Scrawlings of the MagiKarp

Definition. A **map** $f : A \to B$ is a subset $f \subset A \times B$ such that for all $a \in A$, there exists a $b \in B$ such that b is unique with $(a, b) \in f$.

Definition. We write $f(a) = b$ if $(a, b) \in f$. A is the **domain** of f and B is the **codomain**.

Definition. A **binary operation** on A is a map $\star : A \times A \to A$ such that $\star(a_1, a_2) = a_1 \star a_2$ for $a_1, a_2 \in A$.

Definition. A binary operation $\star$ is **associative** on A if for all $a, b, c \in A$, $a \star (b \star c) = (a \star b) \star c$.

Definition. An element $e \in A$ is an **identity** element of $\star$ if for each $a \in A$, $e \star a = a \star e = a$.

Definition. An element $a \in A$ has an **inverse** under $\star$ if there exists a $b \in A$ such that $a \star b = b \star a = e$.

Definition. A set A with an associative binary operation $\star$ is a **group** if A has an identity element under $\star$ and every $a \in A$ has an inverse.

Definition

A group is a pair $(G, \star)$ where G is a set and $\star$ is a binary operation on G such that

1. For all $a, b, c \in A$, $a \star (b \star c) = (a \star b) \star c$.
2. There exists an $e \in G$ such that $a \star e = e \star a = a$ for all $a \in G$.
3. For all $a \in G$, there exists a $b \in G$ such that $a \star b = b \star a = e$.

Definition. A group $(G, \star)$ is **abelian** or commutative if for all $g, h \in G$, $g \star h = h \star g$.

Theorem

Let $(G, \star)$ be a group.

1. e is unique.
2. g^{-1} is unique.
3. $\forall g \in G, \left(g^{-1}\right)^{-1} = g$.
4. $\forall g, h \in G, (g \star h)^{-1} = h^{-1} \star g^{-1}$.

Proof

We may prove each part separately.

1. Suppose e, e' are identity elements. Then for all $a \in G$,

$$a \star e = e \star a = a \tag{i}$$
$$a \star e' = e' \star a = a \tag{ii}$$

By (i), $e' = e \star e'$ and by (ii), $e = e \star e'$. Therefore, $e = e'$.

2. Supposed $a \star b = b \star a = e$, then

$$\begin{aligned} b &= b \star e \\ &= b \star (a \star a^{-1}) \\ &= (b \star a) \star a^{-1} \\ &= e \star a^{-1} \\ &= a^{-1} \end{aligned}$$

Thus, $b = a^{-1}$.

3. $g^{-1} \star \left(g^{-1}\right)^{-1} = e = g^{-1} \star g$. By (ii), $g = \left(g^{-1}\right)^{-1}$.

4. Consider $(a \star b) \star (b^{-1} \star a^{-1})$.

$$\begin{aligned} (a \star b) \star (b^{-1} \star a^{-1}) &= a \star (b \star b^{-1}) \star a^{-1} \\ &= a \star e \star a^{-1} \\ &= a \star a^{-1} \\ &= e \end{aligned}$$

Thus, $(b^{-1} \star a^{-1}) = (a \star b)^{-1}$.

Definition. Let $[n] = \{1, 2, \ldots, n\}$. The **symmetric group** denoted S_n of degree n is the set of all bijections on $[n]$ under the operation of composition.

$$S_n = \{\sigma : [n] \to [n] \mid \sigma \text{ is a bijection}\}$$

Definition. The **order** of $(G, \star)$ is the number of elements in G denoted $|G|$.

Definition. Let $n \geq 2$. The **dihedral group** of index n is the group of all symmetries of a regular polygon P_n with n vertices in the Euclidean plane.

Symmetries of P_n consist of rotations and reflections.

Choose a vertex v. Let L_0 be the line from the center of P_n through v. Let L_k be L_0 rotated by $\frac{\pi k}{n}$ for $1 \leq k \leq n$. Let σ_k be a reflection about L_k. Let ρ_k be a rotation about $\frac{2\pi k}{n}$, $1 \leq k \leq n$.

