Study of a Passenger-taxi Queueing System with Nonzero Matching Time

Ying Shi
School of Information
Zhejiang University of Finance & Economics
Hangzhou, China
Email: angelamacau2015@sina.com

Zhaotong Lian
Faculty of Business Administration
University of Macau
Macau SAR, China
Email: lianzt@umac.mo

Weixin Shang
Faculty of Business
Lingnan University
Hong Kong SAR, China
Email: shangwx@ln.edu.hk

***Abstract*—In this paper, we study a passenger-taxi queueing systems in which nonzero matching time is considered. We assume that passengers and taxi drivers who arrive at a taxi station independently form two queues, and the waiting space for taxis is limited. We use a two-dimensional Markov process to model the system. By using matrix-analytic method, we give a sufficient condition that ensures the existence of steady-state probabilities and present an algorithm to calculate the joint steady-state probabilities. Further, we consider the tail distributions of sojourn times of passengers and taxi drivers, which can be computed by algorithms we present in this paper. And by running Matlab programs, we give the numerical results about the performance measures of the system.**

I. Introduction

Queueing theory is a mathematical theory and method that study the random phenomena and the working processes of random service systems, which have been widely used in many areas. Among a variety of queueing systems, double-sided queueing system are very common and have a wide applications in computer science, perishable inventory system, organ allocation system. For example, passengers and taxis wait for matching in a taxi station; different types of components are piled for matching in an assembly line; organ donors and patients wait for matching in a health care. In this paper, we shed light on double-sided matching problems between passengers and taxis.

Taxis are an important complement to public transportation since they can offer door-to-door service within 24 hours a day. They are convenient and efficient. The expected passengers waiting time and taxi utilization are important indexes in transportation system in a city, which not only affect business conditions of taxi companies but also the decision of the government on taxi market. In the last several decades, a lot of researchers paid attention to the study of passenger-taxi queueing system. The double-sided passenger-taxi queue problem was first proposed by Kendall [1], who introduced an example about passengers and taxis: in an orderly taxi rank, on one side a queue is formed by the arrival of a stream of passengers who wait for taxis, while on the other side, a queue of taxis waits for passengers. Kendall pointed that the size of the queue was the difference between the arrival rates of passengers and taxis. He also pointed that the mean of queue length was equal to zero when the two arrival rates were equal and the variance of queue length increases infinitely with the time. In addition, based on results in Skellam [2], Kendall gave the probability distribution of the queue length and proved that the double-sided queue did not have steady-state probability under the constant arrival rates. In 1961, Dobbie [3] considered the nonhomogeneous Poisson arrival of passengers and taxis and obtained transient probability by using Laurent generating function. Jain [4] examined a double-ended queueing system in which only finite capacity for taxis were available and the inter-arrival distribution of taxis was general. He obtained the time-dependent probability generating function. For the same model as in Jain [4], Bhat [5] considered the control problem that the system is controlled by calling extra taxis whenever the total number of passengers lost to the systems reaches a certain predetermined number. He gave the transient and steady-state probability of the process by using renewal theoretic arguments and obtained the optimal value of the control variable to minimize the total cost. Giveen [6] further discussed the double-sided queueing system with nonhomogeneous Poisson arrival process and showed that the steady-state probabilities exist only if the mean and the variance of the queue length remain finite as time become infinite. Kashyap [7], [8] considered the general arrival of passengers and the Poisson arrival of taxis and assumed that each taxi can take a fixed number of passengers. The probability of an empty state was obtained by the supplementary variable technique. Mendoza et al. [9] imposed thresholds on both sides of the queues to assure stability in the absence of abandonment and investigated minimization of the expected total cost. Kim et al. [10] utilized a simulation approach to investigate the system so that the model was similar to the actual situation. The authors also conducted sensitivity analysis and found that batch size considerably affected the performance of the model. Moreover, if expectations of the batch size distributions were the same, then the model was insensitive to the types of such distributions. Crescenzo et al. [11] considered a double-ended queue with catastrophes and repairs and obtained steady-state and failure-state probabilities. In addition, the authors constructed a continuous approximation and pointed out that the means of the process and continuous approximation process matched.

