Iterated self-applicable truth

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14th March 2024

Contents

1 Overview

Should *is true* be *is justifiable* or *verifiable*?

2 A theory

Language $\mathscr{L}_\Omega$: extend language of PRA by binary predicate T. Assuming an ordinal notation system of a sufficiently large ordinal κ, and $\eta < \kappa$, $T_\eta s$ means $T(\ulcorner \eta \urcorner, s)$.

For $\beta < \kappa$, define a theory T_β extending HA by axioms, for each $\eta \leq \kappa$:

$$\forall \ulcorner A \urcorner \ulcorner B \urcorner (T_\eta \ulcorner A \to B \urcorner \to (T_\eta \ulcorner A \urcorner \to T_\eta \ulcorner B \urcorner)) \tag{I}$$

$$\forall \ulcorner A(x) \urcorner (\forall n T_\eta \ulcorner A(\dot{n}) \urcorner \to T_\eta \ulcorner \forall x A(x) \urcorner \tag{U}$$

$$\forall \xi \leq \beta (T_\eta \ulcorner T_\xi \dot{x} \urcorner \to T_\eta x) \tag{D}$$

$$\forall \xi < \eta (T_\eta x \to T_\eta \ulcorner T_\xi \dot{x} \urcorner) \tag{R}$$

and rules of inference

$$A \vdash T_\beta \ulcorner A \urcorner \tag{Nec}$$

$$T_\beta \ulcorner A \urcorner \vdash A \tag{Conec}$$

What gives the strength is that (D) 'collapses' all internal truth predicates to the current level (η). In contrast, (R) only permits 'expanding' the internal level by lower level predicates.

With $\mathsf{T}^c_\beta = \mathsf{T}_\beta + \text{LEM}$, the thesis argues that

Theorem 1. *For every $p < \omega$,*
$$\|\mathsf{T}^c_p\| = \vartheta(\Omega^{p+1} \cdot \omega).$$

Theorem 2. *For $\omega \leq \beta < \vartheta\Omega^\Omega$,*
$$\vartheta(\Omega^\beta \cdot \omega) \leq \|\mathsf{T}^c_\beta\| \leq \vartheta(\Omega^{\beta+1} \cdot \omega).$$

Theorem 3. *The limit of the autonomous progression of $\{\mathsf{T}^c_\beta : \beta < \Omega\}$ is the large Veblen ordinal, $\vartheta\Omega^\Omega$.*

It seems likely that the same claims hold for (intuitionistic) T_β.

3 From the thesis: Theories with multiple self-applicable truth predicates

chap:ext

Truth is often used as a means of reflection; a tool by which one may obtain principles, schemata etc. that were implicit, but not necessary explicit, in the acceptance of some axiomatic system. Feferman, for example, views the theory of truth Ref (see **??**) as an operation which, when applied to a theory S, answers the question "which statements in the base language … ought to be accepted if one has accepted the basic axioms and rules of [S]?" [**?**, p. 2]. The Friedman-Sheard theories A to I can also be viewed as operations which have been applied to PRA: one adds to PRA a (new) predicate T, formalising the acceptance of PRA; on top of this one adds some subset of the Optional Axioms, for example adding ∀-Inf formalises the acceptance of ω-logic, while the axiom T-Del formalises closure under the rule T-Elim.

Viewing theories of truth as operations provides a natural way to describe the general processes behind their construction and allows one to possibly iterate the operation. In this section we will look at this specifically from the perspective of the Friedman-Sheard theory F.

One way of arguing for the naturalness of S_3 is to view it as formalising the acceptance of S_2. Within S_3 one has T-Rep and T-Del, formalising the rules of inference T-Intro and T-Elim of S_2, and thus
$$\mathrm{S}_3 \vdash \forall x(\mathsf{Bew}_{\mathrm{S}_2}(x) \wedge \mathsf{Sent}_{\mathscr{L}_\mathsf{T}}(x) \to \mathsf{T}(x)).$$

However, S_2 contains the rule T-Intro, so it seems reasonable that the theory attempting to formalise its acceptance should also be closed under T-Intro. But the presence of T-Rep, T-Del and T-Elim in S_3 means this is not possible, so perhaps S_3 is not such a natural theory after all.

Since closure of F under ¬T-Elim is vacuous, F and S_2 are identical as theories. Therefore, S_3 can be seen as formalising the acceptance of F, although one might expect in this case, to also add
$$\mathsf{T}(\ulcorner \neg \mathsf{T}(\dot{x}) \urcorner) \to \mathsf{T}(\underset{\cdot}{\neg} x)$$

as an axiom.[1] Still, the resulting theory cannot be closed under T-Intro, as one would like.

If one were to stratify the language, in much the same way as one would to form a Tarskian hierarchy of truth predicates, the problem can be circumvented. Recall $\mathsf{F} = \mathrm{Base}_T$ + T-Intro + T-Elim + ¬T-Elim + T-Del + ∀-Inf. Let F_0 denote F formulated with the predicate T_0 in place of T, and suppose

[1] As the $\mathscr{L}_\mathsf{T}$-structure $\mathfrak{M}$ used in **??** also satisfies the axiom $\mathsf{T}(\ulcorner \neg \mathsf{T}(\dot{x}) \urcorner) \to \mathsf{T}(\underset{\cdot}{\neg} x)$, the extension of S_3 obtained by adding this axiom is also consistent.

T_1 is a (new) unary predicate symbol. The theory formalising acceptance of F_0, which we shall denote by F_1, would then contain the following axioms

$$\mathsf{val}(x) \wedge \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \rightarrow \mathsf{T}_1(x), \tag{1}$$

{exteqn:FFax0}

$$\mathrm{Ax}_{\mathsf{F}_0}(x) \wedge \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \rightarrow \mathsf{T}_1(x), \tag{2}$$

{exteqn:FFax1}

$$\mathsf{T}_1(x) \wedge \mathsf{T}_1(x \underset{\cdot}{\rightarrow} y) \rightarrow \mathsf{T}_1(y), \tag{3}$$

{exteqn:FFax2}

$$(\forall x\, \mathsf{T}_1(\ulcorner A(\dot{x}) \urcorner)) \rightarrow \mathsf{T}_1(\ulcorner \forall x A(x) \urcorner), \tag{4}$$

{exteqn:FFaxw}

$$\mathsf{T}_1(x) \rightarrow \mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x}) \urcorner), \tag{5}$$

{exteqn:FFax3}

$$\mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x}) \urcorner) \rightarrow \mathsf{T}_1(x), \tag{6}$$

{exteqn:FFax4}

$$\mathsf{T}_1(\ulcorner \neg \mathsf{T}_0(\dot{x}) \urcorner) \rightarrow \mathsf{T}_1(\underset{\cdot}{\neg} x). \tag{7}$$

{exteqn:FFax5}

Equations (1) and (2) state the acceptance of all axioms of F_0 (logical and non-logical), whereas eq. (3) formalises *modus ponens* in F_0. These three also combine to imply the axioms of $\mathrm{Base}_{\mathsf{T}_1}$. Equations (5) to (7) express the acceptance of the rules T_0-Intro, T_0-Elim and $\neg\mathsf{T}_0$-Elim, respectively, in F_0, while eq. (4) closes the predicate under ω-logic.

The predicate T_1 is viewed as an extension of the predicate T_0 and as such we would expect it to satisfy the relevant axioms of F, that is, we also have

$$\mathsf{T}_1(\ulcorner \mathsf{T}_1(\dot{x}) \urcorner) \rightarrow \mathsf{T}_1(x),$$

and closure under T_1-Elim, T_1-Intro and $\neg\mathsf{T}_1$-Elim. Combining also the axioms of F_0 it is then easy to deduce

$$\forall x(\mathsf{Bew}_{\mathsf{F}_0}(x) \wedge \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \rightarrow \mathsf{T}_1(x)). \tag{8}$$

{exteqn:F1refle

We are happy with the thought of T_0 being a self-applicable truth predicate, and so far there is nothing to stop T_1 also being self-applicable. Moreover, T_0 may meaningfully occur in the scope of the predicate T_1. Thus we have described the first step in a hierarchy of self-applicable truth predicates. But, should the predicate T_1 be allowed to occur in the scope of T_0? After all, the motivation behind working with theories that contain their own truth predicate is in their ability to reason about themselves. Since T_1-Intro may apply to arbitrary sentences in $\mathscr{L}_{\mathsf{T}_0,\mathsf{T}_1}$, the question of whether or not T_0 can meaningfully apply to sentences containing T_1 is essentially decided by how we restrict the quantifiers in eqs. (3) to (7) (in particular eq. (5) pertaining to T_0-Intro) for inclusion in F_1: if we restrict them to range over only codes of $\mathscr{L}_{\mathsf{T}_0}$-sentences we will have no non-trivial occurrences of this inter-applicability.

As is consistent with our earlier chapters, we view our theories as being presented in a Hilbert style deduction system, with certain axioms and rules of inference which are treated in their broadest sense. Namely, we consider a rule of inference of a theory S to be applicable to any extension of the language, logic or axioms of S. Thus, if one imagines the theory F_0 being first formulated in the language $\mathscr{L} \cup \{\mathsf{T}_0, \mathsf{T}_1\}$, and only then completing the reflection step to F_1 by adding the axioms and rules pertaining to T_1, it seems natural to suppose the predicates T_0 and T_1 are inter-applicable. Since T_0-Intro was applicable in F_0 to formulae containing the predicate T_1, so should T_0 in F_1. Thus, we expect F_1 to have the axiom T_0-Imp,

$$\forall x \forall y[\mathsf{T}_0(x) \wedge \mathsf{T}_0(x \underset{\cdot}{\rightarrow} y) \rightarrow \mathsf{T}_0(y)],$$

as opposed to its relativised form

$$\forall x \forall y[\mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \wedge \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(y) \rightarrow (\mathsf{T}_0(x) \wedge \mathsf{T}_0(x \underset{\cdot}{\rightarrow} y) \rightarrow \mathsf{T}_0(y))]; \tag{9}$$

{exteqn:T0relat

and, more importantly, that F_1 contains the unrelativised axioms

$$\begin{gathered}\forall x[\mathsf{T}_1(x) \to \mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x})\urcorner)],\\ \forall x[\mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x})\urcorner) \to \mathsf{T}_1(x)],\\ \forall x[\mathsf{T}_1(\ulcorner \neg\mathsf{T}_0(\dot{x})\urcorner) \to \mathsf{T}_1(\underset{\cdot}{\neg} x)].\end{gathered} \tag{10}$$

{exteqn:F1ax1}

This provides, for example,

$$\forall x[\mathsf{T}_1(\ulcorner \mathsf{T}_1(\dot{x})\urcorner) \leftrightarrow \mathsf{T}_1(\ulcorner \mathsf{T}_0(\ulcorner \mathsf{T}_1(\dot{x})\urcorner)\urcorner)],$$

which, by T_1-Intro and eq. (8), yields $\mathsf{T}_0(\ulcorner \mathsf{T}_1(\ulcorner A\urcorner)\urcorner)$ for every theorem A of F_0. Thus our informal interpretation leads us to the curious situation where we have two self-applicable predicates T_0, T_1 which may also be applied to each other.

If, on the other hand, we had considered relativising the quantifiers as in eq. (9), T_1 may apply to the language $\mathscr{L}_{\mathsf{T}_0,\mathsf{T}_1}$ whereas T_0 may only meaningfully apply to $\mathscr{L}_{\mathsf{T}_0}$ and one would obtain the base of a strict hierarchy of self-applicable truth predicates. In this case we can no longer argue that T_0 represents a truth predicate for the whole language but only of the sub-language $\mathscr{L}_{\mathsf{T}_0}$. The reason for choosing a self-applicable notion of truth in the first place was that it may be treated as a truth predicate for the entire language, including any subsequent extension. Thus, T_0 *should* be applicable to sentences involving the predicate T_1 and we find ourselves returning to the world of two inter-applicable truth predicates.

So far we have argued that in F_1 the interpretation of T_0 should be closed under T_0-Elim and $\neg\mathsf{T}_0$-Elim while the interpretation of T_1 should be closed under T_0-Intro, T_0-Elim, $\neg\mathsf{T}_0$-Elim, T_1-Elim and $\neg\mathsf{T}_1$-Elim. However, we desire T_1 to be viewed as an extension of T_0 so as to allow for closure of F_1 under a form of *truth introduction* and this fact should be recognised by the theory. That is, from the point of view of the theory F_1, there should be no essential difference between the predicates T_0 and T_1. Thus we propose to also include the principle

$$\mathsf{T}_0(\ulcorner \mathsf{T}_1(\dot{x})\urcorner) \to \mathsf{T}_0(x) \tag{11}$$

{exteqn:T1Del}

as an axiom of F_1 (in fact as an axiom of F_0). Equation (11) expresses that the interpretation of T_0 is closed under the rule T_1-Elim. This is vacuously valid in the theory F_0. It also serves to confirm the inter-applicability of the two predicates by allowing meaningful inferences regarding T_1 under a T_0 predicate. If we accept eq. (11) we should also accept $\mathsf{T}_0(\ulcorner \neg\mathsf{T}_1(\dot{x})\urcorner) \to \mathsf{T}_0(\underset{\cdot}{\neg} x)$, but as we shall see, this axiom will be trivially satisfied by our model.

We have only described the step F_0 to F_1, but one can imagine repeating this, first adding an additional truth predicate T_2 to F_1 and then reflecting upon it to form the theory F_2.[2] This process may be continued into the transfinite to form a hierarchy of theories, supporting a hierarchy of inter-applicable truth predicates.[3]

The description of F_1 and F_2 presented above is purely motivational and requires making explicit, which the next definition achieves. As we pass beyond finite iterations and consider the construction

[2]A more precise way to describe the construction of F_2 may be given as follows: first imagine formulating F_0 with three predicates, T_0, T_1, and T_2. One then formulates F_1 in this language by reflecting upon F_0, and only then is F_2 formulated by reflecting upon F_1. In this sense we view F_0 as not being formulated in a fixed language, but rather in a language that may be expanded as and when we see fit.

[3]The inter-applicability of the predicates calls into question whether what we obtain is truly a "hierarchy of truth predicates" as we describe. Our model construction, in the next section, will show that one can view the truth predicates as based on a hierarchy, although not, perhaps, in a manner one might first imagine.

of theories F_ω, $\mathsf{F}_{\varepsilon_0}$, *etc.*, one requires the ability to internally quantify over the indices of truth predicates in the language. We therefore need to fix some computable ordinal κ from the outset and only consider iterating the construction up to ordinals $\alpha < \kappa$. In fact to maintain as much similarity with our previous work as possible (for example to ensure a primitive recursive Gödel numbering) we will insist κ is primitive recursively definable. One could consider taking $\kappa = \Gamma_0$ and utilising the encoding chosen in **??**, but as we shall see $\|\mathsf{F}_1\| \geq \Gamma_0$ (see theorem 6 below) and so we will require the construction of a larger class of ordinals to perform a sufficient proof-theoretic analysis. Suitable choices for κ will ultimately depend on our analysis and have no substantial role in defining the theories. Since the precise definition of κ is not essential for the definition, we shall assume for the time being that κ represents some fixed primitive recursive ordinal.

Definition 1. For $\beta < \kappa$, let $\mathscr{L}_\beta$ denote the language of PRA augmented by predicates T_ξ for each $\xi \leq \beta$. Let F_β be the theory formulated in the language $\mathscr{L}_\beta$, extending PA with the schema of induction for $\mathscr{L}_\beta$, and for each $\eta \leq \beta$

$$\begin{array}{cr l}
\forall x(\mathsf{T}_\eta(x_1) \land \mathsf{T}_\eta(x_1 \mathbin{\dot{\to}} x_2) \to \mathsf{T}_\eta(x_2)), & (\mathsf{T}_\eta\text{-Imp}) & \texttt{\{exteqn:T-Imp\}} \\
\forall \ulcorner A(x) \urcorner [(\forall n \mathsf{T}_\eta(\ulcorner A(\dot{n}) \urcorner)) \to \mathsf{T}_\eta(\ulcorner \forall x\, A(x) \urcorner)], & (\mathsf{T}_\eta\text{-}\forall\text{-Inf}) & \texttt{\{exteqn:T-UInf} \\
\forall \xi \leq \bar{\beta}\, \forall x(\mathsf{T}_\eta(\ulcorner \mathsf{T}_{\dot{\xi}}(\dot{x}) \urcorner) \to \mathsf{T}_\eta(x)), & (\mathsf{T}_\eta\text{-Del}) & \texttt{\{exteqn:T-dag1} \\
\forall \xi < \bar{\eta}\, \forall x(\mathsf{T}_\eta(x) \to \mathsf{T}_\eta(\ulcorner \mathsf{T}_{\dot{\xi}}(\dot{x}) \urcorner), & (\mathsf{T}_\eta^-\text{-Rep}) & \texttt{\{exteqn:T-dag2} \\
\forall \xi \leq \bar{\beta}\, \forall x(\mathsf{T}_\eta(\ulcorner \neg \mathsf{T}_{\dot{\xi}}(\dot{x}) \urcorner) \to \mathsf{T}_\eta(\dot{\neg} x)), & (\neg\mathsf{T}_\eta\text{-Del}) & \texttt{\{exteqn:T-dag3}
\end{array}$$

as well as the rules of inference

$$\begin{array}{cr l}
\text{from } A \text{ infer } \mathsf{T}_\beta(\ulcorner A \urcorner), & (\mathsf{T}_\beta\text{-Intro}) & \texttt{\{exteqn:T-Intro} \\
\text{from } \neg\mathsf{T}_\beta(\ulcorner A \urcorner) \text{ infer } \neg A, & (\neg\mathsf{T}_\beta\text{-Elim}) & \texttt{\{exteqn:notT-El} \\
\text{from } \mathsf{T}_\beta(\ulcorner A \urcorner) \text{ infer } A. & (\mathsf{T}_\beta\text{-Elim}) & \texttt{\{exteqn:T-Elim}
\end{array}$$

Define $\mathsf{F}_{<\beta} = \bigcup_{\eta<\beta} \mathsf{F}_\beta$. Let us denote by F_β^n the collection of theorems of F_β provable with at most n (serial) applications of T_β-Intro and no restriction on the number of applications of T_β-Elim. Thus F_β^0 denotes the theory F_β without T_β-Intro, and F_β^n is a sub-theory of F_β^{n+1} for every n.

The theory F_0 is identical to F and the definition of F_β fits the informal description we gave of 'F viewed as an operation applied to $\mathsf{F}_{<\beta}$'. Also, F_β^0 is a conservative extension of PA as, with no applications of T_β-Intro, all predicates T_η in F_β^0 may be interpreted trivially to show neither $\mathsf{F}_\beta^0 \vdash \mathsf{T}_\beta(s)$ nor $\mathsf{F}_\beta^0 \vdash \neg\mathsf{T}_\beta(s)$ may hold.

Although F_1 may be borne from a notion of truth and acceptance, it is by no means necessarily consistent. Indeed, with its multiple truth predicates and inter-applicability, the reader would be duly justified to view this construction with some scepticism. The process of reflection that led us to F_1, however, is almost identical to that which describes the theory S_3. As a result it may not be surprising to know that F_1 is consistent and in fact F_1^1 proves the same arithmetical statements as the theory S_3. Unfortunately we do not at this time have the suitable machinery to prove their equivalence (this will have to wait until section 4), but the following remark should motivate the connection.

extrem:S3 **Remark 1.** It is natural to first consider embedding F_1^1 into S_3. As the predicate T_1 can be interpreted vacuously in F_1^0 (as can T_0), one may expect the interpretation given by

$$\mathsf{T}_1(s)^* = \mathsf{T}_0(s)^* = \mathsf{T}(g(s))$$

to suffice, where g is a primitive recursive function chosen, by the primitive recursion theorem, such that

$$\begin{aligned} g(x) &= x, \text{ if } x \text{ is the code of an arithmetical literal,} \\ g(\ulcorner \mathsf{T}_1(s) \urcorner) &= \ulcorner s \neq s \urcorner, \\ g(\ulcorner \mathsf{T}_0(s) \urcorner) &= \ulcorner \mathsf{T}(g(s)) \urcorner, \\ g(\ulcorner A \circ B \urcorner) &= g(\ulcorner A \urcorner) \mathbin{\underset{\cdot}{\circ}} g(\ulcorner B \urcorner), \text{for } \circ \in \{\wedge, \vee\} \\ g(\ulcorner QxA(x) \urcorner) &= \underset{\cdot}{\forall} x\, g(\ulcorner A(x) \urcorner). \end{aligned}$$

This interpretation will translate the axioms T_1-Del, T_1-$\forall$-Inf and T_1^--Rep of F_1 to the axioms T-Del, $\forall$-Inf and T-Rep respectively of S_3. Since S_3 does not have the axiom $\neg$T-Del (nor can we expect to derive it in S_3) one cannot derive the translation of $\neg\mathsf{T}_1$-Del from F_1^1. Fortunately, the model construction of **??** can easily incorporate the additional axiom $\neg$T-Del and rule $\neg$T-Elim with minimal adjustments: one may consistently add the rule pertaining to $\neg$T-Del to each of the theories $\mathscr{F}_\alpha$ and expand **??** to also prove $\mathscr{F}_\alpha \vdash \neg\mathsf{T}(\ulcorner A \urcorner)$ implies $\mathscr{F}_\alpha \vdash \neg A$, thus showing $\mathsf{S}_3 + \neg$T-Del $+ \neg$T-Elim is a consistent theory.

