

Система уравнений:

$$\begin{aligned}
& \frac{\partial \theta}{\partial t} - a \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) + b\kappa_a(\theta^4 - \varphi) = 0, \\
& -\alpha \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) + \kappa_a(\varphi - \theta^4) = 0, \\
& \Omega = \{(x, y) : 0 < x < L, 0 < y < L, 0 < z < L\}, \\
& -a \frac{\partial \theta}{\partial x} + \beta(\theta - \theta_b) \Big|_{x=0} = 0, \quad a \frac{\partial \theta}{\partial x} + \beta(\theta - \theta_b) \Big|_{x=L} = 0, \\
& -a \frac{\partial \theta}{\partial y} + \beta(\theta - \theta_b) \Big|_{y=0} = 0, \quad a \frac{\partial \theta}{\partial y} + \beta(\theta - \theta_b) \Big|_{y=L} = 0, \\
& -a \frac{\partial \theta}{\partial z} + \beta(\theta - \theta_b) \Big|_{z=0} = 0, \quad a \frac{\partial \theta}{\partial z} + \beta(\theta - \theta_b) \Big|_{z=L} = 0, \\
& -\alpha \frac{\partial \varphi}{\partial x} + u(\varphi - \theta_b^4) \Big|_{x=0} = 0, \quad \alpha \frac{\partial \varphi}{\partial x} + u(\varphi - \theta_b^4) \Big|_{x=L} = 0, \\
& -\alpha \frac{\partial \varphi}{\partial y} + u(\varphi - \theta_b^4) \Big|_{y=0} = 0, \quad \alpha \frac{\partial \varphi}{\partial y} + u(\varphi - \theta_b^4) \Big|_{y=L} = 0, \\
& -\alpha \frac{\partial \varphi}{\partial z} + u(\varphi - \theta_b^4) \Big|_{z=0} = 0, \quad \alpha \frac{\partial \varphi}{\partial z} + u(\varphi - \theta_b^4) \Big|_{z=L} = 0, \\
& \theta|_{t=0} = \theta_0.
\end{aligned}$$

Введем сетку

$$\begin{aligned}
\bar{\omega} &= \{(x_i, y_j, z_k, t_m) : x_i = ih, y_j = jh, z_k = kh, t_m = m\tau, i = \overline{0, N}, j = \overline{0, N}, k = \overline{0, N}, m = \overline{0, M}\}, \\
h &= L/N, \quad \tau = T/M.
\end{aligned}$$

Схема Кранка-Николсон (метод трапеций) с линеаризацией методом Ньютона. Граничные условия аппроксимируем со вторым порядком, аппроксимация граничных условий может быть получена методом фиктивных точек [1, с. 306].

Разностная схема имеет следующий вид (в уравнении для φ нет полусуммы!).

$0 < i < N, 0 < j < N, 0 < k < N$:

$$\begin{aligned}
& \frac{\theta_{i,j,k}^{m+1} - \theta_{i,j,k}^m}{\tau} + \frac{1}{2} \left[-\frac{a}{h^2}(\theta_{i-1,j,k}^{m+1} - 2\theta_{i,j,k}^{m+1} + \theta_{i+1,j,k}^{m+1}) - \frac{a}{h^2}(\theta_{i,j-1,k}^{m+1} - 2\theta_{i,j,k}^{m+1} + \theta_{i,j+1,k}^{m+1}) - \right. \\
& \quad \left. - \frac{a}{h^2}(\theta_{i,j,k-1}^{m+1} - 2\theta_{i,j,k}^{m+1} + \theta_{i,j,k+1}^{m+1}) + b\kappa_a((\theta_{i,j,k}^{m+1})^4 - \varphi_{i,j,k}^{m+1}) \right] + \\
& + \frac{1}{2} \left[-\frac{a}{h^2}(\theta_{i-1,j,k}^m - 2\theta_{i,j,k}^m + \theta_{i+1,j,k}^m) - \frac{a}{h^2}(\theta_{i,j-1,k}^m - 2\theta_{i,j,k}^m + \theta_{i,j+1,k}^m) - \right. \\
& \quad \left. - \frac{a}{h^2}(\theta_{i,j,k-1}^m - 2\theta_{i,j,k}^m + \theta_{i,j,k+1}^m) + b\kappa_a((\theta_{i,j,k}^m)^4 - \varphi_{i,j,k}^m) \right] = 0, \\
& -\frac{\alpha}{h^2}(\varphi_{i-1,j,k}^{m+1} - 2\varphi_{i,j,k}^{m+1} + \varphi_{i+1,j,k}^{m+1}) - \frac{\alpha}{h^2}(\varphi_{i,j-1,k}^{m+1} - 2\varphi_{i,j,k}^{m+1} + \varphi_{i,j+1,k}^{m+1}) - \\
& -\frac{\alpha}{h^2}(\varphi_{i,j,k-1}^{m+1} - 2\varphi_{i,j,k}^{m+1} + \varphi_{i,j,k+1}^{m+1}) + \kappa_a(\varphi_{i,j,k}^{m+1} - (\theta_{i,j,k}^{m+1})^4) = 0.
\end{aligned}$$

