Iterated self-applicable truth

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1 Overview

Should *is true* be *is justifiable* or *verifiable*?

2 A theory

Language $\mathscr{L}_\Omega$: extend language of PRA by binary predicate T. Assuming an ordinal notation system of a sufficiently large ordinal κ, and $\eta < \kappa$, $T_\eta s$ means $T(\ulcorner \eta \urcorner, s)$.

For $\beta < \kappa$, define a theory T_β extending HA by axioms, for each $\eta \leq \kappa$:

$$\text{Some axiom like: } \mathsf{Tr}_{\mathsf{eq}}(x) \to T_\eta x$$

$$\mathsf{valid}(x) \wedge \mathsf{Sent}(x) \to T_\eta x$$

$$\forall \ulcorner A \urcorner \ulcorner B \urcorner (T_\eta \ulcorner A \to B \urcorner \to (T_\eta \ulcorner A \urcorner \to T_\eta \ulcorner B \urcorner)) \tag{I}$$

$$\forall \ulcorner A(x) \urcorner (\forall n T_\eta \ulcorner A(\dot{n}) \urcorner \to T_\eta \ulcorner \forall x A(x) \urcorner \tag{U}$$

$$\forall \xi \leq \bar{\beta} (T_\eta \ulcorner T_\xi \dot{x} \urcorner \to T_\eta x) \tag{D}$$

$$\forall \xi < \bar{\eta} (T_\eta x \to T_\eta \ulcorner T_\xi \dot{x} \urcorner) \tag{R}$$

and rules of inference

$$A \vdash T_\beta \ulcorner A \urcorner \tag{Nec}$$
$$T_\beta \ulcorner A \urcorner \vdash A \tag{Conec}$$

What gives the strength is that (D) 'collapses' all internal truth predicates to the current level (η). In contrast, (R) only permits 'expanding' the internal level by lower level predicates.

We argue that

Theorem 2.1. *For every $p < \omega$,*
$$\|\mathsf{T}_p\| = \vartheta(\Omega^{p+1} \cdot \omega).$$

Theorem 2.2. *For $\omega \le \beta < \vartheta\Omega^\Omega$,*
$$\vartheta(\Omega^\beta \cdot \omega) \le \|\mathsf{T}_\beta\| \le \vartheta(\Omega^{\beta+1} \cdot \omega).$$

Theorem 2.3. *The limit of the autonomous progression of $\{\mathsf{T}_\beta : \beta < \Omega\}$ is the large Veblen ordinal, $\vartheta\Omega^\Omega$.*

It seems likely that the same claims hold for (intuitionistic) T_β.

The ordinal analysis of T_β theories is derived from the analysis of a classical extension of the theories in the first author's PhD thesis [2]. Some arguments require changing to accommodate the intuitionistic base (T-elimination), and some have been streamlined (?).

3 Theories with multiple self-applicable truth predicates — material from the thesis

chap:ext

Truth is often used as a means of reflection; a tool by which one may obtain principles, schemata etc. that were implicit, but not necessary explicit, in the acceptance of some axiomatic system. Feferman, for example, views the theory of truth Ref (see **??**) as an operation which, when applied to a theory S, answers the question "which statements in the base language ... ought to be accepted if one has accepted the basic axioms and rules of [S]?" [4, p. 2]. The Friedman-Sheard theories A to I can also be viewed as operations which have been applied to PRA: one adds to PRA a (new) predicate T, formalising the acceptance of PRA; on top of this one adds some subset of the Optional Axioms, for example adding ∀-Inf formalises the acceptance of ω-logic, while the axiom T-Del formalises closure under the rule T-Elim.

Viewing theories of truth as operations provides a natural way to describe the general processes behind their construction and allows one to possibly iterate the operation. In this section we will look at this specifically from the perspective of the Friedman-Sheard theory F.

One way of arguing for the naturalness of S_3 is to view it as formalising the acceptance of S_2. Within S_3 one has T-Rep and T-Del, formalising the rules of inference T-Intro and T-Elim of S_2, and thus

$$\mathsf{S}_3 \vdash \forall x(\mathsf{Bew}_{\mathsf{S}_2}(x) \wedge \mathsf{Sent}_{\mathscr{L}_\mathsf{T}}(x) \to \mathsf{T}(x)).$$

However, S_2 contains the rule T-Intro, so it seems reasonable that the theory attempting to formalise its acceptance should also be closed under T-Intro. But the presence of T-Rep, T-Del and T-Elim in S_3 means this is not possible, so perhaps S_3 is not such a natural theory after all.

Since closure of F under $\neg$T-Elim is vacuous, F and S_2 are identical as theories. Therefore, S_3 can be seen as formalising the acceptance of F, although one might expect in this case, to also add

$$\mathsf{T}(\ulcorner\neg\mathsf{T}(\dot{x})\urcorner) \to \mathsf{T}(\underset{\cdot}{\neg}x)$$

as an axiom.[1] Still, the resulting theory cannot be closed under T-Intro, as one would like.

If one were to stratify the language, in much the same way as one would to form a Tarskian hierarchy of truth predicates, the problem can be circumvented. Recall $\mathsf{F} = \mathsf{Base}_T$ + T-Intro + T-Elim + $\neg$T-Elim + T-Del + $\forall$-Inf. Let HST_0 denote F formulated with the predicate T_0 in place of T, and suppose T_1 is a (new) unary predicate symbol. The theory formalising acceptance of HST_0, which we shall denote by HST_1, would then contain the following axioms

$$\mathsf{val}(x) \wedge \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \to \mathsf{T}_1(x), \tag{1}$$
{eqn:FFax0}

$$\mathrm{Ax}_{\mathsf{HST}_0}(x) \wedge \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \to \mathsf{T}_1(x), \tag{2}$$
{eqn:FFax1}

$$\mathsf{T}_1(x) \wedge \mathsf{T}_1(x \mathbin{\underset{\cdot}{\to}} y) \to \mathsf{T}_1(y), \tag{3}$$
{eqn:FFax2}

$$(\forall x\, \mathsf{T}_1(\ulcorner A(\dot{x})\urcorner)) \to \mathsf{T}_1(\ulcorner \forall x A(x)\urcorner), \tag{4}$$
{eqn:FFaxw}

$$\mathsf{T}_1(x) \to \mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x})\urcorner), \tag{5}$$
{eqn:FFax3}

$$\mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x})\urcorner) \to \mathsf{T}_1(x), \tag{6}$$
{eqn:FFax4}

$$\mathsf{T}_1(\ulcorner \neg\mathsf{T}_0(\dot{x})\urcorner) \to \mathsf{T}_1(\underset{\cdot}{\neg}x). \tag{7}$$
{eqn:FFax5}

(**??**) state the acceptance of all axioms of HST_0 (logical and non-logical), whereas (3) formalises *modus ponens* in HST_0. These three also combine to imply the axioms of $\mathsf{Base}_{\mathsf{T}_1}$. (**??**) express the acceptance of the rules nec_0, conec_0 and $\neg\mathrm{conec}_0$, respectively, in HST_0, while (4) closes the predicate under ω-logic.

The predicate T_1 is viewed as an extension of the predicate T_0 and as such we would expect it to satisfy the relevant axioms of F, that is, we also have

$$\mathsf{T}_1(\ulcorner \mathsf{T}_1(\dot{x})\urcorner) \to \mathsf{T}_1(x),$$

and closure under conec_1, nec_1 and $\neg\mathrm{conec}_1$. Combining also the axioms of HST_0 it is then easy to deduce

$$\forall x(\mathsf{Bew}_{\mathsf{HST}_0}(x) \wedge \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \to \mathsf{T}_1(x)). \tag{8}$$
{eqn:F1reflecti

We are happy with the thought of T_0 being a self-applicable truth predicate, and so far there is nothing to stop T_1 also being self-applicable. Moreover, T_0 may meaningfully occur in the scope of the predicate T_1. Thus we have described the first step in a hierarchy of self-applicable truth predicates. But, should the predicate T_1 be allowed to occur in the scope of T_0? After all, the motivation behind working with theories that contain their own truth predicate is in their ability to reason about themselves. Since nec_1 may apply to arbitrary sentences in $\mathscr{L}_{\mathsf{T}_0,\mathsf{T}_1}$, the question of whether or not T_0 can meaningfully apply to sentences containing T_1 is essentially decided by how we restrict the quantifiers in (**??**) (in particular (5) pertaining to nec_0) for inclusion in HST_1: if we restrict them to range over only codes of $\mathscr{L}_{\mathsf{T}_0}$-sentences we will have no non-trivial occurrences of this inter-applicability.

[1] As the $\mathscr{L}_{\mathsf{T}}$-structure $\mathfrak{M}$ used in **??** also satisfies the axiom $\mathsf{T}(\ulcorner\neg\mathsf{T}(\dot{x})\urcorner) \to \mathsf{T}(\underset{\cdot}{\neg}x)$, the extension of S_3 obtained by adding this axiom is also consistent.

As is consistent with our earlier chapters, we view our theories as being presented in a Hilbert style deduction system, with certain axioms and rules of inference which are treated in their broadest sense. Namely, we consider a rule of inference of a theory S to be applicable to any extension of the language, logic or axioms of S. Thus, if one imagines the theory HST_0 being first formulated in the language $\mathscr{L} \cup \{\mathsf{T}_0, \mathsf{T}_1\}$, and only then completing the reflection step to HST_1 by adding the axioms and rules pertaining to T_1, it seems natural to suppose the predicates T_0 and T_1 are inter-applicable. Since nec_0 was applicable in HST_0 to formulae containing the predicate T_1, so should T_0 in HST_1. Thus, we expect HST_1 to have the axiom T_0-Imp,

$$\forall x \forall y [\mathsf{T}_0(x) \land \mathsf{T}_0(x \dot{\to} y) \to \mathsf{T}_0(y)],$$

as opposed to its relativised form

$$\forall x \forall y [\mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(x) \land \mathsf{Sent}_{\mathscr{L}_{\mathsf{T}_0}}(y) \to (\mathsf{T}_0(x) \land \mathsf{T}_0(x \dot{\to} y) \to \mathsf{T}_0(y))]; \tag{9}$$ {eqn:T0relative

and, more importantly, that HST_1 contains the unrelativised axioms

$$\begin{gathered} \forall x [\mathsf{T}_1(x) \to \mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x}) \urcorner)], \\ \forall x [\mathsf{T}_1(\ulcorner \mathsf{T}_0(\dot{x}) \urcorner) \to \mathsf{T}_1(x)], \\ \forall x [\mathsf{T}_1(\ulcorner \neg \mathsf{T}_0(\dot{x}) \urcorner) \to \mathsf{T}_1(\dot{\neg} x)]. \end{gathered} \tag{10}$$ {eqn:F1ax1}

This provides, for example,

$$\forall x [\mathsf{T}_1(\ulcorner \mathsf{T}_1(\dot{x}) \urcorner) \leftrightarrow \mathsf{T}_1(\ulcorner \mathsf{T}_0(\ulcorner \mathsf{T}_1(\dot{x}) \urcorner) \urcorner)],$$

which, by nec_1 and (8), yields $\mathsf{T}_0(\ulcorner \mathsf{T}_1(\ulcorner A \urcorner) \urcorner)$ for every theorem A of HST_0. Thus our informal interpretation leads us to the curious situation where we have two self-applicable predicates T_0, T_1 which may also be applied to each other.

If, on the other hand, we had considered relativising the quantifiers as in (9), T_1 may apply to the language $\mathscr{L}_{\mathsf{T}_0, \mathsf{T}_1}$ whereas T_0 may only meaningfully apply to $\mathscr{L}_{\mathsf{T}_0}$ and one would obtain the base of a strict hierarchy of self-applicable truth predicates. In this case we can no longer argue that T_0 represents a truth predicate for the whole language but only of the sub-language $\mathscr{L}_{\mathsf{T}_0}$. The reason for choosing a self-applicable notion of truth in the first place was that it may be treated as a truth predicate for the entire language, including any subsequent extension. Thus, T_0 *should* be applicable to sentences involving the predicate T_1 and we find ourselves returning to the world of two inter-applicable truth predicates.

So far we have argued that in HST_1 the interpretation of T_0 should be closed under conec_0 and $\neg\mathrm{conec}_0$ while the interpretation of T_1 should be closed under nec_0, conec_0, $\neg\mathrm{conec}_0$, conec_1 and $\neg\mathrm{conec}_1$. However, we desire T_1 to be viewed as an extension of T_0 so as to allow for closure of HST_1 under a form of *truth introduction* and this fact should be recognised by the theory. That is, from the point of view of the theory HST_1, there should be no essential difference between the predicates T_0 and T_1. Thus we propose to also include the principle

$$\mathsf{T}_0(\ulcorner \mathsf{T}_1(\dot{x}) \urcorner) \to \mathsf{T}_0(x) \tag{11}$$ {eqn:T1Del}

as an axiom of HST_1 (in fact as an axiom of HST_0). (11) expresses that the interpretation of T_0 is closed under the rule conec_1. This is vacuously valid in the theory HST_0. It also serves to confirm the inter-applicability of the two predicates by allowing meaningful inferences regarding T_1 under a T_0 predicate. If we accept (11) we should also accept $\mathsf{T}_0(\ulcorner\neg\mathsf{T}_1(\dot{x})\urcorner) \to \mathsf{T}_0(\dot{\neg}x)$, but as we shall see, this axiom will be trivially satisfied by our model.

We have only described the step HST_0 to HST_1, but one can imagine repeating this, first adding an additional truth predicate T_2 to HST_1 and then reflecting upon it to form the theory HST_2.[2] This process may be continued into the transfinite to form a hierarchy of theories, supporting a hierarchy of inter-applicable truth predicates.[3]

The description of HST_1 and HST_2 presented above is purely motivational and requires making explicit, which the next definition achieves. As we pass beyond finite iterations and consider the construction of theories HST_ω, $\mathsf{HST}_{\varepsilon_0}$, *etc.*, one requires the ability to internally quantify over the indices of truth predicates in the language. We therefore need to fix some computable ordinal κ from the outset and only consider iterating the construction up to ordinals $\alpha < \kappa$. In fact to maintain as much similarity with our previous work as possible (for example to ensure a primitive recursive Gödel numbering) we will insist κ is primitive recursively definable. One could consider taking $\kappa = \Gamma_0$ and utilising the encoding chosen in **??**, but as we shall see $\|\mathsf{F}_1\| \geq \Gamma_0$ (see theorem 6.7 below) and so we will require the construction of a larger class of ordinals to perform a sufficient proof-theoretic analysis. Suitable choices for κ will ultimately depend on our analysis and have no substantial role in defining the theories. Since the precise definition of κ is not essential for the definition, we shall assume for the time being that κ represents some fixed primitive recursive ordinal.

Definition 3.1. For $\beta < \kappa$, let $\mathscr{L}_\beta$ denote the language of PRA augmented by predicates T_ξ for each $\xi < \beta$. Let HST_β be the theory formulated in the language $\mathscr{L}_{\beta+1}$, extending PA with the schema of induction for $\mathscr{L}_\beta$, and for each $\eta \leq \beta$

$$\forall x(\mathsf{T}_\eta(x_1) \wedge \mathsf{T}_\eta(x_1 \dot{\to} x_2) \to \mathsf{T}_\eta(x_2)), \qquad (\mathrm{imp}_\eta) \quad \{\mathtt{ax:imp}\}$$
$$\forall \ulcorner A(x)\urcorner[(\forall n \mathsf{T}_\eta(\ulcorner A(\dot{n})\urcorner)) \to \mathsf{T}_\eta(\ulcorner \forall x\, A(x)\urcorner)], \qquad (\mathrm{uni}_\eta) \quad \{\mathtt{ax:uni}\}$$
$$\forall \xi \leq \bar{\beta}\, \forall x(\mathsf{T}_\eta(\ulcorner \mathsf{T}_{\dot{\xi}}(\dot{x})\urcorner) \to \mathsf{T}_\eta(x)), \qquad (\mathrm{del}_\eta) \quad \{\mathtt{ax:del}\}$$
$$\forall \xi < \bar{\eta}\, \forall x(\mathsf{T}_\eta(x) \to \mathsf{T}_\eta(\ulcorner \mathsf{T}_{\dot{\xi}}(\dot{x})\urcorner), \qquad (\mathrm{rep}_\eta) \quad \{\mathtt{ax:rep}\}$$

as well as the rules of inference

$$\text{from } A \text{ infer } \mathsf{T}_\beta(\ulcorner A\urcorner), \qquad (\mathrm{nec}_\beta) \quad \{\mathtt{ax:nec}\}$$
$$\text{from } \mathsf{T}_\beta(\ulcorner A\urcorner) \text{ infer } A. \qquad (\mathrm{conec}_\beta) \quad \{\mathtt{ax:conec}\}$$

Define $\mathsf{HST}_{<\beta} = \bigcup_{\eta<\beta} \mathsf{HST}_\beta$. We denote by HST^n_β the collection of theorems of HST_β provable with at most n (serial) applications of (nec_β) and no restriction on the number of applications of (conec_β). Thus HST^0_β denotes the theory HST_β without (nec_β), and HST^n_β is a sub-theory of HST^{n+1}_β for every n.

The theory HST_0 is identical to F and the definition of HST_β fits the informal description we gave of 'F viewed as an operation applied to $\mathsf{HST}_{<\beta}$'. Also, HST^0_β is a conservative extension

[2] A more precise way to describe the construction of HST_2 may be given as follows: first imagine formulating HST_0 with three predicates, T_0, T_1, and T_2. One then formulates HST_1 in this language by reflecting upon HST_0, and only then is HST_2 formulated by reflecting upon HST_1. In this sense we view HST_0 as not being formulated in a fixed language, but rather in a language that may be expanded as and when we see fit.

[3] The inter-applicability of the predicates calls into question whether what we obtain is truly a "hierarchy of truth predicates" as we describe. Our model construction, in the next section, will show that one can view the truth predicates as based on a hierarchy, although not, perhaps, in a manner one might first imagine.

of HA as, with no applications of (nec_β), all predicates T_η in HST^0_β may be interpreted trivially to show neither $\mathsf{HST}^0_\beta \vdash \mathsf{T}_\beta s$ nor $\mathsf{HST}^0_\beta \vdash \neg\mathsf{T}_\beta s$ may hold.