The problem between taxis and passengers has been attracted the attention of researchers in other fields, for example, Hai et al. [12], Hai et al. [13] and Wong et al. [14]. In [12], the authors proposed an equilibrium model to characterize the bilateral searching and meeting between taxis and customers in road networks. Further the authors proposed a meeting function to spell out the meeting frictions that arise endogenously as a result of the distinct spatial feature of the meetings and the taxicustomer moving decisions. The authors showed that the stationary competitive equilibrium could be achieved at fixed fare prices when the demand of customers matches the supply of taxis at prevailing searching and waiting times in every location. In [13], the authors investigated an aggregate taxi market with search frictions. They firstly proved taxi service quality that was measured in terms of customer waiting time and market profitability that was measured in terms of average taxi profit could both increase with taxi fleet size if and only if the meeting function exhibited increasing returns to scale. Then they showed that taxi profits at social optimum were not necessarily negative but instead depended on the returns to scale of the meeting function. Specifically, taxi service made a profit at social optimum when there were decreasing returns to scale and broke even when there were constant returns to scale. Besides, they proved that taxi utilization rate and service quality remained unchanged and were equal to those at social optimum if there were constant returns to scale; they would both decrease (increase) along the Pareto frontier from the social optimum to the monopoly optimum when there were increasing (decreasing) returns to scale. In [14], the bi-level decisions of vacant taxi drivers in customer-search were modeled by using the sequential logit approach. The authors discussed the potential implications of implementing the proposed taxi policies, determined the factors that affected the choices of vacant taxi drivers on traveling towards the nearest taxi stand to search for customers and those vacant taxi drivers who wait at taxi stands, and provided a sub-model for developing micro-simulation models to evaluate the performance of road traffic networks with taxi services as well as developing simulation-based optimization models to answer policy questions related to taxi services.

To the best or our knowledge, no researchers have studied the passenger-taxi queueing systems with nonzero matching time since the system is very complicated. In this paper, we attempt to investigate the influence of matching times on the performance of system. Although we cannot obtain the explicit expressions of the steady-state probabilities, we give an algorithm to compute the joint steady-state probabilities. Then analyze the tail distributions of sojourn times of passengers and taxi drivers. We give the approximate algorithms to compute the tail probabilities.

The rest of the paper is organized as follows. In Section II, we introduce the system in detail and construct a two-dimensional Markov process to model the system. Besides, in this section, an algorithm is presented to compute the joint stead-state probabilities. In Section III, we discuss the tail distributions of sojourn times of passengers and taxis. In Section IV, we present a numerical example to show the performance measures of the model. Finally, we give a concluding remark in the last section.

II. Model Description

In this section, we investigate a passenger-taxi queueing system with finite waiting space for taxis. In this system, we assume that passengers and taxi drivers arrive to the taxi station according to Poisson processes with rates λ_1 and λ_2, respectively. When the matching time between passengers and taxis is long enough related to the inter-arrival time of passengers and taxis, we cannot ignore the matching time, which means passengers and taxi drivers can simultaneously wait in their respective waiting lines. To the best of our knowledge, no literatures has studied this case. For convenience, we assume the matching time follows exponential distribution with rate μ.

The waiting room for taxi drivers is limited by a buffer N. In this case, the system can be modeled as a two-dimensional Markov process $\{(I_1(t), I_2(t)), t \geq 0\}$, where $I_1(t)$ $(I_2(t))$ represents the number of passengers (taxi drivers, respectively) in the system at time t. And the state space is $S_2 = \{(i,j), i \in E_1, j \in E_2\}$, where $E_1 = \{0, 1, 2, \cdots\}$ and $E_2 = \{0, 1, 2, \cdots, N\}$.

