Real Analysis Notes

Rational Numbers and Bounds

If $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

provided that $\frac{c}{d} \neq \frac{0}{1}$.

Note. Strictly speaking, we need to show that these operations are **well-defined** or that they don't depend on the choice of representatives from the equivalence classes.

Definition. Suppose S is an ordered set, and $E \subseteq S$. If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say E is **bounded above** and we call β an **upper bound**. The terms **bounded below** and **lower bound** are defined similarly.

Definition. Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose there exists $\alpha \in S$ such that α is an upper bound for E and if $\gamma < \alpha$, then γ is not an upper bound for E, then α is the **least upper bound** of E or the **supremum** of E, and we write $\alpha = \sup E$. The **greatest lower bound** and **infimum** ($\inf E$) are defined similarly.

Example. Consider the set $\{r \in \mathbb{Q} : r^2 < 2\}$, which has no supremum in $\mathbb{Q}$.

Definition. An ordered set S has the **least-upper-bound property** if the following is true: if $E \subseteq S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Proposition. If an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

Definition. There exists an ordered field $\mathbb{R}$ (called the **real numbers**) which has the least-upper-bound property, and it contains an isomorphic copy of $\mathbb{Q}$.

Note. Finite ordered fields do not exist. Consider $0 \leq 1 \leq 1+1 \leq \ldots$ which can't be a finite chain.

Dedekind Cuts

1. Define the elements of $\mathbb{R}$ as subsets of $\mathbb{Q}$ called **cuts**, where a cut is a subset α of $\mathbb{Q}$ such that
   (a) α is a nonempty proper subset of $\mathbb{Q}$ ($\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$).
   (b) If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.

   (c) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$ (can't be in the set and be an upper bound).

2. Define an order on $\mathbb{R}$ where $\alpha < \beta$ if and only if α is a proper subset of β.

3. Show that the ordered set $\mathbb{R}$ has the least-upper-bound property. To do this, suppose A is a nonempty subset of $\mathbb{R}$ that is bounded above. Let γ be the union of all $\alpha \in A$. Then show $\gamma \in \mathbb{R}$ and $\gamma = \sup A$.

4. For $\alpha, \beta \in \mathbb{R}$, define the sum $\alpha + \beta$ to be the set of all sums $r + s$ where $r \in \alpha$ and $s \in \beta$. Define $0^* = \{t \in \mathbb{Q} : t < 0\}$ then show axioms for addition in fields hold for $\mathbb{R}$, and that 0^* is the additive identity.

5. Show that if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$. This is part of showing that $\mathbb{R}$ is an ordered field.

6. For $\alpha, \beta \in \mathbb{R}$, where $\alpha > 0^*$ and $\beta > 0^*$, define the product $\alpha\beta$ to be $\{p \in \mathbb{Q} : q \leq rs,\ r \in \alpha,\ s \in \beta,\ r > 0,\ s > 0\}$. Note that $\alpha\beta > 0^*$ if $\alpha > 0^*$ and $\beta > 0^*$, which is part of showing that $\mathbb{R}$ is an ordered field.

7. Extend the definition of multiplication to all of $\mathbb{R}$ by setting, for all $\alpha, \beta \in \mathbb{R}$, $\alpha 0^* = 0^* \alpha = 0^*$ and
$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \alpha < 0^*, \beta < 0^* \\ -[(-\alpha)(\beta)] & \alpha < 0^*, \beta > 0^* \\ -[(\alpha)(-\beta)] & \alpha > 0^*, \beta < 0^* \end{cases}$$
then prove the distributive law.

8. Associate to each $r \in \mathbb{Q}$ the real number $r^* = \{t \in \mathbb{Q} : t < r\}$ and let $\mathbb{Q}^* = \{r^* : r \in \mathbb{Q}\}$. These are the rational cuts in $\mathbb{R}$.

9. Show that $\mathbb{Q}$ is isomorphic to $\mathbb{Q}^*$ as ordered fields.

Properties of Real Numbers

Theorem. Any two ordered fields with the least upper-bound-property are isomorphic.

Theorem. If $x, y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the **Archimedean property** of $\mathbb{R}$.