Although HST_1 may be borne from a notion of truth and acceptance, it is by no means necessarily consistent. Indeed, with its multiple truth predicates and inter-applicability, the reader would be duly justified to view this construction with some scepticism. The process of reflection that led us to HST_1, however, is almost identical to that which describes the theory S_3. As a result it may not be surprising to know that HST_1 is consistent and in fact HST^1_1 proves the same arithmetical statements as the theory S_3. Unfortunately we do not at this time have the suitable machinery to prove their equivalence (this will have to wait until section 8), but the following remark should motivate the connection.

rem:S3 **Remark 1.** It is natural to first consider embedding HST^1_1 into S_3. As the predicate T_1 can be interpreted vacuously in HST^0_1 (as can T_0), one may expect the interpretation given by

$$\mathsf{T}_1 s^* = \mathsf{T}_0 s^* = \mathsf{T}(gs)$$

to suffice, where g is a primitive recursive function chosen, by the primitive recursion theorem, such that

$$\begin{aligned}
g(x) &= x, \text{ if } x \text{ is the code of an arithmetical literal,}\\
g(\ulcorner \mathsf{T}_1 s \urcorner) &= \ulcorner s \neq s \urcorner,\\
g(\ulcorner \mathsf{T}_0 s \urcorner) &= \ulcorner \mathsf{T}(gs) \urcorner,\\
g(\ulcorner A \circ B \urcorner) &= g(\ulcorner A \urcorner) \underset{\cdot}{\circ}\, g(\ulcorner B \urcorner), \text{for } \circ \in \{\wedge, \vee\}\\
g(\ulcorner QxA(x) \urcorner) &= \underset{\cdot}{\forall} x\, g(\ulcorner A(x) \urcorner).
\end{aligned}$$

This interpretation will translate the axioms del_1, uni_1 and rep_1 of HST_1 to the axioms T-Del, $\forall$-Inf and T-Rep respectively of S_3. Since S_3 does not have the axiom $\neg$T-Del (nor can we expect to derive it in S_3) one cannot derive the translation of $\neg\mathrm{del}_1$ from HST^1_1. Fortunately, the model construction of **??** can easily incorporate the additional axiom $\neg$T-Del and rule $\neg$T-Elim with minimal adjustments: one may consistently add the rule pertaining to $\neg$T-Del to each of the theories $\mathscr{T}_\alpha$ and expand **??** to also prove $\mathscr{T}_\alpha \vdash \neg\mathsf{T}(\ulcorner A \urcorner)$ implies $\mathscr{T}_\alpha \vdash \neg A$, thus showing $\mathsf{S}_3 + \neg$T-Del $+ \neg$T-Elim is a consistent theory.

The problem with this interpretation manifests when dealing with applications of conec_1 in HST^1_1. Suppose $\mathsf{HST}^1_1 \vdash \mathsf{T}_1(\ulcorner A \urcorner)$. If A does not contain the predicate T_1, $\ulcorner A^* \urcorner = g(\ulcorner A \urcorner)$ and an application of T-Elim in S_3 suffices. If, however, A contains T_1, the interpretation of $\mathsf{T}_1(\ulcorner A \urcorner)$ and A under * are very different; indeed, there will be sentences B for which $\mathsf{S}_3 \vdash \mathsf{T}(g(\ulcorner B \urcorner))$ but $\mathsf{S}_3 \nvdash B^*$ (for example, take B to be $\neg\mathsf{T}_1(\ulcorner C \urcorner)$ where C is any statement provable in S_3. Then $\mathsf{S}_3 \vdash B^*$ implies $\mathsf{S}_3 + \neg$T-Elim $\vdash \neg C$, which contradicts the consistency of $\mathsf{S}_3 + \neg$T-Elim. On the other hand, $g(\ulcorner B \urcorner) = \ulcorner \neg(s \neq s) \urcorner$ for some s, so $\mathsf{S}_3 \vdash \mathsf{T}(g(\ulcorner B \urcorner))$ holds). If A were an axiom of HST^0_1 though, $\mathsf{S}_3 \vdash A^*$; moreover, if one views T_1-Imp, del_1 and T_1-Rep as a rule of inference, as in $\mathscr{T}$, one could deduce closure under conec_1 by induction on the height of a derivation. Viewing the axioms of HST^1_1 in this manner is reminiscent of the use of the infinitary system $\mathscr{F}_\infty$ in the analysis of F; thus it seems natural to delay a further investigation of this connection until we have first analysed HST_1 in detail.

Although the addition of full nec_1 to HST^1_1 (forming HST_1) creates a theory markedly stronger than S_3, it is not straightforward to embed S_3 into HST_1. Such an embedding would require stratifying $\mathscr{L}_\mathsf{T}$ to involve the two predicates T_0 and T_1 of $\mathscr{L}_1$. The most obvious way to proceed would appear to involve replacing the outermost predicate by T_1 and all others by T_0, thus mapping the axioms $\mathsf{T}(\ulcorner\text{T-Imp}\urcorner)$, $\mathsf{T}(\ulcorner\forall\text{-Inf}\urcorner)$ and $\mathsf{T}(\ulcorner\text{T-Del}\urcorner)$ of S_3 to theorems of HST^1_1. This could be achieved by picking a primitive recursive function f which recursively substitutes $\mathsf{T}_0(fs)$ for $\mathsf{T}s$, and considering the interpretation * of $\mathscr{L}_\mathsf{T}$ into $\mathscr{L}_1$ given by

$$(\mathsf{T}s)^* = \mathsf{T}_1(fs).$$

This interpretation maps the axioms $\forall$-Inf, T-Imp, T-Del and T-Rep by instances of axioms uni$_1$, T_1-Imp, del$_1$ and rep$_1$ respectively, but since $f(\ulcorner A\urcorner)$ need not equal $\ulcorner A^*\urcorner$, applications of T-Elim in S_3 do not translate into inferences we can readily recognise as holding in HST_1.

We now move to the task of proving the consistency of HST_β for arbitrary β. This will be obtained by generalising the consistency argument for S_3 to theories with multiple predicates and is presented in section 4 below. Following this we will perform a detailed ordinal analysis of these theories, first determining lower bounds on their proof-theoretic strength, and second upper bounds. Lower bounds on the strength of the theories HST_β will be obtained in section 6 by extending the well-ordering results previously established for S_2 and S_3. Upper bounds are determined in section 7 where we attempt to directly formalise the model constructions of section 4 in a manner similar to our analysis of F. However, much care will be required when dealing with a hierarchy of truth predicates and the transfinite iterations of T-Intro caused by the interaction between axioms uni$_\eta$and rep$_\eta$.

The axiom del$_\eta$implies the interpretation of the predicate T_η is closed under T_ξ-Elim for every $\xi \leq \eta$. Likewise (rep$_\eta$) implies T_η is closed under T_ξ-Intro for each $\xi < \eta$. From this observation we may deduce the following propositions.

nec-and-conec **Proposition 3.2.** HST_β *is closed under (conec$_\eta$) for every $\eta < \kappa$ and under (nec$_\eta$) for every $\eta < \beta$.*

Proof. Let A be a sentence. If $\mathsf{T}_\eta\ulcorner A\urcorner$ is a theorem of HST_β then so is $\mathsf{T}_\beta\ulcorner\mathsf{T}_\eta\ulcorner A\urcorner\urcorner$ (nec$_\beta$), $\mathsf{T}_\beta\ulcorner A\urcorner$ (del$_\beta$), whence A (conec$_\beta$). The converse implications hold if $\eta < \beta$. ☐

global-reflect **Proposition 3.3.** $\mathsf{HST}^1_\beta \vdash \forall\eta < \bar{\beta}\forall x(\mathsf{Bew}_{\mathsf{HST}_\eta}(x) \wedge \mathsf{Sent}_{\mathscr{L}_\kappa}(x) \rightarrow \mathsf{T}_\beta x)$.

Proof. All axioms of HST_η are axioms of HST^0_β and, by (nec$_\beta$), we have

$$\mathsf{HST}^1_\beta \vdash \forall x(\mathsf{Ax}_\eta(x) \wedge \mathsf{Sent}_{\mathscr{L}_\kappa}(x) \rightarrow \mathsf{T}_\beta x),$$

where $\mathsf{Ax}_\eta(x)$ expresses that x is a non-logical axiom of HST_η. To complete the proof, we observe that the axioms (imp$_\beta$), (del$_\beta$), (rep$_\beta$) internalise the rules of modus ponens, (conec$_\eta$) and (nec$_\eta$) of HST_η, respectively. ☐

4 Consistency

sec:Fbconsis

At first glance the theory HST_β could easily look suspect, after all it contains del$_\eta$, conec$_\eta$, nec$_\eta$and an axiom that appears extremely close to T_η-Rep, for each $\eta \leq \beta$. As the motivation behind the theories HST_β comes from abstracting the transition from PA to F one might expect that if HST_β is consistent, models of HST_β may be constructed by extending models of F. This is indeed the case; moreover, the extension we defined for establishing the consistency of S_3 makes a suitable base from which to start the construction. We will only sketch the consistency argument as it will be subsumed by our work in section 7 where we determine an upper bound on the proof-theoretic strength of HST_β.

Let Ω denote the first uncountable ordinal. For the remainder of this section Greek letters, ρ, σ, τ will be used to range over elements of $\mathbb{O}$, the class of all ordinals; letters α, β, etc., will range over countable ordinals; we reserve the letters η, ξ for indices of the truth predicates and so represent ordinals below κ. Suppose

$$\rho = \Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_n} \cdot \beta_n,$$

with $\alpha_0 > \cdots > \alpha_n$ and $\beta_i < \Omega$ for each $i \leq n$. We denote by $\rho|_\gamma$ the ordinal $\Omega^{\alpha_0}\cdot\beta_0+\cdots+\Omega^{\alpha_k}\cdot\beta_k$ where $k < n$ is the least such that $\alpha_k > \gamma \geq \alpha_{k+1}$, or $k = n$ if $\alpha_n > \gamma$. An ordinal ρ is called an Ω-*limit* if $\rho = \rho_0 + \Omega^\eta \cdot \alpha$ and either α or η is a limit ordinal.

Definition 4.1 (Sequent)**.** Let $\kappa \in \mathbb{O}$ be fixed. A (κ-)sequent is an expression $\Gamma \Rightarrow A$ where $\Gamma \cup \{A\}$ is a finite of sentences of $\mathscr{L}_\kappa$.

Definition 4.2 (Semiformal calculus)**.** For each $\rho \in \mathbb{O}$, a Tait-style sequent calculus $\vdash_\rho$ is introduced for κ-sequents, defined by the following rules.One- or two-sided?

- Ax.1. $\vdash_\rho \Gamma \Rightarrow A$ whenever A is a true atomic literal,
- Ax.2$_\eta$. $\vdash_\rho \Gamma, \mathsf{T}_\eta s \Rightarrow \mathsf{T}_\eta s$ for every closed term s and $\eta < \kappa$,
- Ax.3$_\eta$. $\vdash_\rho \Gamma, \mathsf{T}_\eta s \Rightarrow A$ if $s^\mathbb{N}$ is not the code of an $\mathscr{L}_\kappa$-sentence.
- The usual arithmetical rules for $\wedge$, $\vee$ and $\exists$,
- The ω rule: $\dfrac{\vdash_\rho \Gamma \Rightarrow A(\underline{n}) \quad \text{for every } n < \omega}{\vdash_\rho \Gamma \Rightarrow \forall x A(x)}\,\omega$
- The following six rules for every $\eta < \kappa$.

$$\text{imp}_\eta \ \frac{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta s \quad \vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta (s \dot{\rightarrow} t)}{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta t}\,\text{imp}_\eta$$

$$\text{del}_\eta \ \frac{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi s \urcorner) \quad \xi < \beta}{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta s}\,\text{del}_\eta$$

$$\text{rep}_\eta \ \frac{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta s \quad \xi < \eta}{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi s \urcorner)}\,\text{rep}_\eta$$

$$\text{nec}_\eta \ \frac{\vdash_\sigma \Gamma \Rightarrow A \quad \sigma < \rho|_\eta}{\vdash_\rho \Delta, \mathsf{T}_\eta \ulcorner \Gamma \urcorner \Rightarrow \mathsf{T}_\eta \ulcorner A \urcorner}\,\text{nec}_\eta$$

$$\text{uni}_\eta \ \frac{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta \ulcorner A(\underline{n}) \urcorner \text{ for all } n \quad \rho|_\eta \text{ not an } \Omega\text{-limit}}{\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta \ulcorner \forall x A(x) \urcorner}\,\text{uni}_\eta$$

Moreover, for each ρ define the set of $\mathscr{L}_\kappa$-sentences

$$\mathscr{T}_{<\rho} = \{A : \ \vdash_\sigma \ \Rightarrow A \text{ for some } \sigma < \rho\}.$$

Before we proceed with analysing the role of the truth predicates in $\mathscr{T}_\rho$, it is important to note that a rule of *modus ponens*, or cut is lacking from our definition. However, it is not hard to show the cut rule is admissible: a derivation of the form $\mathsf{T}_\eta s, \Gamma \Rightarrow A$ with $\mathsf{T}_\eta s$ principal *must* be an instance of an axiom, hence given derivations of $\mathsf{T}_\eta s, \Gamma \Rightarrow A$ and $\Delta \Rightarrow \mathsf{T}_\eta s$ one can easily obtain instead a derivation of Γ. This argument is essentially identical to that employed in the Cut Elimination theorem in [1] for the analysis of the theory F (see [1, Thm. 3.7]).

thm:FFCutElim **Lemma 4.3** (Admissibility of cut). *Suppose* $\vdash_\rho \Gamma \Rightarrow \mathsf{T}_\eta s$, *and* $\vdash_\rho \Delta, \mathsf{T}_\eta s \Rightarrow A$. *Then* $\vdash_\rho \Gamma, \Delta \Rightarrow A$.

If $\rho < \Omega$, then $\rho|_0 = 0$ and so no applications the rule nec_η is available in $\vdash_\rho$ for any η. Thus, $\mathscr{T}_{<\Omega}$ is trivially closed under conec_η for every η. Simple but not 'trivial' Moreover, for every $\rho < \Omega$, only applications of nec_0 have been permitted in $\vdash_{\Omega\cdot\rho}$. Thus, we can establish by induction on $n < \omega$ that in derivations in $\vdash_{\Omega\cdot n}$ the predicate T_0 may be interpreted by the $\mathscr{T}_{<\Omega\cdot n}$ which happens to be closed under conec_0. Since all other truth predicates that can occur in $\vdash_{\Omega\cdot n}$-derivable sequents may be interpreted vacuously, we conclude $\vdash_{\Omega\cdot n}$ is closed under conec_η for every $\eta < \kappa$. Hence HST_0^n may be interpreted in $\mathscr{T}_{\Omega\cdot n}$.

This suggests that for $n < \omega$, $\mathscr{T}_{\Omega\cdot n}$, like $\mathscr{T}_n$, reconstructs the theories Th_n used by Friedman and Sheard to prove the consistency of F. At the first limit ordinal, we obtain $\mathscr{T}_{<\Omega\cdot\omega}$, a set of $\mathscr{L}_\kappa$-sentences closed under nec_0 and conec_0, and containing del_0, rep_0 (which holds vacuously) and all other axioms of HST_0.

To proceed with the analysis of HST_1, we first consider HST_1^0 which, without the rule nec_1, is vacuously closed under conec_1 and $\neg\mathrm{conec}_1$. In HST_1^1, the situation differs from previous case; we need to interpret the predicate T_1 as a theory closed under ω-logic (due to uni_1), nec_0 (due to rep_1), conec_0 and conec_1 (due to del_1). Moreover, we need to find an interpretation of T_0 closed under ω-logic, conec_0, and now also conec_1 (as implied by the axiom del_0). The properties we established for the set $\mathscr{T}_{<\Omega}$ in **??** motivate us to consider $\mathscr{T}_{<\Omega^2}$, a set closed under nec_0, conec_0 and, by a similar argument as before, ω-logic (*cf.* the proof of **??**). For every $\alpha < \Omega$, the predicate T_1 may be interpreted vacuously in $\mathscr{T}_{<\Omega\cdot\alpha}$, so $\mathscr{T}_{<\Omega^2}$ is also closed under conec_1. Thus $\mathscr{T}_{<\Omega^2}$ provides a consistent interpretation of both predicates T_0 and T_1 in HST_1^1.

The next step is to consider T_1 in HST_1^2. Two applications of nec_1 are permitted and one can derive sentences of the form $\mathsf{T}_1(\ulcorner \mathsf{T}_1(\ulcorner A \urcorner)\urcorner)$ whenever $\mathsf{HST}_1^0 \vdash A$, suggesting a shift to $\mathscr{T}_{\Omega^2}$, where one can derive $\mathsf{T}_1(\ulcorner A \urcorner)$ whenever $A \in \mathscr{T}_{<\Omega^2}$, might yield a suitable interpretation for T_1. However, $\mathscr{T}_{\Omega^2}$ is not closed under nec_0 (only the systems $\mathscr{T}_{<\rho+\Omega\cdot\sigma}$ for limit ordinals σ are), leading us instead to consider $\mathscr{T}_{<\Omega^2+\Omega\cdot\omega}$ which is closed under nec_0, but not ω-logic; there will be sentences $A \in \mathscr{T}_{<\Omega^2}$ for which $\mathscr{T}_{\Omega^2+\Omega\cdot n} \vdash \mathsf{T}_0(f(n, \ulcorner A \urcorner))$ for each $n < \omega$, where $f(0, n) = \ulcorner \bar{n} \urcorner$ and $f(m+1, n) = \ulcorner \mathsf{T}_0(f(m, n))\urcorner$, but the sentence $\forall x\, \mathsf{T}_0(f(x, \ulcorner A \urcorner)$ is not contained in $\mathscr{T}_{<\Omega^2+\Omega\cdot\omega}$. Indeed to obtain both closure under ω-logic and nec_0 we must move to the theory $\mathscr{T}_{<\Omega^2\cdot 2}$. We also require the interpretation to be closed under conec_0. To manage this we repeat the same argument as before, but starting from $\mathscr{T}_{\Omega^2}$ in place of $\mathscr{T}_\Omega$. We know $\mathscr{T}_{\Omega^2}$ is closed under conec_0 since the predicate T_0 can be consistently interpreted as the set $\mathscr{T}_{<\Omega^2}$. This leads us to successively deduce the theories $\mathscr{T}_{\Omega^2+\Omega\cdot n}$ are closed under conec_0 for each $n < \omega$. Note, we can still interpret T_1 in $\mathscr{T}_{\Omega^2+\Omega\cdot n}$ by the set $\mathscr{T}_{<\Omega^2}$ as there has been no further applications of nec_1. In $\mathscr{T}_{\Omega^2+\Omega\cdot\omega}$ we aim to interpret T_0 by $\mathscr{T}_{<\Omega^2+\Omega\cdot\omega}$, which unlike $\mathscr{T}_{\Omega^2+\Omega\cdot n}$ is not closed under ω-logic; however, $\mathscr{T}_{\Omega^2+\Omega\cdot\omega}$ is not closed under (uni_1) so this does not pose a problem. Thus we may continue through the construction of $\mathscr{T}_{<\Omega^2\cdot 2}$ determining each theory $\mathscr{T}_{\Omega^2+\Omega\cdot\alpha}$ for $\alpha < \Omega$ is closed under conec_0.