Define the level n as the number of passengers in the system. Utilize the level-dependent method proposed by Lian et al. [15] to arrange the states in the standard ascending order. Therefore, all states can be written as

$$
\begin{array}{ll}
\text{Level } 0: & (0,0), (0,1), (0,2), \cdots, (0,N); \\
\text{Level } 1: & (1,0), (1,1), (1,2), \cdots, (1,N); \\
\text{Level } 2: & (2,0), (2,1), (2,2), \cdots, (2,N); \\
\text{Level } 3: & (3,0), (3,1), (3,2), \cdots, (3,N); \\
\vdots & \qquad\qquad \vdots
\end{array}
$$

The infinitesimal generator matrix Q is written as follows:

$$
Q = \begin{pmatrix}
B_0 & A & & & \\
C & B_1 & A & & \\
 & C & B_1 & A & \\
 & & \ddots & \ddots & \ddots
\end{pmatrix}, \tag{1}
$$

where matrices $A, C, B_0, B_1 \in \mathcal{R}^{(N+1)\times(N+1)}$,

$$
B_0 = \begin{pmatrix}
-(\lambda_1+\lambda_2) & \lambda_2 & & & & \\
 & -(\lambda_1+\lambda_2) & \lambda_2 & & & \\
 & & \ddots & \ddots & & \\
 & & & & -(\lambda_1+\lambda_2) & \lambda_2 \\
 & & & & & -\lambda_1
\end{pmatrix},
$$

$$
A = diag(\lambda_1, \lambda_1, \lambda_1, \cdots, \lambda_1),
$$

$$
C = \begin{pmatrix}
0 & & & & \\
\mu & 0 & & & \\
 & \mu & 0 & & \\
 & & \ddots & \ddots & \\
 & & & \mu & 0
\end{pmatrix}.
$$

Let $\pi = \{\pi_0, \pi_1, \cdots\}$ be the stationary probability vector of Q, where $\pi_i = \{\pi_{i,0}, \pi_{i,1}, \cdots, \pi_{i,N}\}$ $(i \in E_1)$. Following the results in Neuts [16], we have the following results:

Theorem 1. *The process Q is positive recurrent if and only if the solution R of the matrix-quadratic equation*

$$
R^2C + RB_1 + A = 0, \tag{2}
$$

has all the eigenvalues strictly less than one, and the π_0 is the positive solution of the following equations:

$$
\begin{cases}
\pi_0(B_0 + RC) & = \mathbf{0}, \\
\pi_0(I - R)^{-1}e & = 1,
\end{cases} \tag{3}
$$

where I is an identity matrix, and $e = (1, 1, \cdots, 1)^T$ is a $(N+1) \times (N+1)$ column vector. Thus, the joint stationary probability π_i $(i \in E_1)$ is given by

$$\pi_i = \pi_0 R^i. \tag{4}$$

Proposition 1. *The system is stable if and only if there exists a matrix R such that equation (2) holds.*

III. Sojourn Time of the Queueing System

The system is a double-sided queueing system in which both passengers and taxi drivers need to wait in their respective waiting line. So, in the next two subsections, we will investigate the sojourn times of passengers and taxi drivers, respectively.

A. Sojourn time of passengers

In order to obtain the sojourn time of passengers, we firstly construct a quasi-death process $\{(I_1(t), I_2(t)), t \geqslant 0\}$ with state space $S_2 = \{(i,j), i \in E_1, j \in E_2\}$ in which states $(0,j)$ $\forall j \in E_2$ are absorbing states corresponding to the states that there are no passengers in the system.

Let $W_{i,j}(x) = P\{W_1 > x | I_1(0) = i, I_2(0) = j, i \in E_1, j \in E_2\}$ denotes the conditional tail distribution of the sojourn time of a newly-arrived passenger who observers i passengers and j taxis in the system upon arrival. Thus, the condition sojourn time of a passenger who observers the system state is (i,j) is equivalent to the first passage time from state (i,j) to the absorbing states $(0,k)$ $(\forall\ k \in E_2)$.

Denote $\mathbf{W}_i(x) = \{W_{i,0}(x), W_{i,1}(x), \cdots, W_{i,N}(x)\}$ $(i \geqslant 1)$ and
$\mathcal{W}(x) = \{\mathbf{W}_1(x)^T, \mathbf{W}_2(x)^T, \cdots\}^T$ where the superscript T means transposition.