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$ and suppose the Archimedean property is false. Then y would be an upper bound of A. But then A would have a least upper bound. Say $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound. Thus, $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. But then $\alpha < (m+1)x$, which contradicts the fact that α is an upper bound of A. Thus, the Archimedean property must be true.

Theorem. If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$. We say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Theorem. For every positive real number x and every positive integer n, there is exactly one positive real number y such that $y^n = x$.

Proof. There is at most one since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$. Let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$. Then E is nonempty since $t = \frac{x}{1+x} \implies 0 < t < 1 \implies t^n < t < x \implies t \in E$. We also know E is bounded above since $t > 1 + x \implies t^n > t > x \implies t \notin E$ and t is an upper bound. Define $y = \sup E$. We can then show that $y^n < x$ and $y^n > x$ each lead to contradictions.

Question. Given a real number in decimal form, what is its associated Dedekind cut?

Cardinality of Sets

Definition. Let A and B be sets. If there is a bijection from A to B, then we say A and B have the same **cardinality** (or 'size') and write $A \sim B$. We also write $|A| = |B|$ where $|A|$ denotes the cardinality of A.

Definition. Let $\mathbb{N}$ denote the natural numbers $\{1, 2, 3, \ldots\}$, also denoted $\mathbb{Z}^+$. For $n \in \mathbb{N}$, let $J_n = \{1, 2, \ldots, n\}$ and $J_0 = \varnothing$. For any set A,

1. A is **finite** if $A \sim J_n$ for some $n \in \mathbb{N} \cup \{0\}$.
2. A is **infinite** if it is not finite.
3. A is **countable** if $A \sim \mathbb{N}$.
4. A is **uncountable** if A is neither finite nor countable.
5. A is **at most countable** if A is finite or countable.

Note. We can put an order on the cardinalities where $|A| \leq |B|$ if and only if there exists an injection from A to B.

Proposition. Every infinite subset of a countable set is countable.

Proposition. Let $\{E_n\}$ where $n \in \mathbb{Z}^+$ be a sequence of countable sets. If $S = \cup_{n=1}^{\infty} E_n$, then S is countable.

Proof. Let the elements of E_i be as follows

$$\begin{aligned}
E_1 &= \{x_{11}, x_{12}, x_{13}, \ldots\} \\
E_2 &= \{x_{21}, x_{22}, x_{23}, \ldots\} \\
&\ldots \\
E_i &= \{x_{i1}, x_{i2}, x_{i3}, \ldots\} \\
&\ldots
\end{aligned}$$

We can traverse these elements diagonally to get $S = \{x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, \ldots\}$. Since S is at most countable and $E_1 \subseteq S$ is countable, we have that S is countable.

Proposition. Let A be a countable set and B_n be the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_k \in A$ for $k = 1, 2, \ldots, n$. Then B_n is countable for all $n \in \mathbb{N}$.

Theorem. Let A be the set of all sequences of 0's and 1's. Then A is uncountable.

Proof. Let $E = \{e_1, e_2, \ldots\}$ be a countable subset of A. For each e_i, we analyze its ith digit. We then construct $e \in A$ such that the ith digit of e is the opposite of the ith digit of e_i. For example, if we have

$$\begin{aligned} e_1 &= (\boxed{0}, 1, 0, 1, 1, 1, 0, 1, \ldots) \\ e_2 &= (1, \boxed{1}, 0, 1, 0, 1, 1, 0, \ldots) \\ e_3 &= (0, 0, \boxed{1}, 1, 0, 0, 1, 1, \ldots) \\ e_4 &= (1, 0, 1, \boxed{0}, 1, 0, 1, 1, \ldots) \\ &\ldots \end{aligned}$$

Then $e = (1, 0, 0, 1, \ldots)$. Since $e \notin E$ but $e \in A$, every countable subset of A is a proper subset of A. Thus, A is uncountable.

Metric Spaces

Definition. A set X, whose elements we will call **points**, is a **metric space** if there is a function $d : X \times X \to \mathbb{R}$ such that $\forall p, q \in X$

1. $d(p, q) > 0$ if $p \neq q$, and $d(p, p) = 0$.
2. $d(p, q) = d(q, p)$.
3. $d(p, q) \leq d(p, r) + d(r, q)$, $\forall r \in X$ (triangle inequality).