The argument above highlights that the predicates T_1 and T_0 in HST_1^n may be interpreted as the set $\mathscr{T}_{<\Omega^2\cdot n}$, and hence HST_1 naturally embeds into $\mathscr{T}_{<\Omega^2\cdot\omega}$. If we wanted to proceed beyond this and construct models for HST_2, we could imagine constructing a sequence of systems

$$\mathscr{T}_{\Omega^2\cdot\omega}, \mathscr{T}_{\Omega^2\cdot\omega+\Omega}, \dots, \mathscr{T}_{\Omega^2\cdot\omega+\Omega\cdot\alpha}, \dots, \mathscr{T}_{\Omega^2\cdot(\omega+1)}, \dots, \mathscr{T}_{\Omega^2\cdot\alpha}, \dots$$

to obtain $\mathscr{T}_{<\Omega^3}$, an interpretation of the predicate T_2 in HST^1_2. The ability to recognise each theory $\mathscr{T}_{\Omega^2\cdot\alpha+\Omega\cdot\gamma}$ as closed under conec_1 and conec_0, however, is essential for the interpretation of T_2-Del in HST^1_2. As already argued, the set $\mathscr{T}_{<\Omega^2\cdot\alpha+\Omega\cdot\gamma}$ provides an interpretation of T_0 in $\mathscr{T}_{\Omega^2\cdot\alpha+\Omega\cdot\gamma}$; but unless γ is a limit ordinal, this need not be closed under nec_0, so cannot interpret the predicate T_1. The answer is to interpret T_1 in $\mathscr{T}_{\Omega^2\cdot\alpha+\Omega\cdot\gamma}$ as the set $\mathscr{T}_{<\Omega^2\cdot\alpha}$ for *every* $\gamma < \Omega$. Only when we pass to $\mathscr{T}_{\Omega^2\cdot(\alpha+1)}$ do we alter the interpretation of T_1 (in this case it is changed to the set $\mathscr{T}_{<\Omega^2\cdot(\alpha+1)}$). It is for exactly this reason that the rule $\mathrm{uni}_\eta$was restricted so as to apply to $\mathscr{T}_\rho$ only if $\rho|_\eta$ is not an Ω-limit; the set $\mathscr{T}_{<\Omega^2\cdot\alpha}$ will not be closed under ω-logic if α is a limit ordinal.

Once one has constructed $\mathscr{T}_{<\Omega^3}$ and verified that it is closed under T_n-Elim for $n = 0, 1, 2$, one would then embark on the construction of a further sequence of systems

$$\mathscr{T}_{\Omega^3}, \dots, \mathscr{T}_{\Omega^3+\Omega\cdot\alpha}, \dots, \mathscr{T}_{\Omega^3+\Omega^2}, \dots, \mathscr{T}_{\Omega^3+\Omega^2\cdot 2}, \dots, \mathscr{T}_{\Omega^3+\Omega^2\cdot\alpha}, \dots, \mathscr{T}_{\Omega^3\cdot 2}, \dots$$

and subsequently $\mathscr{T}_{<\Omega^3\cdot\omega}$, a theory into which HST_2 embeds. In general, we expect HST_β to embed into $\mathscr{T}_{\Omega^{\beta+1}\cdot\omega}$ for each β.

The next lemma deals with the task of determining the theory $\mathscr{T}_\rho$ is closed under $\mathrm{conec}_\eta$for every $\eta < \kappa$. Before that, however, we require a result on the behaviour of Ω-limits.

lem:ordOlimit **Proposition 4.4.** *If ρ is not an Ω-limit and $\sigma_n < \rho$ for every $n < \omega$,*

$$\sup_{n<\omega} \sigma_n < \rho.$$

Proof. Suppose ρ is not an Ω-limit and $\sigma_n < \rho$ for every $n < \omega$. Then $\rho > 0$ and there are ordinals ρ_0, α_0 such that $\rho = \rho_0 + \Omega^{\alpha_0} \cdot \Omega$. This means we can associate an ordinal $\delta_n < \Omega$ to each $n < \omega$ so that $\sigma_n < \rho_0 + \Omega^{\alpha_0} \cdot \delta_n$. The set $\{\delta_n : n < \omega\}$ is a countable set of countable ordinals, and hence is bounded in Ω, whence

$$\begin{aligned}
\sup_n \sigma_n &\leq \sup_n\{\rho_0 + \Omega^{\alpha_0} \cdot \delta_n\} \\
&\leq \rho_0 + \Omega^{\alpha_0} \cdot (\sup_n \delta_n) \\
&< \rho_0 + \Omega^{\alpha_0} \cdot \Omega \\
&= \rho.
\end{aligned}$$

□

A sequent Γ is called *T-positive* if all occurrences of a predicate T_η in Γ for any $\eta < \kappa$ are positive. Define, for each ordinal ρ, an $\mathscr{L}_\kappa$-structure $\mathfrak{M}_\rho$ according to the following criterion.

$$\mathfrak{M}_\rho \models \mathsf{T}_\eta s \text{ iff } s^{\mathbb{N}} \in \mathscr{T}_{<\rho|_\eta}.$$

thm:FFTElim **Theorem 4.5** (T-Elimination theorem)**.** *Suppose $\rho \in \mathbb{O}$.*

1. *For every T-positive sequent Γ, $\vdash_\rho \Gamma$ implies $\mathfrak{M}_\rho \models \bigvee \Gamma$;*
2. *For any $\eta < \kappa$, $\vdash_\rho \mathsf{T}_\eta s$ implies there is a sentence A with $s^{\mathbb{N}} = \ulcorner A \urcorner$ and $\vdash_\rho A$;*

Proof. We proceed by transfinite induction on ρ. For (i), one has a *subsidiary* induction on the height of the derivation. The base case is easy to deal with. For the induction step we argue according to the last rule applied in the derivation $\vdash_\rho \Gamma$. Whichever rule was applied, the sequent(s) in the premise must also be T-positive and we may apply the subsidiary induction hypothesis to them.

If the last rule was one of the arithmetical rules, that is, $(\vee_i)$, $(\wedge)$, (ω) or $(\exists)$, $\mathfrak{M}_\rho \models \bigvee\Gamma$ is an immediate consequence of the subsidiary induction hypothesis, and in the case of the weakening rule, $\mathfrak{M}_\rho \models \bigvee\Gamma$ follows from the fact that Γ is T-positive. If the last applied rule was nec_η, $\mathsf{T}_\eta(\ulcorner A\urcorner)$ is contained in Γ and $\vdash_\sigma A$ for some $\sigma < \rho|_\eta$, so $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner A\urcorner)$. For the remaining rules, the subsidiary induction hypothesis implies $\mathfrak{M}_\rho \models \bigvee\Gamma \vee (A_0 \wedge A_1)$ for some suitable choice of A_0, A_1. Of course, if $\mathfrak{M}_\rho \models \bigvee\Gamma$ we are done, so we may assume $\mathfrak{M}_\rho \models A_0 \wedge A_1$.

imp_η. If the last rule applied was imp_η, we may assume A_0 is $\mathsf{T}_\eta(s_0)$ and A_1 is $\mathsf{T}_\eta(s_0 \dot{\to} s_1)$, while Γ contains $\mathsf{T}_\eta(s_1)$. By the above, we may assume $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s_0) \wedge \mathsf{T}_\eta(s_0 \dot{\to} s_1)$. Thus, $s_0^{\mathbb{N}}$ and $s_1^{\mathbb{N}}$ are Gödel numbers of $\mathscr{L}_\kappa$-sentences, say B_0 and B_1 respectively, and there is some $\sigma < \rho|_\eta$ so that $\vdash_\sigma B_0$ and $\vdash_\sigma \neg B_0, B_1$. Admissibility of the cut rule (lemma 4.3) yields $\vdash_\sigma B_1$, and hence $\mathfrak{M}_\rho \models \mathsf{T}_\eta(s_1)$.

del_η. In the case the last applied rule is del_η, we may identify A_0 as $\mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi s\urcorner)$ for some $\xi < \kappa$ and term s; moreover, $\mathsf{T}_\eta s$ is contained in Γ. $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi s\urcorner)$ implies $\vdash_\sigma \mathsf{T}_\xi s$ for some $\sigma < \rho|_\eta$. Since $\sigma < \rho$, the *main* induction hypothesis may be applied, whence $s^{\mathbb{N}} = \ulcorner A\urcorner$ for some A and $\vdash_\sigma A$. Thus $\mathfrak{M}_\rho \models \mathsf{T}_\eta s$ and $\mathfrak{M}_\rho \models \bigvee\Gamma$.

rep_η. Here we have $\mathfrak{M}_\rho \models \mathsf{T}_\eta s$ and $\mathsf{T}_\eta(\ulcorner \mathsf{T}_\xi s\urcorner)$ is in Γ for some $\xi < \eta$. By definition this implies $s^{\mathbb{N}} = \ulcorner A\urcorner$ for some sentence A and $\vdash_\sigma A$ for some $\sigma < \rho|_\eta$, whence $\mathscr{T}_{\sigma+\Omega^{\xi+1}} \vdash \mathsf{T}_\xi s$ is derivable. But since $\xi < \eta$ and $\sigma < \rho|_\eta$, we have $\sigma + \Omega^{\xi+1} < \rho|_\eta$, and so $\mathfrak{M}_\rho \models \bigvee\Gamma$.

uni_η. The assumption is that $\mathfrak{M}_\rho \models \forall x\, \mathsf{T}_\eta(\ulcorner A(\dot{x})\urcorner)$. This entails the existence of, for every $n < \omega$, an ordinal $\sigma_n < \rho|_\eta$ such that $\mathscr{T}_{\sigma_n} \vdash A(\bar{n})$. Weakening and the ω-rule yields $\vdash_\sigma \forall x\, A(x)$, where $\sigma = \sup_n \sigma_n$, but one need not in general have $\sigma < \rho|_\eta$.[4] Due to the restriction on applications of uni_η, however, $\rho|_\eta$ is not an Ω-limit, thus by proposition 4.4, $\sigma < \rho|_\eta$ and so $\mathfrak{M}_\rho \models \mathsf{T}_\eta(\ulcorner \forall x A(x)\urcorner)$, whence $\mathfrak{M}_\rho \models \bigvee\Gamma$.

This completes the proof of (i).

(ii) is now a consequence of (i). If $\vdash_\rho \mathsf{T}_\eta s$, (i) implies $\mathfrak{M}_\rho \models \mathsf{T}_\eta s$, whence $s^{\mathbb{N}} = \ulcorner A\urcorner$ for some $\mathscr{L}_\kappa$-sentence A and $\vdash_\sigma A$ for some $\sigma < \rho|_\eta$. By weakening, $\vdash_\rho A$, as desired.

Observe that in the case of every rule of inference in the system $\mathscr{T}_\rho$, T-positive premises yield T-positive consequents. Therefore $\mathscr{T}_\eta \vdash \Gamma$ implies $\bigvee\Gamma$ is satisfied in the *everything is true* $\mathscr{L}_\kappa$ structure, so $\vdash_\rho \neg\mathsf{T}_\eta s$ is impossible and (iii) holds vacuously. □

prop:FFembed **Proposition 4.6.** *Let A be any axiom of* HST_β. *Then* $\mathscr{T}_{\Omega^{\beta+1}} \vdash A$.

Proof. One can derive each of the axioms via the corresponding rule and $\mathrm{Ax.2}_\eta$, as in **??**. In the case of uni_ηnote $\Omega^{\beta+1}|_\eta$ is not an Ω-limit for any $\eta \leq \beta$. □

thm:FFconsis **Theorem 4.7.** *The theory* HST_β *is consistent for every* $\beta < \kappa$.

[4]For example, suppose $\rho|_\eta = \rho_0 + \Omega^\xi$ and ξ is a limit ordinal. If $\sigma_n = \rho_0 + \Omega^{\xi_n}$, where $\xi = \sup_n \xi_n$ and $\xi_n < \xi$ for every $n < \omega$, one has $\sigma_n < \rho|_\eta$, but $\sup_n \sigma_n = \rho|_\eta$.

Proof. Lemma 4.3, theorem 4.5 and the previous proposition provide the means to deduce, by induction on n, that HST^n_β embeds into $\mathscr{T}_{\Omega^{\beta+1}\cdot n}$. Thus every sentential theorem of HST_β is contained in $\mathscr{T}_{<\Omega^\beta\cdot\omega}$. However, clearly the empty sequent is not derivable in $\mathscr{T}_\rho$ for any ρ, so HST_β must be consistent. □

5 An ordinal notation system for impredicative theories

ec:moreordinals

To carry out an ordinal analysis of HST_β we require the current set of ordinal terms, OT, to be extended to cover a larger segment of the ordinals. We will make use of an ordinal notation system for the Bachmann-Howard ordinal introduced by Rathjen and Weiermann [5]. This ordinal has proved significant in the analysis of certain impredicative systems such as the theory of inductive definitions, ID_1 [3]. It will turn out that the theories HST_β are substantially weaker than ID_1, but this notation system is still a natural one to consider. The key to generating notations for characteristic ordinals beyond Γ_0 is the use of constructions referencing certain 'external points'. In our case the 'external point' will be Ω, the first uncountable ordinal.

In order to generate unique representations for ordinals we will introduce a normal form for non-ε-ordinals, based on the Cantor normal form. We write $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ if $\alpha = \omega^\gamma + \delta$ and either $\delta = 0$ and $\gamma < \alpha$, or $\delta = \omega^{\delta_1} + \cdots + \omega^{\delta_k}$, $\gamma \geq \delta_1 \geq \ldots \geq \delta_k$ and $k \geq 1$. Let $\varepsilon_{\Omega+1}$ be the first ε-ordinal larger than Ω. For each $\alpha < \varepsilon_{\Omega+1}$ we denote by α^* the largest ε-ordinal below Ω used in the normal form presentation for α; that is,

1. $0^* = \Omega^* = 0$,
2. $\alpha^* = \alpha$, if $\alpha < \Omega$ is an ε-ordinal,
3. $\alpha^* = \max\{\gamma^*, \delta^*\}$, if $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$.

[Elaborate that the following is constructive.]

Define sets of ordinals $C_k(\alpha, \beta)$, and a function $\vartheta : \mathbb{O} \to \Omega$ by transfinite recursion on $\alpha \in \mathbb{O}$ as follows.

(C1) $\{0, \Omega\} \cup \beta \subseteq C_k(\alpha, \beta)$,

(C2) $\gamma, \delta \in C_k(\alpha, \beta)$ and $\xi =_{\mathrm{NF}} \omega^\gamma + \delta$ implies $\xi \in C_{k+1}(\alpha, \beta)$,

(C3) $\xi \in C_k(\alpha, \beta)$ and $\xi < \alpha$ implies $\vartheta\xi \in C_{k+1}(\alpha, \beta)$,

(C4) $C(\alpha, \beta) = \bigcup_{k<\omega} C_k(\alpha, \beta)$,

(ϑ1) $\vartheta\alpha = \min\{\xi < \Omega : C(\alpha, \xi) \cap \Omega \subseteq \xi \wedge \alpha \in C(\alpha, \xi)\}$.

The next two propositions shed some light on the role the function ϑ plays in generating intitial segments of $\mathbb{O}$.

prop:theta1 **Proposition 5.1.** *$\vartheta\alpha$ is defined for every $\alpha < \varepsilon_{\Omega+1}$.*

Proof. Let $\gamma_0 = \alpha^* + 1$. By rules (C1) and (C2) we may deduce $\alpha \in C(\alpha, \gamma_0)$. Suppose $\gamma_k < \Omega$ has been defined. As $C(\alpha, \gamma_k)$ has a countable definition, it contains only countably many elements; thus $C(\alpha, \gamma_k) \cap \Omega$ is bounded in Ω. Let $\gamma_{k+1} < \Omega$ be such that $C(\alpha, \gamma_0) \cap \Omega \subseteq \gamma_{k+1}$

and define $\gamma = \sup_{k<\omega} \gamma_k$. Since $\{\gamma_k : k < \omega\}$ is a countable set of countable ordinals, it too must be bounded in Ω, so $\gamma < \Omega$. Since $\alpha < \gamma_0 \leq \gamma$, also $\alpha^* \in C(\alpha, \gamma)$. Finally,

$$C(\alpha, \gamma) \subseteq \bigcup_{n<\omega} C(\alpha, \gamma_n),$$

so $C(\alpha, \gamma) \cap \Omega \subseteq \gamma$ and $\vartheta\alpha \leq \gamma$ by (ϑ1). □

The argument in the proof above provides a means to approximate the ordinal $\vartheta\alpha$ from below. Define $\gamma_0 = \alpha^* + 1$ and $\gamma_{m+1} = \min\{\xi < \Omega : C(\alpha, \gamma_m) \cap \Omega \subseteq \xi\}$; then $\vartheta\alpha \leq \sup_m \gamma_m$. By (C2) it is clear that each γ_m is a limit ordinal, whence we may deduce

$$[(\forall\delta \in C(\alpha, \gamma_m) \cap \Omega)\forall\xi < \delta\, F(\xi)] \rightarrow \forall\delta < \gamma_{m+1}\, F(\delta)$$

for every formula F.

The function ϑ works by "collapsing" ordinals below $\varepsilon_{\Omega+1}$ into countable ordinals, thus allowing one to represent ordinals beyond Γ_0. Moreover, the condition "$\alpha \in C(\alpha, \xi)$" in ($\vartheta$1) ensures the function $\alpha \mapsto \vartheta\alpha$ is strictly increasing on Ω. Thus, unlike the Veblen functions φ_α, one never has $\vartheta\beta = \vartheta\vartheta\beta$. In the following proposition we show ϑ is in fact injective on $\varepsilon_{\Omega+1}$ and show that relation '$\alpha < \beta$' may be decided purely on the normal form presentation for α and β.

rop:thetaitems1 **Proposition 5.2.** *For all ordinals α, β and γ the following holds.*

1. $\alpha \in C(\alpha, \vartheta\alpha)$,
2. $\vartheta\alpha = C(\alpha, \vartheta\alpha) \cap \Omega$ *and* $\vartheta\alpha \notin C(\alpha, \vartheta\alpha)$,
3. $\vartheta\alpha$ *is an* ε*-ordinal,*
4. $\gamma \in C(\alpha, \beta)$ *if and only if* $\gamma^* \in C(\alpha, \beta)$,
5. $\alpha < \vartheta\beta$ *if and only if* $\alpha < \Omega$ *and* $\alpha^* < \vartheta\beta$,
6. $\alpha^* < \vartheta\alpha$,
7. $\vartheta\alpha = \vartheta\beta$ *if and only if* $\alpha = \beta$,
8. itm:theta-comp $\vartheta\alpha < \vartheta\beta$ *if and only if* $(\alpha < \beta \wedge \alpha^* < \vartheta\beta) \vee (\beta < \alpha \wedge \vartheta\alpha \leq \beta^*)$.
9. itm:theta-mono *If* $\alpha <^* \beta$ *then* $\vartheta\alpha < \vartheta\beta$. *Useful: switch with #8*

Proof. Update numbering Proposition 5.1 ensures $\vartheta\alpha$ is defined for every $\alpha < \varepsilon_{\Omega+1}$, so (i) is a direct consequence of (ϑ1). By (C1), $\vartheta\alpha \subsetneq C(\alpha, \vartheta\alpha)$ whence (ii) also follows from (ϑ1). (ii) then implies (iii) since, as a result of (C2), $\gamma < \vartheta\alpha$ only if $\omega^\gamma < \vartheta\alpha$.