Thus, the defective infinitesimal generator matrix is given by

$$Q_1 = \begin{pmatrix} B_1 & A & & & \\ C & B_1 & A & & \\ & C & B_1 & A & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5}$$

Based on the Kolomogorov's Backward Equations, we have

$$\frac{d}{dx}\mathcal{W}(x) = Q_1 \mathcal{W}(x). \tag{6}$$

Thus, we have

$$\mathcal{W}(x) = \exp\{Q_1 x\}\mathbf{1} = \sum_{l=0}^{\infty} \frac{\hbar^l x^l}{l!} e^{-\hbar x} \big(I + \frac{Q_1}{\hbar}\big)^l \mathbf{1}, \tag{7}$$

where $\hbar = \lambda_1 + \lambda_2 + \mu$.

Denote $W_1(x) = P\{W_1 > x\}$ $(x \geqslant 0)$ be the probability the sojourn time of passengers in the stationary state is greater than x. Thus, we have

$$W_1(x) = \alpha \mathcal{W}(x) = \alpha \sum_{l=0}^{\infty} \frac{\hbar^l x^l}{l!} e^{-\hbar x} \mathcal{T}_1^l \mathbf{1}, \tag{8}$$

where $\alpha = (\pi_1, \pi_2, \cdots)$, $\mathcal{T}_1 = I + \frac{Q_1}{\hbar}$:

$$\mathcal{T}_1 = \begin{pmatrix} T_0 & T_1 & & & \\ T_2 & T_0 & T_1 & & \\ & T_2 & T_0 & T_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{9}$$

In this model, the number of possible passengers is unbounded. Based on the ideal of burte-force approach, we can choose a sufficiently large integer M_1 such that the probability of having more than M_1 passengers in the taxi station is negligible, which means there exists a positive number $M_1 > 0$ such that

$$0 \leqslant \alpha \mathcal{T}_1^l \mathbf{1} - \tilde{\alpha}_{M_1} \tilde{\mathcal{T}}_{1,M_1}^l \mathbf{1} < \epsilon_1, \quad \forall \epsilon_1 > 0, \forall l > 0, \tag{10}$$

where $\tilde{\alpha}_{M_1}$ and $\tilde{\mathcal{T}}_{1,M_1}$ are the truncations of α and $\mathcal{T}_1$, respectively,

$$\tilde{\alpha}_{M_1} = (\pi_1, \pi_2, \cdots, \pi_{M_1}), \tag{11}$$

$$\tilde{\mathcal{T}}_{1,M_1} = \begin{pmatrix} T_0 & T_1 & & & & \\ T_2 & T_0 & T_1 & & & \\ & T_2 & T_0 & T_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & T_2 & T_0 & T_1 \end{pmatrix}_{M_1(N+1)\times M_1(N+1)} \tag{12}$$

Thus, the probability of the sojourn time of passengers that is greater than x is approximately equivalent to

$$W_1(x) \approx \tilde{\alpha}_{M_1} \sum_{l=0}^{M_1} \frac{\hbar^l x^l}{l!} e^{-\hbar x} \tilde{\mathcal{T}}_{1,M_1}^l \mathbf{1}. \tag{13}$$

B. Sojourn time of taxi drivers

Taxis are efficient vehicles to complement public transportation. The sojourn time of taxi drivers is also of great importance to the utilization rate of taxis and profit of taxi companies. So it is necessary to investigate the sojourn time of taxi drivers.

Similarly, we construct a quasi-death process $\{(I_1(t), I_2(t)), t \geqslant 0\}$ with state space $S_2 = \{(i,j), i \in E_1, j \in E_2\}$ in which states $(i,0)$ $\forall i \in E_1$ are absorbing states corresponding to the states that there are no taxis in the system.

Let $\bar{W}_{i,j}(y) = P\{W_2 > y | I_1(0) = i, I_2(0) = j, i \in E_1, j \in E_2\}$ denotes the conditional tail distribution of the sojourn time of a newly-arrived taxi driver who observers i passengers and j taxis in the system upon arrival. Thus, the condition sojourn time of a passenger who observers the system state is (i,j) is equivalent to the first passage time from state (i,j) to the absorbing states $(\bar{k}, 0)$ $(\forall\ \bar{k} \in E_1)$.