$i = 0, 0 < j < N, 0 < k < N$:

$$\begin{aligned}
& \frac{1}{2} \left[-\frac{a}{h}(\theta_{1,j,k}^{m+1} - \theta_{0,j,k}^{m+1}) + \beta(\theta_{0,j,k}^{m+1} - \theta_b) - \frac{a}{2h}(\theta_{0,j-1,k}^{m+1} - 2\theta_{0,j,k}^{m+1} + \theta_{0,j+1,k}^{m+1}) - \right. \\
& \quad \left. - \frac{a}{2h}(\theta_{0,j,k-1}^{m+1} - 2\theta_{0,j,k}^{m+1} + \theta_{0,j,k+1}^{m+1}) + \frac{b\kappa_a h}{2}((\theta_{0,j,k}^{m+1})^4 - \varphi_{0,j,k}^{m+1}) \right] + \\
& \quad + \frac{1}{2} \left[-\frac{a}{h}(\theta_{1,j,k}^m - \theta_{0,j,k}^m) + \beta(\theta_{0,j,k}^m - \theta_b) - \frac{a}{2h}(\theta_{0,j-1,k}^m - 2\theta_{0,j,k}^m + \theta_{0,j+1,k}^m) - \right. \\
& \quad \left. - \frac{a}{2h}(\theta_{0,j,k-1}^m - 2\theta_{0,j,k}^m + \theta_{0,j,k+1}^m) + \frac{b\kappa_a h}{2}((\theta_{0,j,k}^m)^4 - \varphi_{0,j,k}^m) \right] + \frac{h}{2\tau}(\theta_{0,j,k}^{m+1} - \theta_{0,j,k}^m) = 0, \\
& -\frac{\alpha}{h}(\varphi_{1,j,k}^{m+1} - \varphi_{0,j,k}^{m+1}) + u_{0,j,k}(\varphi_{0,j,k}^{m+1} - \theta_b^4) - \frac{\alpha}{2h}(\varphi_{0,j-1,k}^{m+1} - 2\varphi_{0,j,k}^{m+1} + \varphi_{0,j+1,k}^{m+1}) - \\
& \quad - \frac{\alpha}{2h}(\varphi_{0,j,k-1}^{m+1} - 2\varphi_{0,j,k}^{m+1} + \varphi_{0,j,k+1}^{m+1}) + \frac{\kappa_a h}{2}(\varphi_{0,j,k}^{m+1} - (\theta_{0,j,k}^{m+1})^4) = 0.
\end{aligned}$$

$i = 0, j = 0, 0 < k < N$:

