau}).\end{aligned} \tag{B.1}$$

We compute

$$\int_0^1\mathrm{d}\tau(1-\tau)^kJ_0(b\sqrt{\tau})=\frac{2k!}{b}\left(\frac{2}{b}\right)^kJ_{k+1}(b)$$

so that κ_L reads

$$\kappa_L = \left(\frac{b}{a}\right)^{2L} \left[\int_0^1 \mathrm{d}\tau J_0\left(a\sqrt{\tau}\right) J_0\left(b\sqrt{\tau}\right) - \frac{2}{b}\sum_{k=0}^{L-1}\left(\frac{a}{b}\right)^k J_k(a)J_{k+1}(b)\right].$$

We prove the equality $\kappa_L = \kappa'_L$ by induction. For $L = 0$, the second sum term vanishes and we trivially obtain an equality

$$\kappa_0 = \int_0^1 \mathrm{d}\tau J_0\left(a\sqrt{\tau}\right) J_0\left(b\sqrt{\tau}\right) = \kappa'_0, \tag{B.2}$$

which also agrees with theorem 3.1 given in [17]. We assume the validity of $\kappa_L = \left(\frac{b}{a}\right)^L \int_0^1 \mathrm{d}\tau J_L(a\sqrt{\tau})J_L(b\sqrt{\tau})$ and consider κ_{L+1}:

$$\begin{aligned}\kappa_{L+1} &= \frac{b^2}{a^2}\left(\frac{b}{a}\right)^{2L}\left[\int_0^1 \mathrm{d}s J_0\left(a\sqrt{s}\right) J_0\left(b\sqrt{s}\right) - \frac{2}{b}\sum_{k=0}^{L-1}\left(\frac{a}{b}\right)^k J_k(a)J_{k+1}(b) - \frac{2}{b}\left(\frac{a}{b}\right)^L J_L(a)J_{L+1}(b)\right] \\ &= \frac{b^2}{a^2}\kappa_L - \frac{2}{b}\left(\frac{b}{a}\right)^{L+2} J_L(a)J_{L+1}(b).\end{aligned}$$

On the κ_L term we use an identity

$$\int_0^1 \mathrm{d}s J_L(a\sqrt{s})J_L(b\sqrt{s}) = 2\frac{bJ_{L-1}(b)J_L(a) - aJ_{L-1}(a)J_L(b)}{a^2 - b^2} \tag{B.3}$$

and utilize recurrence relations $J_{L-1}(x) = \frac{2L}{x}J_L(x) - J_{L+1}(x)$ to eliminate J_{L-1} resulting in

$$\begin{aligned}\kappa_{L+1} &= 2\left(\frac{b}{a}\right)^{L+2}\left[\frac{bJ_{L-1}(b)J_L(a) - aJ_{L-1}(a)J_L(b)}{a^2 - b^2} - \frac{b - a^2/b}{b^2 - a^2}J_L(a)J_{L+1}(b)\right] \\ &= \left(\frac{b}{a}\right)^{L+1}\int_0^1 \mathrm{d}s J_{L+1}(a\sqrt{s})J_{L+1}(b\sqrt{s}) = \kappa'_{L+1}.\end{aligned}$$

□

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