# STAT0032: Exercise Sheet #1

The exercises in this sheet are a sample of questions on probability that are an incentive for further revision, if necessary.

Solve these assorted questions concerning probability.

1. Two fair dice are thrown. Let $X$ be the smallest of the two numbers obtained (or the common value if the same number is obtained on both dice). Find the probability mass function of $X$. Find $P(X > 3)$.

2. Let $X$ be a random variable with expectation $\mu$ and variance $\sigma^2$. Find the expectation and variance of the random variable $Y = (X - \mu)/\sigma$.

3. On a coral reef, $S$ species of fish are present in proportions $p_1, \ldots, p_S$. A biologist wishes to take a sample of the fish, and wants to know how many species of fish she should expect to find in a sample of a given size.

   (i) Suppose a sample of size $n$ is taken. Let $X_i$ ($i = 1, \ldots, S$) be a random variable taking the value 1 if species $i$ occurs, 0 otherwise. Find an expression for $E(X_i)$ (assume that, first, the sample is small relative to the population of any fish species, so that taking the sample has a negligible effect on the proportions of fish remaining; second, species are distributed randomly so that successive fish in the sample can be regarded as independent draws from the population).

   (ii) Now let $Y$ be the number of fish species present in the sample. Express $Y$ in terms of the $X_i$'s, and deduce that the expected number of species is

   $$S - \sum_{i=1}^{S} (1 - p_i)^n.$$

   Are you making any further assumptions in obtaining this result? Check that the formula gives the correct result for a couple of different sample sizes where the answer is "obvious".

4. For each case below, state whether the binomial distribution is suitable. If not, give your reasons; if it would, state the values of parameters $n$ and $p$.

   (i) The number of sixes obtained in three successive throws of a fair die.

   (ii) The number of girls in the families of British prime ministers.

   (iii) The number of aces in a hand of four cards dealt from a standard pack of cards.

(iv) The number of students in a class of 40 whose birthday falls on a Sunday this year.

(v) The number of throws of a fair coin until the first head is obtained.

5. Consider a jury trial in which it takes 8 out of the 12 jurors to convict. That is, in order for the defendant to be convicted, at least 8 of the 12 jurors must vote him guilty. Assume that jurors act independently and each returns a guilty verdict with probability $p$. Let $\alpha$ denote the probability that the defendant is guilty. What is the probability that the jury renders a correct decision?

6. An exam paper consists of ten multiple choice questions, each offering four choices of which only one is correct. If a candidate chooses his answers completely at random, what is the probability that

   (i) he gets at least 8 questions right,

   (ii) the last of the ten questions is the eighth one he gets right,

   (iii) in six such exams, he gets at least 8 questions right in at most one exam?

7. If $X$ is a geometric random variable with parameter $p$, show that $P(X = n + k \mid X > n) = P(X = k)$, $k = 1, 2, 3, \dots$. In the light of the interpretation of a geometric random variable in terms of independent Bernoulli trials, explain why this result is "obvious".

8. Suppose that the number of distinct uranium deposits in a given area is a Poisson random variable with parameter $\mu = 10$. If, in a fixed period of time, each deposit is independently discovered with probability $1/50$, find the probability that (i) exactly one, (ii) at least one and, (iii) at most one deposit is discovered during that time.

9. The proportion of time during a 40-hour week that an industrial robot is in operation is modelled by a random variable $X$ with probability density function

$$f(x) = \begin{cases} cx, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

   where $c$ is a constant. Find (i) $c$, (ii) $P(X < 12)$ and (iii) $P(X > 1/3 \mid X < 1/2)$.

10. The probability density function of $X$, the lifetime in hours of a certain type of electronic device, is given by

$$f(x) = \begin{cases} 10/x^2, & \text{if } x > 10, \\ 0, & \text{otherwise.} \end{cases}$$

   (i) Find $P(X > 15)$.

   (ii) Assume that the life times of the devices are independent. What is the probability that, out of 5 such devices, at least 4 will function for at least 15 hours?