Definition. A subset $S \subseteq G$ of a group $(G, \star)$ is a set of **generators**, denoted $\langle S \rangle = G$, if and only if every element of G can be written as a finite product of elements of S and their inverses.

Definition. Any equation satisfied by generators is called a **relation**.

Definition. A **presentation** of G, denoted $\langle S \mid R \rangle$, is a set of generators of G and relations such that any other relation can be derived by those given.

Example.

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$$

Definition. The cycles $\sigma = (\sigma_1\ \sigma_2\ \ldots\ \sigma_n)$ and $\tau = (\tau_1\ \tau_2\ \ldots\ \tau_n)$ are **disjoint** if $\sigma_i \neq \tau_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Definition. A cycle of length 2 is called a **transposition**.

Definition. An expression of the form $(a_1\ a_2\ \ldots\ a_m)$ is called a **cycle of length m** or an **m-cycle**.

Proposition. Let $\alpha = (a_1\ a_2\ \ldots\ a_m)$ and $\beta = (b_1\ b_2\ \ldots\ b_n)$. If $a_i \neq b_j$ for any i, j, then $\alpha\beta = \beta\alpha$.

Proposition. Every permutation can be written as a product of disjoint cycles.

Proposition. A cycle of length n has order n.

Proposition. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be disjoint cycles. Then,

$$|\alpha_1\alpha_2 \ldots \alpha_n| = \operatorname{lcm}(|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|)$$

Proposition. Every permutation is S_n is a product of 2-cycles (which are not necessarily disjoint).

Proposition. If $\alpha = \beta_1\beta_2 \ldots \beta_r = \gamma_1\gamma_2 \ldots \gamma_s$ where β_i, γ_j are transpositions, then r and s have the same parity.

Definition. If r and s are both odd, α is called an **odd permutation**. If r and s are both even, α is called an **even permutation**.

Definition. The set of even permutations in S_n form a group called the **alternating group**, denoted A_n.

Note. $|A_n| = \frac{n!}{2}$ for $n > 1$.

> **Definition**
>
> Let $(G, \star)$ and $(G', *)$ be groups. A map of sets $\varphi : G \to G'$ is a **group homomorphism** if for all $a, b \in G$,
>
> $$\varphi(a \star b) = \varphi(a) * \varphi(b)$$

Example. The following are two very simple examples of homomorphisms.

Trivial Homomorphism

$$\varphi : G \to G', \varphi(g) = e, \forall g \in G$$

Identity Homomorphism

$$\varphi : G \to G', \varphi(g) = g, \forall g \in G$$

Definition. If $\varphi : G \to G'$ is a homomorphism, the **domain** of φ is $\text{Dom}(\varphi) = G$, the **codomain** of φ is $\text{Codom}(\varphi) = G'$, the **range** or **image** of φ is $\varphi(G) = \{\varphi(g) : g \in G\} \subseteq G'$ denoted $\text{Range}(\varphi)$ or $\text{Im}(\varphi)$.

Definition

A homomorphism which is bijective is called an **isomorphism**.

$\varphi : G \to G'$ is an isomorphism if and only if there exists $\psi : G' \to G$ such that ψ is a homomorphism and $\varphi \circ \psi = 1_{G'}$, $\psi \circ \varphi = 1_G$, i.e. ψ is an inverse homomorphism to φ. We say G is isomorphic to G' by $G \cong G'$ or $\phi : G \xrightarrow{\sim} G'$.

Definition

Let $(G, \star)$ be a group. A subset $H \subseteq G$ is a **subgroup** if $(H, *)$ is also a group.

If $H \neq \emptyset$ and $H \subseteq G$, $H \leq G$ or H is a subgroup of G if and only if

1. H is closed under $\star$ ($\forall h_1, h_2 \in H, h_1 \star h_2 \in H$).
2. H is closed under inverses ($h \in H \Rightarrow h^{-1} \in H$).