(iv). Suppose $\gamma \in C_k(\alpha, \beta)$. If $k = 0$, $\gamma^* \in C(\alpha, \beta)$ is immediate by (C1), so suppose $k > 0$. We show $\gamma^* \in C(\alpha, \beta)$ by examining the normal form of γ. If γ is either 0 or Ω, $\gamma^* = 0 \in C(\alpha, \beta)$ by (C1), and if γ is an ε-ordinal, $\gamma^* = \gamma$ and we are done. Otherwise $\gamma =_{\mathrm{NF}} \omega^\xi + \delta$ and $\delta, \xi \in C_{k-1}(\alpha, \beta)$. The induction hypothesis implies $\delta^*, \xi^* \in C(\alpha, \beta)$, whence $\gamma^* \in C(\alpha, \beta)$. The converse direction holds by a similar argument.

(v) is an immediate consequence of (iv) and (ii); (vi) holds on account of (i), (ii) and (iv). To show (vii) suppose $\vartheta\alpha = \vartheta\beta$ but $\alpha < \beta$. Then $C(\alpha, \vartheta\alpha) \subseteq C(\beta, \vartheta\beta)$, so $\alpha \in C(\beta, \vartheta\beta)$ by (i), whence $\vartheta\alpha \in C(\beta, \vartheta\beta) \cap \Omega$ by (C3). Thus $\vartheta\beta \in C(\beta, \vartheta\beta) \cap \Omega$ contradicting (ii).

(viii). Assume $\alpha < \beta$. By (vi), $\vartheta\alpha < \vartheta\beta$ implies $\alpha^* < \vartheta\beta$. Also $\alpha^* < \vartheta\beta$ implies $\alpha^* \in C(\beta, \vartheta\beta)$, whence $\vartheta\alpha \in C(\beta, \vartheta\beta) \cap \Omega$ and so $\vartheta\alpha < \vartheta\beta$ by (ii). Thus

$$\alpha < \beta \rightarrow (\vartheta\alpha < \vartheta\beta \leftrightarrow \alpha^* < \vartheta\beta). \tag{12}$$

{eqn:theta1.1}

Now suppose $\beta < \alpha$. By the same argument we obtain

$$\beta < \alpha \rightarrow (\vartheta\beta < \vartheta\alpha \leftrightarrow \beta^* < \vartheta\alpha),$$

and so, by (vii),

$$\beta < \alpha \rightarrow (\vartheta\alpha < \vartheta\beta \leftrightarrow \vartheta\alpha \leq \beta^*). \tag{13}$$

{eqn:theta1.2}

Combining (13) and (12) gives (viii). □

We can now proceed with defining a primitive recursive set of ordinal terms for use in the later analysis of HST_β.

Definition 5.3. Define a subset $\mathrm{OT}_\Omega \subseteq \mathbb{N}$, an encoding τ of ordinals into OT_Ω and a rank function $|.|$ on ordinals by recursion according to the following rules.

1. $\tau(0) = 0 \in \mathrm{OT}_\Omega$, $\tau(\Omega) = \langle 0, 1 \rangle \in \mathrm{OT}_\Omega$, and $|0| = |\Omega| = 0$,
2. If $\alpha = \vartheta\alpha_0$ and $\tau(\alpha_0) \in \mathrm{OT}_\Omega$, $\tau(\alpha) = \langle 1, \tau(\alpha_0) \rangle \in \mathrm{OT}_\Omega$ and $|\alpha| = |\alpha_0| + 1$,
3. If $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ and $\tau(\gamma), \tau(\delta) \in \mathrm{OT}_\Omega$, $\tau(\alpha) = \langle 2, \tau(\gamma), \tau(\delta) \rangle \in \mathrm{OT}_\Omega$ and $|\alpha| = \max\{|\gamma|, |\delta|\} + 1$.

It should be noted that the definition of $x \in \mathrm{OT}_\Omega$ and $|\alpha|$ are primitive recursive.

We now want to define an ordering $<_\vartheta$ on OT_Ω such that $\tau(\alpha) <_\vartheta \tau(\beta)$ if and only if $\alpha < \beta$. Conditions (iii) and (vii) of proposition 5.2 ensures every ordinal built up from the constants $0, \Omega$ and functions $\alpha, \beta \mapsto \omega^\alpha + \beta$ and $\alpha \mapsto \vartheta\alpha$ has a unique representation. We may therefore dispense with the function τ and identify members of OT_Ω with the ordinals they represent, as was the case with OT.

Define the relation $\alpha <_\vartheta \beta$ on OT_Ω by recursion on the value of $|\alpha| + |\beta|$. The conditions involved in comparing two ordinals $\vartheta\xi_0$ and $\vartheta\xi_1$ will be taken from (viii) of proposition 5.2. Let $\alpha <_\vartheta \beta$ if and only if one of the following conditions hold.

1. $\alpha = 0$ and $\beta \neq 0$;
2. $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ and either:
   a) $\beta = \Omega$ and $\gamma <_\vartheta \beta$,
   b) $\beta =_{\mathrm{NF}} \omega^{\gamma_0} + \delta_0$ and $\gamma <_\vartheta \gamma_0$, or $\gamma = \gamma_0 \wedge \delta <_\vartheta \delta_0$, or
   c) $\beta = \vartheta\xi$ and $\gamma <_\vartheta \beta$;
3. $\alpha = \vartheta\xi$ and either:
   a) $\beta = \Omega$,

b) $\beta =_{\mathrm{NF}} \omega^{\gamma} + \delta$ and $\alpha \leq_{\vartheta} \gamma$,[5] or

c) $\beta = \vartheta\eta$ and either, $\xi <_{\vartheta} \eta \wedge \xi^* <_{\vartheta} \beta$, or $\eta <_{\vartheta} \xi \wedge \alpha \leq_{\vartheta} \eta^*$.

($\gamma \leq_{\vartheta} \delta$ abbreviates $\gamma <_{\vartheta} \delta$ or $\gamma = \delta$.)

Since the function $\alpha \mapsto \alpha^*$ is primitive recursive, the relation $<_{\vartheta}$ is also primitive recursive.

Before we proceed with the analysis of HST_{β}, we will show how the ordinals $\alpha \geq \Omega$ in OT_{Ω} enable the generation of the φ_{α} functions for $\alpha < \Gamma_0$ and that OT_{Ω} properly extends OT.

Let $\Omega \cdot 0 = 0$ and if $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ and $\alpha_0 \geq \cdots \geq \alpha_n$, define

$$\Omega \cdot \alpha = \omega^{\Omega+\alpha_0} + \cdots + \omega^{\Omega+\alpha_n},$$
$$\Omega^{\beta} \cdot \alpha = \omega^{\Omega\cdot\beta+\alpha_0} + \cdots + \omega^{\Omega\cdot\beta+\alpha_n}.$$

prop:thetaord **Proposition 5.4.** $\mathrm{OT}_{\Omega} \cap \Omega$ *forms an initial segment of the ordinals.*

Proof. Suppose $\delta \in \mathrm{OT}_{\Omega} \cap \Omega$. We prove $\alpha \in \mathrm{OT}_{\Omega}$ for every $\alpha < \delta$ by transfinite induction on α. Suppose

$$\forall \xi < \alpha (\xi \in \mathrm{OT}_{\Omega}). \tag{14}$$ {eqn:ordOT1}

If $\alpha = 0$ or $\alpha =_{\mathrm{NF}} \omega^{\gamma} + \delta$, (14), (C1) and (C2) imply $\alpha \in \mathrm{OT}_{\Omega}$, so assume α is an ε-ordinal. In search of a contradiction, assume further that $\alpha \notin \mathrm{OT}_{\Omega}$. We prove $\alpha = \vartheta\xi$ for some $\xi \in \mathrm{OT}_{\Omega}$. Pick β to be the least ordinal in $\{\gamma \leq \delta : \gamma \in \mathrm{OT}_{\Omega} \wedge \alpha < \gamma\}$. Then β must be an ε-ordinal as otherwise $\beta = \omega^{\gamma_0} + \gamma_1$ for some $\gamma_0, \gamma_1 < \beta$, whence $\alpha \leq \max\{\gamma, \delta\} \in \mathrm{OT}_{\Omega} \cap \Omega$. Since $\beta \in \mathrm{OT}_{\Omega}$, $\beta = \vartheta\xi$ for some $\xi \in \mathrm{OT}_{\Omega}$. Now, as $\xi^* < \vartheta\xi$ we have $\xi^* < \alpha$. Thus $\xi \in C(\xi, \alpha)$.

Moreover, we claim $C(\xi, \alpha) \cap \Omega \subseteq \alpha$. The argument proceeds by induction on the definition of $\eta \in C(\xi, \alpha) \cap \Omega$. In case $\eta = 0$ or $\eta =_{\mathrm{NF}} \omega^{\gamma} + \delta$, it is immediate that $\eta < \alpha$, so suppose $\eta = \vartheta\zeta$. Then $\zeta < \xi$ and, since

$$\eta \in C(\xi, \alpha) \cap \Omega \subseteq C(\xi, \vartheta\xi) \cap \Omega = \vartheta\xi,$$

$\eta < \beta$. By (14) and the fact $\xi \in \mathrm{OT}_{\Omega}$, we obtain $C(\xi, \alpha) \subseteq \mathrm{OT}_{\Omega}$, so $\eta = \vartheta\zeta \in \mathrm{OT}_{\Omega}$, whence $\eta < \alpha$ by the choice of β.

Thus we have shown $\xi \in C(\xi, \alpha)$ and $C(\xi, \alpha) \cap \Omega \subseteq \alpha$, whence $\vartheta\xi \leq \alpha$ by (ϑ1), and $\alpha = \beta \in \mathrm{OT}_{\Omega}$. □

rop:thetaitems2 **Proposition 5.5.**

1. *$\vartheta\Omega$ is the least ordinal closed under the function $\gamma \mapsto \vartheta\gamma$.*

2. *$\vartheta\Omega^2$ is the least ordinal closed under the function $\gamma \mapsto \vartheta(\Omega \cdot \gamma)$.*

3. *$\vartheta\Omega^3$ is the least ordinal closed under the function $\gamma \mapsto \vartheta(\Omega^2 \cdot \gamma)$.*

[5]Recall that if γ is an ε-ordinal, $\beta =_{\mathrm{NF}} \omega^{\gamma} + \delta$ only if $\delta > 0$, whence $\vartheta\xi <_{\vartheta} \beta$ if $\vartheta\xi = \gamma$.

Proof. (i). Let $\gamma_0 = 0$ and $\gamma_{m+1} = \vartheta\gamma_m$; we claim

$$\sup_{n<\omega} \gamma_n = \vartheta\Omega,$$

whence it is clear (i) holds. $\gamma_0 < \vartheta\Omega$ holds trivially and, if $\gamma_m < \vartheta\Omega$, we have $(\gamma_m)^* < \vartheta\Omega$ and so $\gamma_{m+1} = \vartheta\gamma_m < \vartheta\Omega$ by (viii) of proposition 5.2; thus $\sup_n \gamma_n \leq \vartheta\Omega$.

To show $\vartheta\Omega \leq \sup_n \gamma_n$ we prove $\beta < \vartheta\Omega$ implies $\beta < \gamma_m$ for some m by induction on the rank of β. Suppose $\beta < \vartheta\Omega$. Both $\vartheta\Omega$ and $\sup_n \gamma_n$ are ε-ordinals, so the case $\beta =_{\mathrm{NF}} \omega^{\beta_0} + \beta_1$, holds by the induction hypothesis. If $\beta = \vartheta\xi$ we deduce $\xi < \Omega$ and $\xi^* < \vartheta\Omega$, since $\beta < \vartheta\Omega$ and $\Omega^* = 0$. ξ^* has rank strictly less than β, so the induction hypothesis yields an $m < \omega$ such that $\xi^* < \gamma_m$. Proposition 5.2 (v) then entails $\xi < \gamma_m$. Moreover, since $\gamma_m = (\gamma_m)^* < \vartheta\gamma_m$, by proposition 5.2 (vi), $\gamma_m < \gamma_{m+1}$, so $\xi^* < \vartheta\gamma_m$. Therefore $\beta = \vartheta\xi < \vartheta\gamma_m = \gamma_{m+1}$.

(ii). Let $\gamma_0 = 0$ and $\gamma_{m+1} = \vartheta(\Omega \cdot \gamma_m)$; we claim

$$\sup_{n<\omega} \gamma_n = \vartheta\Omega^2,$$

whence (ii) holds. Let $\alpha = \sup_n \gamma_n$. Naturally, $\gamma_0 < \vartheta\Omega^2$, and if $\gamma_m < \vartheta\Omega^2$,

$$(\Omega \cdot \gamma_m)^* = \gamma_m^* < \vartheta\Omega^2,$$

so $\gamma_{m+1} < \vartheta\Omega^2$ by proposition 5.2 (viii). Thus $\alpha \leq \vartheta\Omega^2$.

To show the converse, we prove $\beta < \vartheta\Omega^2$ implies $\beta < \alpha$ by induction on the rank of β. Suppose $\beta < \vartheta\Omega^2$ and $\beta = \vartheta\xi$ for some ξ. As $(\Omega^2)^* = 0$ and $\beta < \vartheta\Omega^2$, proposition 5.2 (viii) implies $\xi < \Omega^2$ and $\xi^* < \vartheta\Omega^2$, whence the induction hypothesis implies $\xi^* < \gamma_m$ for some m. Since $\xi < \Omega^2$, there are $\delta_0, \delta_1 < \Omega$ such that $\xi = \Omega \cdot \delta_0 + \delta_1$, whence $\delta_0^* \leq \xi^* < \gamma_m$. So $\xi < \Omega \cdot \gamma_m$ and $\beta < \gamma_{m+1}$ by proposition 5.2 (viii).

(iii) involves a near identical argument as (ii). Pick $\gamma_0 = 0$ and $\gamma_{m+1} = \vartheta(\Omega^2 \cdot \gamma_m)$. That $\sup_n \gamma_n \leq \vartheta\Omega^3$ is easily established using proposition 5.2. For the converse direction, $\vartheta\Omega^3 \leq \sup_n \gamma_n$, we suppose $\beta < \vartheta\Omega^3$ and seek to determine $\beta < \gamma_m$ for some m. If $\beta = \vartheta\xi < \vartheta\Omega^3$, we may assume $\xi^* < \gamma_m$ for some m, whence $\xi < \Omega^2 \cdot \gamma_m$, and so $\beta < \gamma_{m+1}$. □

Proposition 5.5 allows us to identify some characteristic ordinals in terms of both the Veblen and ϑ functions.

rop:thetaitems3 **Corollary 5.6.**

1. $\alpha < \varphi 20$ *implies* $\varepsilon_\alpha = \vartheta\alpha$.
2. $\vartheta\Omega = \varphi 20$.
3. $\vartheta\Omega^2 = \Gamma_0$.
4. $\vartheta(\Omega^2 + \Omega)$ *is the least fixed point of the function* $\xi \mapsto \Gamma_\xi$, *which enumerates the class* $\{\xi : \xi = \varphi\xi 0\}$.

Proof. (i) is argued by transfinite induction on $\alpha < \varphi 20$. Suppose $\vartheta\beta = \varepsilon_\beta$ for every $\beta < \alpha$. We will begin by showing a) $C(\alpha, \varepsilon_\alpha) \cap \Omega \subseteq \varepsilon_\alpha$, and b) $\alpha \in C(\alpha, \varepsilon_\alpha)$, allowing us to deduce $\vartheta\alpha \leq \varepsilon_\alpha$ by (ϑ1).

a) is shown by a further induction on the construction of $C(\alpha, \varepsilon_\alpha)$. It is trivial that $\beta \in C_0(\alpha, \varepsilon_\alpha) \cap \Omega$ entails $\beta < \varepsilon_\alpha$, and that ε_α is closed under applications of rule (C2). Moreover, the induction hypothesis implies that for $\beta < \alpha$, $\vartheta\beta < \varepsilon_\alpha$, thus (C3) is also dealt with, and $C(\alpha, \varepsilon_\alpha) \cap \Omega \subseteq \varepsilon_\alpha$.

b) uses the fact $\alpha < \varphi 20$, whence $\alpha < \varepsilon_\alpha$ and $\alpha \in C(\alpha, \varepsilon_\alpha)$ by (C1).

To see $\varepsilon_\alpha \leq \vartheta\alpha$, assume otherwise. Then $\vartheta\alpha = \varepsilon_\beta$ for some $\beta < \alpha$ by proposition 5.2 (iii). The induction hypothesis yields $\vartheta\alpha = \vartheta\beta$, contradicting $\beta < \alpha$.

(ii) is an immediate consequence of (i) and proposition 5.5.

(iii). The proof for (i) above can be extended to $\alpha > \varphi 20$, but then one can at best show $\varepsilon_\alpha \leq \vartheta\alpha \leq \varepsilon_{\alpha+1}$ for $\alpha < \Omega$.[6] One can then prove

$$\varphi 2\alpha \leq \vartheta(\Omega + \alpha) \leq \varphi 2(\alpha + 1)$$

for $\alpha < \Omega$ by transfinite induction on α, using the definition of ϑ. This can easily be extended to deduce, in general,

$$\varphi\alpha\beta \leq \vartheta(\Omega \cdot \alpha + \beta) \leq \varphi(\alpha + 1)(\beta + 1)$$

for $\alpha, \beta < \Gamma_0$, from which proposition 5.5 (ii) implies $\vartheta\Omega^2 = \Gamma_0$.

(iv). Let Δ_0 denote the least fixed point of the function $\xi \mapsto \Gamma_\xi$. Following from (iii) above, $\vartheta(\Omega^2 + \alpha) = \Gamma_\alpha$ for $\alpha < \Delta_0$. Since $\vartheta(\Omega^2 + \Omega)$ is the least ordinal closed under the function $\alpha \mapsto \vartheta(\Omega^2 + \alpha)$, we deduce $\vartheta(\Omega^2 + \Omega) = \Delta_0$. □

In this notation system, $\vartheta\Omega^3$ represents the *Ackermann ordinal*, $\vartheta\Omega^\Omega$ denotes the *Veblen ordinal* and $\vartheta\varepsilon_{\Omega+1}$ is the *Bachmann-Howard ordinal* where

$$\vartheta\varepsilon_{\Omega+1} = \sup\{\vartheta\Omega, \vartheta\Omega^\Omega, \vartheta\Omega^{\Omega^\Omega}, \dots\} = \sup_{\alpha \in \mathrm{OT}_\Omega} \vartheta\alpha.$$

Having established an ordinal notation system suitable for the analysis of the theories HST_β, we may now fix the language of HST_β. Since the proof-theoretic strength of each theory HST_β with $\beta < \vartheta\varepsilon_{\Omega+1}$ will not exceed $\vartheta\varepsilon_{\Omega+1}$, we may pick $\kappa = \vartheta\varepsilon_{\Omega+1}$ and suppose the theories HST_β are formulated in the language $\mathscr{L}_\kappa$.

We require a few further technical results about ordinals before we can proceed with the analysis. Suppose $\beta = \Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_n} \cdot \beta_n$ such that $\alpha_0 > \cdots > \alpha_n$ and $\beta_i < \Omega$ for each $i \leq n$. Recall from the previous section that $\beta|_\gamma$ denotes the ordinal $\Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_k} \cdot \beta_k$ where $k < n$ is the least such that $\alpha_k > \gamma \geq \alpha_{k+1}$, or $k = n$ if $\alpha_n > \gamma$.

