Scrawlings of the MagiKarp

Definition. A **map** $f : A \to B$ is a subset $f \subset A \times B$ such that for all $a \in A$, there exists a $b \in B$ such that b is unique with $(a, b) \in f$.

Definition. We write $f(a) = b$ if $(a, b) \in f$. A is the **domain** of f and B is the **codomain**.

Definition. A **binary operation** on A is a map $\star : A \times A \to A$ such that $\star(a_1, a_2) = a_1 \star a_2$ for $a_1, a_2 \in A$.

Definition. A binary operation $\star$ is **associative** on A if for all $a, b, c \in A$, $a \star (b \star c) = (a \star b) \star c$.

Definition. An element $e \in A$ is an **identity** element of $\star$ if for each $a \in A$, $e \star a = a \star e = a$.

Definition. An element $a \in A$ has an **inverse** under $\star$ if there exists a $b \in A$ such that $a \star b = b \star a = e$.

Definition. A set A with an associative binary operation $\star$ is a **group** if A has an identity element under $\star$ and every $a \in A$ has an inverse.

Definition

A group is a pair $(G, \star)$ where G is a set and $\star$ is a binary operation on G such that

1. For all $a, b, c \in A$, $a \star (b \star c) = (a \star b) \star c$.
2. There exists an $e \in G$ such that $a \star e = e \star a = a$ for all $a \in G$.
3. For all $a \in G$, there exists a $b \in G$ such that $a \star b = b \star a = e$.

Definition. A group $(G, \star)$ is **abelian** or commutative if for all $g, h \in G$, $g \star h = h \star g$.

Theorem

Let $(G, \star)$ be a group.

1. e is unique.
2. g^{-1} is unique.
3. $\forall g \in G$, $\left(g^{-1}\right)^{-1} = g$.
4. $\forall g, h \in G$, $(g \star h)^{-1} = h^{-1} \star g^{-1}$.

Proof

We may prove each part separately.

1. Suppose e, e' are identity elements. Then for all $a \in G$,

$$a \star e = e \star a = a \tag{i}$$
$$a \star e' = e' \star a = a \tag{ii}$$

   By (i), $e' = e \star e'$ and by (ii), $e = e \star e'$. Therefore, $e = e'$.

2. Supposed $a \star b = b \star a = e$, then

$$\begin{aligned} b &= b \star e \\ &= b \star (a \star a^{-1}) \\ &= (b \star a) \star a^{-1} \\ &= e \star a^{-1} \\ &= a^{-1} \end{aligned}$$

   Thus, $b = a^{-1}$.

3. $g^{-1} \star \left(g^{-1}\right)^{-1} = e = g^{-1} \star g$. By (ii), $g = \left(g^{-1}\right)^{-1}$.

4. Consider $(a \star b) \star (b^{-1} \star a^{-1})$.

$$\begin{aligned} (a \star b) \star (b^{-1} \star a^{-1}) &= a \star (b \star b^{-1}) \star a^{-1} \\ &= a \star e \star a^{-1} \\ &= a \star a^{-1} \\ &= e \end{aligned}$$

   Thus, $(b^{-1} \star a^{-1}) = (a \star b)^{-1}$.

Definition. Let $[n] = \{1, 2, \ldots, n\}$. The **symmetric group** denoted S_n of degree n is the set of all bijections on $[n]$ under the operation of composition.

$$S_n = \{\sigma : [n] \to [n] \mid \sigma \text{ is a bijection}\}$$

Definition. The **order** of $(G, \star)$ is the number of elements in G denoted $|G|$.

Definition. Let $n \geq 2$. The **dihedral group** of index n is the group of all symmetries of a regular polygon P_n with n vertices in the Euclidean plane.

Symmetries of P_n consist of rotations and reflections.

Choose a vertex v. Let L_0 be the line from the center of P_n through v. Let L_k be L_0 rotated by $\frac{\pi k}{n}$ for $1 \leq k \leq n$. Let σ_k be a reflection about L_k. Let ρ_k be a rotation about $\frac{2\pi k}{n}$, $1 \leq k \leq n$.