Note. The following is notation for arbitrary and abelian groups.

$$x \star y \to xy \text{ for arbitrary } G, x + y \text{ for abelian } G$$
$$e \to 1 \text{ for arbitrary } G, 0 \text{ for abelian } G$$

For an arbitrary subset $A \subseteq G$, and $g \in G$,

$$gA = \{ga : a \in A\} \qquad Ag = \{ag : a \in A\} \qquad gAg^{-1} = \{gag^{-1} : a \in A\}$$

Theorem (Subgroup Criterion)

Let $\emptyset \neq H \subseteq G$, $H \leq G$ if and only if $\forall x, y \in H$, $xy^{-1} \in H$.

Definition. Let $A \subseteq G$ be any subset. The **centralizer** of A in G is $C_G(A) = \{g \in G : gag^{-1} = a\}$ and it is the set of elements in G which commute with all elements of A.

Proposition. $C_G(A) \leq G$

Proof. First we show that the centralizer is not empty. $1a = a1 = a$, $\forall a \in A \Rightarrow 1 \in C_G(A)$ $\Rightarrow C_G(A) \neq 0$ so the centralizer of A is not empty. Let $x, y \in C_G(A)$. We want to show that $xy^{-1} \in C_G(A)$ or that $xy^{-1} \in C_G(A)$. We do this by showing that $\left(xy^{-1}\right) a \left(xy^{-1}\right)^{-1} = a$.

$$\begin{aligned}
\left(xy^{-1}\right) a \left(xy^{-1}\right)^{-1} &= xy^{-1}ayx^{-1} \\
&= x\left(y^{-1}ay\right)x^{-1} \\
&= xax^{-1} && (y \in C_G(A)) \\
&= a && (x \in C_G(A))
\end{aligned}$$

Since this subset satisfies the Subgroup Criterion, the centralizer $C_G(A)$ is a subgroup of G.

Definition. The **center** of a group G is denoted $Z(G) = \{g \in G : gx = xg, \forall x \in G\}$. $Z(G) = C_G(G) \leq G$. $Z(G)$ is the set of elements of G which commute with all elements in G. If G is abelian, $Z(G) = G$.

Definition. The **normalizer** of A in G is $N_G(A) = \{g \in G : gAg^{-1} = A\}$ or $\{g \in G : gag^{-1} = a' \in A\}$.

Proposition. $C_G(A) \leq N_G(A) \leq G$

Definition. A **group action** of a group G on a set A is a map $G \times A \to A$ such that $(g_1 g_2) \cdot a = g_1 \cdot (g_2 \cdot a), \forall g_1, g_2 \in G, \forall a \in A$ and $1 \cdot a = a, \forall a \in A$. It is denoted $G \circlearrowleft A$.

Definition. Suppose $G \circlearrowleft A$, the stabilizer of $a \in A$ in G is $G_a = \{g \in G : g \cdot a = a\}$. $G_a \leq G$.

Definition

An **equivalence relation** $\mathcal{E}$ on a set S is a subset $\mathcal{E} \subseteq S \times S$ which is reflexive, symmetric, and transitive. We write $(a, b) \in \mathcal{E} \Leftrightarrow a\ \mathcal{E}\ b$ or $a \sim b$.

1. $a \sim a$
2. $a \sim b \Leftrightarrow b \sim a$
3. $a \sim b, b \sim c \Rightarrow a \sim c$

Definition. The **equivalence class** of $a \in S$ is $[a] = \{b \in S : a \sim b\}$

Definition. The **quotient set** of S under $\sim$ is $S/\!\sim = \{[a] : a \in S\}$.

Example. $\mathbb{Q} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b \neq 0\}/\!\sim, (a, b) \sim (c, d) \Rightarrow ad = bc$.

Definition. The quotient set comes equipped with the **projection map** $\pi : S \to S/\!\sim$ where $a \mapsto [a] = \pi(a)$. This map is surjective by definition.

Definition

A group G' is a **quotient group** of a group G if

1. $G' = G/\!\sim$, G' is the quotient set of G under an equivalence relation $\sim$.
2. The projection map $\pi : G \to G' = G/\!\sim$ is a group homomorphism.

Definition. Let $\varphi : G \to G'$ be a homomorphism and let $g' \in G'$. The **fiber** over g' is $\varphi^{-1}(g') = \{g \in G : \varphi(g) = g'\}$.

Proposition

All quotient groups come from subgroups.