The following observations are immediate consequences of the definition.

prop:morditems **Proposition 5.7.** *For all ordinals $\alpha, \beta < \varepsilon_{\Omega+1}$ and $\gamma, \delta < \Omega$,*

1. *$\gamma < \delta$ implies $\alpha|_\gamma \leq \alpha|_\delta$,*
2. *$\alpha < \beta$ implies $\alpha|_\gamma \leq \beta|_\gamma$.*
3. *$\delta \leq \gamma$ implies $(\alpha|_\gamma)|_\delta = \alpha|_\gamma$,*
4. *$\beta < \alpha|_\gamma$ if and only if $\beta + \Omega^{\gamma+1} \leq \alpha$,*
5. *$\beta < \alpha|_\gamma$ and $\delta \leq \gamma$ implies $\beta + \Omega^\delta < \alpha|_\gamma$.* item:morditem4

[6] $\vartheta\alpha = \varepsilon_{\alpha+1}$ for $\alpha = \varphi 20$ for example.

6 Lower bounds on the proof-theoretic ordinal

sec:Fblower

We will now establish lower bounds for the theories HST_β. This will be achieved by extending the well-ordering proofs used in our analysis of F (**??**) and S_3 (**??**). Recall $\kappa = \vartheta\varepsilon_{\Omega+1}$.

[Remark that case distinctions below are all decidable]

HST_0 is identical to F, for which an optimal lower bound was established in **??**. However, because of the change in ordinal notation system and the reflective nature of the theories HST_β it will be useful to provide a new proof of the result. For each $\xi < \kappa$ let $\mathsf{wo}_\xi(x)$ denote the formula

$$\forall\ulcorner A(x)\urcorner\forall y \prec x\, \mathsf{T}_\xi(\ulcorner \mathsf{TI}(\dot{y}, A)\urcorner).$$

Let $F_0(\rho)$ denote the formula $\mathsf{wo}_0(\rho^*) \wedge \forall\sigma < \rho[\mathsf{wo}_0(\sigma^*) \rightarrow \mathsf{wo}_0(\vartheta\sigma)]$ and $F_0^\rho(\alpha)$ denote $\alpha < \Omega \rightarrow F_0(\rho+\alpha)$. We begin with a technical lemma.

lem:F0tech **Lemma 6.1.** $\mathsf{HST}_0^1 \vdash \forall\rho[\mathsf{T}_0(\ulcorner F_0(\dot{\rho})\urcorner) \rightarrow \mathsf{wo}_0(\vartheta\rho)]$.

Proof. Argue within HST_0^1, and assume

$$\mathsf{T}_0(\ulcorner F_0(\dot{\rho})\urcorner). \tag{15}$$ {eqn:F0tech1}

Let $\gamma_0 = \rho^* + 1$ and $\gamma_{m+1} = C(\rho, \gamma_m) \cap \Omega$. Moreover, let $\mathsf{wo}_0^1(x)$ denote $\mathsf{T}_0(\ulcorner \mathsf{wo}_0(\dot{x})\urcorner)$. (15) implies $\mathsf{wo}_0^1(\gamma_0)$ and $\forall\sigma < \rho[\mathsf{wo}_0^1(\sigma^*) \rightarrow \mathsf{wo}_0^1(\vartheta\sigma))]$, whence it is easy to deduce $\forall m[\mathsf{wo}_0^1(\gamma_m) \rightarrow \mathsf{wo}_0^1(\gamma_{m+1})]$ and thus $\mathsf{wo}_0^1(\vartheta\rho)$. By del_0, $\mathsf{wo}_0(\vartheta\rho)$ holds. □

:F0wellordering **Lemma 6.2.** *For every $m < \omega$, $\mathsf{HST}_0^m \vdash F_0(\Omega\cdot\bar{m})$.*

Proof. $\mathsf{HST}_0^0 \vdash F_0(\bar{0})$ holds vacuously, so suppose $m = n+1 > 0$ and

$$\mathsf{HST}_0^n \vdash F_0(\Omega\cdot\bar{n}). \tag{16}$$ {eqn:4.1}

The first step is to establish $\mathsf{HST}_0^n \vdash \mathsf{Prog}F_0^{\Omega\cdot\bar{n}}$. Argue informally within HST_0^n, assuming $\forall\nu < \mu\, F_0^{\Omega\cdot\bar{n}}(\nu)$ for some μ, that is,

$$\forall\nu < \mu(\mathsf{wo}_0(\nu^*) \wedge \forall\sigma < \Omega\cdot\bar{n}+\nu[\mathsf{wo}_0(\sigma^*) \rightarrow \mathsf{wo}_0(\vartheta\sigma)]). \tag{17}$$ {eqn:4.3}

We want to show $\mathsf{wo}_0(\mu^*)$ and $\forall\sigma < \Omega\cdot\bar{n}+\mu[\mathsf{wo}_0(\sigma^*) \rightarrow \mathsf{wo}_0(\vartheta\sigma)]$. The former is obvious since the fact $\mathsf{wo}_0(\varepsilon_\alpha)$ is progressive in α is provable in HST_0^1 (*cf.* **??**). To prove the latter, assume $\mathsf{wo}_0(\sigma^*)$ for some $\sigma < \Omega\cdot\bar{n}+\mu$. If $\mu = 0$ or is a limit ordinal, $\mathsf{wo}_0(\vartheta\sigma)$ is immediate given (17). Otherwise $\mu = \nu+1$ for some ν, whence we may assume $\sigma = \Omega\cdot\bar{n}+\nu$. Let $\gamma_0 = \sigma^*+1$ and $\gamma_{m+1} = C(\sigma, \gamma_m) \cap \Omega$. Then $\gamma_m < \Omega$ for each m and

$$\vartheta\sigma \leq \sup_m \gamma_m$$

by proposition 5.1. $\mathsf{wo}_0(\gamma_0)$ is a consequence of $\mathsf{wo}_0(\sigma^*)$, so suppose

$$\mathsf{wo}_0(\gamma_m) \tag{18}$$ {eqn:4.4}

with the aim of showing $\mathsf{wo}_0(\gamma_{m+1})$ by induction on the recursive definition of $C(\sigma, \gamma_m)$. Assume $\mathsf{wo}_0(\alpha)$ holds for every $\alpha \in C_k(\sigma, \gamma_m) \cap \Omega$ and suppose $\beta \in C_{k+1}(\sigma, \gamma_m) \cap \Omega$.

(C1). $\beta \leq \gamma_m$, so $\mathsf{wo}_0(\beta)$ is a result of (18).

(C2). $\beta =_{\text{NF}} \omega^\delta + \eta$, and $\delta, \eta \in C_k(\sigma, \gamma_m)$. Since also $\delta, \eta < \Omega$ the induction hypothesis yields $\mathsf{wo}_0(\delta) \wedge \mathsf{wo}_0(\eta)$ and so $\mathsf{wo}_0(\beta)$.

(C3). $\beta =_{\text{NF}} \vartheta\xi$ and $\xi \in C_k(\sigma, \gamma_m) \cap \sigma$. Thus, $\xi^* \in C_k(\sigma, \gamma_m) \cap \Omega$ and therefore $\mathsf{wo}_0(\xi^*)$ by the induction hypothesis. If $\xi < \Omega \cdot n$, $\mathsf{wo}_0(\vartheta\xi)$ is a consequence of (16), otherwise $\Omega \cdot n \leq \xi < \sigma$ and $\mathsf{wo}_0(\vartheta\xi)$ is implied by (17).

Thus we may deduce $\forall \alpha < \gamma_{m+1}\, \mathsf{wo}_0(\alpha)$, hence $\mathsf{wo}_0(\gamma_{m+1})$, and so $\mathsf{wo}_0(\vartheta\sigma)$, concluding the proof of

$$\mathsf{HST}_0^n \vdash \mathsf{Prog} F_0^{\Omega \cdot \bar{n}}. \tag{19}$$ {eqn:4.2}

An application of nec_0 entails $\mathsf{HST}_0^m \vdash \mathsf{T}_0(\ulcorner \mathsf{Prog} F_0^{\Omega \cdot \bar{n}} \urcorner)$, so

$$\mathsf{HST}_0^m \vdash \forall \alpha[\mathsf{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F_0(\Omega \cdot \bar{n} + \dot{\alpha}) \urcorner)],$$

and hence, by lemma 6.1,

$$\mathsf{HST}_0^m \vdash \forall \alpha(\mathsf{wo}_0(\alpha) \to \mathsf{wo}_0 \vartheta(\Omega \cdot \bar{n} + \alpha)). \tag{20}$$ {eqn:4.6}

To obtain $\mathsf{HST}_0^m \vdash F_0(\Omega \cdot \bar{m})$ and complete the proof we argue within HST_0^m. Firstly, $\mathsf{wo}_0((\Omega \cdot \bar{m})^*)$ holds trivially as $(\Omega \cdot \bar{m})^* = 0$. Secondly, if $\sigma < \Omega \cdot \bar{m}$, we have either $\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\vartheta\sigma)$ by (16), or $\sigma = \Omega \cdot \bar{n} + \zeta$ for some $\zeta < \Omega$, whence $\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\zeta)$ and $\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\vartheta\sigma)$ results from (20). □

or:F0lowerbound **Corollary 6.3.** *$\|\mathsf{HST}_0^m\| \geq \vartheta(\Omega \cdot m)$ and $\|\mathsf{HST}_0\| \geq \vartheta(\Omega \cdot \omega)$.*

Proof. Let $\gamma_0 = 1$ and $\gamma_{k+1} = C(\Omega \cdot m, \gamma_k) \cap \Omega$. Then $\vartheta(\Omega \cdot m) \leq \sup_{k<\omega} \gamma_k$ and $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\gamma}_0)$ holds. Moreover, if $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\alpha})$ for every $\alpha < \gamma_k$ we may deduce $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\alpha})$ for every $\alpha < \gamma_{k+1}$ by induction on the definition of γ_{k+1} thus: suppose $\beta \in C_{k+1}(\Omega \cdot m, \gamma_k) \cap \Omega$ and $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\alpha})$ for every $\alpha \in C_k(\Omega \cdot m, \gamma_k) \cap \Omega$. If β was enumerated into $C_{k+1}(\Omega \cdot m, \gamma_k)$ by either (C1) or (C2), $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\beta})$ is easily obtained from the induction hypothesis. If, however, $\beta = \vartheta\xi$ for some $\xi \in C_k(\Omega \cdot m, \gamma_k) \cap \Omega \cdot m$, $\xi = \Omega \cdot n + \alpha$ for some $n < m$, $\alpha \in C_k(\Omega \cdot m, \gamma_k) \cap \Omega$ and $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\xi}^*)$ by the induction hypothesis, whence lemma 6.2 implies $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\beta})$.

Since $\vartheta(\Omega \cdot m) \leq \sup_{k<\omega} \gamma_k$, we obtain $\mathsf{HST}_0^m \vdash \mathsf{wo}_0(\bar{\alpha})$ for every $\alpha < \vartheta(\Omega \cdot m)$ and so $\mathsf{HST}_0^m \vdash \mathsf{TI}(<\vartheta(\Omega \cdot m))$ by an application of conec_0. □

We will now extend the well-ordering proof above to determine lower bounds on the strength of each theory HST_β. This will be done in stages, first for $\beta = 1$, then for arbitrary $\beta < \omega$ and finally for transfinite levels of the hierarchy. In doing so we will find ourselves migrating from the function $\alpha \mapsto \vartheta\alpha$ to the function $\alpha \mapsto \vartheta(\Omega \cdot \alpha)$, and eventually to functions $\alpha \mapsto \vartheta(\Omega^\beta \cdot \alpha)$.

Before proceeding directly with HST_1 we require a slightly more general form of lemma 6.2. As its proof makes no explicit use of the fact m is finite, nor any application of nec_0 in showing $\mathsf{Prog} F_0^{\Omega \cdot \bar{m}}$ given $F_0(\Omega \cdot \bar{m})$, we may readily deduce the following generalisation.

genwellordering **Proposition 6.4.** $\mathsf{HST}_0^1 \vdash \forall \rho[F_0(\rho) \to \mathsf{Prog} F_0^\rho]$.

Proof. Argue inside HST^1_0 and assume $F_0(\rho)$ and $\forall \nu < \mu \, F^{\rho}_0(\nu)$, that is,

$$\mathsf{wo}_0(\rho^*), \tag{21}$$
{eqn:F03.1}
$$\forall \sigma < \rho[\mathsf{wo}_0(\sigma^*) \rightarrow \mathsf{wo}_0 \vartheta\sigma], \tag{22}$$
{eqn:F03.2}
$$\forall \nu < \mu(\mathsf{wo}_0(\nu^*) \wedge \forall \sigma < \rho + \nu[\mathsf{wo}_0(\sigma^*) \rightarrow \mathsf{wo}_0 \vartheta\sigma]), \tag{23}$$
{eqn:F03.3}

for some $\mu < \Omega$; we want to prove $\mathsf{wo}_0(\mu^*)$ and $\forall \tau < \rho + \mu[\mathsf{wo}_0(\tau^*) \rightarrow \mathsf{wo}_0(\vartheta\tau)]$. The former holds immediately given (23) so assume

$$\mathsf{wo}_0(\tau^*), \tag{24}$$
{eqn:F03.4}

for some $\tau < \rho + \mu$. We will prove $\mathsf{wo}_0(\vartheta\tau)$.

Let $\gamma_0 = \tau^* + 1$ and $\gamma_{m+1} = C(\tau, \gamma_m) \cap \Omega$. That $\mathsf{wo}_0(\gamma_m)$ holds for each m will be established by induction on m. From (24) one has $\mathsf{wo}_0(\gamma_0)$. Assume $\mathsf{wo}_0(\gamma_m)$. In order to show $\mathsf{wo}_0(\gamma_{m+1})$, assume $\mathsf{wo}_0(\alpha)$ for every $\alpha \in C_k(\tau, \gamma_m) \cap \Omega$. Pick an arbitrary $\alpha_0 \in C_{k+1}(\tau, \gamma_m) \cap \Omega$. If α_0 was enumerated into the set by either (C1) or (C2), $\mathsf{wo}_0(\alpha_0)$ is immediate. Otherwise $\alpha_0 = \vartheta\xi$ for some $\xi \in C_k(\tau, \gamma_m) \cap \tau$ and, since $\xi^* \in C_k(\tau, \gamma_m) \cap \Omega$, we have $\mathsf{wo}_0(\xi^*)$. If $\xi < \rho$, (22) provides $\mathsf{wo}_0(\vartheta\xi)$. Otherwise $\tau = \rho + \nu$ for some $\nu < \mu$ and $\mathsf{wo}_0(\vartheta\xi)$ holds due to (23). Either way $\mathsf{wo}_0(\alpha_0)$, and so $\mathsf{wo}_0(\alpha)$ for every $\alpha < \gamma_{m+1}$, whence $\mathsf{wo}_0(\gamma_{m+1})$.

Since $\vartheta\tau \geq \sup_m \gamma_m$ we obtain $\mathsf{wo}_0(\vartheta\tau)$. □

Proposition 6.4 is plays a key role in the analysis of HST^m_0 and also HST^1_1. Lemma 6.1 entails

$$\begin{aligned} \mathsf{HST}^1_0 \vdash \mathsf{T}_0(\ulcorner \mathsf{Prog} F^{\dot\rho}_0 \urcorner) &\rightarrow \forall \alpha[\mathsf{wo}_0(\alpha) \rightarrow \mathsf{T}_0(\ulcorner F^{\dot\rho}_0(\dot\alpha)\urcorner)]). \\ &\rightarrow \forall \alpha[\mathsf{wo}_0(\alpha) \rightarrow \mathsf{wo}_0 \vartheta(\rho + \alpha)]), \end{aligned}$$

so $\mathsf{HST}^1_1 \vdash \forall \rho[\mathsf{T}_1(\ulcorner F_0(\dot\rho)\urcorner) \rightarrow \mathsf{T}_1(\ulcorner F_0(\dot\rho + \Omega)\urcorner)]$. This amounts to proving

$$\mathsf{HST}^1_1 \vdash \forall \rho[F_1(\rho) \rightarrow \mathsf{Prog} F^{\rho}_1] \tag{25}$$
{eqn:F11.1}

where $F_1(\rho)$ is the formula $\mathsf{T}_1(\ulcorner F_0(\rho)\urcorner)$ and $F^{\rho}_1(\alpha)$ denotes $\alpha < \Omega \rightarrow F_1(\rho + \Omega \cdot \alpha)$. (25) is sufficient to deduce a lower bound on the strength of the theory HST^1_1.

r:F11lowerbound **Corollary 6.5.** $\|\mathsf{HST}^1_1\| \geq \vartheta\Omega^2$.

Proof. Since $\mathsf{HST}_0 \vdash F_0(\bar{0})$, (25) implies

$$\mathsf{HST}^1_1 \vdash \mathsf{Prog} F^{\bar{0}}_1. \tag{26}$$
{eqn:F11lower1}

Let $\sigma_0 = 1$ and $\sigma_{m+1} = \vartheta(\Omega \cdot \sigma_m)$. By proposition 5.5 (ii), $\vartheta\Omega^2 = \sup_m \sigma_m$, so it suffices to show $\mathsf{HST}^1_1 \vdash \mathsf{wo}_1(\bar\sigma_m)$ for each m. This is trivial for $m = 0$; for $m = n + 1$ argue within HST^1_1 assuming $\mathsf{wo}_1(\bar\sigma_n)$. Then $\mathsf{wo}_1(\bar\sigma_n + 1)$ and so $\mathsf{T}_1(\ulcorner F_0(\Omega \cdot \bar\sigma_n)\urcorner)$ by an application of conec_1 and (26). Lemma 6.1 yields $\mathsf{T}_1(\ulcorner \mathsf{wo}_0 \vartheta(\Omega \cdot \bar\sigma_n)\urcorner)$ and so $\mathsf{wo}_1(\bar\sigma_m)$ holds. □

Within HST^2_1, the above proof may be replicated under a T_1 predicate, allowing one to reach ordinals beyond $\vartheta\Omega^2$, as the next proposition demonstrates.