The problem with this interpretation manifests when dealing with applications of T_1-Elim in F_1^1. Suppose $\mathsf{F}_1^1 \vdash \mathsf{T}_1(\ulcorner A \urcorner)$. If A does not contain the predicate T_1, $\ulcorner A^* \urcorner = g(\ulcorner A \urcorner)$ and an application of T-Elim in S_3 suffices. If, however, A contains T_1, the interpretation of $\mathsf{T}_1(\ulcorner A \urcorner)$ and A under * are very different; indeed, there will be sentences B for which $\mathsf{S}_3 \vdash \mathsf{T}(g(\ulcorner B \urcorner))$ but $\mathsf{S}_3 \nvdash B^*$ (for example, take B to be $\neg\mathsf{T}_1(\ulcorner C \urcorner)$ where C is any statement provable in S_3. Then $\mathsf{S}_3 \vdash B^*$ implies $\mathsf{S}_3 + \neg$T-Elim $\vdash \neg C$, which contradicts the consistency of $\mathsf{S}_3 + \neg$T-Elim. On the other hand, $g(\ulcorner B \urcorner) = \ulcorner \neg(s \neq s) \urcorner$ for some s, so $\mathsf{S}_3 \vdash \mathsf{T}(g(\ulcorner B \urcorner))$ holds). If A were an axiom of F_1^0 though, $\mathsf{S}_3 \vdash A^*$; moreover, if one views T_1-Imp, T_1-Del and T_1-Rep as a rule of inference, as in $\mathscr{F}$, one could deduce closure under T_1-Elim by induction on the height of a derivation. Viewing the axioms of F_1^1 in this manner is reminiscent of the use of the infinitary system $\mathscr{F}_\infty$ in the analysis of F; thus it seems natural to delay a further investigation of this connection until we have first analysed F_1 in detail.

Although the addition of full T_1-Intro to F_1^1 (forming F_1) creates a theory markedly stronger than S_3, it is not straightforward to embed S_3 into F_1. Such an embedding would require stratifying $\mathscr{L}_\mathsf{T}$ to involve the two predicates T_0 and T_1 of $\mathscr{L}_1$. The most obvious way to proceed would appear to involve replacing the outermost predicate by T_1 and all others by T_0, thus mapping the axioms $\mathsf{T}(\ulcorner$T-Imp$\urcorner)$, $\mathsf{T}(\ulcorner\forall$-Inf$\urcorner)$ and $\mathsf{T}(\ulcorner$T-Del$\urcorner)$ of S_3 to theorems of F_1^1. This could be achieved by picking a primitive recursive function f which recursively substitutes $\mathsf{T}_0(f(s))$ for $\mathsf{T}(s)$, and considering the interpretation * of $\mathscr{L}_\mathsf{T}$ into $\mathscr{L}_1$ given by

$$(\mathsf{T}(s))^* = \mathsf{T}_1(f(s)).$$

This interpretation maps the axioms $\forall$-Inf, T-Imp, T-Del and T-Rep by instances of axioms T_1-$\forall$-Inf, T_1-Imp, T_1-Del and T_1^--Rep respectively, but since $f(\ulcorner A \urcorner)$ need not equal $\ulcorner A^* \urcorner$, applications of T-Elim in S_3 do not translate into inferences we can readily recognise as holding in F_1.

We now move to the task of proving the consistency of F_β for arbitrary β. This will be obtained by generalising the consistency argument for S_3 to theories with multiple predicates and is presented in section 3.1 below. Following this we will perform a detailed ordinal analysis of these theories, first determining lower bounds on their proof-theoretic strength, and second upper bounds. Lower bounds on the strength of the theories F_β will be obtained in section 3.3 by extending the well-ordering results previously established for S_2 and S_3. Upper bounds are determined in section 3.4 where we attempt to directly formalise the model constructions of section 3.1 in a manner similar to our analysis of

F. However, much care will be required when dealing with a hierarchy of truth predicates and the transfinite iterations of T-Intro caused by the interaction between axioms T_η-$\forall$-Inf and T^-_η-Rep.

The axiom T_η-Del implies the interpretation of the predicate T_η is closed under T_ξ-Elim for every $\xi \leq \eta$. Likewise T^-_η-Rep implies T_η is closed under T_ξ-Intro for each $\xi < \eta$. From this observation we may deduce the following proposition, in which Bew_η abbreviates the formalised provability predicate for F_η.

tprop:FFreflec **Proposition 2.** *For every $\eta < \beta$, F_β is closed under T_η-Intro and T_η-Elim. Moreover,*

$$\mathsf{F}^1_\beta \vdash \forall x(\mathrm{Bew}_\eta(x) \wedge \mathsf{Sent}_{\mathscr{L}_\eta}(x) \rightarrow \mathsf{T}_\beta(x)).$$

Proof. All axioms of F_η are axioms of F^0_β, and by T_β-Intro we have

$$\mathsf{F}^1_\beta \vdash \forall x(\mathrm{Ax}_\eta(x) \wedge \mathsf{Sent}_{\mathscr{L}_\eta}(x) \rightarrow \mathsf{T}_\beta(x)),$$

where $\mathrm{Ax}_\eta(x)$ expresses that x is a sentential axiom of F_η. To complete the proof, we observe that F^1_β has the axioms T_β-Imp, T_β-Del, T^-_β-Rep and $\neg\mathsf{T}_\beta$-Del formalising applications of *modus ponens*, T_η-Elim, T_η-Intro and $\neg\mathsf{T}_\eta$-Elim respectively in F_η. □

3.1 Consistency

extsec:Fbconsis

At first glance the theory F_β could easily look suspect, after all it contains T_η-Del, T_η-Elim, T_η-Intro and an axiom that appears extremely close to T_η-Rep, for each $\eta \leq \beta$. As the motivation behind the theories F_β comes from abstracting the transition from PA to F one might expect that if F_β is consistent, models of F_β may be constructed by extending models of F. This is indeed the case; moreover, the extension we defined for establishing the consistency of S_3 makes a suitable base from which to start the construction. We will only sketch the consistency argument as it will be subsumed by our work in section 3.4 where we determine an upper bound on the proof-theoretic strength of F_β.

Let Ω denote the first uncountable ordinal. For the remainder of this section Greek letters, ρ, σ, τ will be used to range over elements of $\mathbb{O}$, the class of all ordinals; letters α, β, etc., will range over countable ordinals; we reserve the letters η, ξ for indices of the truth predicates and so represent ordinals below κ. Suppose

$$\rho = \Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_n} \cdot \beta_n,$$

with $\alpha_0 > \cdots > \alpha_n$ and $\beta_i < \Omega$ for each $i \leq n$. We denote by $\rho|_\gamma$ the ordinal $\Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_k} \cdot \beta_k$ where $k < n$ is the least such that $\alpha_k > \gamma \geq \alpha_{k+1}$, or $k = n$ if $\alpha_n > \gamma$. An ordinal ρ is called an Ω*-limit* if $\rho = \rho_0 + \Omega^\eta \cdot \alpha$ and either α or η is a limit ordinal.

Define a system $\mathscr{F}^*_\rho$ for $\rho \in \mathbb{O}$, formulated in a Tait-style sequent calculus in the language $\mathscr{L}_\kappa$. The system has three axioms.

- Ax.1. $\mathscr{F}^*_\rho \vdash \Gamma, A$ whenever A is a true atomic literal,
- Ax.2$_\eta$. $\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta(s), \neg\mathsf{T}_\eta(s)$ for every closed term s and $\eta < \kappa$,
- Ax.3$_\eta$. $\mathscr{F}^*_\rho \vdash \Gamma, \neg\mathsf{T}_\eta(s)$ if $s^{\mathbb{N}}$ is not the code of an $\mathscr{L}_\kappa$-sentence.

The rules of this system are the usual arithmetical rules ($\wedge$), ($\vee_0$), ($\vee_1$) and ($\exists$), plus the following six rules for every $\eta < \kappa$.

- $$\frac{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta(s_0) \quad \mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta(s_0 \dot{\to} s_1)}{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta(s_1)}\mathrm{Imp}_\eta$$
- $$\frac{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi s \urcorner) \quad \xi < \beta}{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta s}\mathrm{Del}_\eta$$
- $$\frac{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta s \quad \xi < \eta}{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi s \urcorner)}\mathrm{Rep}_\eta$$
- $$\frac{\mathscr{F}^*_\sigma \vdash A \quad \sigma < \rho|_\eta}{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta \ulcorner A \urcorner}\mathrm{Intro}_\eta$$
- $$\frac{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta \ulcorner A(\underline{n}) \urcorner \text{ for all } n \quad \rho|_\eta \text{ not an } \Omega\text{-limit}}{\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta \ulcorner \forall x A(x) \urcorner}\mathrm{U}_\eta$$
- $$\frac{\mathscr{F}^*_\rho \vdash \Gamma, A(\underline{n}) \quad \text{for every } n \in \mathbb{N}}{\mathscr{F}^*_\rho \vdash \Gamma, \forall x A(x)}\omega$$

Moreover, for each ρ define the set of (codes of) $\mathscr{L}_\kappa$-sentences

$$\mathscr{F}^*_{<\rho} = \{\ulcorner A \urcorner : \mathscr{F}^*_\sigma \vdash A \text{ for some } \sigma < \rho\}.$$

Before we proceed with analysing the role of the truth predicates in $\mathscr{F}^*_\rho$, it is important to note that a rule of *modus ponens*, or cut is lacking from our definition. However, it is not hard to show the cut rule is, in fact, admissible in $\mathscr{F}^*_\rho$: a derivation of the form $\Gamma, \neg\mathsf{T}_\eta(s)$ with $\neg\mathsf{T}_\eta(s)$ active *must* be an instance of an axiom, hence if one has derivations of $\Gamma, \neg\mathsf{T}_\eta(s)$ and $\Gamma, \mathsf{T}_\eta(s)$ one can easily obtain instead a derivation of Γ. This argument is essentially identical to that employed in the Cut-Elimination theorem for $\mathscr{F}_\infty$, **??**.

xtthm:FFCutElim **Lemma 3** (Admissibility of cut). *Suppose $\eta < \kappa$, $\mathscr{F}^*_\rho \vdash \Gamma, \mathsf{T}_\eta(s)$, and $\mathscr{F}^*_\rho \vdash \Delta, \neg\mathsf{T}_\eta(s)$ for some ordinal ρ. Then $\mathscr{F}^*_\rho \vdash \Gamma, \Delta$.*

If $\alpha < \Omega$, $\alpha|_0 = 0$ and so no applications the rule T_η-Intro is not available in $\mathscr{F}^*_\alpha$ for any η. Thus, $\mathscr{F}^*_{<\Omega}$ is trivially closed under T_η-Elim for every η. Moreover, for every $\alpha < \Omega$, only applications of T_0-Intro have been permitted in $\mathscr{F}^*_{\Omega\cdot\alpha}$. Thus, we can establish by induction on $n < \omega$ that the predicate T_0 in $\mathscr{F}^*_{\Omega\cdot n}$ may be interpreted as the theory $\mathscr{F}^*_{<\Omega\cdot n}$ and that $\mathscr{F}^*_{\Omega\cdot n}$ is closed under T_0-Elim (this is identical to the argument showing $\mathscr{F}_\infty\overset{\alpha}{\underset{n+1}{\mid}} \mathsf{T}(\ulcorner A \urcorner)$ for some α implies $\mathscr{F}_\infty\overset{\beta}{\underset{n}{\mid}} A$ for some β). Since all other truth predicates in $\mathscr{F}^*_{\Omega\cdot n}$ may be interpreted vacuously, we conclude $\mathscr{F}^*_{\Omega\cdot n}$ is closed under T_η-Elim for every $\eta < \kappa$. One may also interpret every predicate simply by $\mathbb{N}$, so, in fact, $\mathscr{F}^*_{\Omega\cdot n}$ is also closed under $\neg\mathsf{T}_\eta$-Elim for every η. Hence F^n_0 may be interpreted in $\mathscr{F}^*_{\Omega\cdot n}$.

This suggests that for $n < \omega$, $\mathscr{F}^*_{\Omega\cdot n}$, like $\mathscr{F}_n$, reconstructs the theories Th_n used by Friedman and Sheard to prove the consistency of F. At the first limit ordinal, we obtain $\mathscr{F}^*_{<\Omega\cdot\omega}$, a set of $\mathscr{L}_\kappa$-sentences closed under T_0-Intro, T_0-Elim and $\neg\mathsf{T}_0$-Elim, and containing T_0-Del, T_0^--Rep (which holds vacuously), $\neg\mathsf{T}_0$-Del and all other axioms of F_0.

To proceed with the analysis of F_1, we first consider F^0_1 which, without the rule T_1-Intro, is vacuously closed under T_1-Elim and $\neg\mathsf{T}_1$-Elim. In F^1_1, the situation differs from previous case; we need to interpret the predicate T_1 as a theory closed under ω-logic (due to T_1-$\forall$-Inf), T_0-Intro (due to T_1^--Rep), T_0-Elim and T_1-Elim (due to T_1-Del), as well as $\neg\mathsf{T}_0$-Elim and $\neg\mathsf{T}_1$-Elim (due to $\neg\mathsf{T}_1$-Del). Moreover, we need to find an interpretation of T_0 closed under ω-logic, T_0-Elim and $\neg\mathsf{T}_0$-Elim, and now also T_1-Elim and $\neg\mathsf{T}_1$-Elim (as implied by the axioms T_0-Del and $\neg\mathsf{T}_0$-Del). The properties we established for the set $\mathscr{F}_{<\Omega}$ in **??** motivate us to consider $\mathscr{F}^*_{<\Omega^2}$, a set closed under T_0-Intro, T_0-Elim, $\neg\mathsf{T}_0$-Elim and, by a similar argument as before, ω-logic (*cf.* the proof of **??**). For every $\alpha < \Omega$, the predicate T_1 may be interpreted vacuously in $\mathscr{F}^*_{<\Omega\cdot\alpha}$, so $\mathscr{F}^*_{<\Omega^2}$ is also closed under T_1-Elim. Thus $\mathscr{F}^*_{<\Omega^2}$ provides a consistent interpretation of both predicates T_0 and T_1 in F^1_1.

The next step is to consider T_1 in F^2_1. Two applications of T_1-Intro are permitted and one can derive sentences of the form $\mathsf{T}_1(\ulcorner\mathsf{T}_1(\ulcorner A\urcorner)\urcorner)$ whenever $\mathsf{F}^0_1 \vdash A$, suggesting a shift to $\mathscr{F}^*_{\Omega^2}$, where one can derive $\mathsf{T}_1(\ulcorner A\urcorner)$ whenever $A \in \mathscr{F}^*_{<\Omega^2}$, might yield a suitable interpretation for T_1. However, $\mathscr{F}^*_{\Omega^2}$ is not closed under T_0-Intro (only the systems $\mathscr{F}^*_{<\rho+\Omega\cdot\sigma}$ for limit ordinals σ are), leading us instead to consider $\mathscr{F}^*_{<\Omega^2+\Omega\cdot\omega}$ which is closed under T_0-Intro, but not ω-logic; there will be sentences $A \in \mathscr{F}^*_{<\Omega^2}$ for which $\mathscr{F}^*_{\Omega^2+\Omega\cdot n} \vdash \mathsf{T}_0(f(n, \ulcorner A\urcorner))$ for each $n < \omega$, where $f(0,n) = \ulcorner\bar{n}\urcorner$ and $f(m+1,n) = \ulcorner\mathsf{T}_0(f(m,n))\urcorner$, but the sentence $\forall x\, \mathsf{T}_0(f(x, \ulcorner A\urcorner))$ is not contained in $\mathscr{F}^*_{<\Omega^2+\Omega\cdot\omega}$. Indeed to obtain both closure under ω-logic and T_0-Intro we must move to the theory $\mathscr{F}^*_{<\Omega^2\cdot 2}$. We also require the interpretation to be closed under T_0-Elim. To manage this we repeat the same argument as before, but starting from $\mathscr{F}^*_{\Omega^2}$ in place of $\mathscr{F}^*_{\Omega}$. We know $\mathscr{F}^*_{\Omega^2}$ is closed under T_0-Elim since the predicate T_0 can be consistently interpreted as the set $\mathscr{F}^*_{<\Omega^2}$. This leads us to successively deduce the theories $\mathscr{F}^*_{\Omega^2+\Omega\cdot n}$ are closed under T_0-Elim for each $n < \omega$. Note, we can still interpret T_1 in $\mathscr{F}^*_{\Omega^2+\Omega\cdot n}$ by the set $\mathscr{F}^*_{<\Omega^2}$ as there has been no further applications of T_1-Intro. In $\mathscr{F}^*_{\Omega^2+\Omega\cdot\omega}$ we aim to interpret T_0 by $\mathscr{F}^*_{<\Omega^2+\Omega\cdot\omega}$, which unlike $\mathscr{F}^*_{\Omega^2+\Omega\cdot n}$ is not closed under ω-logic; however, $\mathscr{F}^*_{\Omega^2+\Omega\cdot\omega}$ is not closed under (T_1-$\forall$-Inf) so this does not pose a problem. Thus we may continue through the construction of $\mathscr{F}^*_{<\Omega^2\cdot 2}$ determining each theory $\mathscr{F}^*_{\Omega^2+\Omega\cdot\alpha}$ for $\alpha < \Omega$ is closed under T_0-Elim.

The argument above highlights that the predicates T_1 and T_0 in F^n_1 may be interpreted as the set $\mathscr{F}^*_{<\Omega^2\cdot n}$, and hence F_1 naturally embeds into $\mathscr{F}^*_{<\Omega^2\cdot\omega}$. If we wanted to proceed beyond this and construct models for F_2, we could imagine constructing a sequence of systems

$$\mathscr{F}^*_{\Omega^2\cdot\omega}, \mathscr{F}^*_{\Omega^2\cdot\omega+\Omega}, \ldots, \mathscr{F}^*_{\Omega^2\cdot\omega+\Omega\cdot\alpha}, \ldots, \mathscr{F}^*_{\Omega^2\cdot(\omega+1)}, \ldots, \mathscr{F}^*_{\Omega^2\cdot\alpha}, \ldots$$

to obtain $\mathscr{F}^*_{<\Omega^3}$, an interpretation of the predicate T_2 in F^1_2. The ability to recognise each theory $\mathscr{F}^*_{\Omega^2\cdot\alpha+\Omega\cdot\gamma}$ as closed under T_1-Elim and T_0-Elim, however, is essential for the interpretation of T_2-Del in F^1_2. As already argued, the set $\mathscr{F}^*_{<\Omega^2\cdot\alpha+\Omega\cdot\gamma}$ provides an interpretation of T_0 in $\mathscr{F}^*_{\Omega^2\cdot\alpha+\Omega\cdot\gamma}$; but unless γ is a limit ordinal, this need not be closed under T_0-Intro, so cannot interpret the predicate T_1. The answer is to interpret T_1 in $\mathscr{F}^*_{\Omega^2\cdot\alpha+\Omega\cdot\gamma}$ as the set $\mathscr{F}^*_{<\Omega^2\cdot\alpha}$ for *every* $\gamma < \Omega$. Only when we pass to $\mathscr{F}^*_{\Omega^2\cdot(\alpha+1)}$ do we alter the interpretation of T_1 (in this case it is changed to the set $\mathscr{F}^*_{<\Omega^2\cdot(\alpha+1)}$). It is for exactly this reason that the rule T_η-$\forall$-Inf was restricted so as to apply to $\mathscr{F}^*_\rho$ only if $\rho|_\eta$ is not an Ω-limit; the set $\mathscr{F}^*_{<\Omega^2\cdot\alpha}$ will not be closed under ω-logic if α is a limit ordinal.

Once one has constructed $\mathscr{F}^*_{<\Omega^3}$ and verified that it is closed under T_n-Elim for $n = 0, 1, 2$, one would then embark on the construction of a further sequence of systems

$$\mathscr{F}^*_{\Omega^3}, \ldots, \mathscr{F}^*_{\Omega^3+\Omega\cdot\alpha}, \ldots, \mathscr{F}^*_{\Omega^3+\Omega^2}, \ldots, \mathscr{F}^*_{\Omega^3+\Omega^2\cdot 2}, \ldots, \mathscr{F}^*_{\Omega^3+\Omega^2\cdot\alpha}, \ldots, \mathscr{F}^*_{\Omega^3\cdot 2}, \ldots$$

and subsequently $\mathscr{F}^*_{<\Omega^3\cdot\omega}$, a theory into which F_2 embeds. In general, we expect F_β to embed into $\mathscr{F}^*_{\Omega^{\beta+1}\cdot\omega}$ for each β.

