Система уравнений:

$$\frac{\partial\theta}{\partial t} - a\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right) + b\kappa_a(\theta^4 - \varphi) = 0,$$

$$-\alpha\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) + \kappa_a(\varphi - \theta^4) = 0,$$

$$\Omega = \{(x, y)\colon 0 < x < L,\ 0 < y < L,\ 0 < z < L\},$$

$$-a\frac{\partial\theta}{\partial x} + \beta(\theta - \theta_b)\bigg|_{x=0} = 0, \quad a\frac{\partial\theta}{\partial x} + \beta(\theta - \theta_b)\bigg|_{x=L} = 0,$$

$$-a\frac{\partial\theta}{\partial y} + \beta(\theta - \theta_b)\bigg|_{y=0} = 0, \quad a\frac{\partial\theta}{\partial y} + \beta(\theta - \theta_b)\bigg|_{y=L} = 0,$$

$$-a\frac{\partial\theta}{\partial z} + \beta(\theta - \theta_b)\bigg|_{z=0} = 0, \quad a\frac{\partial\theta}{\partial z} + \beta(\theta - \theta_b)\bigg|_{z=L} = 0,$$

$$-\alpha\frac{\partial\varphi}{\partial x} + u(\varphi - \theta_b^4)\bigg|_{x=0} = 0, \quad \alpha\frac{\partial\varphi}{\partial x} + u(\varphi - \theta_b^4)\bigg|_{x=L} = 0,$$

$$-\alpha\frac{\partial\varphi}{\partial y} + u(\varphi - \theta_b^4)\bigg|_{y=0} = 0, \quad \alpha\frac{\partial\varphi}{\partial y} + u(\varphi - \theta_b^4)\bigg|_{y=L} = 0,$$

$$-\alpha\frac{\partial\varphi}{\partial z} + u(\varphi - \theta_b^4)\bigg|_{z=0} = 0, \quad \alpha\frac{\partial\varphi}{\partial z} + u(\varphi - \theta_b^4)\bigg|_{z=L} = 0,$$

$$\theta|_{t=0} = \theta_0.$$

Введем сетку

$$\overline{\omega} = \{(x_i, y_j, z_k, t_m)\colon x_i = ih, y_j = jh, z_k = kh, t_m = m\tau, i = \overline{0, N}, j = \overline{0, N}, k = \overline{0, N}, m = \overline{0, M}\},$$

$$h = L/N, \quad \tau = T/M.$$

Схема Кранка-Николсон (метод трапеций) с линеаризацией методом Ньютона. Граничные условия аппроксимируем со вторым порядком, аппроксимация граничных условий может быть получена методом фиктивных точек [1, с. 306].

Разностная схема имеет следующий вид (в уравнении для φ нет полусуммы!).

$0 < i < N$, $0 < j < N$, $0 < k < N$:

$$\frac{\theta_{i,j,k}^{m+1} - \theta_{i,j,k}^{m}}{\tau} + \frac{1}{2}\Big[-\frac{a}{h^2}(\theta_{i-1,j,k}^{m+1} - 2\theta_{i,j,k}^{m+1} + \theta_{i+1,j,k}^{m+1}) - \frac{a}{h^2}(\theta_{i,j-1,k}^{m+1} - 2\theta_{i,j,k}^{m+1} + \theta_{i,j+1,k}^{m+1}) -$$
$$-\frac{a}{h^2}(\theta_{i,j,k-1}^{m+1} - 2\theta_{i,j,k}^{m+1} + \theta_{i,j,k+1}^{m+1}) + b\kappa_a(4(\widetilde{\theta}_{i,j,k}^{m+1})^3\theta_{i,j,k}^{m+1} - 3(\widetilde{\theta}_{i,j,k}^{m+1})^4 - \varphi_{i,j,k}^{m+1})\Big] +$$
$$+\frac{1}{2}\Big[-\frac{a}{h^2}(\theta_{i-1,j,k}^{m} - 2\theta_{i,j,k}^{m} + \theta_{i+1,j,k}^{m}) - \frac{a}{h^2}(\theta_{i,j-1,k}^{m} - 2\theta_{i,j,k}^{m} + \theta_{i,j+1,k}^{m}) -$$
$$-\frac{a}{h^2}(\theta_{i,j,k-1}^{m} - 2\theta_{i,j,k}^{m} + \theta_{i,j,k+1}^{m}) + b\kappa_a((\theta_{i,j,k}^{m})^4 - \varphi_{i,j,k}^{m})\Big] = 0,$$