:F1wellordering **Lemma 6.6.** *For each m, $\mathsf{HST}^{m+1}_1 \vdash F_1(\Omega^2 \cdot \bar{m})$ and $\mathsf{HST}^{m+1}_1 \vdash \mathsf{Prog} F^{\Omega^2 \cdot \bar{m}}_1$.*

Proof. $\mathsf{HST}_1^1 \vdash F_1(\bar{0})$ holds trivially, so suppose $m = n+1$ and $\mathsf{HST}_1^m \vdash F_1(\Omega^2 \cdot \bar{n})$. (25) yields $\mathsf{HST}_1^m \vdash \mathsf{Prog} F_1^{\Omega^2 \cdot \bar{n}}$, whence an application of nec_1 and T_1^--Rep implies

$$\mathsf{HST}_1^{m+1} \vdash \mathsf{T}_1(\ulcorner \forall \alpha[\mathsf{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F_1(\Omega^2 \cdot \bar{n} + \Omega \cdot \dot{\alpha})\urcorner)]\urcorner). \tag{27}$$ {eqn:F1well.1}

However, arguing within HST_0, from $F_0(\rho)$ one obtains $\mathsf{wo}_0(\vartheta\rho)$, so (27) entails

$$\mathsf{HST}_1^{m+1} \vdash \mathsf{T}_1(\ulcorner \forall \alpha[\mathsf{wo}_0(\alpha) \to \mathsf{wo}_0\vartheta(\Omega^2 \cdot \bar{n} + \Omega \cdot \dot{\alpha})]\urcorner),$$

and thus $\mathsf{HST}_1^{m+1} \vdash F_1(\Omega^2 \cdot \bar{m})$ as required. □

m:F1lowerbound **Theorem 6.7.** *Suppose $m < \omega$. Then every theorem of $\mathsf{PA} + \mathsf{TI}({<}\vartheta(\Omega^2 \cdot m))$ is derivable in HST_1^m. Moreover, every theorem of $\mathsf{PA} + \mathsf{TI}({<}\vartheta(\Omega^2 \cdot \omega))$ is derivable in HST_1.*

Proof. Since $\vartheta 0 = \varepsilon_0$ and HST_1^0 extends PA, the case $m = 0$ holds, so suppose $m = n+1 > 0$. If $\mathsf{HST}_1^m \vdash \mathsf{wo}_0(\bar{\alpha})$, lemma 6.6 implies

$$\mathsf{HST}_1^m \vdash \mathsf{wo}_0\vartheta(\Omega^2 \cdot \bar{n} + \Omega \cdot \bar{\alpha}),$$

whereby if $\sigma_0 = 1$ and $\sigma_{k+1} = \vartheta(\Omega^2 \cdot n + \Omega \cdot \sigma_k)$, $\mathsf{HST}_1^m \vdash \mathsf{wo}_0(\bar{\sigma}_k)$ for every k.

Thus we require to show $\vartheta(\Omega^2 \cdot m) \leq \sup_k \sigma_k$. This is proved by induction on the rank of $\alpha < \vartheta(\Omega^2 \cdot m)$. If $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta < \vartheta(\Omega^2 \cdot m)$, the induction hypothesis immediately implies $\alpha < \sigma_k$ for some k. Otherwise

$$\alpha = \vartheta\xi < \vartheta(\Omega^2 \cdot m)$$

and there are two cases to consider:

1. $\xi < \Omega^2 \cdot m$ and $\xi^* < \vartheta(\Omega^2 \cdot m)$; or
2. $\xi > \Omega^2 \cdot m$ but $\vartheta\xi \leq (\Omega^2 \cdot m)^*$.

b) cannot hold since $(\Omega^2 \cdot m)^* = 0$, so $\xi^* < \vartheta(\Omega^2 \cdot m)$. As ξ^* has rank strictly smaller than α the induction hypothesis implies $\xi^* < \sigma_k$ for some k. But then $\xi < \Omega^2 \cdot n + \Omega \cdot \sigma_k$ and $\alpha < \sigma_{k+1}$.

The second part of the theorem is easily established using the fact $\vartheta(\Omega^2 \cdot \omega) = \sup_k \vartheta(\Omega^2 \cdot k)$. □

We can now turn our attention to the theories HST_p for $p < \omega$. Lemma 6.6 essentially shows $\mathsf{HST}_1 \vdash F_1(\Omega^2 \cdot \bar{\alpha})$ implies $\mathsf{HST}_1 \vdash F_1(\Omega^2 \cdot (\bar{\alpha} + 1))$. This can be extended to show $\mathsf{HST}_1 \vdash \forall \nu < \bar{\mu}\, F_1(\Omega^2 \cdot \nu)$ implies $\mathsf{HST}_1 \vdash F_1(\Omega^2 \cdot \bar{\mu})$, whence

$$\mathsf{HST}_2^1 \vdash \mathsf{Prog} F_2^{\bar{0}} \tag{28}$$ {eqn:Fbwell1}

where $F_2^\rho(\alpha)$ is the formula $\alpha < \Omega \wedge \mathsf{T}_2(\ulcorner F_0(\rho + \Omega^2 \cdot \alpha)\urcorner)$.

(28) suffices to deduce a lower bound for HST_2^1 and acts as the base step in the analysis of HST_2 and ultimately HST_p, which follows a generalised form of the procedure used in lemma 6.6.

Let $F_p(\rho)$, for $0 < p < \omega$, be the formula $\mathsf{T}_p(\ulcorner F_0(\dot{\rho})\urcorner)$, that is

$$\mathsf{T}_p(\ulcorner \mathsf{wo}_0(\rho^*) \wedge \forall \sigma < \dot{\rho}[\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\vartheta\sigma)]\urcorner),$$

and denote by $F_p^\rho(\alpha)$ the formula $\alpha < \Omega \wedge F_p(\rho + \Omega^{\bar{p}} \cdot \alpha)$.

Fpwellordering1 **Lemma 6.8.** *For each $p < \omega$, $\mathsf{HST}^1_p \vdash \forall\rho[F_p(\rho) \to \mathsf{Prog}F^\rho_p]$ and, for $m < \omega$, $\mathsf{HST}^{m+1}_p \vdash F_p(\Omega^{\bar{p}+1}\cdot\bar{m})$.*

Proof. The proof proceeds by induction on $p < \omega$. The case of $p = 0$ has already been shown in proposition 6.4 so suppose $p = q + 1 > 0$. Argue informally within HST^1_p. Assume

$$F_p(\rho), \tag{29}$$ {eqn:Fpwell1}

$$\forall\nu < \mu\, F^\rho_p(\nu) \tag{30}$$ {eqn:Fpwell2}

for some $\mu < \Omega$. If $\mu = 0$, of course $F^\rho_p(\mu)$ holds by (29), and if μ is a limit ordinal, (30) implies $\mathsf{T}_p(\ulcorner\forall\nu < \dot{\mu}\, F_0(\rho + \Omega^{\bar{p}}\cdot\mu)\urcorner)$, whence $F^\rho_p(\mu)$ is immediate. This leaves only the case in which μ is a successor ordinal. But for every ordinal τ,

$$\begin{aligned}
F_p(\tau) &\to \mathsf{T}_p(\ulcorner F_q(\dot{\tau})\urcorner),\\
&\to \mathsf{T}_p(\ulcorner \mathsf{Prog}F^{\dot{\tau}}_q\urcorner),\\
&\to \mathsf{T}_p(\ulcorner \mathsf{T}_0(\ulcorner\mathsf{Prog}F^{\dot{\tau}}_q\urcorner)\urcorner),\\
&\to \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F^{\dot{\tau}}_q(\dot{\alpha})\urcorner)]\urcorner),\\
&\to \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F_0(\dot{\tau} + \Omega^{\bar{q}}\cdot\dot{\alpha})\urcorner)]\urcorner),\\
&\to \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \to \mathsf{wo}_0(\vartheta(\dot{\tau} + \Omega^{\bar{q}}\cdot\alpha))]\urcorner).
\end{aligned} \tag{31}$$ {eqn:Fpwell3}

The second implication holds on account of the induction hypothesis, while the final holds due to del_0 and lemma 6.1. Given that if $\tau \le \sigma < \tau + \Omega^p$ there exists some $\zeta < \Omega$ such that $\sigma < \tau + \Omega^q\cdot\zeta$ and $\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\zeta)$ (pick $\zeta = (\sigma^*)^{\varepsilon+}$), (31) entails $F_p(\tau) \to \mathsf{T}_p(\ulcorner F_0(\dot{\tau} + \Omega^{\bar{p}})\urcorner)$, define $\alpha^{\varepsilon+}$ that is $\forall\tau[F^\tau_p(\alpha) \to F^\tau_p(\alpha + 1)]$.

For the second part, the case $m = 0$ is immediate, so suppose $m = n + 1$ and $\mathsf{HST}^m_p \vdash F_p(\Omega^{\bar{p}+1}\cdot\bar{n})$. Then $\mathsf{HST}^{m+1}_p \vdash \mathsf{T}_p(\ulcorner\mathsf{Prog}F^{\Omega^{\bar{p}+1}\cdot\bar{n}}_p\urcorner)$, from which we deduce

$$\mathsf{HST}^{m+1}_p \vdash \mathsf{T}_p(\ulcorner\forall\alpha[\mathsf{wo}_0(\alpha) \to \mathsf{T}_0(\ulcorner F_p(\Omega^{\bar{p}+1}\cdot\bar{n} + \Omega^{\bar{p}}\cdot\dot{\alpha})\urcorner)]\urcorner)$$

and hence obtain $\mathsf{HST}^{m+1}_p \vdash F_p(\Omega^{\bar{p}+1}\cdot\bar{m})$. □

hm:Fplowerbound **Theorem 6.9.** *For every $m < \omega$ and $p < \omega$, $\mathsf{HST}^m_p \vdash \mathsf{TI}(<\vartheta(\Omega^{p+1}\cdot m))$ and $\mathsf{HST}_p \vdash \mathsf{TI}(<\vartheta(\Omega^{p+1}\cdot\omega))$.*

Proof. For every p, the base case, $m = 0$, is immediate since HST^0_p extends PA formulated in the language $\mathscr{L}_p$. Otherwise $m = n + 1 > 0$ and the previous lemma shows $\mathsf{HST}^m_p \vdash \mathsf{Prog}F^{\Omega^{\bar{p}+1}\cdot\bar{n}}_p$. Given $\mathsf{HST}^m_p \vdash \mathsf{wo}_p(\bar{\alpha})$, one obtains $\mathsf{HST}^m_p \vdash F_p(\Omega^{\bar{p}+1}\cdot\bar{n} + \Omega^{\bar{p}}\cdot\bar{\alpha})$, and so

$$\mathsf{HST}^m_p \vdash \mathsf{wo}_p(\vartheta(\Omega^{\bar{p}+1}\cdot\bar{n} + \Omega^{\bar{p}}\cdot\bar{\alpha})),$$

by unravelling the definition of F_p and lemma 6.1.

Let $\sigma_0 = 1$ and $\sigma_{k+1} = \vartheta(\Omega^{p+1}\cdot n + \Omega^p\cdot\sigma_k)$. The previous paragraph establishes $\mathsf{HST}^m_p \vdash \mathsf{TI}(<\sigma_k)$ for every k, so all that remains is to show $\vartheta(\Omega^{p+1}\cdot m) \le \sup_k \sigma_k$, which proceeds by induction on the rank of $\alpha < \vartheta(\Omega^{p+1}\cdot m)$. If $\alpha = 0$ we are done, and if $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$, the induction hypothesis implies $\alpha < \sigma_k$ for some k. Thus, suppose

$$\alpha = \vartheta\xi < \vartheta(\Omega^{p+1}\cdot m)$$

for which there are two cases to consider.

1. $\xi < \Omega^{p+1} \cdot m$ and $\xi^* < \vartheta(\Omega^{p+1} \cdot m)$; or

2. $\xi > \Omega^{p+1} \cdot m$ but $\vartheta\xi \leq (\Omega^{p+1} \cdot m)^*$.

Since $(\Omega^{p+1} \cdot m)^* = 0$, b) is impossible, and so $\xi^* < \sigma_k$ for some k by the induction hypothesis. Then $\xi < \Omega^{p+1} \cdot n + \Omega^p \cdot \sigma_k$, whence $\alpha < \vartheta(\Omega^{p+1} \cdot n + \Omega^p \cdot \sigma_k) = \sigma_{k+1}$.

The second part of the theorem is an immediate consequence of the fact

$$\vartheta(\Omega^{p+1} \cdot \omega) = \sup_{k<\omega} \vartheta(\Omega^{p+1} \cdot k).$$

□

Finally, we extend the well-ordering proofs to theories HST_β for $\beta \geq \omega$. For $\beta = \omega$ this involves generalising the above proof so that one may derive

$$\mathsf{HST}_\omega \vdash \forall p < \omega\, \mathsf{T}_\omega(\ulcorner \mathsf{Prog} F_p^{\bar{0}} \urcorner), \tag{32}$$ {eqn:Fbetawello

whence $\mathsf{HST}_\omega \vdash \forall p < \omega\, \mathsf{T}_\omega(\ulcorner \forall\alpha[\mathsf{wo}_0(\alpha) \to \mathsf{wo}_0 \vartheta(\Omega^p \cdot \alpha)] \urcorner)$ and also

$$\mathsf{HST}_\omega \vdash \forall \sigma < \Omega^\omega[\mathsf{wo}_\omega(\sigma^*) \to \mathsf{wo}_\omega(\vartheta\sigma))].$$

(32) is not difficult to establish as the proof of lemma 6.8 is clearly uniform in $p < \omega$ and thus can be repeated under a T_ω predicate in HST_ω. But in order to lift the construction to arbitrary $\beta \in \mathrm{OT}_\Omega \cap \Omega$ we require a more general argument.

For each $\beta \in \mathrm{OT}_\Omega$ let $G_\beta(\rho)$ denote the formula

$$\mathsf{T}_\beta(\ulcorner F_0(\dot{\rho}) \urcorner),$$

and let $G_\beta^\rho(\alpha)$ abbreviate $\alpha < \Omega \wedge \forall \eta < \bar{\beta}\, G_\beta(\rho + \Omega^\eta \cdot \alpha)$.

tawellordering1 **Proposition 6.10.** *For each $\beta \in \mathrm{OT}_\Omega \cap \Omega$, $\mathsf{HST}_\beta^1 \vdash \forall\rho[G_\beta(\rho) \to \mathsf{Prog} G_\beta^\rho]$ and for every $m < \omega$, $\mathsf{HST}_\beta^{m+1} \vdash G_\beta(\Omega^{\bar{\beta}} \cdot \bar{m})$.*

The proof of proposition 6.10 is by transfinite induction on β and requires, for a given β, the following technical lemmata.

lem:Fbetatech1 **Lemma 6.11.** $\mathsf{HST}_\beta^1 \vdash \forall\rho[G_\beta(\rho) \to \forall\eta < \bar{\beta}\, \mathsf{T}_\beta(\ulcorner \mathsf{Prog} G_{\dot{\eta}}^{\dot{\rho}} \urcorner)]$.

lem:Fbetatech2 **Lemma 6.12.** $\mathsf{HST}_\beta^1 \vdash \mathsf{T}_\beta(\ulcorner \forall\tau\forall\eta < \bar{\beta}\, [F_0(\tau) \wedge \mathsf{T}_0(\ulcorner \mathsf{Prog} G_{\dot{\eta}}^{\dot{\tau}} \urcorner) \to F_0(\tau + \Omega^\eta)] \urcorner)$.

Proof. The two lemmata hold vacuously when $\beta = 0$. For $\beta > 0$

$$\mathsf{HST}_\beta^1 \vdash G_\beta(\rho) \leftrightarrow \forall\eta < \bar{\beta}\, \mathsf{T}_\beta(\ulcorner G_\eta(\dot{\rho}) \urcorner),$$

so the first lemma would result from replicating the proof of (the transfinite induction hypothesis of) proposition 6.10 under a T_β predicate. This is possible as the proof of the proposition, which is presented below, is uniform in $\eta < \beta$.

In order to establish lemma 6.12, argue within HST^1_β *under* the scope of a T_β predicate. Fix $\eta < \bar{\beta}$, some arbitrary τ and assume

$$\mathsf{T}_0(\ulcorner \mathsf{Prog} G^{\dot{\tau}}_{\dot{\eta}} \urcorner), \tag{33}$$
{eqn:Fbetatech2

$$F_0(\tau). \tag{34}$$
{eqn:Fbetatech3

(33) entails

$$\forall \alpha[\mathsf{wo}_0(\alpha) \to \forall \xi < \eta \, \mathsf{T}_0(\ulcorner G_\eta(\dot{\tau} + \Omega^{\dot{\xi}} \cdot \dot{\alpha}) \urcorner)],$$

so $\forall \alpha[\mathsf{wo}_0(\alpha) \to \forall \xi < \eta \, \mathsf{T}_0(\ulcorner F_0(\dot{\tau} + \Omega^{\dot{\xi}} \cdot \dot{\alpha}) \urcorner)]$ by del_0 and

$$\forall \xi < \eta \forall \alpha[\mathsf{wo}_0(\alpha) \to \mathsf{wo}_0(\vartheta(\tau + \Omega^{\xi} \cdot \alpha))] \tag{35}$$
{eqn:Fbetatech4

by lemma 6.1. If $\tau \leq \sigma < \tau + \Omega^\eta$, there exists some $\zeta < \Omega$ and $\xi < \eta$ such that $\sigma < \tau + \Omega^\xi \cdot \zeta$ and $\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\zeta)$ (pick $\zeta = (\sigma^*)^{\varepsilon+}$), whence $\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\vartheta\sigma)$ results from (35) as $\vartheta\sigma < \vartheta(\tau + \Omega^\xi \cdot \zeta)$. If, however, $\sigma < \tau$, (34) implies $\mathsf{wo}_0(\sigma^*) \to \mathsf{wo}_0(\vartheta\sigma)$. Thus, $F_0(\tau + \Omega^\eta)$ holds. □

of proposition 6.10. Argue informally within HST^1_β. Fix some arbitrary ρ and assume $G_\beta(\rho)$. Lemma 6.11 yields $\forall \eta < \bar{\beta} \, \mathsf{T}_\beta(\ulcorner \mathsf{Prog} G^{\dot{\rho}}_{\dot{\eta}} \urcorner)$ from which, using T^-_β-Rep and lemma 6.12, one obtains $\forall \eta < \bar{\beta} \, \mathsf{T}_\beta(\ulcorner F_0(\rho + \Omega^\eta) \urcorner)$, that is, $\forall \eta < \bar{\beta} \, G_\beta(\rho + \Omega^\eta)$. Therefore

$$\mathsf{HST}^1_p \vdash \forall \rho[G^\rho_\beta(\alpha) \to G^\rho_\beta(\alpha + 1)]. \tag{36}$$
{eqn:Fbetawellc

If μ is a limit ordinal, $\forall \nu < \mu \, G^\rho_\beta(\nu)$ and uni_βimplies

$$\forall \eta < \bar{\beta} \, \mathsf{T}_\beta(\ulcorner \forall \nu < \dot{\mu} \, F_0(\dot{\rho} + \Omega^{\dot{\eta}} \cdot \nu) \urcorner),$$

from which $\mathsf{T}_\beta(\ulcorner F_0(\dot{\rho} + \Omega^{\dot{\eta}} \cdot \dot{\mu}) \urcorner)$ is easily deduced. Thus (36) entails

$$\mathsf{HST}^1_\beta \vdash \forall \rho[G_\beta(\rho) \to \mathsf{Prog} G^\rho_\beta].$$

The second half of proposition 6.10 is proved by induction on $m < \omega$. The case $m = 0$ holds since $F_0(\bar{0})$ is vacuously true. If $m = n + 1$ and $\mathsf{HST}^m_\beta \vdash G_\beta(\Omega^{\bar{\beta}} \cdot \bar{n})$,

$$\mathsf{HST}^{m+1}_\beta \vdash \mathsf{T}_\beta(\ulcorner \mathsf{Prog} G^{\Omega^{\bar{\beta}} \cdot \bar{n}}_\beta \urcorner),$$

so $\mathsf{HST}^{m+1}_\beta \vdash G_\beta(\Omega^{\bar{\beta}} \cdot \bar{m})$ by an argument similar to lemma 6.12. □

Much as in the finite case, proposition 6.10 suffices to obtain lower bounds on the theories HST^m_β. In theorem 6.13 below we prove $\|\mathsf{HST}_\beta\| \geq \vartheta(\Omega^\beta \cdot \omega)$. This is achieved by showing the proof-theoretic ordinal of HST^m_β is closed under the operations $\alpha \mapsto \vartheta(\Omega^\eta \cdot \alpha)$ for every $\eta < \beta$ and extends the strength of HST^n_β for $n < m$. The method will only work if β is not "too large", that is $\vartheta\Omega^\beta$ is indeed the *least* ordinal closed under the above operations. By taking β to be no larger than the Veblen ordinal $\vartheta\Omega^\Omega$, we can ensure that this is the case.