The next lemma deals with the task of determining the theory $\mathscr{F}^*_\rho$ is closed under T_η-Elim for every $\eta < \kappa$. Before that, however, we require the following result regarding the behaviour of Ω-limits.

xtlem:ordOlimit **Proposition 4.** *If ρ is not an Ω-limit and $\sigma_n < \rho$ for every $n < \omega$,*

$$\sup_{n<\omega} \sigma_n < \rho.$$

Proof. Suppose ρ is not an Ω-limit and $\sigma_n < \rho$ for every $n < \omega$. Then $\rho > 0$ and there are ordinals ρ_0, α_0 such that $\rho = \rho_0 + \Omega^{\alpha_0} \cdot \Omega$. This means we can associate an ordinal $\delta_n < \Omega$ to each $n < \omega$ so that $\sigma_n < \rho_0 + \Omega^{\alpha_0} \cdot \delta_n$. The set $\{\delta_n : n < \omega\}$ is a countable set of countable ordinals, and hence is bounded in Ω, whence

$$\begin{aligned}
\sup_n \sigma_n &\leq \sup_n \{\rho_0 + \Omega^{\alpha_0} \cdot \delta_n\} \\
&\leq \rho_0 + \Omega^{\alpha_0} \cdot (\sup_n \delta_n) \\
&< \rho_0 + \Omega^{\alpha_0} \cdot \Omega \\
&= \rho.
\end{aligned}$$

□

A sequent Γ is called *T-positive* if all occurrences of a predicate T_η in Γ for any $\eta < \kappa$ are positive. Define, for each ordinal ρ, an $\mathscr{L}_\kappa$-structure $\mathfrak{M}_\rho$ according to the following criterion.

$$\mathfrak{M}_\rho \models \mathsf{T}_\eta(s) \text{ iff } s^{\mathbb{N}} \in \mathscr{F}^*_{<\rho|_\eta}.$$

extthm:FFTElim **Theorem 4** (T-Elimination theorem). *Suppose $\rho \in \mathbb{O}$.*

1. *For every T-positive sequent Γ, $\mathscr{F}^*_\rho \vdash \Gamma$ implies $\mathfrak{M}_\rho \models \bigvee \Gamma$;*
2. *For any $\eta < \kappa$, $\mathscr{F}^*_\rho \vdash \mathsf{T}_\eta(s)$ implies there is a sentence A with $s^{\mathbb{N}} = \ulcorner A \urcorner$ and $\mathscr{F}^*_\rho \vdash A$;*
3. *For any $\eta < \kappa$, $\mathscr{F}^*_\rho \vdash \neg\mathsf{T}_\eta(s)$ implies there is a sentence A with $s^{\mathbb{N}} = \ulcorner A \urcorner$ and $\mathscr{F}^*_\rho \vdash \neg A$.*

Proof. We proceed by transfinite induction on ρ. For (i), one has a *subsidiary* induction on the height of the derivation. The base case is easy to deal with. For the induction step we argue according to the last rule applied in the derivation $\mathscr{F}^*_\rho \vdash \Gamma$. Whichever rule was applied, the sequent(s) in the premise must also be T-positive and we may apply the subsidiary induction hypothesis to them.

If the last rule was one of the arithmetical rules, that is, $(\vee_i)$, $(\wedge)$, (ω) or $(\exists)$, $\mathfrak{M}_\rho \models \bigvee \Gamma$ is an immediate consequence of the subsidiary induction hypothesis, and in the case of the weakening rule, $\mathfrak{M}_\rho \models \bigvee \Gamma$ follows from the fact that Γ is T-positive. If the last applied rule was T_η-Intro, $\mathsf{T}_\eta(\ulcorner A \urcorner)$ is contained in Γ and $\mathscr{F}^*_\sigma \vdash A$ for some $\sigma < \rho|_\eta$, so $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner A \urcorner)$. For the remaining rules, the subsidiary induction hypothesis implies $\mathfrak{M}_\rho \models \bigvee \Gamma \vee (A_0 \wedge A_1)$ for some suitable choice of A_0, A_1. Of course, if $\mathfrak{M}_\rho \models \bigvee \Gamma$ we are done, so we may assume $\mathfrak{M}_\rho \models A_0 \wedge A_1$.

T_η-Imp. If the last rule applied was T_η-Imp, we may assume A_0 is $\mathsf{T}_\eta(s_0)$ and A_1 is $\mathsf{T}_\eta(s_0 \dot{\rightarrow} s_1)$, while Γ contains $\mathsf{T}_\eta(s_1)$. By the above, we may assume $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s_0) \wedge \mathsf{T}_\eta(s_0 \dot{\rightarrow} s_1)$. Thus, $s_0^{\mathbb{N}}$ and $s_1^{\mathbb{N}}$ are Gödel numbers of $\mathscr{L}_\kappa$-sentences, say B_0 and B_1 respectively, and there is some $\sigma < \rho|_\eta$ so that $\mathscr{F}^*_\sigma \vdash B_0$ and $\mathscr{F}^*_\sigma \vdash \neg B_0, B_1$. Admissibility of the cut rule (lemma 3) yields $\mathscr{F}^*_\sigma \vdash B_1$, and hence $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s_1)$.

T_η-Del. In the case the last applied rule is T_η-Del, we may identify A_0 as $\mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi(s) \urcorner)$ for some $\xi < \kappa$ and term s; moreover, $\mathsf{T}_\eta(s)$ is contained in Γ. $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi(s) \urcorner)$ implies $\mathscr{F}^*_\sigma \vdash \mathsf{T}_\xi(s)$ for some $\sigma < \rho|_\eta$. Since $\sigma < \rho$, the *main* induction hypothesis may be applied, whence $s^{\mathbb{N}} = \ulcorner A \urcorner$ for some A and $\mathscr{F}^*_\sigma \vdash A$. Thus $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s)$ and $\mathfrak{M}_\rho \models \bigvee \Gamma$.

T^-_η-Rep. Here we have $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s)$ and $\mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi(s) \urcorner)$ is in Γ for some $\xi < \eta$. By definition this implies $s^{\mathbb{N}} = \ulcorner A \urcorner$ for some sentence A and $\mathscr{F}^*_\sigma \vdash A$ for some $\sigma < \rho|_\eta$, whence $\mathscr{F}^*_{\sigma+\Omega^{\xi+1}} \vdash \mathsf{T}_\xi(s)$ is derivable. But since $\xi < \eta$ and $\sigma < \rho|_\eta$, we have $\sigma + \Omega^{\xi+1} < \rho|_\eta$, and so $\mathfrak{M}_\rho \models \bigvee \Gamma$.

$\neg\mathsf{T}_\eta$-Del. This case is similar to T_η-Del above; by the main induction hypothesis we know, for every $\sigma < \rho$ and $\xi < \kappa$, that $\mathscr{F}^*_\sigma$ is closed under $\neg\mathsf{T}_\xi$-Elim, thus $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner\neg\mathsf{T}_\xi(s)\urcorner)$ implies $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\underset{\cdot}{\neg} s)$ as desired.

T_η-$\forall$-Inf. The assumption is that $\mathfrak{M}_\rho \models \forall x\, \mathsf{T}_\eta(\ulcorner A(\dot{x})\urcorner)$. This entails the existence of, for every $n < \omega$, an ordinal $\sigma_n < \rho|_\eta$ such that $\mathscr{F}^*_{\sigma_n} \vdash A(\bar{n})$. Weakening and the ω-rule yields $\mathscr{F}^*_\sigma \vdash \forall x\, A(x)$, where $\sigma = \sup_n \sigma_n$, but one need not in general have $\sigma < \rho|_\eta$.[4] Due to the restriction on applications of T_η-$\forall$-Inf, however, $\rho|_\eta$ is not an Ω-limit, thus by proposition 4, $\sigma < \rho|_\eta$ and so $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner \forall x A(x)\urcorner)$, whence $\mathfrak{M}_\rho \models \bigvee \Gamma$.

This completes the proof of (i).

(ii) is now a consequence of (i). If $\mathscr{F}^*_\rho \vdash \mathsf{T}_\eta(s)$, (i) implies $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s)$, whence $s^{\mathbb{N}} = \ulcorner A \urcorner$ for some $\mathscr{L}_\kappa$-sentence A and $\mathscr{F}^*_\sigma \vdash A$ for some $\sigma < \rho|_\eta$. By weakening, $\mathscr{F}^*_\rho \vdash A$, as desired.

Observe that in the case of every rule of inference in the system $\mathscr{F}^*_\rho$, T-positive premises yield T-positive consequents. Therefore $\mathscr{F}^*_\eta \vdash \Gamma$ implies $\bigvee \Gamma$ is satisfied in the *everything is true* $\mathscr{L}_\kappa$ structure, so $\mathscr{F}^*_\rho \vdash \neg\mathsf{T}_\eta(s)$ is impossible and (iii) holds vacuously. □

extprop:FFembed **Proposition 5.** *Let A be any axiom of F_β. Then $\mathscr{F}^*_{\Omega^{\beta+1}} \vdash A$.*

Proof. One can derive each of the axioms via the corresponding rule and Ax.2_η, as in **??**. In the case of T_η-$\forall$-Inf note $\Omega^{\beta+1}|_\eta$ is not an Ω-limit for any $\eta \leq \beta$. □

extthm:FFconsis **Theorem 5.** *The theory F_β is consistent for every $\beta < \kappa$.*

Proof. Lemma 3, theorem 4 and the previous proposition provide the means to deduce, by induction on n, that F^n_β embeds into $\mathscr{F}^*_{\Omega^{\beta+1}\cdot n}$. Thus every sentential theorem of F_β is contained in $\mathscr{F}^*_{<\Omega^\beta\cdot\omega}$. However, clearly the empty sequent is not derivable in $\mathscr{F}^*_\rho$ for any ρ, so F_β must be consistent. □

3.2 An ordinal notation system for impredicative theories

ec:moreordinals

To carry out an ordinal analysis of F_β we require the current set of ordinal terms, OT, to be extended to cover a larger segment of the ordinals. We will make use of an ordinal notation system for the Bachmann-Howard ordinal introduced by Rathjen and Weiermann [?]. This ordinal has proved significant in the analysis of certain impredicative systems such as the theory of inductive definitions, ID_1 [?]. It will turn out that the theories F_β are substantially weaker than ID_1, but this notation system is still a natural one to consider. The key to generating notations for characteristic ordinals beyond Γ_0 is the use of constructions referencing certain 'external points'. In our case the 'external point' will be Ω, the first uncountable ordinal.

In order to generate unique representations for ordinals we will introduce a normal form for non-ε-ordinals, based on the Cantor normal form. We write $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ if $\alpha = \omega^\gamma + \delta$ and either $\delta = 0$ and $\gamma < \alpha$, or $\delta = \omega^{\delta_1} + \cdots + \omega^{\delta_k}$, $\gamma \geq \delta_1 \geq \ldots \geq \delta_k$ and $k \geq 1$. Let $\varepsilon_{\Omega+1}$ be the first ε-ordinal larger than Ω. For each $\alpha < \varepsilon_{\Omega+1}$ we denote by α^* the largest ε-ordinal below Ω used in the normal form presentation for α; that is,

1. $0^* = \Omega^* = 0$,

[4]For example, suppose $\rho|_\eta = \rho_0 + \Omega^\xi$ and ξ is a limit ordinal. If $\sigma_n = \rho_0 + \Omega^{\xi_n}$, where $\xi = \sup_n \xi_n$ and $\xi_n < \xi$ for every $n < \omega$, one has $\sigma_n < \rho|_\eta$, but $\sup_n \sigma_n = \rho|_\eta$.

2. $\alpha^* = \alpha$, if $\alpha < \Omega$ is an ε-ordinal,

3. $\alpha^* = \max\{\gamma^*, \delta^*\}$, if $\alpha =_{\rm NF} \omega^\gamma + \delta$.

Define sets of ordinals $C_k(\alpha, \beta)$, and a function $\vartheta\colon \mathbb{O} \to \Omega$ by transfinite recursion on $\alpha \in \mathbb{O}$ as follows.

(C1) $\{0, \Omega\} \cup \beta \subseteq C_k(\alpha, \beta)$,

(C2) $\gamma, \delta \in C_k(\alpha, \beta)$ and $\xi =_{\rm NF} \omega^\gamma + \delta$ implies $\xi \in C_{k+1}(\alpha, \beta)$,

(C3) $\xi \in C_k(\alpha, \beta)$ and $\xi < \alpha$ implies $\vartheta\xi \in C_{k+1}(\alpha, \beta)$,

(C4) $C(\alpha, \beta) = \bigcup_{k<\omega} C_k(\alpha, \beta)$,

(ϑ1) $\vartheta\alpha = \min\{\xi < \Omega : C(\alpha, \xi) \cap \Omega \subseteq \xi \ \wedge \ \alpha \in C(\alpha, \xi)\}$.

The next two propositions shed some light on the role the function ϑ plays in generating intitial segments of $\mathbb{O}$.

extprop:theta1 **Proposition 6.** *$\vartheta\alpha$ is defined for every $\alpha < \varepsilon_{\Omega+1}$.*

Proof. Let $\gamma_0 = \alpha^* + 1$. By rules (C1) and (C2) we may deduce $\alpha \in C(\alpha, \gamma_0)$. Suppose $\gamma_k < \Omega$ has been defined. As $C(\alpha, \gamma_k)$ has a countable definition, it contains only countably many elements; thus $C(\alpha, \gamma_k) \cap \Omega$ is bounded in Ω. Let $\gamma_{k+1} < \Omega$ be such that $C(\alpha, \gamma_0) \cap \Omega \subseteq \gamma_{k+1}$ and define $\gamma = \sup_{k<\omega} \gamma_k$. Since $\{\gamma_k : k < \omega\}$ is a countable set of countable ordinals, it too must be bounded in Ω, so $\gamma < \Omega$. Since $\alpha < \gamma_0 \leq \gamma$, also $\alpha^* \in C(\alpha, \gamma)$. Finally,

$$C(\alpha, \gamma) \subseteq \bigcup_{n<\omega} C(\alpha, \gamma_n),$$

so $C(\alpha, \gamma) \cap \Omega \subseteq \gamma$ and $\vartheta\alpha \leq \gamma$ by (ϑ1). □

The argument in the proof above provides a means to approximate the ordinal $\vartheta\alpha$ from below. Define $\gamma_0 = \alpha^* + 1$ and $\gamma_{m+1} = \min\{\xi < \Omega : C(\alpha, \gamma_m) \cap \Omega \subseteq \xi\}$; then $\vartheta\alpha \leq \sup_m \gamma_m$. By (C2) it is clear that each γ_m is a limit ordinal, whence we may deduce

$$[(\forall \delta \in C(\alpha, \gamma_m) \cap \Omega) \forall \xi < \delta\, F(\xi)] \to \forall \delta < \gamma_{m+1}\, F(\delta)$$

for every formula F.

The function ϑ works by "collapsing" ordinals below $\varepsilon_{\Omega+1}$ into countable ordinals, thus allowing one to represent ordinals beyond Γ_0. Moreover, the condition "$\alpha \in C(\alpha, \xi)$" in ($\vartheta$1) ensures the function $\alpha \mapsto \vartheta\alpha$ is strictly increasing on Ω. Thus, unlike the Veblen functions φ_α, one never has $\vartheta\beta = \vartheta\vartheta\beta$. In the following proposition we show ϑ is in fact injective on $\varepsilon_{\Omega+1}$ and show that relation '$\alpha < \beta$' may be decided purely on the normal form presentation for α and β.

rop:thetaitems1 **Proposition 7.** *For all ordinals α, β and γ the following holds.*

1. *$\alpha \in C(\alpha, \vartheta\alpha)$,*

2. *$\vartheta\alpha = C(\alpha, \vartheta\alpha) \cap \Omega$ and $\vartheta\alpha \notin C(\alpha, \vartheta\alpha)$,*

3. *$\vartheta\alpha$ is an ε-ordinal,*

4. *$\gamma \in C(\alpha, \beta)$ if and only if $\gamma^* \in C(\alpha, \beta)$,*

5. *$\beta < \vartheta\alpha$ if and only if $\beta < \Omega$ and $\beta^* < \vartheta\alpha$,*

6. *$\alpha^* < \vartheta\alpha$,*

7. *$\vartheta\alpha = \vartheta\beta$ if and only if $\alpha = \beta$,*

8. *$\vartheta\alpha < \vartheta\beta$ if and only if $(\alpha < \beta \wedge \alpha^* < \vartheta\beta) \vee (\beta < \alpha \wedge \vartheta\alpha \leq \beta^*)$.*

Proof. Proposition 6 ensures $\vartheta\alpha$ is defined for every $\alpha < \varepsilon_{\Omega+1}$, so (i) is a direct consequence of (ϑ1). By (C1), $\vartheta\alpha \subseteq C(\alpha, \vartheta\alpha)$ whence (ii) also follows from (ϑ1). (ii) then implies (iii) since, as a result of (C2), $\gamma < \vartheta\alpha$ only if $\omega^\gamma < \vartheta\alpha$.

(iv). Suppose $\gamma \in C_k(\alpha, \beta)$. If $k = 0$, $\gamma^* \in C(\alpha, \beta)$ is immediate by (C1), so suppose $k > 0$. We show $\gamma^* \in C(\alpha, \beta)$ by examining the normal form of γ. If γ is either 0 or Ω, $\gamma^* = 0 \in C(\alpha, \beta)$ by (C1), and if γ is an ε-ordinal, $\gamma^* = \gamma$ and we are done. Otherwise $\gamma =_{\mathrm{NF}} \omega^\xi + \delta$ and $\delta, \xi \in C_{k-1}(\alpha, \beta)$. The induction hypothesis implies $\delta^*, \xi^* \in C(\alpha, \beta)$, whence $\gamma^* \in C(\alpha, \beta)$. The converse direction holds by a similar argument.

(v) is an immediate consequence of (iv) and (ii); (vi) holds on account of (i), (ii) and (iv). To show (vii) suppose $\vartheta\alpha = \vartheta\beta$ but $\alpha < \beta$. Then $C(\alpha, \vartheta\alpha) \subseteq C(\beta, \vartheta\beta)$, so $\alpha \in C(\beta, \vartheta\beta)$ by (i), whence $\vartheta\alpha \in C(\beta, \vartheta\beta) \cap \Omega$ by (C3). Thus $\vartheta\beta \in C(\beta, \vartheta\beta) \cap \Omega$ contradicting (ii).

(viii). Assume $\alpha < \beta$. By (vi), $\vartheta\alpha < \vartheta\beta$ implies $\alpha^* < \vartheta\beta$. Also $\alpha^* < \vartheta\beta$ implies $\alpha^* \in C(\beta, \vartheta\beta)$, whence $\vartheta\alpha \in C(\beta, \vartheta\beta) \cap \Omega$ and so $\vartheta\alpha < \vartheta\beta$ by (ii). Thus

$$\alpha < \beta \rightarrow (\vartheta\alpha < \vartheta\beta \leftrightarrow \alpha^* < \vartheta\beta). \tag{12}$$ {exteqn:theta1.

Now suppose $\beta < \alpha$. By the same argument we obtain

$$\beta < \alpha \rightarrow (\vartheta\beta < \vartheta\alpha \leftrightarrow \beta^* < \vartheta\alpha),$$

and so, by (vii),

$$\beta < \alpha \rightarrow (\vartheta\alpha < \vartheta\beta \leftrightarrow \vartheta\alpha \leq \beta^*). \tag{13}$$ {exteqn:theta1.

Combining eqs. (12) and (13) gives (viii). □

We can now proceed with defining a primitive recursive set of ordinal terms for use in the later analysis of F_β.

Definition 8. Define a subset $\mathrm{OT}_\Omega \subseteq \mathbb{N}$, an encoding τ of ordinals into OT_Ω and a rank function $|.|$ on ordinals by recursion according to the following rules.

1. $\tau(0) = 0 \in \mathrm{OT}_\Omega$, $\tau(\Omega) = \langle 0, 1 \rangle \in \mathrm{OT}_\Omega$, and $|0| = |\Omega| = 0$,

2. If $\alpha = \vartheta\alpha_0$ and $\tau(\alpha_0) \in \mathrm{OT}_\Omega$, $\tau(\alpha) = \langle 1, \tau(\alpha_0) \rangle \in \mathrm{OT}_\Omega$ and $|\alpha| = |\alpha_0| + 1$,

3. If $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ and $\tau(\gamma), \tau(\delta) \in \mathrm{OT}_\Omega$, $\tau(\alpha) = \langle 2, \tau(\gamma), \tau(\delta) \rangle \in \mathrm{OT}_\Omega$ and $|\alpha| = \max\{|\gamma|, |\delta|\} + 1$.

It should be noted that the definition of $x \in \mathrm{OT}_\Omega$ and $|\alpha|$ are primitive recursive.

We now want to define an ordering $<_\vartheta$ on OT_Ω such that $\tau(\alpha) <_\vartheta \tau(\beta)$ if and only if $\alpha < \beta$. Conditions (iii) and (vii) of proposition 7 ensures every ordinal built up from the constants 0, Ω and functions $\alpha, \beta \mapsto \omega^\alpha + \beta$ and $\alpha \mapsto \vartheta\alpha$ has a unique representation. We may therefore dispense with the function τ and identify members of OT_Ω with the ordinals they represent, as was the case with OT.