Proof

Let $\varphi : G \to G'$ be a homomorphism, then φ induces an equivalence relation on G. Let $x \sim y \Leftrightarrow \varphi(x) = \varphi(y)$. But φ is a group homomorphism, so $\varphi(x) = \varphi(y) \Leftrightarrow \varphi(x)\varphi(y)^{-1} = 1_{G'} \Leftrightarrow \varphi(x)\varphi(y^{-1}) = 1 \Leftrightarrow \varphi(xy^{-1}) = 1$. So $x \sim y \Leftrightarrow \varphi(xy^{-1}) = 1$. Let $K = \{g \in G : \varphi(g) = 1\}$. Then $x \sim y \Leftrightarrow xy^{-1} \in K$. Recall $K = \text{Ker}(\varphi) \leq G$.

Let G' be a quotient group of G. Then $x \sim y \Leftrightarrow [x] = [y] \Leftrightarrow \pi(x) = \pi(y)$ where $\pi : G \to G'$ is the projection. But $\pi(x) = \pi(y) \Leftrightarrow xy^{-1} \in \text{Ker}(\varphi)$.

Definition. The **right coset** of a subgroup H of a group G by the element $x \in G$ is $Hx = \{hx : h \in H\}$. The **left coset**, denoted xH is denoted similarly.

Proposition. Let $\varphi : G \to G'$ be a homomorphism and $K = \text{Ker}(\varphi)$. Then $xKx^{-1} \subseteq K, \forall x \in G$.

Proof. We must show $\varphi(xkx^{-1}) = 1_{G'}$ for $x \in G, k \in K$. Then, $\varphi(xkx^{-1}) = \varphi(x)\varphi(k)\varphi(x^{-1}) = \varphi(x)\varphi(x)^{-1} = 1_{G'}$.

Definition

The subgroup $N \leq G$ is **normal** if $xNx^{-1} \subseteq N$ for all $x \in G$. It is denoted $N \trianglelefteq G$.

Proposition. $\text{Ker}(\varphi) \trianglelefteq G$ for any homomorphism $\varphi : G \to G'$.

Theorem

Let $N \leq G$. Then the following are equivalent.

1. $N \trianglelefteq G$ ($xNx^{-1} \subseteq N, \forall x \in G$)
2. $xNx^{-1} = N$
3. $xN = Nx$
4. $\forall x, y \in G,\ xy^{-1} \in N \Leftrightarrow y^{-1}x \in N$

Proof

(1) $\Rightarrow$ (2) Assume $\forall x \in G$, $xNx^{-1} \subseteq N$. We want to show $xNx^{-1} = N$. We do this by showing $N \subseteq xNx^{-1}$. Let $x \in G$, $n_0 \in N$. We show $n_0 \in xNx^{-1}$. Note that $x \in G \Rightarrow x^{-1} \in G$. Thus, $x^{-1}N\left(x^{-1}\right)^{-1} \subseteq N$ since $N \trianglelefteq G$. Thus there exists n such that $x^{-1}nx = n_1 \in N$. $n_0 = x\left(x^{-1}n_0x\right)x^{-1} = xn_1x^{-1} \in xNx^{-1}$.

(3) $\Rightarrow$ (4) Assume $\forall x \in G$, $xN = Nx$. Let $x, y \in G$. We want to show $xy^{-1} \in N \Leftrightarrow y^{-1}x \in N$. So we must show this is true in both directions. Suppose $xy^{-1} \in N$. Then there exists an $n_1 \in N$ such that $xy^{-1} = n_1$. Thus, $x = n_1y \in Ny = yN$ by assumption. So $x \in yN$. Thus there exists $n_2 \in N$ such that $x = yn_2 \Rightarrow y^{-1}x = n_2 \in N$. Thus, $xy^{-1} \in N \Rightarrow y^{-1}x \in N$. Similarly, $y^{-1}x \in N \Rightarrow xy^{-1} \in N$.

Proposition. Let $H \leq G$. Then, $x \sim y \Leftrightarrow y^{-1}x \in H$ is an equivalence relation on G.

Proof. We want to show $\sim$ is reflexive, symmetric, and transitive.