Fbetalowerbound **Theorem 6.13.** *For every $\beta < \vartheta\Omega^{\Omega}$ and every $m < \omega$, $\mathsf{HST}^{m}_{\beta} \vdash \mathsf{TI}(<\vartheta(\Omega^{\beta} \cdot m))$ and $\mathsf{HST}_{\beta} \vdash \mathsf{TI}(<\vartheta(\Omega^{\beta} \cdot \omega))$.*

Proof. For every $\beta < \vartheta\Omega^{\Omega}$, the base case, $m = 0$, is immediate since HST^{0}_{β} extends PA formulated in the language $\mathscr{L}_{\kappa}$. Otherwise $m = n+1 > 0$ and proposition 6.10 implies $\mathsf{HST}^{m}_{\beta} \vdash \mathsf{Prog}G^{\Omega^{\bar\beta}\cdot\bar n}_{\beta}$. Then, given $\mathsf{HST}^{m}_{\beta} \vdash \mathsf{wo}_{\beta}(\bar\alpha)$, we obtain $\mathsf{HST}^{m}_{\beta} \vdash \forall\eta < \bar\beta\, G_{\beta}(\Omega^{\bar\beta} \cdot \bar n + \Omega^{\bar\eta} \cdot \bar\alpha)$, and so

$$\mathsf{HST}^{m}_{\beta} \vdash \forall\eta < \bar\beta\, \mathsf{wo}_{\beta}\vartheta(\Omega^{\bar\beta} \cdot \bar n + \Omega^{\bar\eta} \cdot \bar\alpha))),$$

by unravelling the definition of G_{β}.

Let $\sigma_0 = 1$ and $\sigma_{k+1} = \sup_{\eta<\beta} \vartheta(\Omega^{\beta} \cdot n + \Omega^{\eta} \cdot \sigma_k)$. The previous paragraph establishes $\mathsf{HST}^{m}_{\beta} \vdash \mathsf{TI}(<\sigma_k)$ for every k. Therefore, all that remains is to show $\vartheta(\Omega^{\beta} \cdot m) \leq \sup_k \sigma_k$, which proceeds by induction on the rank of $\alpha < \vartheta(\Omega^{\beta} \cdot m)$. If $\alpha = 0$ we are done, and if $\alpha =_{\mathrm{NF}} \omega^{\gamma} + \delta$, the induction hypothesis implies $\alpha < \sigma_k$ for some k. Thus, suppose

$$\alpha = \vartheta\xi < \vartheta(\Omega^{\beta} \cdot m).$$

There are two cases to consider.

1. $\xi < \Omega^{\beta} \cdot m$ and $\xi^* < \vartheta(\Omega^{\beta} \cdot m)$; or
2. $\xi > \Omega^{\beta} \cdot m$ but $\vartheta\xi \leq (\Omega^{\beta} \cdot m)^*$.

a) entails $\xi^* < \sigma_k$ for some k by the induction hypothesis (since ξ^* has rank strictly smaller than α). Then $\xi < \Omega^{\beta} \cdot m + \Omega^{\eta} \cdot \sigma_k$ for some $\eta < \beta$, whence $\alpha < \vartheta(\Omega^{\beta} \cdot n + \Omega^{\eta} \cdot \sigma_k) \leq \sigma_{k+1}$.

To manage b) one utilises $\beta < \vartheta(\Omega^{\Omega})$. Since $\vartheta\xi \leq \beta^*$ and $(\Omega^{\Omega})^* = 0$, we have

$$\vartheta\xi < \vartheta(\Omega^{\Omega})$$

and $\xi < \Omega^{\Omega}$. But then $\beta \leq \xi^* < \Omega$ contradicting $\vartheta\xi \leq \beta^*$.

The lower bound on HST_{β} is an immediate consequence of the fact

$$\vartheta(\Omega^{\beta} \cdot \omega) = \sup_{k<\omega} \vartheta(\Omega^{\beta} \cdot k).$$

□

7 Upper bounds on the proof-theoretic ordinal

sec:Fbupper1

We present a ordinal function f on OT such that for all $\rho \in \mathrm{OT}$, all arithmetical theorems of HST_{ρ} are derivable in the classical theory $\mathsf{PA}+\mathsf{TI}(<f(\rho))$. By a slightly more involved argument, following the method of [1], the role of Peano arithmetic can be replaced by Heyting arithmetic, HA.

The proof proceeds by formalising the consistency proof of section 4 within the confines of the theory $\mathsf{PA} + \mathsf{TI}(<f(\rho))$. The main technical difficulty is in formalising the co-Necessitation theorem, theorem 4.5, for two reasons: (1) the proof appeals to transfinite induction beyond the first uncountable ordinal and a subsidiary induction on all countable ordinals, (2) the case

$$\frac{A \text{ true literal}}{\vert\!\frac{\alpha}{\rho}\, \Gamma, A}\text{Ax1} \qquad \frac{\vert\!\frac{\alpha}{\rho}\, \Gamma, A \quad \vert\!\frac{\alpha}{\rho}\, \Gamma, B}{\vert\!\frac{\beta}{\rho}\, \Gamma, A \wedge B}\wedge \qquad \frac{\vert\!\frac{\alpha}{\rho}\, \Gamma, A(\bar{n}) \quad \text{all } n < \omega}{\vert\!\frac{\beta}{\rho}\, \Gamma, \forall x A(x)}\omega$$

$$\frac{s^{\mathbb{N}} = t^{\mathbb{N}}}{\vert\!\frac{\alpha}{\rho}\, \Gamma, \neg \mathsf{T}_\eta s, \mathsf{T}_\eta t}\text{Ax2} \qquad \frac{\vert\!\frac{\alpha}{\rho}\, \Gamma, A, B}{\vert\!\frac{\beta}{\rho}\, \Gamma, A \vee B}\vee \qquad \frac{\vert\!\frac{\alpha}{\rho}\, \Gamma, A(s)}{\vert\!\frac{\beta}{\rho}\, \Gamma, \exists x A(x)}\exists$$

$$\frac{s^{\mathbb{N}} \notin \mathsf{Sent}_\kappa}{\vert\!\frac{\alpha}{\rho}\, \Gamma, \neg \mathsf{T}_\eta s}\text{Ax3}$$

f-L-rules

Figure 1: Axioms and logical rules of the infinitary calculus; in each rule $\beta > \alpha$.

in which the last rule employed was uni_η employs non-finitistic properties of Ω, specifically proposition 4.4.

The first concern is directly alleviated by parameterising the height of derivations in $\mathscr{T}_\rho$ from which a suitable choice of f ensures that applications of the subsidiary induction hypothesis are bounded below $f(\rho)$ and all references to the main induction hypothesis in the co-necessitation theorem are for ordinals of the form $\Omega^{\alpha_0} \cdot \beta_0 + \cdots + \Omega^{\alpha_n} \cdot \beta_n$ where $\alpha_i, \beta_i < f(\delta)$ for all i.

The second point, however, has no immediate alternative; regularity of Ω is needed to ensure that the set $\mathscr{T}_{<\rho}$ is closed under the ω-rule in the case that ρ is not an Ω-limit, as this property is required to establish that $\mathscr{T}_\rho$ is closed under conec_η. Suppose, for instance, that $\rho = \rho' + \Omega^{\xi+1}$, which is not an Ω-limit. If $A(x)$ is a formula of $\mathscr{L}_\kappa$ and $\vdash_{\rho_n} A(\bar{n})$ with $\rho_n < \rho$ for every n, we want to obtain an ordinal $\sigma < \rho$ such that $\vdash_\sigma \forall x\, A(x)$. As each application of nec_η in a derivation contributes to the height of a derivation, a bound on the height of a derivation implicitly bounds its T-rank. That is, if $\vdash_\tau B$ with height bound α, it will follow that $\vdash_\sigma B$ for some $\sigma \le \tau$ such that $\sigma^* \le \alpha$. In particular, if α bounds the height of the derivation $\vdash_{\rho_n} A(\bar{n})$ for each n, there exists an ordinal $\beta < \Omega$ effectively computed from ρ and α such that $\vdash_\sigma \forall x\, A(x)$ where $\sigma = \rho' + \Omega^\xi \cdot \beta$.

Although the above arguments are still vague, the reader may find them helpful motivation for a number of the technical results to follow, in particular in the Bounding Lemma (**??**).

In what follows it will be important to compare ordinals beyond Ω by their construction as well in addition to their magnitude, for which we introduce a new ordering $<^*$, defined on OT_Ω by

$$\rho <^* \sigma \text{ iff } \rho < \sigma \text{ and } \rho^* \le \sigma^*, \qquad \rho \le^* \sigma \text{ iff } \rho <^* \sigma \text{ or } \rho = \sigma.$$

Definition 7.1. Fix $\kappa = \vartheta\varepsilon_{\Omega+1}$. The relation $\vert\!\frac{\alpha}{}\, \Gamma$ where $\alpha \in \mathrm{OT}_\Omega$ and Γ is an $\mathscr{L}_\kappa$-sequent is defined inductively by the rules in figs. 1 and 2.

Remark on $\sigma = \sigma|_0 + \alpha$ in above definition: $\vert\!\frac{\alpha}{\sigma|_0}\, \Gamma$.

Given $\vert\!\frac{\alpha}{\rho}\, \Gamma$, we refer to Γ as being *derivable* with *T-rank* ρ and *height* α. For a fixed η, the rules (imp_η) to (uni_η) are collectively referred to as T_η*-rules*. The collection as a whole (for all η) is referred to as the *Truth rules*. In each application of a rule or axiom above we specify the *principal*, *minor* and *side* formulae as standard.

The following lemma is an immediate consequence of the definition.

Lemma 7.2 (Weakening). *If* $\vert\!\frac{\alpha}{}\, \Gamma$ *and* $\alpha \le^* \delta$ *then* $\vert\!\frac{\delta}{}\, \Gamma, \Delta$ *for all* Δ.

$$\dfrac{\vdash^{\alpha} \Gamma, \mathsf{T}_\eta s \quad \vdash^{\alpha} \Gamma, \mathsf{T}_\eta(s \dot{\rightarrow} t)}{\vdash^{\beta} \Gamma, \mathsf{T}_\eta t}\,\mathrm{imp}_\eta \qquad \dfrac{\vdash^{\alpha} \Gamma, \mathsf{T}_\eta \ulcorner \mathsf{T}_\xi s \urcorner \quad s^{\mathbb{N}} = t^{\mathbb{N}}}{\vdash^{\beta} \Gamma, \mathsf{T}_\eta t}\,\mathrm{del}_\eta$$

$$\dfrac{\vdash^{\alpha} \Gamma, \mathsf{T}_\eta s \quad t^{\mathbb{N}} = \ulcorner \mathsf{T}_\xi s \urcorner^{\mathbb{N}}}{\vdash^{\beta} \Gamma, \mathsf{T}_\eta t}\,\mathrm{rep}_\eta \qquad \dfrac{\vdash^{\alpha} A \quad \alpha \,\#\, \Omega^{\eta+1} \leq^* \beta \quad t^{\mathbb{N}} = \ulcorner A \urcorner^{\mathbb{N}}}{\vdash^{\beta} \mathsf{T}_\eta t}\,\mathrm{nec}_\eta$$

$$\dfrac{\vdash^{\alpha} \Gamma, \mathsf{T}_\eta \ulcorner A(\bar{n}) \urcorner \quad \text{for all } n < \omega, \alpha|_\eta \text{ not an } \Omega\text{-limit and } t^{\mathbb{N}} = \ulcorner \forall x A(x) \urcorner^{\mathbb{N}}}{\vdash^{\beta} \Gamma, \mathsf{T}_\eta t}\,\mathrm{uni}_\eta$$

f-T-rules Figure 2: Truth rules of the infinitary calculus; in each case $\alpha <^* \beta$.

Verification of the next proposition is straightforward and, hence, omitted.

prop:T8props **Proposition 7.3.** *Update:*

1. *For every $\mathcal{L}_\kappa$-sentence A, $\vdash^{\omega}_{0} \neg A, A$*
2. lem:F8L1itemvi *If A is a false equation, then $\vdash^{\alpha}_{\sigma} \Gamma, A$ implies $\vdash^{\alpha}_{\sigma} \Gamma$.*
3. *If $\vdash^{\alpha} A \vee B$ then $\vdash^{\alpha} A, B$.*
4. lem:F8L1itemviji *If $s^{\mathbb{N}} \notin \mathsf{Sent}_\kappa$, then $\vdash^{\alpha}_{\sigma} \Gamma, \mathsf{T}_\eta s$ implies $\vdash^{\alpha}_{\sigma} \Gamma$.We use assumptions about coding.*
5. lem:F8L1itemx *If $\vdash^{\alpha}_{\sigma} \Gamma, A(s), A(t)$ and $s^{\mathbb{N}} = t^{\mathbb{N}}$ then $\vdash^{\alpha}_{\sigma} \Gamma, A(s)$.*

The design of the calculus give rise to the following observation:

bounding **Lemma 7.4** (Bounding lemma)**.** *Fix $\beta < \kappa$ and suppose Γ is a sequent in the language of arithmetic. For all $\alpha < \Omega^\kappa$, if $\vdash^{\alpha} \Gamma$ then there exists $\alpha' < \vartheta\alpha$ such that $\vdash^{\alpha'} \Gamma$.*

Proof. By induction on the derivation of $\vdash^{\alpha} \Gamma$. As the calculus is cut-free, we observe that no truth-rule occurs in the derivation of a truth-free sequent. □

The design of the calculus means that the cut rule is admissible with the same bounds as the arithmetical fragment.

cut-admiss **Theorem 7.5** (Admissibility of cut)**.** *For all α, if $\vdash^{\alpha} \Gamma, A$ and $\vdash^{\alpha} \Delta, \neg A$ then $\vdash^{\alpha+\vartheta\alpha} \Gamma, \Delta$.*

Explain why this is just the usual argument.

Proof. By the definition, the literals $\neg\mathsf{T}_\eta s$ are principal in only two rules, both axioms. Thus, given $\vdash^{\alpha}_{\sigma} \Gamma, \mathsf{T}_\eta s$ and $\vdash^{\alpha}_{\sigma} \Delta, \neg\mathsf{T}_\eta s$, if the latter holds by virtue of axiom (ax2) with $\neg\mathsf{T}_\eta s$ principal, then $s^{\mathbb{N}} \notin \mathsf{Sent}_\kappa$ and $\vdash^{\alpha}_{\sigma} \Gamma, \Delta$ holds by the previous proposition. If, on the other hand, $\vdash^{\alpha}_{\sigma} \Delta, \neg\mathsf{T}_\eta s$ is an instance of (ax2), then $\vdash^{\alpha}_{\sigma} \Gamma, \Delta$ is either itself an instance of (ax2), or can be obtained from $\vdash^{\alpha}_{\sigma} \Gamma, \mathsf{T}_\eta s$ substitution of equal terms (also proposition 7.3). The remaining cases follow by standard cut-admissibility argument for ω-logic and we refer the reader to, e.g., [1, 2] for analogous arguments. It is important to note, in particular, that the necessitation rules $(\mathrm{nec}_\eta)_{\eta<\kappa}$ do not need to be 'permuted' with the cut rule: if the cut formula is a side formula of the inference then the cut reduces to weakening, and if it is the principal formula of the rule then, by the above argument, this premise of the cut need not be consulted. □

What remains is to establish a counterpart of theorem 4.5, and confirm that the argument relative to proofs of T-rank ρ can be formalised within the theory $\mathsf{PA} + \mathsf{TI}({<}\delta)$ for suitable δ.

We define a collection of standard $\mathscr{L}_\kappa$-structures $\mathfrak{N}_\rho := \langle \mathbb{N}, (\mathscr{T}_{\rho,\eta})_{\eta<\kappa} \rangle$ for $\rho < \Omega^\kappa$. The interpretation of the predicate T_η in $\mathfrak{N}_\rho$ is given by the set

$$\mathscr{T}_{\rho,\eta} := \left\{ \ulcorner A \urcorner \mid \text{there exist } \alpha < \rho|_\eta \text{ such that } \alpha^* < \vartheta\rho \text{ and } \vdash^{\alpha} A \right\}.$$

prop:T8mono **Proposition 7.6.** *If $\rho \leq^* \sigma$ and $\mathfrak{N}_\rho \models A$, then $\mathfrak{N}_\sigma \models A$ for every T-positive formula A.* Defined T-positive?

Proof. If $\rho \leq^* \sigma$ then $\rho|_\eta \leq \sigma_\eta$ for all η and $\vartheta\rho \leq \vartheta\sigma$. Therefore, $\mathscr{T}_{\rho,\eta} \subseteq \mathscr{T}_{\sigma,\eta}$ for every $\eta < \kappa$ and the claim holds. □

The structure $\mathfrak{N}_\rho$ is so chosen to provide a model of axioms of the theory HST_ρ for certain ρ. This claim witnessed by the next result.

soundness-lemma **Lemma 7.7.** *The following hold for every $\eta < \kappa$ and $\rho < \Omega^\kappa$.*

i-soundimp 1. $\mathfrak{N}_\rho \models \forall x \forall y (\mathsf{T}_\eta x \wedge \mathsf{T}_\eta (x \dot{\to} y) \to \mathsf{T}_\eta y)$.

i-soundrep 2. $\mathfrak{N}_\rho \models \forall x (\mathsf{T}_\eta x \to \forall \xi < \bar{\eta}\, \mathsf{T}_\eta \ulcorner \mathsf{T}_\xi \dot{x} \urcorner)$. More coding

3. $\mathfrak{N}_\rho \models \forall \xi < \bar{\kappa} \forall x (\mathsf{T}_\eta \ulcorner \mathsf{T}_\xi \dot{x} \urcorner \to \mathsf{T}_\eta x)$ *if, for every A and σ,*

i-sounddel $$\text{if } \vartheta\sigma < \vartheta\rho \text{ and } \vdash^{\sigma} A \text{ then } \mathfrak{N}_\sigma \models A.$$

i-sounduni 4. *If $\rho|_\eta$ is not an Ω-limit, $\mathfrak{N}_\rho \models \forall x\, \mathsf{T}_\eta \ulcorner A(\dot{x}) \urcorner$ and $\rho <^* \sigma$, then $\mathfrak{N}_\sigma \models \mathsf{T}_\eta \ulcorner \forall x\, A(x) \urcorner$.*

Proof. We treat each case in turn, using appealing to standardness of $\mathfrak{N}_\rho$.