Define the relation $\alpha <_\vartheta \beta$ on OT_Ω by recursion on the value of $|\alpha| + |\beta|$. The conditions involved in comparing two ordinals $\vartheta\xi_0$ and $\vartheta\xi_1$ will be taken from (viii) of proposition 7. Let $\alpha <_\vartheta \beta$ if and only if one of the following conditions hold.

1. $\alpha = 0$ and $\beta \neq 0$;
2. $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ and either:
   a) $\beta = \Omega$ and $\gamma <_\vartheta \beta$,
   b) $\beta =_{\mathrm{NF}} \omega^{\gamma_0} + \delta_0$ and $\gamma <_\vartheta \gamma_0$, or $\gamma = \gamma_0 \wedge \delta <_\vartheta \delta_0$, or
   c) $\beta = \vartheta\xi$ and $\gamma <_\vartheta \beta$;
3. $\alpha = \vartheta\xi$ and either:
   a) $\beta = \Omega$,
   b) $\beta =_{\mathrm{NF}} \omega^\gamma + \delta$ and $\alpha \leq_\vartheta \gamma$,[5] or
   c) $\beta = \vartheta\eta$ and either, $\xi <_\vartheta \eta \wedge \xi^* <_\vartheta \beta$, or $\eta <_\vartheta \xi \wedge \alpha \leq_\vartheta \eta^*$.

   ($\gamma \leq_\vartheta \delta$ abbreviates $\gamma <_\vartheta \delta$ or $\gamma = \delta$.)

Since the function $\alpha \mapsto \alpha^*$ is primitive recursive, the relation $<_\vartheta$ is also primitive recursive.

Before we proceed with the analysis of F_β, we will show how the ordinals $\alpha \geq \Omega$ in OT_Ω enable the generation of the φ_α functions for $\alpha < \Gamma_0$ and that OT_Ω properly extends OT.

Let $\Omega \cdot 0 = 0$ and if $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ and $\alpha_0 \geq \cdots \geq \alpha_n$, define

$$\begin{aligned}\Omega \cdot \alpha &= \omega^{\Omega+\alpha_0} + \cdots + \omega^{\Omega+\alpha_n},\\ \Omega^\beta \cdot \alpha &= \omega^{\Omega\cdot\beta+\alpha_0} + \cdots + \omega^{\Omega\cdot\beta+\alpha_n}.\end{aligned}$$

tprop:thetaord **Proposition 9.** $\mathrm{OT}_\Omega \cap \Omega$ *forms an initial segment of the ordinals.*

Proof. Suppose $\delta \in \mathrm{OT}_\Omega \cap \Omega$. We prove $\alpha \in \mathrm{OT}_\Omega$ for every $\alpha < \delta$ by transfinite induction on α. Suppose

$$\forall\xi < \alpha(\xi \in \mathrm{OT}_\Omega). \tag{14}$$ {exteqn:ordOT1}

If $\alpha = 0$ or $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$, eq. (14), (C1) and (C2) imply $\alpha \in \mathrm{OT}_\Omega$, so assume α is an ε-ordinal. In search of a contradiction, assume further that $\alpha \notin \mathrm{OT}_\Omega$. We prove $\alpha = \vartheta\xi$ for some $\xi \in \mathrm{OT}_\Omega$. Pick β to be the least ordinal in $\{\gamma \leq \delta : \gamma \in \mathrm{OT}_\Omega \wedge \alpha < \gamma\}$. Then β must be an ε-ordinal as otherwise $\beta = \omega^{\gamma_0} + \gamma_1$ for some $\gamma_0, \gamma_1 < \beta$, whence $\alpha \leq \max\{\gamma, \delta\} \in \mathrm{OT}_\Omega \cap \Omega$. Since $\beta \in \mathrm{OT}_\Omega$, $\beta = \vartheta\xi$ for some $\xi \in \mathrm{OT}_\Omega$. Now, as $\xi^* < \vartheta\xi$ we have $\xi^* < \alpha$. Thus $\xi \in C(\xi, \alpha)$.

Moreover, we claim $C(\xi, \alpha) \cap \Omega \subseteq \alpha$. The argument proceeds by induction on the definition of $\eta \in C(\xi, \alpha) \cap \Omega$. In case $\eta = 0$ or $\eta =_{\mathrm{NF}} \omega^\gamma + \delta$, it is immediate that $\eta < \alpha$, so suppose $\eta = \vartheta\zeta$. Then $\zeta < \xi$ and, since

$$\eta \in C(\xi, \alpha) \cap \Omega \subseteq C(\xi, \vartheta\xi) \cap \Omega = \vartheta\xi,$$

[5]Recall that if γ is an ε-ordinal, $\beta =_{\mathrm{NF}} \omega^\gamma + \delta$ only if $\delta > 0$, whence $\vartheta\xi <_\vartheta \beta$ if $\vartheta\xi = \gamma$.

$\eta < \beta$. By eq. (14) and the fact $\xi \in \mathrm{OT}_\Omega$, we obtain $C(\xi, \alpha) \subseteq \mathrm{OT}_\Omega$, so $\eta = \vartheta\zeta \in \mathrm{OT}_\Omega$, whence $\eta < \alpha$ by the choice of β.

Thus we have shown $\xi \in C(\xi, \alpha)$ and $C(\xi, \alpha) \cap \Omega \subseteq \alpha$, whence $\vartheta\xi \leq \alpha$ by (ϑ1), and $\alpha = \beta \in \mathrm{OT}_\Omega$. □

rop:thetaitems2 **Proposition 10.**

1. *$\vartheta\Omega$ is the least ordinal closed under the function $\gamma \mapsto \vartheta\gamma$.*

2. *$\vartheta\Omega^2$ is the least ordinal closed under the function $\gamma \mapsto \vartheta(\Omega \cdot \gamma)$.*

3. *$\vartheta\Omega^3$ is the least ordinal closed under the function $\gamma \mapsto \vartheta(\Omega^2 \cdot \gamma)$.*

Proof. (i). Let $\gamma_0 = 0$ and $\gamma_{m+1} = \vartheta\gamma_m$; we claim

$$\sup_{n<\omega} \gamma_n = \vartheta\Omega,$$

whence it is clear (i) holds. $\gamma_0 < \vartheta\Omega$ holds trivially and, if $\gamma_m < \vartheta\Omega$, we have $(\gamma_m)^* < \vartheta\Omega$ and so $\gamma_{m+1} = \vartheta\gamma_m < \vartheta\Omega$ by (viii) of proposition 7; thus $\sup_n \gamma_n \leq \vartheta\Omega$.

To show $\vartheta\Omega \leq \sup_n \gamma_n$ we prove $\beta < \vartheta\Omega$ implies $\beta < \gamma_m$ for some m by induction on the rank of β. Suppose $\beta < \vartheta\Omega$. Both $\vartheta\Omega$ and $\sup_n \gamma_n$ are ε-ordinals, so the case $\beta =_{\mathrm{NF}} \omega^{\beta_0} + \beta_1$, holds by the induction hypothesis. If $\beta = \vartheta\xi$ we deduce $\xi < \Omega$ and $\xi^* < \vartheta\Omega$, since $\beta < \vartheta\Omega$ and $\Omega^* = 0$. ξ^* has rank strictly less than β, so the induction hypothesis yields an $m < \omega$ such that $\xi^* < \gamma_m$. Proposition 7 (v) then entails $\xi < \gamma_m$. Moreover, since $\gamma_m = (\gamma_m)^* < \vartheta\gamma_m$, by proposition 7 (vi), $\gamma_m < \gamma_{m+1}$, so $\xi^* < \vartheta\gamma_m$. Therefore $\beta = \vartheta\xi < \vartheta\gamma_m = \gamma_{m+1}$.

(ii). Let $\gamma_0 = 0$ and $\gamma_{m+1} = \vartheta(\Omega \cdot \gamma_m)$; we claim

$$\sup_{n<\omega} \gamma_n = \vartheta\Omega^2,$$

whence (ii) holds. Let $\alpha = \sup_n \gamma_n$. Naturally, $\gamma_0 < \vartheta\Omega^2$, and if $\gamma_m < \vartheta\Omega^2$,

$$(\Omega \cdot \gamma_m)^* = \gamma_m^* < \vartheta\Omega^2,$$

so $\gamma_{m+1} < \vartheta\Omega^2$ by proposition 7 (viii). Thus $\alpha \leq \vartheta\Omega^2$.

To show the converse, we prove $\beta < \vartheta\Omega^2$ implies $\beta < \alpha$ by induction on the rank of β. Suppose $\beta < \vartheta\Omega^2$ and $\beta = \vartheta\xi$ for some ξ. As $(\Omega^2)^* = 0$ and $\beta < \vartheta\Omega^2$, proposition 7 (viii) implies $\xi < \Omega^2$ and $\xi^* < \vartheta\Omega^2$, whence the induction hypothesis implies $\xi^* < \gamma_m$ for some m. Since $\xi < \Omega^2$, there are $\delta_0, \delta_1 < \Omega$ such that $\xi = \Omega \cdot \delta_0 + \delta_1$, whence $\delta_0^* \leq \xi^* < \gamma_m$. So $\xi < \Omega \cdot \gamma_m$ and $\beta < \gamma_{m+1}$ by proposition 7 (viii).

(iii) involves a near identical argument as (ii). Pick $\gamma_0 = 0$ and $\gamma_{m+1} = \vartheta(\Omega^2 \cdot \gamma_m)$. That $\sup_n \gamma_n \leq \vartheta\Omega^3$ is easily established using proposition 7. For the converse direction, $\vartheta\Omega^3 \leq \sup_n \gamma_n$, we suppose $\beta < \vartheta\Omega^3$ and seek to determine $\beta < \gamma_m$ for some m. If $\beta = \vartheta\xi < \vartheta\Omega^3$, we may assume $\xi^* < \gamma_m$ for some m, whence $\xi < \Omega^2 \cdot \gamma_m$, and so $\beta < \gamma_{m+1}$. □

Proposition 10 allows us to identify some characteristic ordinals in terms of both the Veblen and ϑ functions.

rop:thetaitems3 **Corollary 11.**

1. $\alpha < \varphi 20$ *implies* $\varepsilon_\alpha = \vartheta\alpha$.

2. $\vartheta\Omega = \varphi 20$.

3. $\vartheta\Omega^2 = \Gamma_0$.

4. $\vartheta(\Omega^2+\Omega)$ *is the least fixed point of the function* $\xi \mapsto \Gamma_\xi$, *which enumerates the class* $\{\xi : \xi = \varphi\xi 0\}$.

Proof. (i) is argued by transfinite induction on $\alpha < \varphi 20$. Suppose $\vartheta\beta = \varepsilon_\beta$ for every $\beta < \alpha$. We will begin by showing a) $C(\alpha, \varepsilon_\alpha) \cap \Omega \subseteq \varepsilon_\alpha$, and b) $\alpha \in C(\alpha, \varepsilon_\alpha)$, allowing us to deduce $\vartheta\alpha \leq \varepsilon_\alpha$ by (ϑ1).

a) is shown by a further induction on the construction of $C(\alpha, \varepsilon_\alpha)$. It is trivial that $\beta \in C_0(\alpha, \varepsilon_\alpha) \cap \Omega$ entails $\beta < \varepsilon_\alpha$, and that ε_α is closed under applications of rule (C2). Moreover, the induction hypothesis implies that for $\beta < \alpha$, $\vartheta\beta < \varepsilon_\alpha$, thus (C3) is also dealt with, and $C(\alpha, \varepsilon_\alpha) \cap \Omega \subseteq \varepsilon_\alpha$.

b) uses the fact $\alpha < \varphi 20$, whence $\alpha < \varepsilon_\alpha$ and $\alpha \in C(\alpha, \varepsilon_\alpha)$ by (C1).

To see $\varepsilon_\alpha \leq \vartheta\alpha$, assume otherwise. Then $\vartheta\alpha = \varepsilon_\beta$ for some $\beta < \alpha$ by proposition 7 (iii). The induction hypothesis yields $\vartheta\alpha = \vartheta\beta$, contradicting $\beta < \alpha$.

(ii) is an immediate consequence of (i) and proposition 10.

(iii). The proof for (i) above can be extended to $\alpha > \varphi 20$, but then one can at best show $\varepsilon_\alpha \leq \vartheta\alpha \leq \varepsilon_{\alpha+1}$ for $\alpha < \Omega$.[6] One can then prove

$$\varphi 2\alpha \leq \vartheta(\Omega + \alpha) \leq \varphi 2(\alpha + 1)$$

for $\alpha < \Omega$ by transfinite induction on α, using the definition of ϑ. This can easily be extended to deduce, in general,

$$\varphi\alpha\beta \leq \vartheta(\Omega \cdot \alpha + \beta) \leq \varphi(\alpha + 1)(\beta + 1)$$

for $\alpha, \beta < \Gamma_0$, from which proposition 10 (ii) implies $\vartheta\Omega^2 = \Gamma_0$.

(iv). Let Δ_0 denote the least fixed point of the function $\xi \mapsto \Gamma_\xi$. Following from (iii) above, $\vartheta(\Omega^2 + \alpha) = \Gamma_\alpha$ for $\alpha < \Delta_0$. Since $\vartheta(\Omega^2 + \Omega)$ is the least ordinal closed under the function $\alpha \mapsto \vartheta(\Omega^2 + \alpha)$, we deduce $\vartheta(\Omega^2 + \Omega) = \Delta_0$. □

In this notation system, $\vartheta\Omega^3$ represents the *Ackermann ordinal*, $\vartheta\Omega^\Omega$ denotes the *Veblen ordinal* and $\vartheta\varepsilon_{\Omega+1}$ is the *Bachmann-Howard ordinal* where

$$\vartheta\varepsilon_{\Omega+1} = \sup\{\vartheta\Omega, \vartheta\Omega^\Omega, \vartheta\Omega^{\Omega^\Omega}, \dots\} = \sup_{\alpha \in \mathrm{OT}_\Omega} \vartheta\alpha.$$

Having established an ordinal notation system suitable for the analysis of the theories F_β, we may now fix the language of F_β. Since the proof-theoretic strength of each theory F_β with $\beta < \vartheta\varepsilon_{\Omega+1}$ will not exceed $\vartheta\varepsilon_{\Omega+1}$, we may pick $\kappa = \vartheta\varepsilon_{\Omega+1}$ and suppose the theories F_β are formulated in the language $\mathscr{L}_\kappa$.

We require a few further technical results about ordinals before we can proceed with the analysis. Suppose $\beta = \Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_n} \cdot \beta_n$ such that $\alpha_0 > \cdots > \alpha_n$ and $\beta_i < \Omega$ for each $i \leq n$. Recall from the previous section that $\beta|_\gamma$ denotes the ordinal $\Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_k} \cdot \beta_k$ where $k < n$ is the least such that $\alpha_k > \gamma \geq \alpha_{k+1}$, or $k = n$ if $\alpha_n > \gamma$.

The following observations are immediate consequences of the definition.

prop:morditems **Proposition 12.** *For all ordinals* $\alpha, \beta < \varepsilon_{\Omega+1}$ *and* $\gamma, \delta < \Omega$,

[6] $\vartheta\alpha = \varepsilon_{\alpha+1}$ for $\alpha = \varphi 20$ for example.

1. $\gamma < \delta$ *implies* $\alpha|_\gamma \leq \alpha|_\delta$,
2. $\alpha < \beta$ *implies* $\alpha|_\gamma \leq \beta|_\gamma$.
3. $\delta \leq \gamma$ *implies* $(\alpha|_\gamma)|_\delta = \alpha|_\gamma$,
4. $\beta < \alpha|_\gamma$ *if and only if* $\beta + \Omega^{\gamma+1} \leq \alpha$,
5. $\beta < \alpha|_\gamma$ *and* $\delta \leq \gamma$ *implies* $\beta + \Omega^\delta < \alpha|_\gamma$.

item:morditem4

3.3 Lower bounds on the proof-theoretic ordinal

extsec:Fblower

We will now establish lower bounds for the theories F_β. This will be achieved by extending the well-ordering proofs used in our analysis of F (**??**) and S_3 (**??**). Recall $\kappa = \vartheta\varepsilon_{\Omega+1}$.

F_0 is identical to F, for which an optimal lower bound was established in **??**. However, because of the change in ordinal notation system and the reflective nature of the theories F_β it will be useful to provide a new proof of the result. For each $\xi < \kappa$ let $\mathrm{wo}_\xi(x)$ denote the formula

$$\forall \ulcorner A(x) \urcorner \forall y \prec x\, \mathsf{T}_\xi(\ulcorner \mathsf{TI}(\dot{y}, A) \urcorner).$$

Let $F_0(\rho)$ denote the formula $\mathrm{wo}_0(\rho^*) \wedge \forall \sigma < \rho[\mathrm{wo}_0(\sigma^*) \rightarrow \mathrm{wo}_0(\vartheta\sigma)]$ and $F_0^\rho(\alpha)$ denote $\alpha < \Omega \rightarrow F_0(\rho + \alpha)$. We begin with a technical lemma.

extlem:F0tech **Lemma 13.** $\mathsf{F}_0^1 \vdash \forall \rho[\mathsf{T}_0(\ulcorner F_0(\dot{\rho}) \urcorner) \rightarrow \mathrm{wo}_0(\vartheta\rho)]$.

Proof. Argue within F_0^1, and assume

$$\mathsf{T}_0(\ulcorner F_0(\dot{\rho}) \urcorner). \tag{15}$$ {exteqn:F0tech1

Let $\gamma_0 = \rho^* + 1$ and $\gamma_{m+1} = C(\rho, \gamma_m) \cap \Omega$. Moreover, let $\mathrm{wo}_0^1(x)$ denote $\mathsf{T}_0(\ulcorner \mathrm{wo}_0(\dot{x}) \urcorner)$. Equation (15) implies $\mathrm{wo}_0^1(\gamma_0)$ and $\forall \sigma < \rho[\mathrm{wo}_0^1(\sigma^*) \rightarrow \mathrm{wo}_0^1(\vartheta\sigma))]$, whence it is easy to deduce $\forall m[\mathrm{wo}_0^1(\gamma_m) \rightarrow \mathrm{wo}_0^1(\gamma_{m+1})]$ and thus $\mathrm{wo}_0^1(\vartheta\rho)$. By T_0-Del, $\mathrm{wo}_0(\vartheta\rho)$ holds. □

:F0wellordering **Lemma 14.** *For every* $m < \omega$, $\mathsf{F}_0^m \vdash F_0(\Omega \cdot \bar{m})$.

Proof. $\mathsf{F}_0^0 \vdash F_0(\bar{0})$ holds vacuously, so suppose $m = n + 1 > 0$ and

$$\mathsf{F}_0^n \vdash F_0(\Omega \cdot \bar{n}). \tag{16}$$ {exteqn:4.1}

The first step is to establish $\mathsf{F}_0^n \vdash \mathsf{Prog} F_0^{\Omega\cdot\bar{n}}$. Argue informally within F_0^n, assuming $\forall \nu < \mu\, F_0^{\Omega\cdot\bar{n}}(\nu)$ for some μ, that is,

$$\forall \nu < \mu(\mathrm{wo}_0(\nu^*) \wedge \forall \sigma < \Omega \cdot \bar{n} + \nu[\mathrm{wo}_0(\sigma^*) \rightarrow \mathrm{wo}_0(\vartheta\sigma)]). \tag{17}$$ {exteqn:4.3}

We want to show $\mathrm{wo}_0(\mu^*)$ and $\forall \sigma < \Omega \cdot \bar{n} + \mu[\mathrm{wo}_0(\sigma^*) \rightarrow \mathrm{wo}_0(\vartheta\sigma)]$. The former is obvious since the fact $\mathrm{wo}_0(\varepsilon_\alpha)$ is progressive in α is provable in F_0^1 (*cf.* **??**). To prove the latter, assume $\mathrm{wo}_0(\sigma^*)$ for some $\sigma < \Omega \cdot \bar{n} + \mu$. If $\mu = 0$ or is a limit ordinal, $\mathrm{wo}_0(\vartheta\sigma)$ is immediate given eq. (17). Otherwise $\mu = \nu + 1$ for some ν, whence we may assume $\sigma = \Omega \cdot \bar{n} + \nu$. Let $\gamma_0 = \sigma^* + 1$ and $\gamma_{m+1} = C(\sigma, \gamma_m) \cap \Omega$. Then $\gamma_m < \Omega$ for each m and

$$\vartheta\sigma \leq \sup_m \gamma_m$$

by proposition 6. $\mathrm{wo}_0(\gamma_0)$ is a consequence of $\mathrm{wo}_0(\sigma^*)$, so suppose

$$\mathrm{wo}_0(\gamma_m) \tag{18}$$ {exteqn:4.4}

with the aim of showing $\mathrm{wo}_0(\gamma_{m+1})$ by induction on the recursive definition of $C(\sigma, \gamma_m)$. Assume $\mathrm{wo}_0(\alpha)$ holds for every $\alpha \in C_k(\sigma, \gamma_m) \cap \Omega$ and suppose $\beta \in C_{k+1}(\sigma, \gamma_m) \cap \Omega$.