Definition. A subset $S \subseteq G$ of a group $(G, \star)$ is a set of **generators**, denoted $\langle S \rangle = G$, if and only if every element of G can be written as a product of elements of S and their inverses.

Definition. Any equation satisfied by generators is called a **relation**.

Definition. A **presentation** of G, denoted $\langle S \mid R \rangle$, is a set of generators of G and relations such that any other relation can be derived by those given.

Example.

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$$

Definition. The cycles $\sigma = (\sigma_1\ \sigma_2\ \dots\ \sigma_n)$ and $\tau = (\tau_1\ \tau_2\ \dots\ \tau_n)$ are **disjoint** if $\sigma_i \neq \tau_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Definition. A cycle of length 2 is called a **transposition**.

Definition. An expression of the form $(a_1\ a_2\ \dots\ a_m)$ is called a **cycle of length m** or an **m-cycle**.

Proposition. Let $\alpha = (a_1\ a_2\ \dots\ a_m)$ and $\beta = (b_1\ b_2\ \dots\ b_n)$. If $a_i \neq b_j$ for any i, j, then $\alpha\beta = \beta\alpha$.

Proposition. Every permutation can be written as a product of disjoint cycles.

Proposition. A cycle of length n has order n.

Proposition. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be disjoint cycles. Then,

$$|\alpha_1\alpha_2 \dots \alpha_n| = \operatorname{lcm}(|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|)$$

Proposition. Every permutation is S_n is a product of 2-cycles (which are not necessarily disjoint).

Proposition. If $\alpha = \beta_1\beta_2 \dots \beta_r = \gamma_1\gamma_2 \dots \gamma_s$ where β_i, γ_j are transpositions, then r and s have the same parity.

Definition. If r and s are both odd, α is called an **odd permutation**. If r and s are both even, α is called an **even permutation**.

Definition. The set of even permutations in S_n form a group called the **alternating group**, denoted A_n.

Note. $|A_n| = \frac{n!}{2}$ for $n > 1$.

Definition

Let $(G, \star)$ and $(G', *)$ be groups. A map of sets $\varphi : G \to G'$ is a **group homomorphism** if for all $a, b \in G$,

$$\varphi(a \star b) = \varphi(a) * \varphi(b)$$

Example. The following are two very simple examples of homomorphisms.

Trivial Homomorphism

$$\varphi : G \to G', \varphi(g) = e, \forall g \in G$$

Identity Homomorphism

$$\varphi : G \to G', \varphi(g) = g, \forall g \in G$$

Definition. If $\varphi : G \to G'$ is a homomorphism, the **domain** of φ is $\text{Dom}(\varphi) = G$, the **codomain** of φ is $\text{Codom}(\varphi) = G'$, the **range** or **image** of φ is $\varphi(G) = \{\varphi(g) : g \in G\} \subseteq G'$ denoted $\text{Range}(\varphi)$ or $\text{Im}(\varphi)$.

Definition

A homomorphism which is bijective is called an **isomorphism**.

$\varphi : G \to G'$ is an isomorphism if and only if there exists $\psi : G' \to G$ such that ψ is a homomorphism and $\varphi \circ \psi = 1_{G'}$, $\psi \circ \varphi = 1_G$, i.e. ψ is an inverse homomorphism to φ. We say G is isomorphic to G' by $G \cong G'$ or $\phi : G \xrightarrow{\sim} G'$.

Definition

Let $(G, \star)$ be a group. A subset $H \subseteq G$ is a **subgroup** if $(H, *)$ is also a group.

If $H \neq \emptyset$ and $H \subseteq G$, $H \leq G$ or H is a subgroup of G if and only if

1. H is closed under $\star$ ($\forall h_1, h_2 \in H, h_1 \star h_2 \in H$).
2. H is closed under inverses ($h \in H \Rightarrow h^{-1} \in H$).