Denote $\bar{\mathbf{W}}_i(y) = \{\bar{W}_{i,1}(y), \bar{W}_{i,2}(y), \cdots, \bar{W}_{i,N}(y)\}$ $(i \in E_1)$ and
$\bar{\mathcal{W}}(y) = \{\mathbf{W}_0(y)^T, \mathbf{W}_1(y)^T, \cdots\}^T$.

Thus, the defective infinitesimal generator matrix is given by

$$Q_2 = \begin{pmatrix} \bar{B}_0 & \bar{A} & & & \\ \bar{C} & \bar{B}_1 & \bar{A} & & \\ & \bar{C} & \bar{B}_1 & \bar{A} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{14}$$

where matrices $\bar{A}, \bar{C}, \bar{B}_0, \bar{B}_1 \in \mathcal{R}^{N \times N}$,

$$\bar{B}_0 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_2 & & & \\ & -(\lambda_1+\lambda_2) & \lambda_2 & & \\ & & \ddots & \ddots & \\ & & & -(\lambda_1+\lambda_2) & \lambda_2 \\ & & & & -\lambda_1 \end{pmatrix},$$

$$\bar{A} = diag(\lambda_1, \lambda_1, \lambda_1, \cdots, \lambda_1),$$

$$
\bar{C} = \begin{pmatrix} 0 & & & & \\ \mu & 0 & & & \\ & \mu & 0 & & \\ & & \ddots & \ddots & \\ & & & \mu & 0 \end{pmatrix}.
$$

Based on the Kolomogorov's Backward Equations, we have

$$
\frac{d}{dx}\bar{\mathcal{W}}(y) = Q_2\bar{\mathcal{W}}(y). \tag{15}
$$

Thus, we have

$$
\bar{\mathcal{W}}(y) = \exp\{Q_2 y\}\mathbf{1} = \sum_{l=0}^{\infty} \frac{\hbar^l y^l}{l!} e^{-\hbar y} \big(I + \frac{Q_2}{\hbar}\big)^l \mathbf{1}, \tag{16}
$$

where $\hbar = \lambda_1 + \lambda_2 + \mu$.

Denote $W_2(y) = P\{W_2 > y\}$ $(y \geqslant 0)$ be the probability of the sojourn time of taxi drivers in the stationary state is greater than y. Thus, we have

$$
W_2(y) = \beta\bar{\mathcal{W}}(y) = \beta \sum_{l=0}^{\infty} \frac{\hbar^l y^l}{l!} e^{-\hbar y} \mathcal{T}_2^l \mathbf{1}, \tag{17}
$$

where $\beta = (\beta_0, \beta_1, \cdots)$, $(\beta_i = \{\pi_{i,1}\pi_2, \cdots, \pi_{i,N}\})$ and $\mathcal{T}_2 = I + \frac{Q_2}{\hbar}$:

$$
\mathcal{T}_2 = \begin{pmatrix} \bar{T}_0 & \bar{T}_1 & & & \\ \bar{T}_2 & \bar{T}_3 & \bar{T}_1 & & \\ & \bar{T}_2 & \bar{T}_3 & \bar{T}_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{18}
$$

Similarly, based on the ideal of burte-force approach, we also can find a sufficiently large integer M_2 such that

$$
0 \leqslant \beta T_2^l \mathbf{1} - \tilde{\beta}_{M_2} \tilde{\mathcal{T}}_{2,M_2}^l \mathbf{1} < \epsilon_2, \quad \forall \epsilon_2 > 0, \tag{19}
$$

where $\tilde{\beta}_{M_2}$ and $\tilde{\mathcal{T}}_{2,M_2}$ are the truncations of β and $\mathcal{T}_2$, respectively,

$$
\tilde{\beta}_{M_2} = (\beta_0, \beta_1, \cdots, \beta_{M_2}), \tag{20}
$$

$$
\tilde{\mathcal{T}}_{2,M_2} = \begin{pmatrix} \bar{T}_0 & \bar{T}_1 & & & & \\ \bar{T}_2 & \bar{T}_3 & \bar{T}_1 & & & \\ & \bar{T}_2 & \bar{T}_0 & \bar{T}_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \bar{T}_2 & \bar{T}_0 & \bar{T}_1 \end{pmatrix}_{(M_2+1)N \times (M_2+1)N} \tag{21}
$$