(C1). $\beta \leq \gamma_m$, so $\mathrm{wo}_0(\beta)$ is a result of eq. (18).

(C2). $\beta =_{\mathrm{NF}} \omega^\delta + \eta$, and $\delta, \eta \in C_k(\sigma, \gamma_m)$. Since also $\delta, \eta < \Omega$ the induction hypothesis yields $\mathrm{wo}_0(\delta) \wedge \mathrm{wo}_0(\eta)$ and so $\mathrm{wo}_0(\beta)$.

(C3). $\beta =_{\mathrm{NF}} \vartheta\xi$ and $\xi \in C_k(\sigma, \gamma_m) \cap \sigma$. Thus, $\xi^* \in C_k(\sigma, \gamma_m) \cap \Omega$ and therefore $\mathrm{wo}_0(\xi^*)$ by the induction hypothesis. If $\xi < \Omega \cdot n$, $\mathrm{wo}_0(\vartheta\xi)$ is a consequence of eq. (16), otherwise $\Omega \cdot n \leq \xi < \sigma$ and $\mathrm{wo}_0(\vartheta\xi)$ is implied by eq. (17).

Thus we may deduce $\forall \alpha < \gamma_{m+1}\, \mathrm{wo}_0(\alpha)$, hence $\mathrm{wo}_0(\gamma_{m+1})$, and so $\mathrm{wo}_0(\vartheta\sigma)$, concluding the proof of

$$\mathsf{F}_0^n \vdash \mathsf{Prog} F_0^{\Omega\cdot\bar{n}}. \tag{19}$$ {exteqn:4.2}

An application of T_0-Intro entails $\mathsf{F}_0^m \vdash \mathsf{T}_0(\ulcorner \mathsf{Prog} F_0^{\Omega\cdot\bar{n}} \urcorner)$, so

$$\mathsf{F}_0^m \vdash \forall\alpha[\mathrm{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F_0(\Omega \cdot \bar{n} + \dot{\alpha})\urcorner)],$$

and hence, by lemma 13,

$$\mathsf{F}_0^m \vdash \forall\alpha(\mathrm{wo}_0(\alpha) \to \mathrm{wo}_0\vartheta(\Omega \cdot \bar{n} + \alpha)). \tag{20}$$ {exteqn:4.6}

To obtain $\mathsf{F}_0^m \vdash F_0(\Omega \cdot \bar{m})$ and complete the proof we argue within F_0^m. Firstly, $\mathrm{wo}_0((\Omega \cdot \bar{m})^*)$ holds trivially as $(\Omega \cdot \bar{m})^* = 0$. Secondly, if $\sigma < \Omega \cdot \bar{m}$, we have either $\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0(\vartheta\sigma)$ by eq. (16), or $\sigma = \Omega \cdot \bar{n} + \zeta$ for some $\zeta < \Omega$, whence $\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0(\zeta)$ and $\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0(\vartheta\sigma)$ results from eq. (20). □

or:F0lowerbound **Corollary 15.** $\|\mathsf{F}_0^m\| \geq \vartheta(\Omega \cdot m)$ *and* $\|\mathsf{F}_0\| \geq \vartheta(\Omega \cdot \omega)$.

Proof. Let $\gamma_0 = 1$ and $\gamma_{k+1} = C(\Omega \cdot m, \gamma_k) \cap \Omega$. Then $\vartheta(\Omega \cdot m) \leq \sup_{k<\omega} \gamma_k$ and $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\gamma}_0)$ holds. Moreover, if $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\alpha})$ for every $\alpha < \gamma_k$ we may deduce $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\alpha})$ for every $\alpha < \gamma_{k+1}$ by induction on the definition of γ_{k+1} thus: suppose $\beta \in C_{k+1}(\Omega \cdot m, \gamma_k) \cap \Omega$ and $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\alpha})$ for every $\alpha \in C_k(\Omega \cdot m, \gamma_k) \cap \Omega$. If β was enumerated into $C_{k+1}(\Omega \cdot m, \gamma_k)$ by either (C1) or (C2), $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\beta})$ is easily obtained from the induction hypothesis. If, however, $\beta = \vartheta\xi$ for some $\xi \in C_k(\Omega \cdot m, \gamma_k) \cap \Omega \cdot m$, $\xi = \Omega \cdot n + \alpha$ for some $n < m$, $\alpha \in C_k(\Omega \cdot m, \gamma_k) \cap \Omega$ and $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\xi}^*)$ by the induction hypothesis, whence lemma 14 implies $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\beta})$.

Since $\vartheta(\Omega \cdot m) \leq \sup_{k<\omega} \gamma_k$, we obtain $\mathsf{F}_0^m \vdash \mathrm{wo}_0(\bar{\alpha})$ for every $\alpha < \vartheta(\Omega \cdot m)$ and so $\mathsf{F}_0^m \vdash \mathsf{TI}({<}\vartheta(\Omega \cdot m))$ by an application of T_0-Elim. □

We will now extend the well-ordering proof above to determine lower bounds on the strength of each theory F_β. This will be done in stages, first for $\beta = 1$, then for arbitrary $\beta < \omega$ and finally for transfinite levels of the hierarchy. In doing so we will find ourselves migrating from the function $\alpha \mapsto \vartheta\alpha$ to the function $\alpha \mapsto \vartheta(\Omega \cdot \alpha)$, and eventually to functions $\alpha \mapsto \vartheta(\Omega^\beta \cdot \alpha)$.

Before proceeding directly with F_1 we require a slightly more general form of lemma 14. As its proof makes no explicit use of the fact m is finite, nor any application of T_0-Intro in showing $\mathsf{Prog} F_0^{\Omega\cdot\bar{m}}$ given $F_0(\Omega \cdot \bar{m})$, we may readily deduce the following generalisation.

genwellordering **Proposition 16.** $\mathsf{F}_0^1 \vdash \forall\rho[F_0(\rho) \to \mathsf{Prog} F_0^{\rho}]$.

Proof. Argue inside F_0^1 and assume $F_0(\rho)$ and $\forall\nu < \mu\, F_0^{\rho}(\nu)$, that is,

$$\mathrm{wo}_0(\rho^*), \tag{21}$$ {exteqn:F03.1}

$$\forall\sigma < \rho[\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0\vartheta\sigma], \tag{22}$$ {exteqn:F03.2}

$$\forall\nu < \mu(\mathrm{wo}_0(\nu^*) \wedge \forall\sigma < \rho+\nu[\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0\vartheta\sigma]), \tag{23}$$ {exteqn:F03.3}

for some $\mu < \Omega$; we want to prove $\mathrm{wo}_0(\mu^*)$ and $\forall\tau < \rho+\mu[\mathrm{wo}_0(\tau^*) \to \mathrm{wo}_0(\vartheta\tau)]$. The former holds immediately given eq. (23) so assume

$$\mathrm{wo}_0(\tau^*), \tag{24}$$ {exteqn:F03.4}

for some $\tau < \rho + \mu$. We will prove $\mathrm{wo}_0(\vartheta\tau)$.

Let $\gamma_0 = \tau^* + 1$ and $\gamma_{m+1} = C(\tau, \gamma_m) \cap \Omega$. That $\mathrm{wo}_0(\gamma_m)$ holds for each m will be established by induction on m. From eq. (24) one has $\mathrm{wo}_0(\gamma_0)$. Assume $\mathrm{wo}_0(\gamma_m)$. In order to show $\mathrm{wo}_0(\gamma_{m+1})$, assume $\mathrm{wo}_0(\alpha)$ for every $\alpha \in C_k(\tau,\gamma_m) \cap \Omega$. Pick an arbitrary $\alpha_0 \in C_{k+1}(\tau,\gamma_m) \cap \Omega$. If α_0 was enumerated into the set by either (C1) or (C2), $\mathrm{wo}_0(\alpha_0)$ is immediate. Otherwise $\alpha_0 = \vartheta\xi$ for some $\xi \in C_k(\tau,\gamma_m) \cap \tau$ and, since $\xi^* \in C_k(\tau,\gamma_m) \cap \Omega$, we have $\mathrm{wo}_0(\xi^*)$. If $\xi < \rho$, eq. (22) provides $\mathrm{wo}_0(\vartheta\xi)$. Otherwise $\tau = \rho + \nu$ for some $\nu < \mu$ and $\mathrm{wo}_0(\vartheta\xi)$ holds due to eq. (23). Either way $\mathrm{wo}_0(\alpha_0)$, and so $\mathrm{wo}_0(\alpha)$ for every $\alpha < \gamma_{m+1}$, whence $\mathrm{wo}_0(\gamma_{m+1})$.

Since $\vartheta\tau \geq \sup_m \gamma_m$ we obtain $\mathrm{wo}_0(\vartheta\tau)$. □

Proposition 16 is plays a key role in the analysis of F_0^m and also F_1^1. Lemma 13 entails

$$\begin{aligned}\mathsf{F}_0^1 \vdash \mathsf{T}_0(\ulcorner \mathsf{Prog} F_0^{\dot\rho} \urcorner) &\to \forall\alpha[\mathrm{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F_0^{\dot\rho}(\dot\alpha) \urcorner)]). \\ &\to \forall\alpha[\mathrm{wo}_0(\alpha) \to \mathrm{wo}_0\vartheta(\rho+\alpha)]),\end{aligned}$$

so $\mathsf{F}_1^1 \vdash \forall\rho[\mathsf{T}_1(\ulcorner F_0(\dot\rho) \urcorner) \to \mathsf{T}_1(\ulcorner F_0(\dot\rho + \Omega) \urcorner)]$. This amounts to proving

$$\mathsf{F}_1^1 \vdash \forall\rho[F_1(\rho) \to \mathsf{Prog} F_1^{\rho}] \tag{25}$$ {exteqn:F11.1}

where $F_1(\rho)$ is the formula $\mathsf{T}_1(\ulcorner F_0(\rho) \urcorner)$ and $F_1^{\rho}(\alpha)$ denotes $\alpha < \Omega \to F_1(\rho + \Omega\cdot\alpha)$. Equation (25) is sufficient to deduce a lower bound on the strength of the theory F_1^1.

r:F11lowerbound **Corollary 17.** $\|\mathsf{F}_1^1\| \geq \vartheta\Omega^2$.

Proof. Since $\mathsf{F}_0 \vdash F_0(\bar 0)$, eq. (25) implies

$$\mathsf{F}_1^1 \vdash \mathsf{Prog} F_1^{\bar 0}. \tag{26}$$ {exteqn:F11lowe

Let $\sigma_0 = 1$ and $\sigma_{m+1} = \vartheta(\Omega\cdot\sigma_m)$. By proposition 10 (ii), $\vartheta\Omega^2 = \sup_m \sigma_m$, so it suffices to show $\mathsf{F}_1^1 \vdash \mathrm{wo}_1(\bar\sigma_m)$ for each m. This is trivial for $m = 0$; for $m = n+1$ argue within F_1^1 assuming $\mathrm{wo}_1(\bar\sigma_n)$. Then $\mathrm{wo}_1(\bar\sigma_n + 1)$ and so $\mathsf{T}_1(\ulcorner F_0(\Omega\cdot\bar\sigma_n) \urcorner)$ by an application of T_1-Elim and eq. (26). Lemma 13 yields $\mathsf{T}_1(\ulcorner \mathrm{wo}_0\vartheta(\Omega\cdot\bar\sigma_n) \urcorner)$ and so $\mathrm{wo}_1(\bar\sigma_m)$ holds. □

Within F_1^2, the above proof may be replicated under a T_1 predicate, allowing one to reach ordinals beyond $\vartheta\Omega^2$, as the next proposition demonstrates.

:F1wellordering **Lemma 18.** *For each m, $\mathsf{F}_1^{m+1} \vdash F_1(\Omega^2\cdot\bar m)$ and $\mathsf{F}_1^{m+1} \vdash \mathsf{Prog} F_1^{\Omega^2\cdot\bar m}$.*

Proof. $\mathsf{F}_1^1 \vdash F_1(\bar{0})$ holds trivially, so suppose $m = n + 1$ and $\mathsf{F}_1^m \vdash F_1(\Omega^2 \cdot \bar{n})$. Equation (25) yields $\mathsf{F}_1^m \vdash \mathrm{Prog} F_1^{\Omega^2 \cdot \bar{n}}$, whence an application of T_1-Intro and T_1^--Rep implies

$$\mathsf{F}_1^{m+1} \vdash \mathsf{T}_1(\ulcorner \forall \alpha[\mathrm{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F_1(\Omega^2 \cdot \bar{n} + \Omega \cdot \dot{\alpha})\urcorner)]\urcorner). \tag{27}$$ {exteqn:F1well.

However, arguing within F_0, from $F_0(\rho)$ one obtains $\mathrm{wo}_0(\vartheta\rho)$, so eq. (27) entails

$$\mathsf{F}_1^{m+1} \vdash \mathsf{T}_1(\ulcorner \forall \alpha[\mathrm{wo}_0(\alpha) \to \mathrm{wo}_0 \vartheta(\Omega^2 \cdot \bar{n} + \Omega \cdot \dot{\alpha})]\urcorner),$$

and thus $\mathsf{F}_1^{m+1} \vdash F_1(\Omega^2 \cdot \bar{m})$ as required. □

m:F1lowerbound **Theorem 6.** *Suppose $m < \omega$. Then every theorem of $\mathsf{PA} + \mathsf{TI}(<\vartheta(\Omega^2 \cdot m))$ is derivable in F_1^m. Moreover, every theorem of $\mathsf{PA} + \mathsf{TI}(<\vartheta(\Omega^2 \cdot \omega))$ is derivable in F_1.*

Proof. Since $\vartheta 0 = \varepsilon_0$ and F_1^0 extends PA, the case $m = 0$ holds, so suppose $m = n+1 > 0$. If $\mathsf{F}_1^m \vdash \mathrm{wo}_0(\bar{\alpha})$, lemma 18 implies

$$\mathsf{F}_1^m \vdash \mathrm{wo}_0 \vartheta(\Omega^2 \cdot \bar{n} + \Omega \cdot \bar{\alpha}),$$

whereby if $\sigma_0 = 1$ and $\sigma_{k+1} = \vartheta(\Omega^2 \cdot n + \Omega \cdot \sigma_k)$, $\mathsf{F}_1^m \vdash \mathrm{wo}_0(\bar{\sigma}_k)$ for every k.

Thus we require to show $\vartheta(\Omega^2 \cdot m) \leq \sup_k \sigma_k$. This is proved by induction on the rank of $\alpha < \vartheta(\Omega^2 \cdot m)$. If $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta < \vartheta(\Omega^2 \cdot m)$, the induction hypothesis immediately implies $\alpha < \sigma_k$ for some k. Otherwise

$$\alpha = \vartheta\xi < \vartheta(\Omega^2 \cdot m)$$

and there are two cases to consider:

1. $\xi < \Omega^2 \cdot m$ and $\xi^* < \vartheta(\Omega^2 \cdot m)$; or
2. $\xi > \Omega^2 \cdot m$ but $\vartheta\xi \leq (\Omega^2 \cdot m)^*$.

b) cannot hold since $(\Omega^2 \cdot m)^* = 0$, so $\xi^* < \vartheta(\Omega^2 \cdot m)$. As ξ^* has rank strictly smaller than α the induction hypothesis implies $\xi^* < \sigma_k$ for some k. But then $\xi < \Omega^2 \cdot n + \Omega \cdot \sigma_k$ and $\alpha < \sigma_{k+1}$.

The second part of the theorem is easily established using the fact $\vartheta(\Omega^2 \cdot \omega) = \sup_k \vartheta(\Omega^2 \cdot k)$. □

We can now turn our attention to the theories F_p for $p < \omega$. Lemma 18 essentially shows $\mathsf{F}_1 \vdash F_1(\Omega^2 \cdot \bar{\alpha})$ implies $\mathsf{F}_1 \vdash F_1(\Omega^2 \cdot (\bar{\alpha} + 1))$. This can be extended to show $\mathsf{F}_1 \vdash \forall \nu < \bar{\mu}\, F_1(\Omega^2 \cdot \nu)$ implies $\mathsf{F}_1 \vdash F_1(\Omega^2 \cdot \bar{\mu})$, whence

$$\mathsf{F}_2^1 \vdash \mathrm{Prog} F_2^{\bar{0}} \tag{28}$$ {exteqn:Fbwell1

where $F_2^\rho(\alpha)$ is the formula $\alpha < \Omega \wedge \mathsf{T}_2(\ulcorner F_0(\rho + \Omega^2 \cdot \alpha)\urcorner)$.

Equation (28) suffices to deduce a lower bound for F_2^1 and acts as the base step in the analysis of F_2 and ultimately F_p, which follows a generalised form of the procedure used in lemma 18.

Let $F_p(\rho)$, for $0 < p < \omega$, be the formula $\mathsf{T}_p(\ulcorner F_0(\dot{\rho})\urcorner)$, that is

$$\mathsf{T}_p(\ulcorner \mathrm{wo}_0(\rho^*) \wedge \forall \sigma < \dot{\rho}[\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0(\vartheta\sigma)]\urcorner),$$

and denote by $F_p^\rho(\alpha)$ the formula $\alpha < \Omega \wedge F_p(\rho + \Omega^{\bar{p}} \cdot \alpha)$.

Fpwellordering1 **Lemma 19.** *For each $p < \omega$, $\mathsf{F}^1_p \vdash \forall\rho[F_p(\rho) \rightarrow \mathsf{Prog}F^\rho_p]$ and, for $m < \omega$, $\mathsf{F}^{m+1}_p \vdash F_p(\Omega^{\bar{p}+1} \cdot \bar{m})$.*

Proof. The proof proceeds by induction on $p < \omega$. The case of $p = 0$ has already been shown in proposition 16 so suppose $p = q + 1 > 0$. Argue informally within F^1_p. Assume

$$F_p(\rho), \tag{29}$$ {exteqn:Fpwell1

$$\forall\nu < \mu\, F^\rho_p(\nu) \tag{30}$$ {exteqn:Fpwell2

for some $\mu < \Omega$. If $\mu = 0$, of course $F^\rho_p(\mu)$ holds by eq. (29), and if μ is a limit ordinal, eq. (30) implies $\mathsf{T}_p(\ulcorner\forall\nu < \dot{\mu}\, F_0(\rho + \Omega^{\bar{p}} \cdot \mu)\urcorner)$, whence $F^\rho_p(\mu)$ is immediate. This leaves only the case in which μ is a successor ordinal. But for every ordinal τ,

$$\begin{aligned} F_p(\tau) &\rightarrow \mathsf{T}_p(\ulcorner F_q(\dot{\tau})\urcorner), \\ &\rightarrow \mathsf{T}_p(\ulcorner\mathsf{Prog}F^{\dot{\tau}}_q\urcorner), \\ &\rightarrow \mathsf{T}_p(\ulcorner\mathsf{T}_0(\ulcorner\mathsf{Prog}F^{\dot{\tau}}_q\urcorner)\urcorner), \\ &\rightarrow \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \rightarrow \mathsf{T}_0(\ulcorner F^{\dot{\tau}}_q(\dot{\alpha})\urcorner)]\urcorner), \\ &\rightarrow \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \rightarrow \mathsf{T}_0(\ulcorner F_0(\dot{\tau} + \Omega^{\bar{q}} \cdot \dot{\alpha})\urcorner)]\urcorner), \\ &\rightarrow \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \rightarrow \mathsf{wo}_0(\vartheta(\dot{\tau} + \Omega^{\bar{q}} \cdot \alpha))]\urcorner). \end{aligned} \tag{31}$$ {exteqn:Fpwell3

The second implication holds on account of the induction hypothesis, while the final holds due to T_0-Del and lemma 13. Given that if $\tau \le \sigma < \tau + \Omega^p$ there exists some $\zeta < \Omega$ such that $\sigma < \tau + \Omega^q \cdot \zeta$ and $\mathsf{wo}_0(\sigma^*) \rightarrow \mathsf{wo}_0(\zeta)$ (pick $\zeta = (\sigma^*)^{\varepsilon+}$), eq. (31) entails $F_p(\tau) \rightarrow \mathsf{T}_p(\ulcorner F_0(\dot{\tau} + \Omega^{\bar{p}})\urcorner)$, that is $\forall\tau[F^\tau_p(\alpha) \rightarrow F^\tau_p(\alpha + 1)]$.