$$-\frac{\alpha}{h^2}(\varphi_{i-1,j,k}^{m+1} - 2\varphi_{i,j,k}^{m+1} + \varphi_{i+1,j,k}^{m+1}) - \frac{\alpha}{h^2}(\varphi_{i,j-1,k}^{m+1} - 2\varphi_{i,j,k}^{m+1} + \varphi_{i,j+1,k}^{m+1}) -$$
$$-\frac{\alpha}{h^2}(\varphi_{i,j,k-1}^{m+1} - 2\varphi_{i,j,k}^{m+1} + \varphi_{i,j,k+1}^{m+1}) + \kappa_a(\varphi_{i,j,k}^{m+1} - 4(\widetilde{\theta}_{i,j,k}^{m+1})^3\theta_{i,j,k}^{m+1} + 3(\widetilde{\theta}_{i,j,k}^{m+1})^4) = 0.$$

$i=0,\ 0<j<N,\ 0<k<N$:

$$\frac{1}{2}\Big[-\frac{a}{h}(\theta^{m+1}_{1,j,k}-\theta^{m+1}_{0,j,k})+\beta(\theta^{m+1}_{0,j,k}-\theta_b)-\frac{a}{2h}(\theta^{m+1}_{0,j-1,k}-2\theta^{m+1}_{0,j,k}+\theta^{m+1}_{0,j+1,k})-$$

$$-\frac{a}{2h}(\theta^{m+1}_{0,j,k-1}-2\theta^{m+1}_{0,j,k}+\theta^{m+1}_{0,j,k+1})+\frac{b\kappa_a h}{2}(4(\widetilde{\theta}^{m+1}_{0,j,k})^3\theta^{m+1}_{0,j,k}-3(\widetilde{\theta}^{m+1}_{0,j,k})^4-\varphi^{m+1}_{0,j,k})\Big]+$$

$$+\frac{1}{2}\Big[-\frac{a}{h}(\theta^{m}_{1,j,k}-\theta^{m}_{0,j,k})+\beta(\theta^{m}_{0,j,k}-\theta_b)-\frac{a}{2h}(\theta^{m}_{0,j-1,k}-2\theta^{m}_{0,j,k}+\theta^{m}_{0,j+1,k})-$$

$$-\frac{a}{2h}(\theta^{m}_{0,j,k-1}-2\theta^{m}_{0,j,k}+\theta^{m}_{0,j,k+1})+\frac{b\kappa_a h}{2}((\theta^{m}_{0,j,k})^4-\varphi^{m}_{0,j,k})\Big]+\frac{h}{2\tau}(\theta^{m+1}_{0,j,k}-\theta^{m}_{0,j,k})=0,$$

$$-\frac{\alpha}{h}(\varphi^{m+1}_{1,j,k}-\varphi^{m+1}_{0,j,k})+u_{0,j,k}(\varphi^{m+1}_{0,j,k}-\theta_b^4)-\frac{\alpha}{2h}(\varphi^{m+1}_{0,j-1,k}-2\varphi^{m+1}_{0,j,k}+\varphi^{m+1}_{0,j+1,k})-$$

$$-\frac{\alpha}{2h}(\varphi^{m+1}_{0,j,k-1}-2\varphi^{m+1}_{0,j,k}+\varphi^{m+1}_{0,j,k+1})+\frac{\kappa_a h}{2}(\varphi^{m+1}_{0,j,k}-4(\widetilde{\theta}^{m+1}_{0,j,k})^3\theta^{m+1}_{0,j,k}+3(\widetilde{\theta}^{m+1}_{0,j,k})^4)=0.$$

$i=0,\ j=0,\ 0<k<N$:

$$\frac{1}{2}\Big[-\frac{a}{h}(\theta^{m+1}_{1,0,k}-\theta^{m+1}_{0,0,k})+\beta(\theta^{m+1}_{0,0,k}-\theta_b)-\frac{a}{h}(\theta^{m+1}_{0,1,k}-\theta^{m+1}_{0,0,k})+\beta(\theta^{m+1}_{0,0,k}-\theta_b)-$$

$$-\frac{a}{2h}(\theta^{m+1}_{0,0,k-1}-2\theta^{m+1}_{0,0,k}+\theta^{m+1}_{0,0,k+1})+\frac{b\kappa_a h}{2}(4(\widetilde{\theta}^{m+1}_{0,0,k})^3\theta^{m+1}_{0,0,k}-3(\widetilde{\theta}^{m+1}_{0,0,k})^4-\varphi^{m+1}_{0,0,k})\Big]+$$