Thus, the probability of the sojourn time of taxi drivers that is greater than y is approximately equivalent to

$$
W_2(y) \approx \tilde{\beta}_{M_2} \sum_{l=0}^{M_2} \frac{\hbar^l y^l}{l!} e^{-\hbar y} \tilde{\mathcal{T}}_{2,M_2}^l \mathbf{1}. \tag{22}
$$

IV. Numerical Results

In this section, we will present several examples to show some characteristics of the passenger-taxi queueing system with nonzero matching time.

Example 4.1 In this example, we assume that passengers and taxi drivers arrive to a taxi station from two sides according to two different Poisson processes independently with arrival rate λ_1 and λ_2 per minute, respectively. And the matching between passengers and taxis follows an exponential matching time with rate μ. And the size of taxis is limited by buffer $N = 4$. The state space is

$$
S_2 = \{(i,0),(i,1),(i,2),(i,3),(i,4), \quad i \in E_1\}.
$$

In order to get the stationary probabilities, we firstly need to find a matrix R such that equation (2) holds. By running an Matlab program, we find that when the buffer $N = 4$ is set, the sufficient condition that guarantee the stability of the system is the matching time μ and the arrival rates of λ_1 and λ_2 should satisfy the following inequality:

$$
\lambda_1 < \mu < \lambda_2. \tag{23}
$$

$\lambda_1 < \mu$ ensures the queue of passengers is stable. When the waiting space for taxis is very small (that is, the buffer of taxis is small enough), the arrival rate of taxis must be larger than the one of passengers, otherwise, passengers will wait for a quite long time.

If we choose $\lambda_1 = 6$, $\lambda_2 = 15$ and $\mu = 10$, according to the discussion in Section II, we have

$$
A = diag(6,6,6,6,6),
$$

$$
\bar{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \end{pmatrix},
$$

$$
B_0 = \begin{pmatrix} -21 & 15 & 0 & 0 & 0 \\ 0 & -21 & 15 & 0 & 0 \\ 0 & 0 & -21 & 15 & 0 \\ 0 & 0 & 0 & -21 & 15 \\ 0 & 0 & 0 & 0 & -6 \end{pmatrix},
$$

$$
B_1 = \begin{pmatrix} -21 & 15 & 0 & 0 & 0 \\ 0 & -22 & 15 & 0 & 0 \\ 0 & 0 & -22 & 15 & 0 \\ 0 & 0 & 0 & -22 & 15 \\ 0 & 0 & 0 & 0 & -16 \end{pmatrix},
$$

According to Theorem II, we can get the matrix

$$
R = \begin{pmatrix} 0.3472 & 0.2084 & 0.1457 & 0.1269 & 0.1190 \\ 0.0333 & 0.2363 & 0.1456 & 0.1126 & 0.1055 \\ 0.0097 & 0.294 & 0.2426 & 0.1693 & 0.1587 \\ 0.0061 & 0.0137 & 0.0431 & 0.2804 & 0.2628 \\ 0.0055 & 0.0118 & 0.0306 & 0.0942 & 0.4634 \end{pmatrix}
$$

with eigenvalues 0.6167, 0.3750, 0.2400, 0.1821,0.1560. And the stationary probability when there is no passenger present in the system is

$$
\pi_0 = (0.0045, 0.0142, 0.0374, 0.0944, 0.2361).
$$

Thus, the stationary probabilities when there are i passengers present in the system are