For the second part, the case $m = 0$ is immediate, so suppose $m = n + 1$ and $\mathsf{F}^m_p \vdash F_p(\Omega^{\bar{p}+1} \cdot \bar{n})$. Then $\mathsf{F}^{m+1}_p \vdash \mathsf{T}_p(\ulcorner\mathsf{Prog}F^{\Omega^{\bar{p}+1}\cdot\bar{n}}_p\urcorner)$, from which we deduce

$$\mathsf{F}^{m+1}_p \vdash \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \rightarrow \mathsf{T}_0(\ulcorner F_p(\Omega^{\bar{p}+1} \cdot \bar{n} + \Omega^{\bar{p}} \cdot \dot{\alpha})\urcorner)]\urcorner)$$

and hence obtain $\mathsf{F}^{m+1}_p \vdash F_p(\Omega^{\bar{p}+1} \cdot \bar{m})$. □

m:Fplowerbound **Theorem 7.** *For every $m < \omega$ and $p < \omega$, $\mathsf{F}^m_p \vdash \mathsf{TI}(<\vartheta(\Omega^{p+1} \cdot m))$ and $\mathsf{F}_p \vdash \mathsf{TI}(<\vartheta(\Omega^{p+1} \cdot \omega))$.*

Proof. For every p, the base case, $m = 0$, is immediate since F^0_p extends PA formulated in the language $\mathscr{L}_p$. Otherwise $m = n + 1 > 0$ and the previous lemma shows $\mathsf{F}^m_p \vdash \mathsf{Prog}F^{\Omega^{\bar{p}+1}\cdot\bar{n}}_p$. Given $\mathsf{F}^m_p \vdash \mathsf{wo}_p(\bar{\alpha})$, one obtains $\mathsf{F}^m_p \vdash F_p(\Omega^{\bar{p}+1} \cdot \bar{n} + \Omega^{\bar{p}} \cdot \bar{\alpha})$, and so

$$\mathsf{F}^m_p \vdash \mathsf{wo}_p(\vartheta(\Omega^{\bar{p}+1} \cdot \bar{n} + \Omega^{\bar{p}} \cdot \bar{\alpha})),$$

by unravelling the definition of F_p and lemma 13.

Let $\sigma_0 = 1$ and $\sigma_{k+1} = \vartheta(\Omega^{p+1} \cdot n + \Omega^p \cdot \sigma_k)$. The previous paragraph establishes $\mathsf{F}^m_p \vdash \mathsf{TI}(<\sigma_k)$ for every k, so all that remains is to show $\vartheta(\Omega^{p+1} \cdot m) \le \sup_k \sigma_k$, which proceeds by induction on the rank of $\alpha < \vartheta(\Omega^{p+1} \cdot m)$. If $\alpha = 0$ we are done, and if $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$, the induction hypothesis implies $\alpha < \sigma_k$ for some k. Thus, suppose

$$\alpha = \vartheta\xi < \vartheta(\Omega^{p+1} \cdot m)$$

for which there are two cases to consider.

1. $\xi < \Omega^{p+1} \cdot m$ and $\xi^* < \vartheta(\Omega^{p+1} \cdot m)$; or
2. $\xi > \Omega^{p+1} \cdot m$ but $\vartheta\xi \leq (\Omega^{p+1} \cdot m)^*$.

Since $(\Omega^{p+1} \cdot m)^* = 0$, b) is impossible, and so $\xi^* < \sigma_k$ for some k by the induction hypothesis. Then $\xi < \Omega^{p+1} \cdot n + \Omega^p \cdot \sigma_k$, whence $\alpha < \vartheta(\Omega^{p+1} \cdot n + \Omega^p \cdot \sigma_k) = \sigma_{k+1}$.

The second part of the theorem is an immediate consequence of the fact

$$\vartheta(\Omega^{p+1} \cdot \omega) = \sup_{k<\omega} \vartheta(\Omega^{p+1} \cdot k).$$

□

Finally, we extend the well-ordering proofs to theories F_β for $\beta \geq \omega$. For $\beta = \omega$ this involves generalising the above proof so that one may derive

$$\mathsf{F}_\omega \vdash \forall p < \omega \, \mathsf{T}_\omega(\ulcorner \mathsf{Prog} F_p^{\bar{0}} \urcorner), \tag{32}$$ {exteqn:Fbetawe

whence $\mathsf{F}_\omega \vdash \forall p < \omega \, \mathsf{T}_\omega(\ulcorner \forall\alpha[\mathrm{wo}_0(\alpha) \rightarrow \mathrm{wo}_0 \vartheta(\Omega^p \cdot \alpha)] \urcorner)$ and also

$$\mathsf{F}_\omega \vdash \forall \sigma < \Omega^\omega [\mathrm{wo}_\omega(\sigma^*) \rightarrow \mathrm{wo}_\omega(\vartheta\sigma))].$$

Equation (32) is not difficult to establish as the proof of lemma 19 is clearly uniform in $p < \omega$ and thus can be repeated under a T_ω predicate in F_ω. But in order to lift the construction to arbitrary $\beta \in \mathrm{OT}_\Omega \cap \Omega$ we require a more general argument.

For each $\beta \in \mathrm{OT}_\Omega$ let $G_\beta(\rho)$ denote the formula

$$\mathsf{T}_\beta(\ulcorner F_0(\dot{\rho}) \urcorner),$$

and let $G^\rho_\beta(\alpha)$ abbreviate $\alpha < \Omega \wedge \forall \eta < \bar{\beta} \, G_\beta(\rho + \Omega^\eta \cdot \alpha)$.

tawellordering1 **Proposition 20.** *For each $\beta \in \mathrm{OT}_\Omega \cap \Omega$, $\mathsf{F}^1_\beta \vdash \forall\rho[G_\beta(\rho) \rightarrow \mathsf{Prog} G^\rho_\beta]$ and for every $m < \omega$, $\mathsf{F}^{m+1}_\beta \vdash G_\beta(\Omega^{\bar{\beta}} \cdot \bar{m})$.*

The proof of proposition 20 is by transfinite induction on β and requires, for a given β, the following technical lemmata.

lem:Fbetatech1 **Lemma 21.** $\mathsf{F}^1_\beta \vdash \forall\rho[G_\beta(\rho) \rightarrow \forall \eta < \bar{\beta} \, \mathsf{T}_\beta(\ulcorner \mathsf{Prog} G^{\dot{\rho}}_{\dot{\eta}} \urcorner)]$.

lem:Fbetatech2 **Lemma 22.** $\mathsf{F}^1_\beta \vdash \mathsf{T}_\beta(\ulcorner \forall\tau\forall\eta < \bar{\beta} \, [F_0(\tau) \wedge \mathsf{T}_0(\ulcorner \mathsf{Prog} G^{\dot{\tau}}_{\dot{\eta}} \urcorner) \rightarrow F_0(\tau + \Omega^\eta)] \urcorner)$.

Proof. The two lemmata hold vacuously when $\beta = 0$. For $\beta > 0$

$$\mathsf{F}^1_\beta \vdash G_\beta(\rho) \leftrightarrow \forall \eta < \bar{\beta} \, \mathsf{T}_\beta(\ulcorner G_\eta(\dot{\rho}) \urcorner),$$

so the first lemma would result from replicating the proof of (the transfinite induction hypothesis of) proposition 20 under a T_β predicate. This is possible as the proof of the proposition, which is presented below, is uniform in $\eta < \beta$.

In order to establish lemma 22, argue within F^1_β *under* the scope of a T_β predicate. Fix $\eta < \bar{\beta}$, some arbitrary τ and assume

$$\mathsf{T}_0(\ulcorner \mathsf{Prog} G^{\dot{\tau}}_{\dot{\eta}} \urcorner), \tag{33}$$ {exteqn:Fbetate

$$F_0(\tau). \tag{34}$$ {exteqn:Fbetate

Equation (33) entails

$$\forall\alpha[\mathrm{wo}_0(\alpha) \to \forall\xi < \eta\, \mathsf{T}_0(\ulcorner G_\eta(\dot\tau + \Omega^{\dot\xi} \cdot \dot\alpha)\urcorner)],$$

so $\forall\alpha[\mathrm{wo}_0(\alpha) \to \forall\xi < \eta\, \mathsf{T}_0(\ulcorner F_0(\dot\tau + \Omega^{\dot\xi} \cdot \dot\alpha)\urcorner)]$ by T_0-Del and

$$\forall\xi < \eta\forall\alpha[\mathrm{wo}_0(\alpha) \to \mathrm{wo}_0(\vartheta(\tau + \Omega^{\xi} \cdot \alpha))] \tag{35}$$

{exteqn:Fbetate

by lemma 13. If $\tau \le \sigma < \tau + \Omega^\eta$, there exists some $\zeta < \Omega$ and $\xi < \eta$ such that $\sigma < \tau + \Omega^\xi \cdot \zeta$ and $\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0(\zeta)$ (pick $\zeta = (\sigma^*)^{\varepsilon+}$), whence $\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0(\vartheta\sigma)$ results from eq. (35) as $\vartheta\sigma < \vartheta(\tau + \Omega^\xi \cdot \zeta)$. If, however, $\sigma < \tau$, eq. (34) implies $\mathrm{wo}_0(\sigma^*) \to \mathrm{wo}_0(\vartheta\sigma)$. Thus, $F_0(\tau + \Omega^\eta)$ holds. □

of proposition 20. Argue informally within F^1_β. Fix some arbitrary ρ and assume $G_\beta(\rho)$. Lemma 21 yields $\forall\eta < \bar\beta\, \mathsf{T}_\beta(\ulcorner \mathrm{Prog}G^{\rho}_{\eta}\urcorner)$ from which, using T^-_β-Rep and lemma 22, one obtains $\forall\eta < \bar\beta\, \mathsf{T}_\beta(\ulcorner F_0(\rho + \Omega^\eta)\urcorner)$, that is, $\forall\eta < \bar\beta\, G_\beta(\rho + \Omega^\eta)$. Therefore

$$\mathsf{F}^1_p \vdash \forall\rho[G^\rho_\beta(\alpha) \to G^\rho_\beta(\alpha + 1)]. \tag{36}$$

{exteqn:Fbetawe

If μ is a limit ordinal, $\forall\nu < \mu\, G^\rho_\beta(\nu)$ and T_β-$\forall$-Inf implies

$$\forall\eta < \bar\beta\, \mathsf{T}_\beta(\ulcorner \forall\nu < \dot\mu\, F_0(\dot\rho + \Omega^{\dot\eta} \cdot \nu)\urcorner),$$

from which $\mathsf{T}_\beta(\ulcorner F_0(\dot\rho + \Omega^{\dot\eta} \cdot \dot\mu)\urcorner)$ is easily deduced. Thus eq. (36) entails

$$\mathsf{F}^1_\beta \vdash \forall\rho[G_\beta(\rho) \to \mathrm{Prog}G^\rho_\beta].$$

The second half of proposition 20 is proved by induction on $m < \omega$. The case $m = 0$ holds since $F_0(\bar 0)$ is vacuously true. If $m = n + 1$ and $\mathsf{F}^m_\beta \vdash G_\beta(\Omega^{\bar\beta} \cdot \bar n)$,

$$\mathsf{F}^{m+1}_\beta \vdash \mathsf{T}_\beta(\ulcorner \mathrm{Prog}G^{\Omega^{\bar\beta}\cdot\bar n}_\beta\urcorner),$$

so $\mathsf{F}^{m+1}_\beta \vdash G_\beta(\Omega^{\bar\beta} \cdot \bar m)$ by an argument similar to lemma 22. □

Much as in the finite case, proposition 20 suffices to obtain lower bounds on the theories F^m_β. In theorem 8 below we prove $\|\mathsf{F}_\beta\| \ge \vartheta(\Omega^\beta \cdot \omega)$. This is achieved by showing the proof-theoretic ordinal of F^m_β is closed under the operations $\alpha \mapsto \vartheta(\Omega^\eta \cdot \alpha)$ for every $\eta < \beta$ and extends the strength of F^n_β for $n < m$. The method will only work if β is not "too large", that is $\vartheta\Omega^\beta$ is indeed the *least* ordinal closed under the above operations. By taking β to be no larger than the Veblen ordinal $\vartheta\Omega^\Omega$, we can ensure that this is the case.

Fbetalowerbound **Theorem 8.** *For every $\beta < \vartheta\Omega^\Omega$ and every $m < \omega$, $\mathsf{F}^m_\beta \vdash \mathsf{TI}(<\vartheta(\Omega^\beta \cdot m))$ and $\mathsf{F}_\beta \vdash \mathsf{TI}(<\vartheta(\Omega^\beta \cdot \omega))$.*

Proof. For every $\beta < \vartheta\Omega^\Omega$, the base case, $m = 0$, is immediate since F^0_β extends PA formulated in the language $\mathscr{L}_\kappa$. Otherwise $m = n + 1 > 0$ and proposition 20 implies $\mathsf{F}^m_\beta \vdash \mathrm{Prog}G^{\Omega^{\bar\beta}\cdot\bar n}_\beta$. Then, given $\mathsf{F}^m_\beta \vdash \mathrm{wo}_\beta(\bar\alpha)$, we obtain $\mathsf{F}^m_\beta \vdash \forall\eta < \bar\beta\, G_\beta(\Omega^{\bar\beta} \cdot \bar n + \Omega^{\bar\eta} \cdot \bar\alpha)$, and so

$$\mathsf{F}^m_\beta \vdash \forall\eta < \bar\beta\, \mathrm{wo}_\beta\vartheta(\Omega^{\bar\beta} \cdot \bar n + \Omega^{\bar\eta} \cdot \bar\alpha))),$$

by unravelling the definition of G_β.

Let $\sigma_0 = 1$ and $\sigma_{k+1} = \sup_{\eta<\beta} \vartheta(\Omega^\beta \cdot n + \Omega^\eta \cdot \sigma_k)$. The previous paragraph establishes $\mathsf{F}^m_\beta \vdash \mathsf{TI}(<\sigma_k)$ for every k. Therefore, all that remains is to show $\vartheta(\Omega^\beta \cdot m) \leq \sup_k \sigma_k$, which proceeds by induction on the rank of $\alpha < \vartheta(\Omega^\beta \cdot m)$. If $\alpha = 0$ we are done, and if $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$, the induction hypothesis implies $\alpha < \sigma_k$ for some k. Thus, suppose

$$\alpha = \vartheta\xi < \vartheta(\Omega^\beta \cdot m).$$

There are two cases to consider.

1. $\xi < \Omega^\beta \cdot m$ and $\xi^* < \vartheta(\Omega^\beta \cdot m)$; or

2. $\xi > \Omega^\beta \cdot m$ but $\vartheta\xi \leq (\Omega^\beta \cdot m)^*$.

a) entails $\xi^* < \sigma_k$ for some k by the induction hypothesis (since ξ^* has rank strictly smaller than α). Then $\xi < \Omega^\beta \cdot m + \Omega^\eta \cdot \sigma_k$ for some $\eta < \beta$, whence $\alpha < \vartheta(\Omega^\beta \cdot n + \Omega^\eta \cdot \sigma_k) \leq \sigma_{k+1}$.

To manage b) one utilises $\beta < \vartheta(\Omega^\Omega)$. Since $\vartheta\xi \leq \beta^*$ and $(\Omega^\Omega)^* = 0$, we have

$$\vartheta\xi < \vartheta(\Omega^\Omega)$$

and $\xi < \Omega^\Omega$. But then $\beta \leq \xi^* < \Omega$ contradicting $\vartheta\xi \leq \beta^*$.

The lower bound on F_β is an immediate consequence of the fact

$$\vartheta(\Omega^\beta \cdot \omega) = \sup_{k<\omega} \vartheta(\Omega^\beta \cdot k).$$

□

3.4 Upper bounds on the proof-theoretic ordinal

extsec:Fbupper1

We now move to the task of determining an upper bound on the strength of the theory F_β. The ultimate aim is to demonstrate that any arithmetical theorem of F_β is derivable in the theory $\mathsf{PA} + \mathsf{TI}(<\delta)$ for a suitable ordinal δ. To achieve this it is natural to attempt to formalise the consistency proof provided in section 3.1. For it to be successful, however, we must show that the arguments involved can be carried out within the confines of $\mathsf{PA}+\mathsf{TI}(<\delta)$. The main difficulty is with formalising the T-Elimination theorem, theorem 4, since:

- The proof makes free use of transfinite induction beyond the first uncountable ordinal.
- Moreover, each subsidiary induction appears to also require transfinite induction beyond δ (albeit this time bounded in Ω) in order to support later steps in the main induction.
- In particular, the case in which the last rule employed was T_η-$\forall$-Inf makes use of non-finitistic properties of Ω; namely proposition 4.

By introducing bounds on the height of the derivations in $\mathscr{F}^*_\rho$ (for a fixed ordinal $\rho \in \mathrm{OT}_\Omega$), we observe that δ can be chosen so as to bound all uses of the subsidiary transfinite induction hypothesis described in the second point. Moreover, δ can be chosen in such a way that all references to the main induction hypothesis in theorem 4 are for ordinals of the form $\Omega^{\alpha_0} \cdot \beta_0 + \Omega^{\alpha_1} \cdot \beta_1 + \cdots + \Omega^{\alpha_n} \cdot \beta_n$ where $\alpha_i, \beta_i < \delta$ for every $i \leq n$; the upshot being that a formalised version of theorem 4 needs only countable induction on some suitable (lexicographical-style) ordering.

The third point, however, has no immediate alternative; we need the regularity of Ω to ensure the set $\mathscr{F}^*_{<\rho}$ is closed under the ω-rule if ρ is not an Ω-limit, a property required to establish that $\mathscr{F}^*_\rho$ is closed under T_η-Elim. Suppose $\rho = \rho' + \Omega^{\xi+1}$ is not an Ω-limit. If $A(x)$ is a formula of $\mathscr{L}_\kappa$ and $\mathscr{F}^*_{\rho_n} \vdash A(\bar{n})$ with $\rho_n < \rho$ for every n, we want to obtain an ordinal $\sigma < \rho$ such that $\mathscr{F}^*_\sigma \vdash \forall x\, A(x)$. As each application of T_η-Intro must increase the height of a derivation, introducing bounds on the height of a derivation indirectly yields bounds on its T-rank. That is, if $\mathscr{F}^*_\tau \vdash B$ with height α, one can prove the existence of a ordinal $\tau_0 \leq \tau$ such that $\mathscr{F}^*_{\tau_0} \vdash B$ and τ_0^* is no larger than the maximum of α and all η such that T_η-Intro was utilised in the derivation $\mathscr{F}^*_\tau \vdash B$. Thus, if α is an ordinal bounding the height of the derivation $\mathscr{F}^*_{\rho_n} \vdash A(\bar{n})$ for every n, we may observe the ordinal ρ_n is "bounded" by α for every n, and so effectively find an ordinal $\gamma < \Omega$ depending on α and ρ, so that if $\sigma = \rho' + \Omega^\xi \cdot \gamma$, $\rho_n < \sigma$ for every n and, hence, $\mathscr{F}^*_\sigma \vdash \forall x\, A(x)$.

Although at present these arguments are somewhat vague, they should explain the motivation behind a number of the technical results to follow. In particular, the remarks we made in relation to the third point are made explicit in the Condensation lemma (lemma 25).

It will be important to compare ordinals greater than Ω by their construction as well as their size. Therefore, we introduce a new ordering $<^*$, defined on OT_Ω by

$$\rho <^* \sigma \leftrightarrow (\rho < \sigma \wedge \rho^* \leq \sigma^*).$$

$\rho \leq^* \sigma$ abbreviates $\rho <^* \sigma \vee \rho = \sigma$.

Definition 23. Fix $\kappa = \vartheta\varepsilon_{\Omega+1}$. Define $\mathscr{T}_\infty \vdash^{\alpha}_{\rho} \Gamma$ for $\alpha, \rho \in \mathrm{OT}_\Omega$ with $\alpha < \Omega$ and Γ an $\mathscr{L}_\kappa$-sequent, according to the following rules: *(Ax.1)*, $(\wedge)$, $(\vee)$, (ω) and $(\exists)$ as in **??**, along with the following for each $\eta < \kappa$ and each $\beta, \gamma < \alpha \in \mathrm{OT}_\Omega \cap \Omega$.

(Ax.2_η) $\vdash^{\alpha}_{\rho} \Gamma, \neg\mathsf{T}_\eta(s_0), \mathsf{T}_\eta(s_1)$ if $s_0^{\mathbb{N}} = s_1^{\mathbb{N}}$.

(Ax.3_η) $\vdash^{\alpha}_{\rho} \Gamma, \neg\mathsf{T}_\eta(s)$ if $s^{\mathbb{N}}$ is not the Gödel number of an $\mathscr{L}_\kappa$-sentence.

(T_η-Imp) $\vdash^{\beta}_{\rho} \Gamma, \mathsf{T}_\eta(s_0)$, $\vdash^{\gamma}_{\rho} \Gamma, \mathsf{T}_\eta(s_1)$ and $s_0^{\mathbb{N}} = (s_1 \dot{\rightarrow} s_2)^{\mathbb{N}}$ implies $\vdash^{\alpha}_{\rho} \Gamma, \mathsf{T}_\eta(s_2)$.

(T_η-Intro) $\vdash^{\alpha}_{\rho} \Gamma, \mathsf{T}_\eta(s)$ whenever $\vdash^{\beta}_{\sigma} A$, with $\sigma \# \Omega^{\eta+1} \leq^* \rho$ and $s^{\mathbb{N}} = \ulcorner A \urcorner$.

extinf:dag1 (T_η-Del) $\vdash^{\beta}_{\rho} \Gamma, \mathsf{T}_\eta(s)$, $s^{\mathbb{N}} = \ulcorner \mathsf{T}_\xi(s_0) \urcorner$ and $\xi < \kappa$ implies $\vdash^{\alpha}_{\rho} \Gamma, \mathsf{T}_\eta(s_0)$.