Definition. The number $d(p, q)$ is the **distance** from p to q, and d is a **metric**.

Note. $\mathbb{R}^k$ is a metric space with the usual metric $d(x, y) = |x - y|$, $x, y \in \mathbb{R}^k$.

Proposition. Every subset Y of a metric space X is also a metric space where we restrict the metric of X to points in Y.

Open and Closed Sets

Definition. Let X be a metric space, $p \in X$, and $E \subseteq X$.

1. Let $r \in \mathbb{R}^+$. The **neighborhood** of p with **radius** r is the set $N_r(p) = \{q \in X : d(p, q) < r\}$.
2. The point p is a **limit point** of E if every neighborhood of p contains a point $q \in E$ and $q \neq p$.
3. If $p \in E$ and p is not a limit point of E, then p is an **isolated point** of E.
4. E is **closed** if every limit point of E is in E.

5. If $p \in E$ and there is an $r \in \mathbb{R}^+$ such that $N_r(p) \subseteq E$, then p is an **interior point** of E.

6. E is **open** if every point of E is an interior point of E.

7. The **complement** of E in X is the set $E^c = \{x \in X : x \notin E\}$.

8. E is **perfect** if it is closed and if every point of E is a limit point of E.

9. E is **bounded** if there exists a number $M \in \mathbb{R}^+$ and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

10. E is **dense** in X if every point of X is in E or a limit of E (or both).

Proposition. Every neighborhood is an open set.

Proof. Consider the neighborhood $E = N_r(p)$, and let $q \in E$. Then $r - d(p,q) \in \mathbb{R}^+$. For all points s such that $d(q,s) < r - d(p,q)$, we have $d(p,s) \leq d(p,q) + d(q,s) < d(p,q) + (r - d(p,q)) = r$, implying $s \in N_r(p)$. Thus, q is an interior point of E, and the result follows.

Proposition. If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.

Proposition. Let X be a metric space and suppose $E \subseteq X$. The set E is open in X if and only if its complement is closed.

Proof. We first prove the forward direction then the backward direction.
($\Rightarrow$) Suppose E is open. If x is a limit point of E^c, then every neighborhood of x contains a point of E^c. In this case, x can't be an interior point of E, and because E is open, $x \in E^c$. Thus, E^c is closed.
($\Leftarrow$) Now suppose E^c is closed. If $x \in E$ then $x \notin E^c$ and is thus not a limit point of E^c. In this case, there is a neighborhood $N(x)$ such that $N(x) \cap E^c = \varnothing$, implying that $N(x) \subseteq E$. Thus, x is an interior point and E is open.

Theorem. Consider the following statements regarding unions and intersections of open and closed sets.

1. For any collection $\{G_\alpha\}$ of open sets, $\cup_\alpha G_\alpha$ is open.

2. For any collection $\{F_\alpha\}$ of closed sets, $\cap_\alpha F_\alpha$ is closed.

3. For any finite collection $G_1, G_2, \ldots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.

4. For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed.

Definition. Let X be a metric space. If $E \subseteq X$, let E' be the set of limit points of E. The **closure** of E is the set $\overline{E} = E \cup E'$.

Proposition. If X is a metric space and $E \subseteq X$, then

1. $\overline{E}$ is closed.

2. $E = \overline{E}$ if and only if E is closed.

3. $\overline{E} \subseteq F$ for every closed subset F of X such that $E \subseteq F$.

Note. $\overline{E}$ is the smallest closed set that contains E.

Compactness

Definition. Let X be a metric space, and let $E \subseteq X$. An **open cover** of E is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \cup_\alpha G_\alpha$.

Definition. If $\{G_\alpha\}$ is an open cover of E, then a subset of $\{G_\alpha\}$ that is also an open cover of E is called a **subcover** of $\{G_\alpha\}$.

Definition. A subset K of a metric space is **compact** if every open cover of K contains a finite subcover.

Proposition. Every finite set in a metric space is compact.

Proposition. Compact subsets are closed.