Note. The following is notation for arbitrary and abelian groups.

$$x \star y \to xy \text{ for arbitrary } G,\ x + y \text{ for abelian } G$$
$$e \to 1 \text{ for arbitrary } G,\ 0 \text{ for abelian } G$$

For an arbitrary subset $A \subseteq G$, and $g \in G$,

$$gA = \{ga : a \in A\} \qquad Ag = \{ag : a \in A\} \qquad gAg^{-1} = \{gag^{-1} : a \in A\}$$

Theorem

Let $\emptyset \neq H \subseteq G$, $H \leq G$ if and only if $\forall x, y \in H$, $xy^{-1} \in H$.

Definition. Let $A \subseteq G$ be any subset. The **centralizer** of A in G is $C_G(A) = \{g \in G : gag^{-1} = a\}$ and it is the set of elements in G which commute with all elements of A.

Proposition. $C_G(A) \leq G$

Proof. First we show that the centralizer is not empty. $1a = a1 = a$, $\forall a \in A \Rightarrow 1 \in C_G(A)$ $\Rightarrow C_G(A) \neq 0$ so the centralizer of A is not empty. Let $x, y \in C_G(A)$. We want to show that $xy^{-1} \in C_G(A)$ or that $xy^{-1} \in C_G(A)$. We do this by showing that $\left(xy^{-1}\right) a \left(xy^{-1}\right)^{-1} = a$.

$$\begin{aligned}
\left(xy^{-1}\right) a \left(xy^{-1}\right)^{-1} &= xy^{-1}ayx^{-1} \\
&= x\left(y^{-1}ay\right)x^{-1} \\
&= xax^{-1} && (y \in C_G(A)) \\
&= a && (x \in C_G(A))
\end{aligned}$$

Since this subset satisfies the Subgroup Criterion, the centralizer $C_G(A)$ is a subgroup of G.

Definition. The **center** of a group G is denoted $Z(G) = \{g \in G : gx = xg, \forall x \in G\}$. $Z(G) = C_G(G) \leq G$. $Z(G)$ is the set of elements of G which commute with all elements in G. If G is abelian, $Z(G) = G$.

Definition. The **normalizer** of A in G is $N_G(A) = \{g \in G : gAg^{-1} = A\}$ or $\{g \in G : gag^{-1} = a' \in A\}$.

Proposition. $C_G(A) \leq N_G(A) \leq G$

Definition. A **group action** of a group G on a set A is a map $G \times A \to A$ such that $(g_1 g_2) \cdot a = g_1 \cdot (g_2 \cdot a), \forall g_1, g_2 \in G, \forall a \in A$ and $1 \cdot a = a, \forall a \in A$. It is denoted $G \circlearrowleft A$.

Definition. Suppose $G \circlearrowleft A$, the stabilizer of $a \in A$ in G is $G_a = \{g \in G : g \cdot a = a\}$. $G_a \leq G$.

Definition

An **equivalence relation** $\mathcal{E}$ on a set S is a subset $\mathcal{E} \subseteq S \times S$ which is reflexive, symmetric, and transitive. We write $(a, b) \in \mathcal{E} \Leftrightarrow a \,\mathcal{E}\, b$ or $a \sim b$.

1. $a \sim a$
2. $a \sim b \Leftrightarrow b \sim a$
3. $a \sim b, b \sim c \Rightarrow a \sim c$

Definition. The **equivalence class** of $a \in S$ is $[a] = \{b \in S : a \sim b\}$

Definition. The **quotient set** of S under $\sim$ is $S/{\sim} = \{[a] : a \in S\}$.

Example. $\mathbb{Q} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b \neq 0\}/{\sim}$, $(a, b) \sim (c, d) \Rightarrow ad = bc$.

Definition. The quotient set comes equipped with the **projection map** $\pi : S \to S/{\sim}$ where $a \mapsto [a] = \pi(a)$. This map is surjective by definition.

Definition

A group G' is a **quotient group** of a group G if

1. $G' = G/{\sim}$, G' is the quotient set of G under an equivalence relation $\sim$.
2. The projection map $\pi : G \to G' = G/{\sim}$ is a group homomorphism.

Definition. Let $\varphi : G \to G'$ be a homomorphism and let $g' \in G'$. The **fiber** over g' is $\varphi^{-1}(g') = \{g \in G : \varphi(g) = g'\}$.

