Real Analysis Notes

If $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

provided that $\frac{c}{d} \neq \frac{0}{1}$.

Strictly speaking, we need to show that these operations are **well-defined** or that they don't depend on the choice of representatives from the equivalence classes.

Definition. Suppose S is an ordered set, and $E \subseteq S$. If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say E is **bounded above** and we call β an **upper bound**. The terms **bounded below** and **lower bound** are defined similarly.

Definition. Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose there exists $\alpha \in S$ such that α is an upper bound for E and if $\gamma < \alpha$, then γ is not an upper bound for E, then α is the **least upper bound** of E or the **supremum** of E, and we write $\alpha = \sup E$. The **greatest lower bound** and **infimum** ($\inf E$) are defined similarly.

Example. Consider the set $\{r \in \mathbb{Q} : r^2 < 2\}$, which has no supremum in $\mathbb{Q}$.

Definition. An ordered set S has the **least-upper-bound property** if the following is true: if $E \subseteq S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

If an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

Definition. There exists an ordered field $\mathbb{R}$ (called the **real numbers**) which has the least-upper-bound property, and it contains an isomorphic copy of $\mathbb{Q}$.

Finite ordered fields do not exist. Consider $0 \leq 1 \leq 1+1 \leq \ldots$ which can't be a finite chain.
Any two ordered fields with the least upper-bound-property are isomorphic.

Theorem. If $x, y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the **Archimedean property** of $\mathbb{R}$.

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$ and suppose the Archimedean property is false. Then y would be an upper bound of A. But then A would have a least upper bound. Say $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound. Thus, $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. But then $\alpha < (m+1)x$, which contradicts the fact that α is an upper bound of A. Thus, the Archimedean property must be true.

Theorem. If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$. We say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Theorem. For every positive real number x and every positive integer n, there is exactly one positive real number y such that $y^n = x$.

Proof. There is at most one since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$. Let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$. Then E is nonempty since $t = \frac{x}{1+x} \implies 0 < t < 1 \implies t^n < t < x \implies t \in E$. We also know E is bounded above since $t > 1 + x \implies t^n > t > x \implies t \notin E$ and t is an upper bound. Define $y = \sup E$. We can then show that $y^n < x$ and $y^n > x$ each lead to contradictions.