1. $x \sim x$: $x^{-1}x = 1 \in H$
2. $x \sim y \Rightarrow y \sim x$: $x \sim y \Leftrightarrow y^{-1}x \in H \Rightarrow x^{-1}y \in H \Leftrightarrow y \sim x$
3. $x \sim y, y \sim z \Rightarrow x \sim z$: $y^{-1}x \in H, z^{-1}y \in H \Rightarrow (z^{-1}y)(y^{-1}x) = z^{-1}x \in H \Leftrightarrow x \sim z$

Thus, $\sim$ is an equivalence relation on G.

Any subgroup gives an equivalence relation.

Definition. An equivalence relation on a set S is the same as a **partition** of S. $P = \{A_1, A_2, \ldots\}$, $A_i \subseteq S$ such that $S \cup_{i \in \mathbb{N}} A_i$, $A_i \cap A_j = \emptyset$, $i \neq j$. $a \sim b \Leftrightarrow a, b \in A_i$.

Proposition. For $H \leq G$, $x \sim y \Leftrightarrow y^{-1}x \in H \Leftrightarrow xH = yH$ $(Hx = Hy)$.

Proof. Suppose $y^{-1}x \in H$. We want to show that $xH = yH$ or $xH \subseteq yH$ and $yH \subseteq xH$. $y^{-1}x \in H$ implies that there exists a $h_1 \in H$ such that $y^{-1}x = h_1$. Thus, $x = yh_1 \Rightarrow x \in yH$. $y^{-1}x \in H \Leftrightarrow x^{-1}y \in H$ which implies that there exists a $h_2 \in H$ such that $x^{-1}y = h_2 \Rightarrow y = xh_2 \in xH$.

Note. $[x] = xH$.

Proposition. For $N \leq G$, let $G/N = \{xN : x \in G\}$. Define $xN \cdot yN = (xy)N$. Then G/N is a group if and only if $N \trianglelefteq G$.

$G/N = G/\sim$ $(x \sim y \Leftrightarrow xN = yN)$

Every quotient group is G/N for some N.

$\pi : G \to G/\sim$, $\mathrm{Ker}(\pi) \trianglelefteq G$, $G/\sim\, = G/\mathrm{Ker}(\pi)$.

Proposition. If $H \leq G$ and G is abelian, then $H \trianglelefteq G$.

If G is a group and $\sim$ is an equivalence relation on G, then the quotient set $G/\sim$ is a quotient group if and only if the projection map $\pi : G \to G/\sim$, $\pi(x) = [x]$ is a homomorphism.

If $N \trianglelefteq G$, then G/N is a quotient group, where $G/N = \{xN : x \in G\}$ and $xN \cdot yN = (xy)N$.

These notions of quotient groups are equivalent.

Proposition. If $\sim$ is an equivalence relation and $G/\sim$ is a quotient group, then there exists a homomorphism $\pi : G \to G/\sim$ and $\text{Ker}(\pi) \trianglelefteq G$.

Proof. $x \sim y \Leftrightarrow \pi(x) = \pi(y) \Leftrightarrow \pi(y^{-1}x) = 1 \Leftrightarrow y^{-1}x \in \text{Ker}(\pi) \Leftrightarrow x\text{Ker}(\pi) = y\text{Ker}(\pi)$.

If $N \trianglelefteq G$, define $x \sim y \Leftrightarrow xN = yN \Leftrightarrow y^{-1}x \in N$. Then, $G/\sim = G/N$, $[x] = xN$, $\pi : G \to G/N$, $\pi(x) = xN$, $\text{Ker}(\pi) = N$.

Proposition. Every subgroup of an abelian group is a normal subgroup.

Definition. $S^n \subseteq \mathbb{R}^{n+1}$, $S^n = \{(x_1, x_2, \ldots, x_{n+1}) : \sum x_i^2 = 1\}$

For $H \leq G$, the relation $x \sim y \Leftrightarrow xH = yH \Leftrightarrow y^{-1}x \in H$ is an equivalence relation and thus partitions G into equivalence classes.

$$G = \bigcup_{x \in G} [x], \ [x] \cap [y] = \emptyset, \ [x] \neq [y]$$
$$G = \bigcup_{x \in G} xH, \ xH \cap yH = \emptyset, \ x \nsim y$$

Proposition. Let $H \leq G$. The number of right cosets of H equals the number of left cosets of H.