$$+\frac{1}{2}\Big[-\frac{a}{h}(\theta^{m}_{1,0,k}-\theta^{m}_{0,0,k})+\beta(\theta^{m}_{0,0,k}-\theta_b)-\frac{a}{h}(\theta^{m}_{0,1,k}-\theta^{m}_{0,0,k})+\beta(\theta^{m}_{0,0,k}-\theta_b)-$$

$$-\frac{a}{2h}(\theta^{m}_{0,0,k-1}-2\theta^{m}_{0,0,k}+\theta^{m}_{0,0,k+1})+\frac{b\kappa_a h}{2}((\theta^{m}_{0,0,k})^4-\varphi^{m}_{0,0,k})\Big]+\frac{h}{2\tau}(\theta^{m+1}_{0,0,k}-\theta^{m}_{0,0,k})=0,$$

$$-\frac{\alpha}{h}(\varphi^{m+1}_{1,0,k}-\varphi^{m+1}_{0,0,k})+u_{0,0,k}(\varphi^{m+1}_{0,0,k}-\theta_b^4)-\frac{\alpha}{h}(\varphi^{m+1}_{0,1,k}-\varphi^{m+1}_{0,0,k})+u_{0,0,k}(\varphi^{m+1}_{0,0,k}-\theta_b^4)-$$

$$-\frac{\alpha}{2h}(\varphi^{m+1}_{0,0,k-1}-2\varphi^{m+1}_{0,0,k}+\varphi^{m+1}_{0,0,k+1})+\frac{\kappa_a h}{2}(\varphi^{m+1}_{0,0,k}-4(\widetilde{\theta}^{m+1}_{0,0,k})^3\theta^{m+1}_{0,0,k}+3(\widetilde{\theta}^{m+1}_{0,0,k})^4)=0.$$

$i=0,\ j=0,\ k=0$:

$$\frac{1}{2}\Big[-\frac{a}{h}(\theta^{m+1}_{1,0,0}-\theta^{m+1}_{0,0,0})+\beta(\theta^{m+1}_{0,0,0}-\theta_b)-\frac{a}{h}(\theta^{m+1}_{0,1,0}-\theta^{m+1}_{0,0,0})+\beta(\theta^{m+1}_{0,0,0}-\theta_b)-$$

$$-\frac{a}{h}(\theta^{m+1}_{0,0,1}-\theta^{m+1}_{0,0,0})+\beta(\theta^{m+1}_{0,0,0}-\theta_b)+\frac{b\kappa_a h}{2}(4(\widetilde{\theta}^{m+1}_{0,0,0})^3\theta^{m+1}_{0,0,0}-3(\widetilde{\theta}^{m+1}_{0,0,0})^4-\varphi^{m+1}_{0,0,0})\Big]+$$

$$+\frac{1}{2}\Big[-\frac{a}{h}(\theta^{m}_{1,0,0}-\theta^{m}_{0,0,0})+\beta(\theta^{m}_{0,0,0}-\theta_b)-\frac{a}{h}(\theta^{m}_{0,1,0}-\theta^{m}_{0,0,0})+\beta(\theta^{m}_{0,0,0}-\theta_b)-$$

$$-\frac{a}{h}(\theta^{m}_{0,0,1}-\theta^{m}_{0,0,0})+\beta(\theta^{m}_{0,0,0}-\theta_b)+\frac{b\kappa_a h}{2}((\theta^{m}_{0,0,0})^4-\varphi^{m}_{0,0,0})\Big]+\frac{h}{2\tau}(\theta^{m+1}_{0,0,0}-\theta^{m}_{0,0,0})=0,$$

$$-\frac{\alpha}{h}(\varphi^{m+1}_{1,0,0}-\varphi^{m+1}_{0,0,0})+u_{0,0,0}(\varphi^{m+1}_{0,0,0}-\theta_b^4)-\frac{\alpha}{h}(\varphi^{m+1}_{0,1,0}-\varphi^{m+1}_{0,0,0})+u_{0,0,0}(\varphi^{m+1}_{0,0,0}-\theta_b^4)-$$

$$-\frac{\alpha}{h}(\varphi^{m+1}_{0,0,1}-\varphi^{m+1}_{0,0,0})+u_{0,0,0}(\varphi^{m+1}_{0,0,0}-\theta_b^4)+\frac{\kappa_a h}{2}(\varphi^{m+1}_{0,0,0}-4(\widetilde{\theta}^{m+1}_{0,0,0})^3\theta^{m+1}_{0,0,0}+3(\widetilde{\theta}^{m+1}_{0,0,0})^4)=0.$$

Для остальных случаев схема записывается симметрично.

Список литературы

[1] Калиткин Н.Н. Численные методы. М.: Наука, 1978.