Real Analysis Notes

Rational Numbers and Bounds

If $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

provided that $\frac{c}{d} \neq \frac{0}{1}$.

Note. Strictly speaking, we need to show that these operations are **well-defined** or that they don't depend on the choice of representatives from the equivalence classes.

Definition. Suppose S is an ordered set, and $E \subseteq S$. If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say E is **bounded above** and we call β an **upper bound**. The terms **bounded below** and **lower bound** are defined similarly.

Definition. Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose there exists $\alpha \in S$ such that α is an upper bound for E and if $\gamma < \alpha$, then γ is not an upper bound for E, then α is the **least upper bound** of E or the **supremum** of E, and we write $\alpha = \sup E$. The **greatest lower bound** and **infimum** ($\inf E$) are defined similarly.

Example. Consider the set $\{r \in \mathbb{Q} : r^2 < 2\}$, which has no supremum in $\mathbb{Q}$.

Definition. An ordered set S has the **least-upper-bound property** if the following is true: if $E \subseteq S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Proposition. If an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

Definition. There exists an ordered field $\mathbb{R}$ (called the **real numbers**) which has the least-upper-bound property, and it contains an isomorphic copy of $\mathbb{Q}$.

Note. Finite ordered fields do not exist. Consider $0 \leq 1 \leq 1+1 \leq \ldots$ which can't be a finite chain.

Dedekind Cuts

1. Define the elements of $\mathbb{R}$ as subsets of $\mathbb{Q}$ called **cuts**, where a cut is a subset α of $\mathbb{Q}$ such that
   (a) α is a nonempty proper subset of $\mathbb{Q}$ ($\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$).
   (b) If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.

(c) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$ (can't be in the set and be an upper bound).

2. Define an order on $\mathbb{R}$ where $\alpha < \beta$ if and only if α is a proper subset of β.

3. Show that the ordered set $\mathbb{R}$ has the least-upper-bound property. To do this, suppose A is a nonempty subset of $\mathbb{R}$ that is bounded above. Let γ be the union of all $\alpha \in A$. Then show $\gamma \in \mathbb{R}$ and $\gamma = \sup A$.

4. For $\alpha, \beta \in \mathbb{R}$, define the sum $\alpha + \beta$ to be the set of all sums $r + s$ where $r \in \alpha$ and $s \in \beta$. Define $0^* = \{t \in \mathbb{Q} : t < 0\}$ then show axioms for addition in fields hold for $\mathbb{R}$, and that 0^* is the additive identity.

5. Show that if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$. This is part of showing that $\mathbb{R}$ is an ordered field.

6. For $\alpha, \beta \in \mathbb{R}$, where $\alpha > 0^*$ and $\beta > 0^*$, define the product $\alpha\beta$ to be $\{p \in \mathbb{Q} : q \leq rs,\ r \in \alpha,\ s \in \beta,\ r > 0,\ s > 0\}$. Note that $\alpha\beta > 0^*$ if $\alpha > 0^*$ and $\beta > 0^*$, which is part of showing that $\mathbb{R}$ is an ordered field.

7. Extend the definition of multiplication to all of $\mathbb{R}$ by setting, for all $\alpha, \beta \in \mathbb{R}$, $\alpha 0^* = 0^* \alpha = 0^*$ and
$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \alpha < 0^*, \beta < 0^* \\ -[(-\alpha)(\beta)] & \alpha < 0^*, \beta > 0^* \\ -[(\alpha)(-\beta)] & \alpha > 0^*, \beta < 0^* \end{cases}$$
then prove the distributive law.

8. Associate to each $r \in \mathbb{Q}$ the real number $r^* = \{t \in \mathbb{Q} : t < r\}$ and let $\mathbb{Q}^* = \{r^* : r \in \mathbb{Q}\}$. These are the rational cuts in $\mathbb{R}$.

9. Show that $\mathbb{Q}$ is isomorphic to $\mathbb{Q}^*$ as ordered fields.

Properties of Real Numbers

Theorem. Any two ordered fields with the least upper-bound-property are isomorphic.