($\neg\mathsf{T}_\eta$-Del) $\vdash^{\beta}_{\rho} \Gamma, \mathsf{T}_\eta(s)$, $s^{\mathbb{N}} = \ulcorner \neg\mathsf{T}_\xi(s_0) \urcorner$ and $\xi < \kappa$ implies $\vdash^{\alpha}_{\rho} \Gamma, \mathsf{T}_\eta(s_1)$, if $s_1^{\mathbb{N}} = (\dot{\neg} s_0)^{\mathbb{N}}$.

extinf:dag2 (T_η^--Rep) $\vdash^{\beta}_{\rho} \Gamma, \mathsf{T}_\eta(s)$, $\xi < \eta$ implies $\vdash^{\alpha}_{\rho} \Gamma, \mathsf{T}_\eta(s_0)$ if $s^{\mathbb{N}} = \ulcorner \mathsf{T}_\xi(s_0) \urcorner$ and $s_0^{\mathbb{N}}$ is the Gödel number of an $\mathscr{L}_\kappa$-sentence.

(T_η-ω) $\vdash^{\beta}_{\rho} \Gamma, \mathsf{T}_\eta(\ulcorner A(\bar{n}) \urcorner)$ for every $n \in \mathbb{N}$, where $A(x)$ is a formula of $\mathscr{L}_\kappa$ with at most x free, implies $\vdash^{\alpha}_{\rho} \Gamma, \mathsf{T}_\eta(s)$ whenever $s^{\mathbb{N}} = \ulcorner \forall x A(x) \urcorner$ and $\rho|_\eta$ is not an Ω-limit.

(Weakening) $\vdash^{\beta}_{\sigma} \Gamma$ implies $\vdash^{\alpha}_{\rho} \Gamma, \Delta$ whenever $\sigma \leq^* \rho$ and Δ is an $\mathscr{L}_{\mathsf{T}}$-sequent.

We refer to ρ as the *T-rank* and α as the *height* of the derivation $\mathscr{T}_\infty \vdash^{\alpha}_{\rho} \Gamma$. For a fixed η, the rules (T_η-Imp) to (T_η-ω), are collectively referred to as T_η*-rules*. The collection as a whole (for any η) is referred to as the *T-rules*. In each application of a rule or axiom above we specify the *active*, *minor* and *side* formulae as usual.

Often when working with hierarchical systems one needs an extended notion of the rank of a formula. In this case, however, the basic definition of rank suffices and as such our work in $\mathscr{T}_\infty$ is greatly simplified. Define the *rank* of an $\mathscr{L}_\kappa$-sentence A, denoted $|A|$, as in **??** with the addition that $|\mathsf{T}_\eta(s)| = |\neg\mathsf{T}_\eta(s)| = 0$ for every $\eta < \kappa$.

We observe the following straightforward results for $\mathscr{T}_\infty$ and, henceforth make free use of them without mention.

extprop:T8props **Proposition 24.**

tlem:F8L1itemvi 1. *$\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, A$ implies $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma$ if A is a false arithmetical literal.*

em:F8L1itemviji 2. *$\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, \mathsf{T}_\eta(s)$ implies $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma$ if $s^{\mathbb{N}}$ is not the code of an $\mathscr{L}_\kappa$-sentence.*

tlem:F8L1itemx 3. *$\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, \mathsf{T}_\eta(s), \mathsf{T}_\eta(t)$ and $s^{\mathbb{N}} = t^{\mathbb{N}}$ implies $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, \mathsf{T}_\eta(s)$.*

tlem:F8L1itemxi 4. *$\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, \neg\mathsf{T}_\eta(s), \neg\mathsf{T}_\eta(t)$ and $s^{\mathbb{N}} = t^{\mathbb{N}}$ implies $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, \neg\mathsf{T}_\eta(s)$.*

tlem:F8L1itemix 5. *$\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, A(t)$ and $s^{\mathbb{N}} = t^{\mathbb{N}}$ implies $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, A(s)$.*

Proof. The proof is essentially identical to the proof of **??**, and hence omitted. □

Although we did not formulate $\mathscr{T}_\infty$ with a cut rule, it is easy to see that such a rule is admissible. The proof proceeds in the same manner as the cut elimination theorems of **??** (*cf.* **??????**).

tthm:T8CutElim **Theorem 9** (Admissibility of cut). *For every $\alpha, \beta < \Omega$ and every $\sigma \in \mathrm{OT}_\Omega$, $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma, A$ and $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Delta, \neg A$ imply $\mathscr{T}_\infty\vdash^{\beta}_{\sigma} \Gamma, \Delta$ for some $\beta < \alpha^{\varepsilon+}$.*[7]

Proof. By the definition of $\mathscr{T}_\infty$, $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Delta, \neg\mathsf{T}_\eta(s)$ holds with $\neg\mathsf{T}_\eta(s)$ active only if $\Delta, \neg\mathsf{T}_\eta(s)$ is an instance of either (Ax.2$_\eta$) or (Ax.3$_\eta$). If further, $\mathscr{T}_\infty\vdash^{\beta}_{\sigma} \Gamma, \mathsf{T}_\eta(s)$ holds, it is easy to deduce in fact $\mathscr{T}_\infty\vdash^{\alpha\#\beta}_{\sigma} \Gamma, \Delta$. Thus the claim holds when A is either $\mathsf{T}_\eta(s)$ or $\neg\mathsf{T}_\eta(s)$ for some $\eta < \kappa$ and term s.

The other cases are proved by induction $|A|$ and their arguments are identical to those used in the Reduction Lemma for $\mathscr{F}_\infty$, **??**. □

The remainder of this section is concerned with proving the equivalent form of the T-Elimination theorem for $\mathscr{F}^*_\rho$, theorem 4, and showing the argument can be formalised within the theory $\mathsf{PA}+\mathsf{TI}(<\delta)$ for some suitable δ. We begin with the Condensation lemma, replacing the use of proposition 4.

extlem:T8conden **Lemma 25** (Condensation lemma). *Let $\Theta^*(\sigma)$ denote the least ordinal η^* such that $\Omega^{\eta+1} > \sigma$. If $\mathscr{T}_\infty\vdash^{\alpha}_{\sigma} \Gamma$ holds for some α, σ and Γ, there is a $\rho \leq \sigma$ with $\rho^* \leq \max\{\alpha^*, \Theta^*(\sigma)\}$ such that $\mathscr{T}_\infty\vdash^{\alpha}_{\rho} \Gamma$.*

Proof. We proceed by transfinite induction on the height of the derivation $\vdash^{\alpha}_{\sigma} \Gamma$. In the case of axioms one may always take $\rho = 0$ and for any rule other than (T_η-ω) or (T_η-Intro) one may apply the induction hypothesis and leave ρ unchanged. In the case (T_η-Intro) was the last rule applied one obtains a formula $\mathsf{T}_\eta(\ulcorner A \urcorner)$ in Γ and a derivation $\vdash^{\alpha_0}_{\tau} A$ for some $\alpha_0 < \alpha$ with $\tau \# \Omega^{\eta+1} \leq^* \sigma$. The

[7]Recall $\alpha^{\varepsilon+}$ is the least ε-ordinal greater than α.

induction hypothesis yields an ordinal $\tau_0 \leq \tau$ fulfilling $\tau_0^* \leq \max\{\alpha_0^*, \Theta^*(\tau)\}$ such that $\vdash^{\alpha_0}_{\tau_0} A$, whence we observe taking $\rho = \tau_0 \# \Omega^{\eta+1}$ suffices, since

$$\begin{aligned}\rho^* &\leq \max\{\tau_0^*, \eta^*\}\\ &\leq \max\{\alpha_0^*, \Theta^*(\tau), \eta^*\}\\ &\leq \max\{\alpha^*, \Theta^*(\sigma)\}.\end{aligned}$$

In the case of (T_η-ω) suppose $\sigma|_\eta$ is not an Ω-limit, Γ contains the sentence $\mathsf{T}_\eta(\ulcorner \forall x\, A(x)\urcorner)$ and there exists $\alpha_0 < \alpha$ such that $\vdash^{\alpha_0}_{\sigma} \Gamma, \mathsf{T}_\eta(\ulcorner A(\bar{n})\urcorner)$ for every n. The induction hypothesis yields a $\tau_0 \leq \sigma$ such that $\vdash^{\alpha_0}_{\tau_0} \Gamma, \mathsf{T}_\eta(\ulcorner A(\bar{n})\urcorner)$ with $\tau_0^* \leq \max\{\alpha_0^*, \Theta^*(\sigma)\}$. If τ_0 is not an Ω-limit take $\rho = \tau_0$, otherwise pick $\rho = \tau_0 + \Omega^\xi$, where ξ is such that $\sigma|_\eta = \sigma_0 + \Omega^\xi$ for some $\sigma_0 < \sigma$. In either case $\rho \leq \sigma$ and

$$\rho^* = \max\{\tau_0^*, \xi^*\} \leq \max\{\alpha^*, \Theta^*(\sigma)\}.$$

□

We will now define a collection of standard $\mathscr{L}_\kappa$-structures $\mathfrak{M}_\rho$ indexed by ordinals $\alpha < \Omega$ and $\rho < \Omega^\kappa$. These play the role of the structures $\mathfrak{M}_\rho$ used in theorem 4. Set

$$\begin{aligned}\mathfrak{M}_\rho \models \mathsf{T}_\eta(s) \text{ iff there is an } \mathscr{L}_\kappa\text{-sentence } A \text{ with } s^{\mathbb{N}} = \ulcorner A \urcorner \text{ and ordinals}\\ \beta, \sigma \text{ such that } \sigma^*, \beta < \vartheta\rho,\ \sigma < \rho|_\eta \text{ and } \mathscr{T}_\infty \vdash^{\beta}_{\sigma} A\,.\end{aligned}$$

extprop:T8mono **Proposition 26.** *If Γ is a T-positive sequent and $\rho \leq^* \sigma$, $\mathfrak{M}_\rho \models \bigvee \Gamma$ implies $\mathfrak{M}_\sigma \models \bigvee \Gamma$.*

Proof. Suppose $\rho \leq^* \sigma$. Since $\rho^* < \vartheta\sigma$, we have $\vartheta\rho \leq \vartheta\sigma$ by proposition 7 (viii), which coupled with the fact $\rho|_\eta \leq \sigma|_\eta$ entails $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s)$ implies $\mathfrak{M}_\sigma \models \mathsf{T}_\eta(s)$ for every $\eta < \kappa$. The remaining cases are now easily verified. □

The next proposition will enable us to pass from derivations in $\mathscr{T}_\infty$ to satisfaction in the structures $\mathfrak{M}_\rho$ for T-positive sequents.

extlem:T8Mprop **Lemma 27.** *The following hold for every $\alpha < \Omega$, $\eta < \kappa$ and $\rho < \Omega^\kappa$.*

exteqn:T8propii 1. $\mathfrak{M}_\rho \models \forall x \forall y(\mathsf{T}_\eta(x) \wedge \mathsf{T}_\eta(x \dot{\to} y) \to \mathsf{T}_\eta(y))$.

xteqn:T8propiii 2. $\mathfrak{M}_\rho \models \forall \xi < \bar{\eta}\, \forall x(\mathsf{T}_\eta(x) \to \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi(\dot{x})\urcorner))$.

3. $\mathfrak{M}_\rho \models \forall \xi < \bar{\kappa} \forall x(\mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi(\dot{x})\urcorner) \to \mathsf{T}_\eta(x))$ *if, for every $\sigma < \rho$ and β such that $\sigma^*, \beta < \vartheta\rho$,*

exteqn:T8propiv $$\mathscr{T}_\infty \vdash^{\beta}_{\sigma} \Gamma \text{ implies } \mathfrak{M}_{\sigma+\beta} \models \bigvee \Gamma.$$

exteqn:T8propv 4. $\mathfrak{M}_\rho \models \forall \xi < \bar{\kappa} \forall \ulcorner A \urcorner(\mathsf{T}_\eta(\ulcorner \neg \mathsf{T}_\xi(\ulcorner A \urcorner)\urcorner) \to \mathsf{T}_\eta(\ulcorner \neg A \urcorner))$.

exteqn:T8propvi 5. *If $\rho|_\eta$ is not an Ω-limit, $\mathfrak{M}_\rho \models \forall x\, \mathsf{T}_\eta(\ulcorner A(\dot{x})\urcorner)$ and $\rho <^* \rho'$, $\mathfrak{M}_{\rho'} \models \mathsf{T}_\eta(\ulcorner \forall x\, A(x)\urcorner)$.*

Proof. 1. We observe that if $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner A \urcorner) \wedge \mathsf{T}_\eta(\ulcorner A \rightarrow B \urcorner)$, there is an $\alpha < \vartheta\rho$ and $\sigma < \rho|_\eta$ such that

$$\vdash^{\alpha}_{\sigma} A \text{ and} \vdash^{\alpha}_{\sigma} \neg A, B$$

holds and $\sigma^* < \vartheta\rho$. Admissibility of the cut rule, theorem 9, yields $\vdash^{\gamma}_{\sigma} B$ for some $\gamma < \alpha^{\varepsilon+}$. But $\rho > \Omega$, so $\gamma < \vartheta\alpha < \vartheta\rho$ and hence $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner B \urcorner)$ by definition.

2. Fix some $\eta < \kappa$. Note $(\rho|_\eta)^* < \vartheta\rho$ for every η, since $\rho^* < \vartheta\rho$. Also $\rho < \Omega^\kappa$, so $\eta^* \leq (\rho|_\eta)^*$. If $\rho|_\eta = 0$, 2 holds vacuously. Otherwise, suppose $\mathscr{T}_\infty \vdash^{\beta}_{\sigma} A$ for some $\mathscr{L}_\kappa$-sentence A, some β and $\sigma < \rho|_\eta$ such that $\beta, \sigma^* < \vartheta\rho$. Then $\sigma \# \Omega^{\xi+1} < \rho|_\eta$ for every $\xi < \eta$, and $(\sigma \# \Omega^{\xi+1})^* \leq \max\{\sigma^*, \eta^*\} < \vartheta\rho$. So $\mathfrak{M}_\rho \models \forall \xi < \bar{\eta}\, \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi(\ulcorner A \urcorner) \urcorner)$ as required.

3. Suppose $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi(s) \urcorner)$ and $\xi < \kappa$; we want to show $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s)$. By definition there are ordinals α and $\sigma < \rho|_\eta$ such that $\sigma^*, \alpha < \vartheta\rho$ and

$$\mathscr{T}_\infty \vdash^{\alpha}_{\sigma} \mathsf{T}_\xi(s).$$

The additional assumption yields $\mathfrak{M}_{\sigma+\alpha} \models \mathsf{T}_\xi(s)$, whence $s^{\mathbb{N}} = \ulcorner A \urcorner$ for some A and there is a β and $\tau < \sigma|_\xi$ such that $\tau^*, \beta < \vartheta(\sigma + \alpha)$ and

$$\mathscr{T}_\infty \vdash^{\beta}_{\tau} A.$$

Since $\sigma^*, \alpha < \vartheta\rho$ and $\sigma + \alpha < \rho$,

$$\vartheta(\sigma + \alpha) < \vartheta\rho,$$

and thus $\tau^*, \beta < \vartheta\rho$. Moreover, $\tau < \rho|_\eta$ so $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s)$ follows by definition.

4 holds vacuously as $\mathscr{T}_\infty \vdash^{\beta}_{\sigma} \neg \mathsf{T}_\xi(s)$ cannot hold for any β, σ and term s.

5. Suppose $\mathfrak{M}_\rho \models \forall x\, \mathsf{T}_\eta(\ulcorner A(\dot{x}) \urcorner)$, $\rho < \rho'$ and $\rho|_\eta$ is not an Ω-limit. By definition, $\rho|_\eta > 0$ and one obtains ordinals α_n and τ_n for each $n \in \mathbb{N}$ such that

$$\begin{aligned} \tau_n &< \rho|_\eta, \\ \alpha_n, \tau_n^* &< \vartheta\rho, \\ \mathscr{T}_\infty &\vdash^{\alpha_n}_{\tau_n} A(\bar{n}). \end{aligned}$$

Our aim is to find ordinals σ, γ such that

$$\sigma < \rho'|_\eta, \tag{37}$$
{exteqn:T8Mprop

$$\gamma, \sigma^* < \vartheta\rho', \tag{38}$$
{exteqn:T8Mprop

$$\mathscr{T}_\infty \vdash^{\gamma+1}_{\sigma} \forall x A(x). \tag{39}$$
{exteqn:T8Mprop

As $\rho|_\eta$ is not an Ω-limit there exists $\xi, \rho_0 < \rho$ such that $\rho|_\eta = \rho_0 + \Omega^{\xi+1}$. Moreover, the Condensation lemma 25 provides ordinals $\sigma_n \leq \tau_n$ such that

$$\sigma_n^* \leq \max\{\alpha_n^*, \Theta^*(\tau_n)\}, \tag{40}$$
{exteqn:T8Mprop

$$\mathscr{T}_\infty \vdash^{\alpha_n}_{\sigma_n} A(\bar{n}), \tag{41}$$
{exteqn:T8Mprop

where $\Theta^*(\tau_n)$ is defined as in the statement of lemma 25. Let $\gamma = \vartheta\rho$ and $\sigma = \rho_0 + \Omega^\xi \cdot \vartheta\rho$. Then eq. (37) clearly holds; moreover,

$$\begin{aligned} \sigma^* &\leq \max\{\rho_0^*, \xi^*, \vartheta\rho\}, \\ &= \vartheta\rho, \end{aligned} \tag{42}$$
{exteqn:T8Mprop

so $\sigma^* = \vartheta\rho < \vartheta\rho'$. Therefore eq. (38) also holds.

To establish eq. (39) and complete the proof, we will show $\sigma_n \leq^* \sigma$ and apply (Weakening) to eq. (41). Fix an arbitrary $n \in \mathbb{N}$. By eq. (40),

$$\begin{aligned}\sigma_n^* &\leq \max\{\alpha_n^*, \Theta^*(\tau_n)\},\\ &\leq \max\{\alpha_n^*, \tau_n^*\},\\ &< \vartheta\rho.\end{aligned}$$

Moreover, $\sigma_n < \sigma$ because either $\sigma_n < \rho_0 < \sigma$ or there is some $\gamma_0 < \Omega$ such that

$$\rho_0 + \Omega^{\xi} \cdot \gamma_0 \leq \sigma_n < \rho_0 + \Omega^{\xi} \cdot (\gamma_0 + 1),$$

whence $\gamma_0^* \leq \sigma_n^* < \vartheta\rho$, so $\gamma_0 < \vartheta\rho$ and $\sigma_n < \sigma$. Hence $\mathfrak{M}_{\rho'} \models \mathsf{T}_\eta(\ulcorner \forall x\, A(x) \urcorner)$ as desired. □

extlem:T8Elim **Theorem 10** (T-Elimination theorem). *For every $\rho \in \mathrm{OT}_\Omega$ and $\alpha \in \mathrm{OT}_\Omega \cap \Omega$,*

1. *$\mathscr{T}_\infty \mid\!\frac{\alpha}{\rho}\, \Gamma$ implies $\mathfrak{M}_{\rho+\alpha} \models \bigvee \Gamma$ if Γ is a T-positive sequent;*
2. *$\mathscr{T}_\infty \mid\!\frac{\alpha}{\rho}\, \mathsf{T}_\eta(s)$ implies there is an $\mathscr{L}_\mathsf{T}$-sentence A with $s^{\mathbb{N}} = \ulcorner A \urcorner$ such that $\mathscr{T}_\infty \mid\!\frac{\beta}{\sigma}\, A$ for some $\beta, \sigma < \rho|_\eta$ with $\sigma^*, \beta < \vartheta(\rho + \alpha)$;*
3. *$\mathscr{T}_\infty \mid\!\frac{\alpha}{\rho}\, \neg\mathsf{T}_\eta(s)$ implies there is an $\mathscr{L}_\mathsf{T}$-sentence A with $s^{\mathbb{N}} = \ulcorner A \urcorner$ such that $\mathscr{T}_\infty \mid\!\frac{\alpha}{\rho}\, \neg A$.*

Proof. By *main* transfinite induction on ρ with a *subsidiary* transfinite induction on α. We begin with (i).