Proposition

All quotient groups come from subgroups.

Proof

Let $\varphi : G \to G'$ be a homomorphism, then φ induces an equivalence relation on G. Let $x \sim y \Leftrightarrow \varphi(x) = \varphi(y)$. But φ is a group homomorphism, so $\varphi(x) = \varphi(y) \Leftrightarrow \varphi(x)\varphi(y)^{-1} = 1_{G'} \Leftrightarrow \varphi(x)\varphi(y^{-1}) = 1 \Leftrightarrow \varphi(xy^{-1}) = 1$. So $x \sim y \Leftrightarrow \varphi(xy^{-1}) = 1$. Let $K = \{g \in G : \varphi(g) = 1\}$. Then $x \sim y \Leftrightarrow xy^{-1} \in K$. Recall $K = \text{Ker}(\varphi) \leq G$.

Let G' be a quotient group of G. Then $x \sim y \Leftrightarrow [x] = [y] \Leftrightarrow \pi(x) = \pi(y)$ where $\pi : G \to G'$ is the projection. But $\pi(x) = \pi(y) \Leftrightarrow xy^{-1} \in \text{Ker}(\varphi)$.

Definition. The **right coset** of a subgroup H of a group G by the element $x \in G$ is $Hx = \{hx : h \in H\}$. The **left coset**, denoted xH is denoted similarly.

Proposition. Let $\varphi : G \to G'$ be a homomorphism and $K = \text{Ker}(\varphi)$. Then $xKx^{-1} \subseteq K, \forall x \in G$.

Proof. We must show $\varphi(xkx^{-1}) = 1_{G'}$ for $x \in G, k \in K$. Then, $\varphi(xkx^{-1}) = \varphi(x)\varphi(k)\varphi(x^{-1}) = \varphi(x)\varphi(x)^{-1} = 1_{G'}$.

Definition

The subgroup $N \leq G$ is **normal** if $xNx^{-1} \subseteq N$ for all $x \in G$. It is denoted $N \trianglelefteq G$.

Proposition. $\text{Ker}(\varphi) \trianglelefteq G$ for any homomorphism $\varphi : G \to G'$.

Theorem

Let $N \leq G$. Then the following are equivalent.

1. $N \trianglelefteq G$ ($xNx^{-1} \subseteq N, \forall x \in G$)
2. $xNx^{-1} = N$
3. $xN = Nx$
4. $\forall x, y \in G,\ xy^{-1} \in N \Leftrightarrow y^{-1}x \in N$

Proof

(1) $\Rightarrow$ (2) Assume $\forall x \in G$, $xNx^{-1} \subseteq N$. We want to show $xNx^{-1} = N$. We do this by showing $N \subseteq xNx^{-1}$. Let $x \in G$, $n_0 \in N$. We show $n_0 \in xNx^{-1}$. Note that $x \in G \Rightarrow x^{-1} \in G$. Thus, $x^{-1}N\left(x^{-1}\right)^{-1} \subseteq N$ since $N \trianglelefteq G$. Thus there exists n such that $x^{-1}nx = n_1 \in N$. $n_0 = x\left(x^{-1}n_0x\right)x^{-1} = xn_1x^{-1} \in xNx^{-1}$.

(3) $\Rightarrow$ (4) Assume $\forall x \in G$, $xN = Nx$. Let $x, y \in G$. We want to show $xy^{-1} \in N \Leftrightarrow y^{-1}x \in N$. So we must show this is true in both directions. Suppose $xy^{-1} \in N$. Then there exists an $n_1 \in N$ such that $xy^{-1} = n_1$. Thus, $x = n_1y \in Ny = yN$ by assumption. So $x \in yN$. Thus there exists $n_2 \in N$ such that $x = yn_2 \Rightarrow y^{-1}x = n_2 \in N$. Thus, $xy^{-1} \in N \Rightarrow y^{-1}x \in N$. Similarly, $y^{-1}x \in N \Rightarrow xy^{-1} \in N$.