Proof. Let $R = \{Hx : x \in G\}$ and $L = \{xH : x \in G\}$. We construct a bijection $L \to R$. Define $f : R \to L$ by $f(Hx) = x^{-1}H$, and define $g : L \to R$ by $g(xH) = Hx^{-1}$. Then f and g are mutually inverse. Hence $R \leftrightarrow L$.

Definition. The number of distinct left cosets of H in G is called the **index** of H in G, and is denoted $[G : H]$.

Theorem (Lagrange's Theorem)
If H is a subgroup of G, $|G| = |H|[G : H]$.

Corollary. In a finite group, the order of every element divides the order of the group.

Corollary. A group of prime order is cyclic.

Corollary. Let G be a finite group and let $a \in G$. Then, $a^{|G|} = 1$.

Let $\varphi : G \to G'$ be a homomorphism. How far is φ from an isomorphism? How can φ fail to be an isomorphism?

1. φ could fail to be injective. ($\text{Ker}(\varphi) \neq \{1\}$)

2. φ could fail to be surjective.

> **Theorem** (First Isomorphism Theorem)
> Let $\varphi : G \to G'$ be a homomorphism. Then $\mathrm{Ker}(\varphi) \trianglelefteq G$, $\mathrm{Im}(\varphi) \leq G'$ and
> $$G/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi)$$

Proposition. There exists an isomorphism $\theta : G/\mathrm{Ker}(\varphi) \to \mathrm{Im}(\varphi)$ such that

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & G' \\
G/\mathrm{Ker}(\varphi) \arrow[r, "\theta"] & \mathrm{Im}(\varphi) \arrow[u, hook, "\iota"']
\end{tikzcd}$$

The curved arrow in the middle means the diagram is commutative, i.e. $\varphi = \iota \cdot \theta \cdot \pi$. The curved arrow means it is injective.

Proof. Define $\theta : G/\mathrm{Ker}(\varphi) \to \mathrm{Im}(\varphi)$ by $\theta(x\mathrm{Ker}(\varphi)) = \varphi(x)$.

First we show that θ is well-defined. Suppose $x\mathrm{Ker}(\varphi) = y\mathrm{Ker}(\varphi)$. Then,

$$\begin{aligned}
x\mathrm{Ker}(\varphi) = y\mathrm{Ker}(\varphi) &\Leftrightarrow y^{-1}x\mathrm{Ker}(\varphi) = \mathrm{Ker}(\varphi) \\
&\Leftrightarrow y^{-1}x \in \mathrm{Ker}(\varphi) \\
&\Leftrightarrow \varphi(y^{-1}x) = 1 \\
&\Leftrightarrow \varphi(y)^{-1}\varphi(x) = 1 \\
&\Leftrightarrow \varphi(x) = \varphi(y) \\
&\Leftrightarrow \theta(x\mathrm{Ker}(\varphi)) = \theta(y\mathrm{Ker}(\varphi))
\end{aligned}$$

Thus, θ is well-defined.

Then, we show that θ is a homomorphism. Let $K = \mathrm{Ker}(\varphi)$.

$$\begin{aligned}
\theta(xKyK) &= \theta(xyK) \\
&= \varphi(xy) \\
&= \varphi(x)\varphi(y) \\
&= \theta(xK)\theta(yK)
\end{aligned}$$

Thus, θ is a homomorphism.

Then, we show that θ is injective.

$$\begin{aligned}\theta(xK) = \theta(yK) &\Leftrightarrow \varphi(x) = \varphi(y)\\ &\Leftrightarrow \varphi(y)^{-1}\varphi(x) = 1\\ &\Leftrightarrow \varphi(y^{-1}x) = 1\\ &\Leftrightarrow y^{-1}x \in K\\ &\Leftrightarrow xK = yK\end{aligned}$$

Thus, θ is injective.