Proof. Let K be a compact subset of a metric space X. We will show that K is closed by showing that K^c is open. Suppose $p \in K^c$. We will show that K^c is open by showing that p is an interior point of K^c. For each $q \in K$, let V_q and W_q be neighborhoods of p and q, of radius less than half the distance between p and q. Since K is compact, there are finitely many points, say $q_1, \ldots, q_n$ in K such that if $W = W_{q_1} \cup W_{q_2} \cup \ldots \cup W_{q_n}$, then $K \subseteq W$. If $V = V_{q_1} \cap V_{q_2} \cap \ldots \cap V_{q_n}$, then V is a neighborhood of p that does not intersect W, which covers K. Thus, $V \subseteq K^c$, and p is therefore an interior point of K^c.

Proposition. Suppose $K \subseteq X \subseteq Y$. Then K is compact relative to X if and only if K is compact relative to Y.

Proposition. Closed subsets of compact sets are compact.

Proposition. If F is closed and K is compact, then $F \cap K$ is compact.

Proposition. If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof. Suppose no point of K is a limit point of E. Then each point $q \in K$ would have a neighborhood V_q that contains at most one point of E (namely q, if $q \in E$). But then no finite subset of $\{V_q\}$ can cover E, and the same is true for K because $E \subseteq K$. This contradicts the fact that K is compact. Thus, the theorem follows.

Theorem

Let E be a subset of $\mathbb{R}^k$ (viewed as a metric space with the usual metric). The following are equivalent.

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E.

Note. The Heine-Borel Theorem is "(1) if and only if (2)" for $\mathbb{R}^k$.

Note. For all metric spaces, "(2) if and only if (3)" holds.

Perfect Sets

Definition. Let X be a metric space, and E be a subset of X. We say E is **perfect** if

1. E is closed and
2. Every point of E is a limit point of E.

Proposition. If P is a nonempty perfect set in $\mathbb{R}^k$, then P is uncountable.

Proof. Since P has limit points, we know P is infinite. Suppose P is countable and define the points of P by $x_1, x_2, \ldots$. Let V_1 be any neighborhood of x_1. If V_1 has radius r, note that $\overline{V_1} = \{y \in \mathbb{R}^k : |y - x_1| \leq r\}$. We will use V_1 to recursively construct a sequence $\{V_n\}$ of neighborhoods as follows. Suppose V_n has been constructed so that $V_n \cap P$ is not empty. Since every point of P is a limit point of P, there is a neighborhood V_{n+1} such that $\overline{V_{n+1}} \subseteq V_n$, $x_n \notin \overline{V_{n+1}}$, and $V_{n+1} \cap P \neq \{\}$. By the last condition, our recursive construction can proceed to give us a sequence $\{V_n\}$ of neighborhoods. Let $K_n = \overline{V_n} \cap P$. Since $\overline{V_n}$ is closed and bounded, $\overline{V_n}$ is compact and K_n is compact. Since $x_n \notin \overline{V_{n+1}}$, no point of P is contained in $\cap_{n=1}^{\infty} K_n$. But since $K_n \subseteq P$, this implies that $\cap_{n=1}^{\infty} K_n$ is empty. But each K_n is not empty by the fact that $V_{n+1} \cap P \neq \{\}$ and $K_n \supseteq K_{n+1}$. But this contradicts the corollary that if $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supseteq K_{n+1}$, then $\cap_{n=1}^{\infty} K_n$ is not empty. The theorem follows.

Proposition. Let $a, b \in \mathbb{R}$ and $a < b$. Then the interval $[a, b]$ is uncountable. Also, $\mathbb{R}$ is uncountable.

Note. There are, however, perfect sets in $\mathbb{R}$ that contain no intervals.

Example. Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$ and let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Removing the middle thirds from these intervals yields $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continuing gives us a sequence $\{E_n\}$ of compact sets such that

1. $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$ and
2. E_n is the union of 2^n disjoint intervals, each of length $\frac{1}{3^n}$.

Then the set $P = \cap_{n=1}^{\infty} E_n$ is called the **Cantor set**. Note that P is compact. Also, it is not empty. P contains no intervals and P is perfect.