First observe that since $\mathfrak{M}_\rho$ is standard all arithmetical rules and axioms are dealt with easily. Proposition 26 deals with (Weakening). Moreover, no instances of (Ax.2$_\eta$) and (Ax.3$_\eta$) are T-positive, so Γ is not an instance of these axioms. Thus we need only consider the T-rules. Suppose $\mathscr{T}_\infty \mid\!\frac{\alpha}{\rho}\, \Gamma$ is derived via a T_η-rule. If the last rule applied was (T_η-Intro), there is an $\mathscr{L}_\kappa$-sentence A and term s with $s^{\mathbb{N}} = \ulcorner A \urcorner$ so that $\mathsf{T}_\eta(s)$ is in Γ, and $\mathscr{T}_\infty \mid\!\frac{\beta}{\sigma}\, A$ holds for some $\beta < \alpha$ and σ such that $\sigma \,\#\, \Omega^{\eta+1} \leq^* \rho$. But then $\sigma < \rho|_\eta$ and $\beta, \sigma^* < \vartheta\rho$, so $\mathfrak{M}_{\rho+\alpha} \models \bigvee \Gamma$.

For the remaining rules, lemma 27 entails $\mathfrak{M}_{\rho+\alpha} \models \bigvee \Gamma$ is a consequence of the subsidiary induction hypothesis and, in the case of (T_η-Del), the main induction hypothesis.

(ii) follows directly as a result of (i) and the definition of $\mathfrak{M}_{\rho+\alpha}$.

It is easy to check that $\mathscr{T}_\infty \mid\!\frac{\alpha}{\rho}\, \neg\mathsf{T}_\eta(s)$ can only hold if $s^{\mathbb{N}}$ is not the code of some $\mathscr{L}_\kappa$-sentence, whence (iii) holds vacuously. □

Having established that $\mathscr{T}_\infty$ is also closed under T_η-Elim for every $\eta < \kappa$ we can proceed with embedding each F_η into $\mathscr{T}_\infty$. This is the focus of the following proposition and theorem.

extprop:T8embed **Proposition 28.** *For each axiom A of F_β, $\mathscr{T}_\infty \mid\!\frac{\alpha}{\Omega^{\beta+1}\cdot n}\, A$, for some $\alpha < \varepsilon_0$ and every $n < \omega$.*

extthm:T8Embed **Lemma 29** (Embedding lemma for $\mathscr{T}_\infty$). *Fix some $\beta < \vartheta\varepsilon_{\Omega+1}$ and suppose $\mathsf{F}_\beta \vdash A$ with A a sentence of $\mathscr{L}_\beta$. Then there are ordinals α and ρ such that $\mathscr{T}_\infty \mid\!\frac{\alpha}{\rho}\, A$, $\rho < \Omega^{\beta+1} \cdot \omega$, $\alpha, \rho^* < \vartheta(\Omega^{\beta+1} \cdot \omega)$ and $\rho = \rho|_0$.*

Proof. We proceed by induction on the length of the derivation $\mathsf{F}_\beta \vdash A$.

Case I. $\beta = 0$. This essentially follows from the embedding of F in $\mathscr{F}_\infty$, **??**, since $\vartheta(\Omega \cdot \omega) = \varphi\omega 0$. Proposition 28 deals with the axioms of F_0. According to the induction hypothesis, an application of *modus ponens* in F_0 means one has

$$\mathscr{T}_\infty \overset{\alpha_0}{\underset{\Omega \cdot m_0}{\vdash}} B, \qquad \mathscr{T}_\infty \overset{\alpha_1}{\underset{\Omega \cdot m_1}{\vdash}} \neg B, A$$

for some sentence B, ordinals $\alpha_0, \alpha_1 < \vartheta(\Omega \cdot (m+1))$, and $m_0, m_1 < \omega$. Weakening implies

$$\mathscr{T}_\infty \overset{\gamma}{\underset{\Omega \cdot m}{\vdash}} B, \qquad \mathscr{T}_\infty \overset{\gamma}{\underset{\Omega \cdot m}{\vdash}} \neg B, A$$

where $\gamma = \max\{\alpha_0, \alpha_1\}$ and $m = \max\{m_0, m_1\}$, whence $\mathscr{T}_\infty \overset{\alpha}{\underset{\Omega \cdot m}{\vdash}} A$ holds for some $\alpha < \gamma^{\varepsilon+}$. Since $\gamma < \vartheta(\Omega \cdot \omega) < \Omega$, $\vartheta\gamma < \vartheta(\Omega \cdot \omega)$ and so $\alpha < \vartheta(\Omega \cdot \omega)$ whence we are done.

An application of T_0-Intro in F_0 corresponds to an application of the rule (T_0-Intro) in $\mathscr{T}_\infty$. It is an immediate consequence of the induction hypothesis that in this case the desired bounds are maintained.

This leaves only applications of T_0-Elim and $\neg\mathsf{T}_0$-Elim in F_0 to consider, for which we employ theorem 10. $\neg\mathsf{T}_0$-Elim is immediate, so suppose $\mathsf{F}_0 \vdash \mathsf{T}_0(\ulcorner A \urcorner)$ for some $\mathscr{L}_\kappa$-sentence A. The induction hypothesis implies

$$\mathscr{T}_\infty \overset{\alpha}{\underset{\Omega \cdot m}{\vdash}} \mathsf{T}_0(\ulcorner A \urcorner) \tag{43}$$

{exteqn:T8embec

for some $\alpha < \vartheta(\Omega \cdot \omega)$ and $m < \omega$. By theorem 10 one obtains some $\gamma < \vartheta(\Omega \cdot m + \alpha)$ and $n < m$ such that

$$\mathscr{T}_\infty \overset{\gamma}{\underset{\Omega \cdot n}{\vdash}} A$$

holds. But $\Omega \cdot m + \alpha < \Omega \cdot \omega$ and $\alpha < \vartheta(\Omega \cdot \omega)$, so $\gamma < \vartheta(\Omega \cdot \omega)$.

Case II. $\beta > 0$. Proposition 28 deals with the axioms of F_β. According to the induction hypothesis, an application of *modus ponens* in F_β means one has

$$\mathscr{T}_\infty \overset{\alpha_0}{\underset{\sigma}{\vdash}} B, \qquad \mathscr{T}_\infty \overset{\alpha_1}{\underset{\tau}{\vdash}} \neg B, A$$

for some sentence B, ordinals $\alpha_0, \alpha_1 < \vartheta(\Omega^{\beta+1} \cdot \omega)$, and $\sigma, \tau < \Omega^{\beta+1} \cdot \omega$ such that $\sigma^*, \tau^* < \vartheta(\Omega^{\beta+1} \cdot \omega)$, $\sigma|_0 = \sigma$ and $\tau|_0 = \tau$. Applying (Weakening) yields

$$\mathscr{T}_\infty \overset{\alpha}{\underset{\rho}{\vdash}} B, \qquad \mathscr{T}_\infty \overset{\alpha}{\underset{\rho}{\vdash}} \neg B, A$$

where $\rho = \sigma \# \tau$ and $\alpha = \max\{\alpha_0, \alpha_1\}$, whence one obtains $\mathscr{T}_\infty \overset{\gamma}{\underset{\rho}{\vdash}} A$ for some $\gamma < \alpha^{\varepsilon+}$. We take $\rho = \sigma \# \tau$ in place of $\rho = \max\{\sigma, \tau\}$ to ensure $\sigma, \tau \leq^* \rho$ as is required for (Weakening). Since $\rho < \Omega^{\beta+1} \cdot \omega$ and $\rho^* = \max\{\sigma^*, \tau^*\}$, we have $\rho^*, \gamma < \vartheta(\Omega^{\beta+1} \cdot \omega)$.

If an application of T_β-Intro was utilised in F_β, A is $\mathsf{T}_\beta(\ulcorner B \urcorner)$ for some sentence B and the induction hypothesis yields

$$\mathscr{T}_\infty \overset{\alpha}{\underset{\rho}{\vdash}} B$$

for some $\alpha < \vartheta(\Omega^{\beta+1} \cdot \omega)$ and $\rho < \Omega^{\beta+1} \cdot \omega$. An application of the rule (T_β-Intro) in $\mathscr{T}_\infty$ yields

$$\mathscr{T}_\infty \overset{\alpha+1}{\underset{\rho \# \Omega^{\beta+1}}{\vdash}} \mathsf{T}_\beta(\ulcorner B \urcorner)\,.$$

$\rho \# \Omega^{\beta+1} < \Omega^{\beta+1} \cdot \omega$ if and only if $\rho < \Omega^{\beta+1} \cdot \omega$, and

$$\begin{aligned}(\rho \# \Omega^{\beta+1})^* &= \max\{\rho^*, \beta^*\}\\ &\leq \vartheta(\Omega^{\beta+1} \cdot \omega),\end{aligned}$$

so the desired bounds are maintained.

This leaves only applications of T_β-Elim and $\neg\mathsf{T}_\beta$-Elim in F_β to consider, for which we employ theorem 10. The case of $\neg\mathsf{T}_\beta$-Elim is immediate given the induction hypothesis, so suppose $\mathsf{F}_\beta \vdash \mathsf{T}_\beta(\ulcorner A \urcorner)$ for some $\mathscr{L}_\kappa$-sentence A. The induction hypothesis entails

$$\mathscr{T}_\infty \overset{\alpha}{\underset{\rho}{\vdash}} \mathsf{T}_\beta(\ulcorner A \urcorner) \tag{44}$$ {exteqn:T8embe

for some $\rho < \Omega^{\beta+1} \cdot \omega$ and $\alpha, \rho^* < \vartheta(\Omega^{\beta+1} \cdot \omega)$. By theorem 10 (ii) one obtains some γ and σ such that $\sigma^*, \gamma < \vartheta(\rho + \alpha)$, $\sigma < \rho|_\beta$ and

$$\mathscr{T}_\infty \overset{\gamma}{\underset{\sigma}{\vdash}} A$$

holds. $\sigma < \Omega^{\beta+1} \cdot \omega$ and, since $\rho^*, \alpha < \vartheta(\Omega^{\beta+1} \cdot \omega)$, one has $\vartheta(\rho + \alpha) < \vartheta(\Omega^{\beta+1} \cdot \omega)$, whence $\gamma, \sigma^* < \vartheta(\Omega^{\beta+1} \cdot \omega)$. □

Lemma 29 allows us to easily determine an upper bound on the strength of the theories F_β. By making use of the remarks at the beginning of this section, we see that the next theorem is provable with an almost identical argument to **??**.

m:FbetaupperbdI **Theorem 11.** *Every arithmetical theorem of* F_β, *for* $\beta < \vartheta\varepsilon_{\Omega+1}$, *is derivable in* $\mathsf{PA} + \mathsf{TI}(<\vartheta(\Omega^{\beta+1} \cdot \omega))$.

4 Conclusion

extsec:conc

By combining the work of sections 3.3 and 3.4 we may determine the strength of the theory F_β for every $\beta < \vartheta\Omega^\Omega$.

thm:Fpstrength **Theorem 12.** *For every* $p < \omega$

$$\|\mathsf{F}_p\| = \vartheta(\Omega^{p+1} \cdot \omega).$$

m:Fbetastrength **Theorem 13.** *For every* $\beta \geq \omega$ *with* $\beta < \vartheta\Omega^\Omega$,

$$\vartheta(\Omega^\beta \cdot \omega) \leq \|\mathsf{F}_\beta\| \leq \vartheta(\Omega^{\beta+1} \cdot \omega).$$

Proof. Theorem 11 provides the upper bounds for the two theorems. The lower bound for theorem 12 results from theorem 7, whereas the lower bound of theorem 13 is a consequence of theorem 8 □

When presented with some hierarchy of theories $\{\mathsf{T}_\alpha : \alpha \in \mathbb{O}\}$ it is natural to ask the limit of the corresponding autonomous progression, that is, the least ordinal not in the set X_T generated by the operation $\|\mathsf{T}_0\| \subseteq X_\mathsf{T}$ and $\alpha \in X_\mathsf{T}$ implies $\|\mathsf{T}_\alpha\| \subseteq X_\mathsf{T}$. Autonomous progressions of ramified theories such as RA_α are well-studied, as are those obtained by iterating reflection principles. In the case $\mathsf{T}_\alpha = \mathsf{F}_\alpha$ this is not difficult to determine given theorem 13.

extthm:Fbauto **Theorem 14.** *The limit of the autonomous progression defined from* $\{\mathsf{F}_\beta : \beta < \Omega\}$ *is the large Veblen ordinal,* $\vartheta\Omega^\Omega$.

Proof. Let $\sigma_0 = 0$ and $\sigma_{m+1} = \vartheta(\Omega^{\sigma_m})$. Theorem 8 implies $\|\mathsf{F}^1_{\sigma_m}\| \geq \sigma_{m+1}$, while theorem 13 entails $\|\mathsf{F}_{\sigma_m}\| < \sigma_{m+2}$, so $X_\mathsf{F} = \sup_m \sigma_m$. It remains to show

$$\vartheta(\Omega^\Omega) = \sup_{m<\omega} \sigma_m.$$

Since $\sigma_m < \Omega$ for every m, we have $\Omega^{\sigma_m} < \Omega^\Omega$. So $\sigma_m < \vartheta(\Omega^\Omega)$ implies $\sigma_{m+1} < \vartheta(\Omega^\Omega)$. Thus $\vartheta(\Omega^\Omega) \geq \sup_m \sigma_m$ is established by induction on m.

For the converse direction we prove $\alpha < \vartheta(\Omega^\Omega)$ implies $\alpha < \sigma_m$ for some m by induction on the rank of α. If $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ for some γ, δ, one easily obtains $\alpha < \sigma_m$ by the induction hypothesis, so suppose

$$\alpha = \vartheta\xi < \vartheta(\Omega^\Omega),$$

for which there are two cases to consider:

1. $\xi < \Omega^\Omega$ and $\xi^* < \vartheta(\Omega^\Omega)$; or
2. $\xi > \Omega^\Omega$ but $\vartheta\xi \leq (\Omega^\Omega)^*$.

Since $(\Omega^\Omega)^* = 0$, the latter is impossible. From the former, however, one obtains $\alpha < \vartheta(\Omega^\Omega)$ via the induction hypothesis. □

Our motivation for defining the theory F_β as we did stemmed from the idea of formalising the acceptance of F. The theory S_3 with just one truth predicate appears to almost achieve this, but the general inability to close S_3 under the rule T-Intro means the truth predicate no longer satisfies the same principles as it did in F. This lead us to consider stratifying the language, viewing the original predicate of F, now T_0, as the base level and gradually extending the language by including predicates T_1, T_2, etc. in such a way that each predicate T_η in the language locally satisfies the same axioms and rules as T_0.

The analysis of the theories F_β reveals that stratification of the language did not lead us as far from the world of a single self-applicable truth predicate as might have first appeared. Indeed, theorems 4 and 5 show the truth predicates of F^1_β may be treated as identical; they can all be interpreted as the set $\mathscr{F}^*_{<\Omega^{\beta+1}}$. Within F^2_β they may all be interpreted as the set $\mathscr{F}^*_{<\Omega^{\beta+1}\cdot 2}$ and, in general, all truth predicates in F_β can be interpreted as the set $\mathscr{F}^*_{<\Omega^{\beta+2}}$ (one cannot simply use $\mathscr{F}^*_{<\Omega^{\beta+1}\cdot\omega}$ for the interpretation of T_β in F_β as the set is not closed under the ω-rule, whereas $\mathscr{F}^*_{<\Omega^{\beta+2}}$ is, as well being closed under T_η-Elim, T_η-Intro for every $\eta \leq \beta$.) The upshot is that we may view each predicate T_η as "extending" the base predicate T_0 as well as T_ξ for $\xi < \eta$. It would be interesting to determine whether the theory F_β can be rewritten in some natural type-free form.

The model constructions employed in the previous section for the analysis of F_1 allow us to obtain an upper bound for the theory S_3 introduced in **??**. Essentially, we stratify the language $\mathscr{L}_\mathsf{T}$ as described in remark 1, interpreting the outermost truth predicate by T_1 and all others by T_0, but by first embedding S_3 in an infinitary theory formulated without T-Elim, we avoid the problems relating to T_0-Elim and T_1-Elim.

thm:S3strength **Theorem 15.** S_3 *proves the same arithmetical statements as* F^1_1 *and hence has proof-theoretic ordinal* Γ_0.

Proof. We define an infinitary proof system $\mathscr{S}_\infty$ based on $\mathscr{T}_\infty$ into which we may embed S_3. Let * be the interpretation of $\mathscr{L}_0$ into $\mathscr{L}_\mathsf{T}$ that recursively interprets the predicate T_0 as T and otherwise commutes with all connectives and quantifiers. Define $\mathscr{S}_\infty \overset{\alpha}{\vdash} \Gamma$ according to the rules (Ax.1), (Ax.2), (Ax.3), ($\wedge$), ($\vee_i$), (ω), ($\exists$), (T-Imp), (T-Del), (T-Rep), (T-ω), and the following additional rule

(Ax.4)$^{\mathscr{F}}$ $\mathscr{T}_\infty \vdash^{\beta}_{\Omega\cdot\gamma} A$ for some $\gamma, \beta < \Omega$ implies $\mathscr{S}_\infty \vdash^{\alpha} \Gamma, \mathsf{T}(\ulcorner A^* \urcorner)$ for any $\alpha > \max\{\beta, \gamma\}$.

The Condensation lemma entails, for $\gamma < \Omega$, that

$$\mathscr{T}_\infty \vdash^{\alpha}_{\Omega\cdot\gamma} \Gamma \text{ implies } \mathscr{S}_\infty \vdash^{\alpha} \Gamma^*. \tag{45}$$

{exteqn:S81}

Define a sequence of $\mathscr{L}_\mathsf{T}$-structures

$$\mathfrak{M}_\alpha = \langle \mathbb{N}, \{\ulcorner A^* \urcorner : \mathscr{T}_\infty \vdash^{\beta}_{\Omega\cdot\gamma} A \text{ for some } \gamma < \alpha \text{ and } \beta < \vartheta(\Omega \cdot \alpha)\} \rangle.$$

We claim

$$\mathscr{S}_\infty \vdash^{\alpha} \Gamma \text{ implies } \mathfrak{M}_\alpha \models \bigvee \Gamma \tag{46}$$

{exteqn:S82}

whenever Γ is T-positive. The proof proceeds by transfinite induction on α. If Γ is an instance of (Ax.4)$^{\mathscr{F}}$, $\mathfrak{M}_\alpha \models \bigvee \Gamma$ holds by definition, while if Γ is derived through an application of (T-Rep), it follows from the induction hypothesis and closure of $\mathscr{T}_\infty$ under (T_0-Intro). If the last applied rule is (T-ω), $\mathfrak{M}_\alpha \models \bigvee \Gamma$ holds by an application of (ω) in $\mathscr{T}_\infty$ and the fact $\vartheta(\Omega \cdot \alpha)$ is increasing in α. Furthermore, by the definition of the function ϑ, we have $\vartheta(\Omega \cdot \delta + \beta) < \vartheta(\Omega \cdot \alpha)$ whenever $\delta < \alpha$ and $\beta < \vartheta(\Omega \cdot \alpha)$; thus theorem 10 implies

$$\mathfrak{M}_\alpha \models \forall \ulcorner A \urcorner (\mathsf{T}(\ulcorner \mathsf{T}(\ulcorner A \urcorner) \urcorner) \rightarrow \mathsf{T}(\ulcorner A \urcorner)),$$

and we may deduce $\mathfrak{M}_\alpha \models \bigvee \Gamma$ from the induction hypothesis if the last rule applied was (T-Del).

Equation (46) can now be utilised along with eq. (45) to conclude

$$\mathscr{S}_\infty \vdash^{\alpha} \mathsf{T}(\ulcorner A \urcorner) \text{ implies } \mathscr{S}_\infty \vdash^{\vartheta(\Omega\cdot\alpha)} A.$$

Since $\vartheta\Omega^2$ is the least ordinal closed under the function $\alpha \mapsto \vartheta(\Omega \cdot \alpha)$ (see proposition 10) we may deduce $\mathsf{S}_3 \vdash A$ implies $\mathscr{S}_\infty \vdash^{\alpha} A$ for some $\alpha < \vartheta\Omega^2$ for any sentence A, whence $\|\mathsf{S}_3\| \leq \vartheta\Omega^2$. Finally, note $\vartheta\Omega^2 = \Gamma_0$ by corollary 11. □

On a final note, one may reasonably consider applying this idea to the other Friedman-Sheard theories (or indeed to other theories of self-referential truth). The theory E is ω-inconsistent (*cf.* ??), so a definition of E_1 cannot consistently contain the axiom T_1-$\forall$-Inf. Thus, since the predicate T_1 could not adhere to the rules governing T_0, this process could not be applied to E. Iterating the operation pertaining to I, however, does lead to a hierarchy of consistent theories. I does not contain the rule T-Intro, so one would not naturally include the principle

$$\mathsf{T}_1(x) \rightarrow \mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x}) \urcorner)$$

as an axiom of I_β. Without such an axiom, though, there would be no dependency between the two predicates: any occurrence of T_0 under T_1 may be treated vacuous, as can any occurrence of T_1 under T_0. In an attempt to resolve the matter one could add to I_1 the axiom $\mathsf{T}_1(\ulcorner A \urcorner)$ whenever A is an axiom of I_0 (I formulated with the predicate T_0). This would appear a natural choice as it agrees with the construction of F_1. It would be interesting to ascertain the strength of the resulting iteration and compare it to our results regarding F_β.