Then, we show that θ is surjective. Let $y \in \text{Im}(\varphi)$. There exists $xK \in G/K$ such that $\theta(xK) = y$. We know there exists an $x \in G$ such that $\varphi(x) = y$. $\theta(xK) = \varphi(x) = y$. Thus, θ is surjective and θ is an isomorphism.

Proposition. Let $a \in G$. If $|a| = \infty$, then $\langle a\rangle \cong (\mathbb{Z}, +)$. If $|a| = n$, then $\langle a\rangle = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Proof. Consider $\mathbb{Z} \xrightarrow{\pi} G$ defined by $\pi(k) = a^k$.

Definition. Let $(A, \star)$ and $(B, *)$ be groups. The **direct product** or **direct sum** of A and B is $A \oplus B = \{(a, b) : a \in A, b \in B\}$ where $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \star a_2, b_1 * b_2) \in A \oplus B$.

Definition. In a group G, define $a \sim b \Leftrightarrow \exists x \in G$ such that $b = xax^{-1}$. This is an equivalence relation and a and b are **conjugates**.

Definition. For any $x \in G$, the **inner automorphism** of G induced by x is $T_x : G \to G$ defined by $T_x(g) = xgx^{-1}$.

Definition. The set of all inner automorphisms of G is a group, called the **inner automorphism group**, and is denoted $\text{Inn}(G) = \{T_x : G \to G \mid x \in G\}$.

Proposition. $G/Z(G) \cong \text{Inn}(G)$

Proof. Consider $\psi : G \to \text{Inn}(G)$ defined by $x \mapsto T_x$. Then, ψ is surjective, i.e. $\text{Im}(\psi) = \text{Inn}(G)$. We then determine the kernel of the homomorphism.

$$\begin{aligned}\text{Ker}(\psi) &= \{x \in G : \psi(x) = 1_G\}\\ &= \{x \in G : T_x(g) = g,\ \forall g \in G\}\\ &= \{x \in G : xgx^{-1} = g,\ \forall g \in G\}\\ &= \{x \in G : xg = gx,\ \forall g \in G\}\\ &= Z(G)\end{aligned}$$

By the first isomorphism theorem, $G/Z(G) \cong \text{Inn}(G)$.

Theorem (Third Isomorphism Theorem)

Let G be a group. Let $A \trianglelefteq G$, $B \trianglelefteq G$. If $A \subseteq B$, then $A \trianglelefteq B$, $B/A \trianglelefteq G/A$, and

$$(G/A)/(B/A) \cong (G/B)$$

Proof

First we establish $A \trianglelefteq B$. $A \leq B$ because $A \leq G$ and $A \subseteq B$.

$$A \trianglelefteq B \Leftrightarrow bAb^{-1} \subseteq A, \ \forall b \in B$$

$$A \trianglelefteq G \Leftrightarrow xAx^{-1} \subseteq A, \ \forall x \in G$$

But $B \subseteq G$ so $b \in G$. Thus, $bAb^{-1} \subseteq A, \forall b \in B$ and $A \trianglelefteq B$. Thus, $A \trianglelefteq B$ and we may construct B/A.

We first show $B/A \leq G/A$. It is closed under multiplication since $(b_1A)(b_2A) = (b_1b_2)A \in B/A$ because B is a group. It is also closed under inverses since $(bA)^{-1} = b^{-1}A \in B/A$.

We then show $B/A \trianglelefteq G/A$ by showing $x(B/A)x^{-1} \subseteq B/A, \forall x \in G/A$. Let $x \in G/A \Leftrightarrow yA$, $y \in G$. We want to show $(yA)(B/A)(yA)^{-1} \subseteq B/A$. Let $z \in (yA)(B/A)(yA)^{-1}$. Then, there exist $a_1, a_2 \in A$, $b_1 \in B$ such that

$$\begin{aligned} z &= (ya_1)(b_1A)(y^{-1}a_2) \\ &= y(a_1b_1)Ay^{-1}a_2 \\ &= y(a_1b_1)y^{-1}Aa_2 \end{aligned}$$

We know $a_2 \in A \Rightarrow Aa_2 = A$ and $A \subseteq B \Rightarrow a_1 \in A \subseteq B \Rightarrow a_1 \in A \Rightarrow a_1b_1 \in B$. Thus, there exists $b_2 \in B$ such that $a_1b_1 = b_2$. We substitute these in to get

$$z = yb_2y^{-1}A$$

We know $B \trianglelefteq G \Rightarrow yBy^{-1} \subseteq B$. Thus, there exists a $b_3 \in B$ such that $yb_2y^{-1} = b_3 \in B$. We then get $z = b_3A$. Since $z = b_3A \in B/A$, $B/A \trianglelefteq G/A$.

