Component Analysis V. Sobol Analysis

(What I've been working on in Matlab)

System Notation

$$\dot{\vec{x}} = M\vec{x} \tag{1}$$

$$\begin{bmatrix} \dot{\vec{x}}_1 \\ \dot{\vec{x}}_2 \\ \dot{\vec{x}}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_1 & B_{1,2} & B_{1,3} & \dots \\ B_{2,1} & A_2 & B_{2,3} & \dots \\ B_{3,1} & B_{3,2} & A_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \\ \vdots \end{bmatrix} \tag{2}$$

If B matrices are all 0 we have a series of completely independent linear systems.

If B matrices are sparse, we have “compartments” that are connected to form the overall system.

Parameter Notation

zoom in on one of those compartments and define parameters

$$\dot{\vec{x}}_i = A_i(\vec{\theta})\vec{x}_i \tag{3}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \theta_1 a_{1,1} & a_{1,2} & \theta_2 a_{1,3} & \dots \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots \\ \theta_3 a_{3,1} & a_{3,2} & \theta_2 a_{3,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{bmatrix} \tag{4}$$

locations of θ_i would depend upon actual model

Two Approaches for Model Reduction

Sobol

- Assume $\theta_i \sim U[0.9, 1.1]$
- Pick number of samples to evaluate, N_{Sobol}
- Compute Sobol index (S_i) and total Sobol index (ST_i) of each parameter

Component

- Run model normally: $\dot{\vec{x}} = M(\vec{\theta})\vec{x}$
- Run model without one component: $\dot{\vec{x}} = M(\vec{\theta})\vec{x}, \theta_i = 0$
- Quantify how much these two runs differ (C_i)
- Repeat for all components $i = 1, 2, \ldots, N_\theta$

Want to show that small S_i or $ST_i \not\Rightarrow C_i$ small

Building the system

inputs: $[N_1, N_2, \ldots],\ I_{max},\ I_{min},\ N_{connections}, x_{total}$

outputs: $\begin{bmatrix} A_1 & B_{1,2} & B_{1,3} & \ldots \\ B_{2,1} & A_2 & B_{2,3} & \ldots \\ B_{3,1} & B_{3,2} & A_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \vec{x_0}$

with

$A_i \in \mathbb{R}^{N_i \times N_i}$

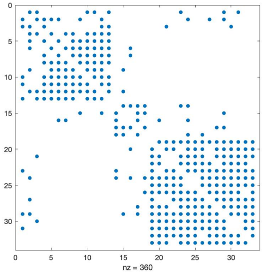
$a_{i,j}, b_{i,j} \in \mathbb{Z} \cup [I_{min}, I_{max}] \qquad a_{i,j} = -a_{j,i} \qquad B_{i,j} = -B_{j,i}^T$

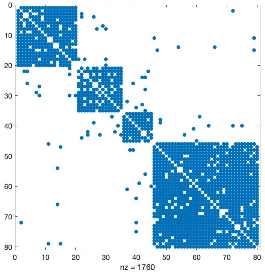
$\sum_{i,j} ||B_{i,j}||_0 = N_{connections}$

$||\vec{x_0}||_1 = x_{total} \qquad |\vec{x_0}| = \vec{x_0}$

Building the system

Example matrices:





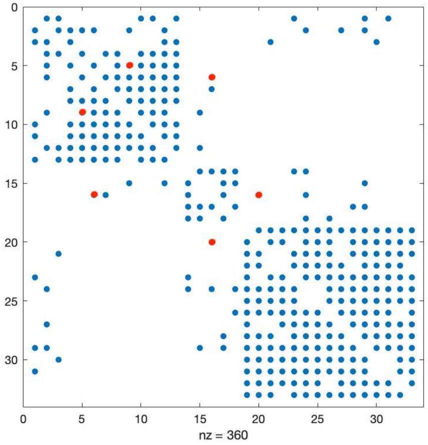
Selecting Parameters

inputs: $N_{\theta in}, N_{\theta out}$ $\qquad$ $N_\theta = N_{\theta in} + N_{\theta out}$

outputs: list of indexes $[i, j]$ to be treated as parameters with
$N_{\theta in}$ parameters chosen "inside" compartments
$N_{\theta out}$ parameters chosen "outside" compartments
All parameters non-zero elements of M

*currently parameters continue skew symmetry of M

Selecting Parameters



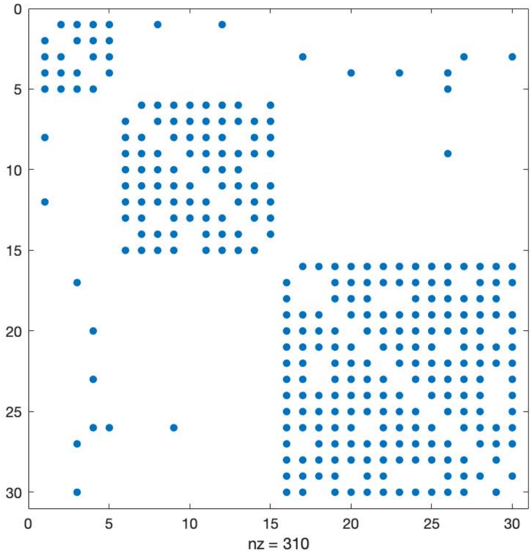
Running the Analysis 1

QoI $=$ final value of final state variable

Component Analysis: $C_i = \frac{|\text{nominal QoI} - \text{altered QoI}|}{\text{nominal QoI}}$

rank parameters by increasing C_i and increasing ST_i, compare lists

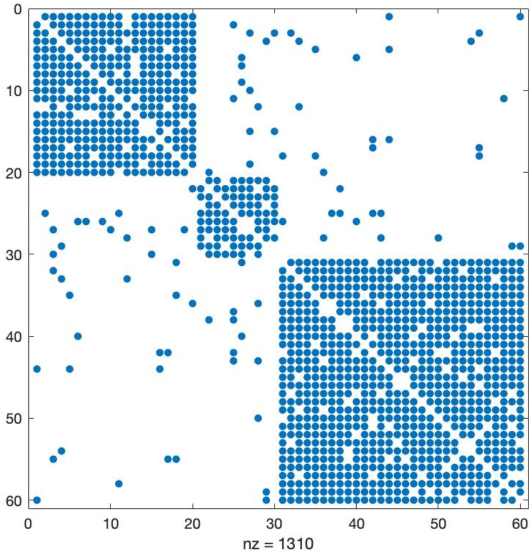
Running the Analysis 1



Running the Analysis 1

Parameter	Ranking by C	Ranking by S	Ranking by ST
inside 1	7	10	10
inside 2	5	5	4
inside 3	8	7	9
inside 4	2	3	3
inside 5	9	6	5
outside 1	1	8	6
outside 2	6	9	8
outside 3	4	2	2
outside 4	10	4	7
outside 5	3	1	1

Running the Analysis 2



Running the Analysis 1

Parameter	Ranking by C	Ranking by S	Ranking by ST
inside 1	3	7	7
inside 2	1	2	3
inside 3	8	8	8
outside 1	7	10	9
outside 2	10	6	6
outside 3	4	3	2
outside 4	2	9	10
outside 5	6	1	1
outside 6	9	5	5
outside 7	5	4	4

Questions

- Keep parameters mirrored?
- Keep M skew-symmetric? ($\dot{\vec{x}} = Me^{-t}\vec{x}$)
- C_i computed using all state variables (inner-outer norm)
- How to compare different systems? (Distributions)
- DIfferent QoI?