

Real Analysis Notes

If $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

provided that $\frac{c}{d} \neq \frac{0}{1}$.

Strictly speaking, we need to show that these operations are **well-defined** or that they don't depend on the choice of representatives from the equivalence classes.

Definition. Suppose S is an ordered set, and $E \subseteq S$. If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say E is **bounded above** and we call β an **upper bound**. The terms **bounded below** and **lower bound** are defined similarly.

Definition. Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose there exists $\alpha \in S$ such that α is an upper bound for E and if $\gamma < \alpha$, then γ is not an upper bound for E , then α is the **least upper bound** of E or the **supremum** of E , and we write $\alpha = \sup E$. The **greatest lower bound** and **infimum** ($\inf E$) are defined similarly.

Example. Consider the set $\{r \in \mathbb{Q} : r^2 < 2\}$, which has no supremum in \mathbb{Q} .

Definition. An ordered set S has the **least-upper-bound property** if the following is true: if $E \subseteq S$, E is not empty, and E is bounded above, then $\sup E$ exists in S .

If an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

Definition. There exists an ordered field \mathbb{R} (called the **real numbers**) which has the least-upper-bound property, and it contains an isomorphic copy of \mathbb{Q} .

Finite ordered fields do not exist. Consider $0 \leq 1 \leq 1 + 1 \leq \dots$ which can't be a finite chain.

Dedekind Cuts

1. Define the elements of \mathbb{R} as subsets of \mathbb{Q} called **cuts**, where a cut is a subset α of \mathbb{Q} such that
 - (a) α is a nonempty proper subset of \mathbb{Q} ($\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$).
 - (b) If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.
 - (c) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$ (can't be in the set and be an upper bound).
2. Define an order on \mathbb{R} where $\alpha < \beta$ if and only if α is a proper subset of β .

3. Show that the ordered set \mathbb{R} has the least-upper-bound property. To do this, suppose A is a nonempty subset of \mathbb{R} that is bounded above. Let γ be the union of all $\alpha \in A$. Then show $\gamma \in \mathbb{R}$ and $\gamma = \sup A$.
4. For $\alpha, \beta \in \mathbb{R}$, define the sum $\alpha + \beta$ to be the set of all sums $r + s$ where $r \in \alpha$ and $s \in \beta$. Define $0^* = \{t \in \mathbb{Q} : t < 0\}$ then show axioms for addition in fields hold for \mathbb{R} , and that 0^* is the additive identity.
5. Show that if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$. This is part of showing that \mathbb{R} is an ordered field.
6. For $\alpha, \beta \in \mathbb{R}$, where $\alpha > 0^*$ and $\beta > 0^*$, define the product $\alpha\beta$ to be $\{p \in \mathbb{Q} : p \leq rs, r \in \alpha, s \in \beta, r > 0, s > 0\}$. Note that $\alpha\beta > 0^*$ if $\alpha > 0^*$ and $\beta > 0^*$, which is part of showing that \mathbb{R} is an ordered field.
7. Extend the definition of multiplication to all of \mathbb{R} by setting, for all $\alpha, \beta \in \mathbb{R}$, $\alpha 0^* = 0^* \alpha = 0^*$ and

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \alpha < 0^*, \beta < 0^* \\ -[(-\alpha)(\beta)] & \alpha < 0^*, \beta > 0^* \\ -[(\alpha)(-\beta)] & \alpha > 0^*, \beta < 0^* \end{cases}$$

then prove the distributive law.

8. Associate to each $r \in \mathbb{Q}$ the real number $r^* = \{t \in \mathbb{Q} : t < r\}$ and let $\mathbb{Q}^* = \{r^* : r \in \mathbb{Q}\}$. These are the rational cuts in \mathbb{R} .
9. Show that \mathbb{Q} is isomorphic to \mathbb{Q}^* as ordered fields.

Any two ordered fields with the least upper-bound-property are isomorphic.

Theorem. If $x, y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the **Archimedean property** of \mathbb{R} .

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$ and suppose the Archimedean property is false. Then y would be an upper bound of A . But then A would have a least upper bound. Say $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound. Thus, $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. But then $\alpha < (m+1)x$, which contradicts the fact that α is an upper bound of A . Thus, the Archimedean property must be true.

Theorem. If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$. We say that \mathbb{Q} is **dense** in \mathbb{R} .

Theorem. For every positive real number x and every positive integer n , there is exactly one positive real number y such that $y^n = x$.

Proof. There is at most one since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$. Let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$. Then E is nonempty since $t = \frac{x}{1+x} \implies 0 < t < 1 \implies t^n < t < x \implies t \in E$. We also know E is bounded above since $t > 1+x \implies t^n > t > x \implies t \notin E$ and t is an upper bound. Define $y = \sup E$. We can then show that $y^n < x$ and $y^n > x$ each lead to contradictions.