Now we prove $(G/A)/(B/A) \cong (G/B)$. We define the homomorphism $\omega : G/A \to G/B$ such that $\omega(xA) = xB$. We show that ω is well-defined. If $xA = yA$, then

$$\begin{aligned} xA = yA &\Leftrightarrow y^{-1}x \in A \subseteq B \\ &\Rightarrow y^{-1}x \in B \\ &\Leftrightarrow xB = yB \\ &\Leftrightarrow \omega(xA) = \omega(yA) \end{aligned}$$

We may then determine the kernel and image of the homomorphism.

$$\text{Im}(\omega) = \{xB : x \in G\} = G/B$$

$$\text{Ker}(\omega) = \{xA : \omega(xA) = B\} = \{xA : xB = B\} = \{xA : x \in B\} = B/A$$

By the first isomorphism theorem, $(G/A)/\text{Ker}(\omega) \cong \text{Im}(\omega)$ so $(G/A)/(B/A) \cong (G/B)$.

Proposition. There is an isomorphism $\theta : (G/A)/(B/A) \to G/B$ such that this diagram commutes.

$$\begin{tikzcd}
& G/A \arrow[dl, "\pi"'] \arrow[dd, "\omega"] \arrow[dr, "\sigma"] & \\
G \arrow[dr, "\rho"] & & (G/A)/(B/A) \arrow[dl, "\theta"] \\
& G/B &
\end{tikzcd}$$

Theorem (Second Isomorphism Theorem)

Let G be a group, $A \leq G$, and $N \trianglelefteq G$. Then $AN \leq G$, $N \trianglelefteq AN$, and $A \cap N \trianglelefteq A$. Also,

$$(AN)/N \cong A/(A \cap N)$$

Proof

Let $\varphi : A \to AN/N$ such that $a \mapsto aN$. Then by the first isomorphism theorem, $(AN)/N \cong A/(A \cap N)$.

Example. We look at an example of the third isomorphism theorem. Let $G = \mathbb{Z}$, $A = 12\mathbb{Z}$, and $B = 4\mathbb{Z}$. We observe that $A \trianglelefteq B \trianglelefteq G$ so the conditions for the third isomorphism theorem are satisfied.

$$G/A = \mathbb{Z}/12\mathbb{Z} = \{0, 1, \ldots, 11\} (\text{mod } 12)$$

$$B/A = 4\mathbb{Z}/12\mathbb{Z} = \{0, 4, 8\} (\text{mod } 12)$$

$$(G/A)/(B/A) = \{0, 1, 2, 3\} (\text{mod } 4) = \mathbb{Z}/4\mathbb{Z}$$

$$(\mathbb{Z}/12\mathbb{Z})/(4\mathbb{Z}/12\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$$

Example. We look at an example of the second isomorphism theorem. Let $G = \mathbb{Z}$, $N = 12\mathbb{Z}$, and $A = 8\mathbb{Z}$.

$$A \cap N = \{0, (2)\, 4, (4)\, 8, \ldots\} = 24\mathbb{Z}$$

$$AN = \{0, (4)\,, (8)\,, \ldots\} = 4\mathbb{Z}$$

$$AN/A = 4\mathbb{Z}/12\mathbb{Z} = \{0, 4, 8\} (\text{mod } 12)$$

$$A/(A \cap N) = 8\mathbb{Z}/24\mathbb{Z} = \{0, 8, 16\} (\text{mod } 24)$$

$$AN/N \cong \mathbb{Z}/3\mathbb{Z} \cong A/(A \cap N)$$