1. Suppose $\vdash^{\alpha} A$ and $\vdash^{\alpha} \neg A \vee B$ for $\alpha < \rho|_\eta$ satisfying $\alpha^* < \vartheta\rho$. Admissibility of cut, theorem 7.5, implies $\vdash^{\alpha+\vartheta\alpha} B$. As $\alpha^* < \vartheta\rho$ and $\alpha < \rho$, we have $\vartheta\alpha < \vartheta\rho$ by proposition 5.2(8). Hence $\mathfrak{N}_\rho \models \mathsf{T}_\eta \ulcorner B \urcorner$.

2. Suppose $\vdash^{\alpha} A$, that $\alpha < \rho|_\eta$ and $\alpha^* < \vartheta\rho$. An application of (nec_η) yields $\vdash^{\delta} \mathsf{T}_\xi \ulcorner A \urcorner$ where $\delta = \alpha \# \Omega^{\xi+1}$. If $\xi < \eta$, then $\delta < \rho|_\eta$ and, since $\xi^* \leq \rho^* < \vartheta\rho$, it follows that $\delta^* < \vartheta\rho$ and $\mathfrak{N}_\rho \models \forall \xi < \bar{\eta}\, \mathsf{T}_\eta \ulcorner \mathsf{T}_\xi \ulcorner A \urcorner \urcorner$.

3. Assume that $\vdash^{\sigma} A$ implies $\mathfrak{N}_\sigma \models A$ for all σ such that $\vartheta\sigma < \vartheta\rho$. Suppose $\mathfrak{N}_\rho \models \mathsf{T}_\eta \ulcorner \mathsf{T}_\xi s \urcorner$ for some $\xi < \kappa$. Let $\alpha < \rho|_\eta$ and B such that $\alpha^* < \vartheta\rho$ and

$$\vdash^{\alpha} \mathsf{T}_\xi s.$$

As $\vartheta\alpha < \vartheta\rho$, we deduce, by the assumption, $\mathfrak{N}_\alpha \models \mathsf{T}_\xi s$, whereby $s^{\mathbb{N}} = \ulcorner B \urcorner$ for some sentence B and there exists $\beta < \alpha|_\xi$ such that $\beta^* < \vartheta\alpha$ and

$$\vdash^{\beta} B.$$

Then $\beta < \alpha|_\xi \leq \alpha \leq \rho|_\eta$, and $\beta^* < \vartheta\alpha < \vartheta\rho$, so $\mathfrak{N}_\rho \models \mathsf{T}_\eta \ulcorner B \urcorner$.

4. Suppose $\rho|_\eta = \rho_0 + \Omega^{\xi+1}$ is not an Ω-limit and $\mathfrak{N}_\rho \models \forall x\, \mathsf{T}_\eta \ulcorner A(\dot{x}) \urcorner$. Let $\rho <^* \rho'$ and $(\alpha_n)_n$ This is Ω-limit
be such that for each n,

- $\alpha_n < \rho|_\eta$,
- $(\alpha_n)^* < \vartheta\rho$,

- $\vdash^{\alpha_n} A(\bar{n})$.

Set $\sigma = \rho_0 + \Omega^{\xi}.\vartheta\rho < \rho|_\eta$. As $\alpha_n <^* \sigma$ for each n, weakening implies that $\vdash^{\sigma} A(\bar{n})$, whence $\vdash^{\sigma+1} \forall x A(x)$. Clearly, $\sigma < \rho'|_\eta$. As $\sigma^* = \vartheta\rho < \vartheta\rho'$ we conclude that $\mathfrak{N}_{\rho'} \models \mathsf{T}_\eta\ulcorner\forall x\, A(x)\urcorner$. □

soundness-thm **Theorem 7.8** (Soundness theorem). *For every $\sigma < \Omega^\kappa$, if $\vdash^{\sigma} \Gamma$ and Γ is T-positive, then $\mathfrak{N}_\sigma \models \bigvee\Gamma$.*

Proof. By transfinite induction on $\vartheta\sigma$. Since $\mathfrak{N}_\sigma$ is standard, the logical rules and axioms (with T-positive end-sequent) are dealt with readily. Thus we need only dispense the truth rules. Suppose $\vdash^{\sigma} \Gamma$ is derived via a truth rule. If the last rule applied was (nec$_\eta$), there is an $\mathscr{L}_\kappa$-sentence A, formula $\mathsf{T}_\eta s \in \Gamma$ where $s^{\mathbb{N}} = \ulcorner A\urcorner$, and ordinal $\alpha <^* \sigma|_\eta$ such that $\vdash^{\alpha} A$. In particular, $\alpha < \sigma|_\eta$ and $\alpha^* \le \sigma^* < \vartheta\sigma$, so $\mathfrak{N}_\sigma \models \bigvee\Gamma$.

For the remaining truth rules, we recall that $\alpha <^* \sigma$ implies $\vartheta\alpha < \vartheta\sigma$, so the induction hypothesis, lemma 7.7 and monotonicity lemma establish each case. In the verification of (del$_\eta$), the induction hypothesis implies the additional hypothesis of lemma 7.7 □

conec-thm **Corollary 7.9** (Conecessitation theorem). *If $\vdash^{\sigma} \mathsf{T}_\eta s$ then there exists an $\mathscr{L}_\mathsf{T}$-sentence A with $s^{\mathbb{N}} = \ulcorner A\urcorner$ and $\vdash^{\sigma+\vartheta\sigma} A$.*

Proof. Theorem 7.8 and weakening. □

Having established that the infinitary calculus is closed under (conec$_\eta$) for all $\eta < \kappa$ it is clear that the calculus subsumes each of the finitary theories HST_η. The final task toward an ordinal analysis is to observe an appropriate upper bound on the embedding.

embed-axioms **Proposition 7.10.** *If A is an axiom of HST_η, then $\vdash^{\Omega^{\eta+1}+\omega.2} A$.*

We omit the proof of proposition 7.10 as it is straightforward. The role of $\Omega^{\eta+1}$ in the bound is to accommodate the application of (nec$_\eta$) needed in deriving the first two axioms of HST_η. Fill in axioms

embed-lemma **Lemma 7.11** (Embedding lemma). *Fix some $\beta < \vartheta\varepsilon_{\Omega+1}$ and suppose $\mathsf{HST}_\beta \vdash A$ with A a sentence of $\mathscr{L}_\kappa$. There exists n and $\alpha < \vartheta(\Omega^{\beta+1}.n)$ such that*

$$\vdash^{\Omega^{\beta+1}.n+\alpha} A.$$

Proof. Let $\sigma_n = \Omega^{\beta+1}.n + \vartheta(\Omega^{\beta+1}.n)$ and note that for all n, $\sigma_n + \vartheta\sigma_n <^* \sigma_{n+1}$.

We show, by induction on the length of HST_β derivations that if $\mathsf{HST}_\beta \vdash A$ then there exists n such that for every closed instantiation A^* of A we have $\vdash^{\sigma_n} A$. Henceforth, we let A^* represent an arbitrary closed instantiations of A.

Proposition 7.10 deals with the axioms of HST_β. In the case of an application of (nec$_\beta$), we may assume that A is $\mathsf{T}_\beta\ulcorner B\urcorner$ for a sentence B and, invoking the induction hypothesis,

$$\vdash^{\sigma_n} B.$$

An application of (nec$_\beta$) yields

$$\vdash^{\sigma_n\#\Omega^{\beta+1}} \mathsf{T}_\beta\ulcorner B\urcorner.$$

As $\sigma_n \# \Omega^{\beta+1} <^* \sigma_{n+1}$ weakening completes the argument.

If the final rule of inference is modus ponens, the induction hypothesis and weakening implies that for some n and sentence B we have

$$\vdash^{\sigma_n} B \quad \text{and} \quad \vdash^{\sigma_n} \neg B, A^*.$$

Admissibility of cut (theorem 7.5) implies

$$\vdash^{\sigma_n + \vartheta\sigma_n} A.$$

An application of weakening yields

$$\vdash^{\sigma_{n+1}} A.$$

This leaves only applications of (conec_β) to consider, which follows an argument analogous to the case of cut. Suppose $\mathsf{HST}_\beta \vdash \mathsf{T}_\beta \ulcorner A \urcorner$ for some $\mathscr{L}_\kappa$-sentence A. The induction hypothesis provides an n such that

$$\vdash^{\sigma_n} \mathsf{T}_\beta \ulcorner A \urcorner.$$

By corollary 7.9,

$$\vdash^{\sigma_n + \vartheta\sigma_n} A.$$

Since $\sigma_n + \vartheta\sigma_n <^* \sigma_{n+1}$ we are done. □

The embedding lemma readily provides an upper bound on the strength of the theories HST_β.

HST-upper-bound **Theorem 7.12.** *Every arithmetical theorem of* HST_β*, for* $\beta < \vartheta\varepsilon_{\Omega+1}$*, is derivable in* $\mathsf{PA} + \mathsf{TI}(<\vartheta(\Omega^{\beta+1} \cdot \omega))$.

8 Conclusion

sec:conc

We have introduced a hierarchy $(\mathsf{HST}_\alpha)_{\alpha \in \mathbb{O}}$ of intuitionistic theories of self-applicable truth indexed by ordinals from a fixed elementary ordinal notation system $\mathbb{O}$ and argued that HST_α formalises an intensional acceptance of the theories HST_β for $\beta < \alpha$.

Presented with such a hierarchy of theories $(\mathsf{T}_\alpha)_{\alpha \in \mathbb{O}}$ it is natural to ask the limit of the corresponding autonomous progression, that is, the least ordinal not in the set X_T generated by the operation $\|\mathsf{T}_0\| \subseteq X_\mathsf{T}$ and $\alpha \in X_\mathsf{T}$ implies $\|\mathsf{T}_\alpha\| \subseteq X_\mathsf{T}$. Autonomous progressions of ramified theories such as RA_α are well-studied, as are those obtained by iterating reflection principles [6]. In the case $\mathsf{T}_\alpha = \mathsf{HST}_\alpha$ this is not difficult to determine given theorem 8.2.

Combining the results of the previous section we determine the strength of the theory HST_β for every $\beta < \vartheta\Omega^\Omega$.

thm:Fpstrength **Theorem 8.1.** *For every* $p < \omega$, $\|\mathsf{HST}_p\| = \vartheta(\Omega^{p+1} \cdot \omega)$.

m:Fbetastrength **Theorem 8.2.** *For every* $\beta \geq \omega$ *with* $\beta < \vartheta\Omega^\Omega$,

$$\vartheta(\Omega^\beta \cdot \omega) \leq \|\mathsf{HST}_\beta\| \leq \vartheta(\Omega^{\beta+1} \cdot \omega).$$

Proof. Theorem 7.12 provides both upper bounds for the two theorems. The lower bounds are a corollary of theorem 6.9 and theorem 8.2 respectively. □

thm:Fbauto **Theorem 8.3.** *The limit of the autonomous progression defined from $\{\mathsf{HST}_\beta : \beta < \Omega\}$ is the large Veblen ordinal, $\vartheta\Omega^\Omega$.*

Proof. Let $\sigma_0 = 0$ and $\sigma_{m+1} = \vartheta(\Omega^{\sigma_m})$. Theorem 6.13 implies $\|\mathsf{HST}^1_{\sigma_m}\| \geq \sigma_{m+1}$, while theorem 8.2 entails $\|\mathsf{HST}_{\sigma_m}\| < \sigma_{m+2}$, so $X_\mathsf{F} = \sup_m \sigma_m$. It remains to show

$$\vartheta(\Omega^\Omega) = \sup_{m<\omega} \sigma_m.$$

Since $\sigma_m < \Omega$ for every m, we have $\Omega^{\sigma_m} < \Omega^\Omega$. So $\sigma_m < \vartheta(\Omega^\Omega)$ implies $\sigma_{m+1} < \vartheta(\Omega^\Omega)$. Thus $\vartheta(\Omega^\Omega) \geq \sup_m \sigma_m$ is established by induction on m.

For the converse direction we prove $\alpha < \vartheta(\Omega^\Omega)$ implies $\alpha < \sigma_m$ for some m by induction on the rank of α. If $\alpha =_{\mathrm{NF}} \omega^\gamma + \delta$ for some γ, δ, one easily obtains $\alpha < \sigma_m$ by the induction hypothesis, so suppose

$$\alpha = \vartheta\xi < \vartheta(\Omega^\Omega),$$

for which there are two cases to consider:

1. $\xi < \Omega^\Omega$ and $\xi^* < \vartheta(\Omega^\Omega)$; or
2. $\xi > \Omega^\Omega$ but $\vartheta\xi \leq (\Omega^\Omega)^*$.

Since $(\Omega^\Omega)^* = 0$, the latter is impossible. From the former, however, one obtains $\alpha < \vartheta(\Omega^\Omega)$ via the induction hypothesis. □

Our motivation for defining the theory HST_β as we did stemmed from the idea of formalising the acceptance of F. The theory S_3 with just one truth predicate appears to almost achieve this, but the general inability to close S_3 under the rule T-Intro means the truth predicate no longer satisfies the same principles as it did in F. This lead us to consider stratifying the language, viewing the original predicate of F, now T_0, as the base level and gradually extending the language by including predicates T_1, T_2, etc. in such a way that each predicate T_η in the language locally satisfies the same axioms and rules as T_0.

The analysis of the theories HST_β reveals that stratification of the language did not lead us as far from the world of a single self-applicable truth predicate as might have first appeared. Indeed, theorem 4.5 and **??** show the truth predicates of HST^1_β may be treated as identical; they can all be interpreted as the set $\mathscr{T}_{<\Omega^{\beta+1}}$. Within HST^2_β they may all be interpreted as the set $\mathscr{T}_{<\Omega^{\beta+1}\cdot 2}$ and, in general, all truth predicates in HST_β can be interpreted as the set $\mathscr{T}_{<\Omega^{\beta+2}}$ (one cannot simply use $\mathscr{T}_{<\Omega^{\beta+1}\cdot\omega}$ for the interpretation of T_β in HST_β as the set is not closed under the ω-rule, whereas $\mathscr{T}_{<\Omega^{\beta+2}}$ is, as well being closed under conec_η, $\mathrm{nec}_\eta$for every $\eta \leq \beta$.) The upshot is that we may view each predicate T_η as "extending" the base predicate T_0 as well as T_ξ for $\xi < \eta$. It would be interesting to determine whether the theory HST_β can be rewritten in some natural type-free form.

The model constructions employed in the previous section for the analysis of HST_1 allow us to obtain an upper bound for the theory S_3 introduced in **??**. Essentially, we stratify the language $\mathscr{L}_\mathsf{T}$ as described in remark 1, interpreting the outermost truth predicate by T_1 and all others by T_0, but by first embedding S_3 in an infinitary theory formulated without T-Elim, we avoid the problems relating to conec_0 and conec_1.

thm:S3strength **Theorem 8.4.** S_3 *proves the same arithmetical statements as* HST^1_1 *and hence has proof-theoretic ordinal* Γ_0.

Proof. We define an infinitary proof system $\mathscr{S}_\infty$ based on $\mathscr{T}_\infty$ into which we may embed S_3. Let * be the interpretation of $\mathscr{L}_0$ into $\mathscr{L}_\mathsf{T}$ that recursively interprets the predicate T_0 as T and otherwise commutes with all connectives and quantifiers. Define $\mathscr{S}_\infty \vdash^{\alpha} \Gamma$ according to the rules (Ax.1), (Ax.2), (Ax.3), ($\wedge$), ($\vee_i$), (ω), ($\exists$), (T-Imp), (T-Del), (T-Rep), (T-ω), and the following additional rule

(Ax.4)$^{\mathscr{T}}$ $\vdash^{\beta}_{\Omega\cdot\gamma} A$ for some $\gamma, \beta < \Omega$ implies $\mathscr{S}_\infty \vdash^{\alpha} \Gamma, \mathsf{T}(\ulcorner A^* \urcorner)$ for any $\alpha > \max\{\beta, \gamma\}$.

The Bounding lemma entails, for $\gamma < \Omega$, that

$$\vdash^{\alpha}_{\Omega\cdot\gamma} \Gamma \text{ implies } \mathscr{S}_\infty \vdash^{\alpha} \Gamma^*. \tag{37}$$ {eqn:S81}

Define a sequence of $\mathscr{L}_\mathsf{T}$-structures

$$\mathfrak{M}_\alpha = \langle \mathbb{N}, \{\ulcorner A^* \urcorner : \vdash^{\beta}_{\Omega\cdot\gamma} A \text{ for some } \gamma < \alpha \text{ and } \beta < \vartheta(\Omega\cdot\alpha)\} \rangle.$$

We claim

$$\mathscr{S}_\infty \vdash^{\alpha} \Gamma \text{ implies } \mathfrak{M}_\alpha \models \bigvee \Gamma \tag{38}$$ {eqn:S82}

whenever Γ is T-positive. The proof proceeds by transfinite induction on α. If Γ is an instance of (Ax.4)$^{\mathscr{T}}$, $\mathfrak{M}_\alpha \models \bigvee \Gamma$ holds by definition, while if Γ is derived through an application of (T-Rep), it follows from the induction hypothesis and closure of $\mathscr{T}_\infty$ under (nec_0). If the last applied rule is (T-ω), $\mathfrak{M}_\alpha \models \bigvee \Gamma$ holds by an application of (ω) in $\mathscr{T}_\infty$ and the fact $\vartheta(\Omega\cdot\alpha)$ is increasing in α. Furthermore, by the definition of the function ϑ, we have $\vartheta(\Omega\cdot\delta+\beta) < \vartheta(\Omega\cdot\alpha)$ whenever $\delta < \alpha$ and $\beta < \vartheta(\Omega\cdot\alpha)$; thus corollary 7.9 implies

$$\mathfrak{M}_\alpha \models \forall \ulcorner A \urcorner (\mathsf{T}(\ulcorner \mathsf{T}(\ulcorner A \urcorner) \urcorner) \to \mathsf{T}(\ulcorner A \urcorner)),$$

and we may deduce $\mathfrak{M}_\alpha \models \bigvee \Gamma$ from the induction hypothesis if the last rule applied was (T-Del).

(38) can now be utilised along with (37) to conclude

$$\mathscr{S}_\infty \vdash^{\alpha} \mathsf{T}(\ulcorner A \urcorner) \text{ implies } \mathscr{S}_\infty \vdash^{\vartheta(\Omega\cdot\alpha)} A.$$

Since $\vartheta\Omega^2$ is the least ordinal closed under the function $\alpha \mapsto \vartheta(\Omega\cdot\alpha)$ (see proposition 5.5) we may deduce $\mathsf{S}_3 \vdash A$ implies $\mathscr{S}_\infty \vdash^{\alpha} A$ for some $\alpha < \vartheta\Omega^2$ for any sentence A, whence $\|\mathsf{S}_3\| \leq \vartheta\Omega^2$. Finally, note $\vartheta\Omega^2 = \Gamma_0$ by corollary 5.6. □

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