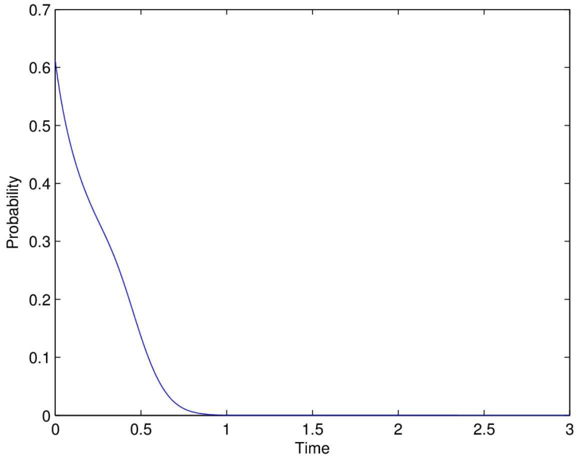
$$
\pi_i = \pi_0 R^i.
$$

Next, we need to consider the tail distribution of sojourn time of passengers. We find that there exists a positive number $M_1 = 15$ such that

$$
0 \leqslant \alpha \mathcal{T}_1^l \mathbf{1} - \tilde{\alpha}_{15} \tilde{\mathcal{T}}_{1,15}^l \mathbf{1} < \epsilon_1, \quad \forall \epsilon_1 > 0, \forall l > 0. \tag{24}
$$

Figure 1 depicts the tail distribution of sojourn time of passengers. From the figure, obviously, we can see that the tail probability of a passenger who needs to wait for taxis is 0.6, which means the probability of each passenger doesn't need to wait and can take a taxi immediately is equal to 0.4. The figure also tells us the tail probability of passengers who need to wait longer than 0.8 minutes is 0, which means almost no passenger needs to wait longer than 0.8 minutes.

Fig. 1. The tail distribution of sojourn time of passengers.

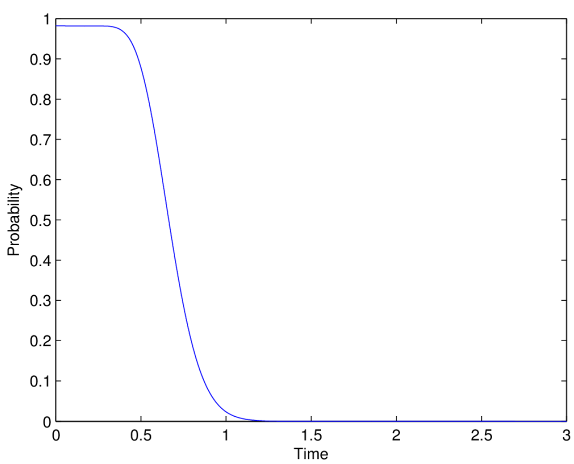


Next, we need to numerically investigate the tail distribution of sojourn time of taxi drivers. We find that there exists a positive number $M_2 = 20$ such that

$$0 \leqslant \beta \mathcal{T}_2^l \mathbf{1} - \tilde{\beta}_{20} \tilde{\mathcal{T}}_{2,20}^l \mathbf{1} < \epsilon_2, \quad \forall \epsilon_2 > 0, \forall l > 0. \tag{25}$$

Figure 2 depicts the tail distribution of sojourn time of taxi drivers. From the figure, we find out that the tail probability of taxi drivers who need to wait for passengers in the system is very close to 1, which means taxi drivers cannot pick up passengers immediately up arrival. The tail probability of taxi drivers who need to wait longer than 1.2 minutes is 0, which means each taxi driver can pick up a passenger every 1.2 minutes on average.

Fig. 2. The tail distribution of sojourn time of taxi drivers.



V. Conclusion

In this chapter, we treat a double-sided passenger-taxi queueing system with limited capacity for taxis. Under the assumption of nonzero matching time, we use a two-dimensional Markov process to model the system. By using matrix-analytic method, we give algorithms to compute the joint stationary probability and tail distributions of sojourn time of passengers and taxi drivers. Finally, we present an example to show the performance measures of the system. We find that the system is stable and there exist the stationary probabilities only if the arrival rates of passengers and taxi drivers and matching rate satisfy: $\lambda_1 < \mu < \lambda_2$.

Acknowledgments

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