Theorem. If $x, y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the **Archimedean property** of $\mathbb{R}$.

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$ and suppose the Archimedean property is false. Then y would be an upper bound of A. But then A would have a least upper bound. Say $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound. Thus, $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. But then $\alpha < (m+1)x$, which contradicts the fact that α is an upper bound of A. Thus, the Archimedean property must be true.

Theorem. If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$. We say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Theorem. For every positive real number x and every positive integer n, there is exactly one positive real number y such that $y^n = x$.

Proof. There is at most one since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$. Let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$. Then E is nonempty since $t = \frac{x}{1+x} \implies 0 < t < 1 \implies t^n < t < x \implies t \in E$. We also know E is bounded above since $t > 1 + x \implies t^n > t > x \implies t \notin E$ and t is an upper bound. Define $y = \sup E$. We can then show that $y^n < x$ and $y^n > x$ each lead to contradictions.

Question. Given a real number in decimal form, what is its associated Dedekind cut?

Cardinality of Sets

Definition. Let A and B be sets. If there is a bijection from A to B, then we say A and B have the same **cardinality** (or 'size') and write $A \sim B$. We also write $|A| = |B|$ where $|A|$ denotes the cardinality of A.

Definition. Let $\mathbb{N}$ denote the natural numbers $\{1, 2, 3, \ldots\}$, also denoted $\mathbb{Z}^+$. For $n \in \mathbb{N}$, let $J_n = \{1, 2, \ldots, n\}$ and $J_0 = \varnothing$. For any set A,

1. A is **finite** if $A \sim J_n$ for some $n \in \mathbb{N} \cup \{0\}$.
2. A is **infinite** if it is not finite.
3. A is **countable** if $A \sim \mathbb{N}$.
4. A is **uncountable** if A is neither finite nor countable.
5. A is **at most countable** if A is finite or countable.

Note. We can put an order on the cardinalities where $|A| \leq |B|$ if and only if there exists an injection from A to B.

Proposition. Every infinite subset of a countable set is countable.

Proposition. Let $\{E_n\}$ where $n \in \mathbb{Z}^+$ be a sequence of countable sets. If $S = \cup_{n=1}^{\infty} E_n$, then S is countable.

Proof. Let the elements of E_i be as follows

$$\begin{aligned}
E_1 &= \{x_{11}, x_{12}, x_{13}, \ldots\} \\
E_2 &= \{x_{21}, x_{22}, x_{23}, \ldots\} \\
&\cdots \\
E_i &= \{x_{i1}, x_{i2}, x_{i3}, \ldots\} \\
&\cdots
\end{aligned}$$

We can traverse these elements diagonally to get $S = \{x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, \ldots\}$. Since S is at most countable and $E_1 \subseteq S$ is countable, we have that S is countable.

Proposition. Let A be a countable set and B_n be the set of all n-tuples $(a_1, a_2, \dots, a_n)$ where $a_k \in A$ for $k = 1, 2, \dots, n$. Then B_n is countable for all $n \in \mathbb{N}$.

Theorem. Let A be the set of all sequences of 0's and 1's. Then A is uncountable.

Proof. Let $E = \{e_1, e_2, \dots\}$ be a countable subset of A. For each e_i, we analyze its ith digit. We then construct $e \in A$ such that the ith digit of e is the opposite of the ith digit of e_i. For example, if we have

$$
\begin{aligned}
e_1 &= (\boxed{0}, 1, 0, 1, 1, 1, 0, 1, \dots) \\
e_2 &= (1, \boxed{1}, 0, 1, 0, 1, 1, 0, \dots) \\
e_3 &= (0, 0, \boxed{1}, 1, 0, 0, 1, 1, \dots) \\
e_4 &= (1, 0, 1, \boxed{0}, 1, 0, 1, 1, \dots) \\
&\cdots
\end{aligned}
$$

Then $e = (1, 0, 0, 1, \dots)$. Since $e \notin E$ but $e \in A$, every countable subset of A is a proper subset of A. Thus, A is uncountable.