$$\begin{aligned}
& \frac{1}{2} \left[-\frac{a}{h}(\theta_{1,0,k}^{m+1} - \theta_{0,0,k}^{m+1}) + \beta(\theta_{0,0,k}^{m+1} - \theta_b) - \frac{a}{h}(\theta_{0,1,k}^{m+1} - \theta_{0,0,k}^{m+1}) + \beta(\theta_{0,0,k}^{m+1} - \theta_b) - \right. \\
& \quad \left. - \frac{a}{2h}(\theta_{0,0,k-1}^{m+1} - 2\theta_{0,0,k}^{m+1} + \theta_{0,0,k+1}^{m+1}) + \frac{b\kappa_a h}{2}((\theta_{0,0,k}^{m+1})^4 - \varphi_{0,0,k}^{m+1}) \right] + \\
& \quad + \frac{1}{2} \left[-\frac{a}{h}(\theta_{1,0,k}^m - \theta_{0,0,k}^m) + \beta(\theta_{0,0,k}^m - \theta_b) - \frac{a}{h}(\theta_{0,1,k}^m - \theta_{0,0,k}^m) + \beta(\theta_{0,0,k}^m - \theta_b) - \right. \\
& \quad \left. - \frac{a}{2h}(\theta_{0,0,k-1}^m - 2\theta_{0,0,k}^m + \theta_{0,0,k+1}^m) + \frac{b\kappa_a h}{2}((\theta_{0,0,k}^m)^4 - \varphi_{0,0,k}^m) \right] + \frac{h}{2\tau}(\theta_{0,0,k}^{m+1} - \theta_{0,0,k}^m) = 0, \\
& -\frac{\alpha}{h}(\varphi_{1,0,k}^{m+1} - \varphi_{0,0,k}^{m+1}) + u_{0,0,k}(\varphi_{0,0,k}^{m+1} - \theta_b^4) - \frac{\alpha}{h}(\varphi_{0,1,k}^{m+1} - \varphi_{0,0,k}^{m+1}) + u_{0,0,k}(\varphi_{0,0,k}^{m+1} - \theta_b^4) - \\
& \quad - \frac{\alpha}{2h}(\varphi_{0,0,k-1}^{m+1} - 2\varphi_{0,0,k}^{m+1} + \varphi_{0,0,k+1}^{m+1}) + \frac{\kappa_a h}{2}(\varphi_{0,0,k}^{m+1} - (\theta_{0,0,k}^{m+1})^4) = 0.
\end{aligned}$$

$i = 0, j = 0, k = 0$:

$$\begin{aligned}
& \frac{1}{2} \left[-\frac{a}{h}(\theta_{1,0,0}^{m+1} - \theta_{0,0,0}^{m+1}) + \beta(\theta_{0,0,0}^{m+1} - \theta_b) - \frac{a}{h}(\theta_{0,1,0}^{m+1} - \theta_{0,0,0}^{m+1}) + \beta(\theta_{0,0,0}^{m+1} - \theta_b) - \right. \\
& \quad \left. - \frac{a}{h}(\theta_{0,0,1}^{m+1} - \theta_{0,0,0}^{m+1}) + \beta(\theta_{0,0,0}^{m+1} - \theta_b) + \frac{b\kappa_a h}{2}((\theta_{0,0,0}^{m+1})^4 - \varphi_{0,0,0}^{m+1}) \right] + \\
& \quad + \frac{1}{2} \left[-\frac{a}{h}(\theta_{1,0,0}^m - \theta_{0,0,0}^m) + \beta(\theta_{0,0,0}^m - \theta_b) - \frac{a}{h}(\theta_{0,1,0}^m - \theta_{0,0,0}^m) + \beta(\theta_{0,0,0}^m - \theta_b) - \right. \\
& \quad \left. - \frac{a}{h}(\theta_{0,0,1}^m - \theta_{0,0,0}^m) + \beta(\theta_{0,0,0}^m - \theta_b) + \frac{b\kappa_a h}{2}((\theta_{0,0,0}^m)^4 - \varphi_{0,0,0}^m) \right] + \frac{h}{2\tau}(\theta_{0,0,0}^{m+1} - \theta_{0,0,0}^m) = 0, \\
& -\frac{\alpha}{h}(\varphi_{1,0,0}^{m+1} - \varphi_{0,0,0}^{m+1}) + u_{0,0,0}(\varphi_{0,0,0}^{m+1} - \theta_b^4) - \frac{\alpha}{h}(\varphi_{0,1,0}^{m+1} - \varphi_{0,0,0}^{m+1}) + u_{0,0,0}(\varphi_{0,0,0}^{m+1} - \theta_b^4) - \\
& \quad - \frac{\alpha}{h}(\varphi_{0,0,1}^{m+1} - \varphi_{0,0,0}^{m+1}) + u_{0,0,0}(\varphi_{0,0,0}^{m+1} - \theta_b^4) + \frac{\kappa_a h}{2}(\varphi_{0,0,0}^{m+1} - (\theta_{0,0,0}^{m+1})^4) = 0.
\end{aligned}$$

Для остальных случаев схема записывается симметрично.

Список литературы

- [1] Калиткин Н.Н. Численные методы. М.: Наука, 1978.