STAT0032: Exercise Sheet #7

The exercises in this sheet focus on nonlinear regression. As in the previous sheet, some of the questions are from James et al., "An Introduction to Statistical Learning" (ISLR).

1. (Exercise 1 of Chapter 7, ISLR.) It was mentioned in the chapter that a cubic regression spline with one knot at ξ can be obtained using a basis of the form $x, x^2, x^3, (x-\xi)_+^3$, where $(x-\xi)_+^3 = (x-\xi)^3$ if $x > \xi$ and equals 0 otherwise. We will now show that a function of the form

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x-\xi)_+^3$$

is indeed a cubic regression spline, regardless of the values of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$.

(a) Find a cubic polynomial

$$f_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$$

such that $f(x) = f_1(x)$ for all $x \leq \xi$. Express a_1, b_1, c_1, d_1 in terms of β_0, β_1, β_2, β_3, β_4.

(b) Find a cubic polynomial

$$f_2(x) = a_2 + b_2 x + c_2 x^2 + d_2 x^3$$

such that $f(x) = f_2(x)$ for all $x > \xi$. Express a_2, b_2, c_2, d_2 in terms of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$. We have now established that $f(x)$ is a piecewise polynomial.

(c) Show that $f_1(\xi) = f_2(\xi)$. That is, $f(x)$ is continuous at ξ.

(d) Show that $f_1'(\xi) = f_2'(\xi)$. That is, $f'(x)$ is continuous at ξ.

(e) Show that $f_1''(\xi) = f_2''(\xi)$. That is, $f''(x)$ is continuous at ξ.

Therefore, $f(x)$ is indeed a cubic spline.

Hint: Parts (d) and (e) of this problem require knowledge of single-variable calculus. As a reminder, given a cubic polynomial

$$f_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3,$$

the first derivative takes the form

$$f_1'(x) = b_1 + 2c_1 x + 3d_1 x^2$$

and the second derivative takes the form

$$f_1''(x) = 2c_1 + 6d_1 x.$$

2. (Exercise 2 of Chapter 7, ISLR.) Suppose that a curve $\hat{g}$ is computed to smoothly fit a set of n points using the following formula:

$$\hat{g} = \arg\min_{g} \left(\sum_{i=1}^{n} (y_i - g(x_i))^2 + \lambda \int [g^{(m)}(x)]^2 dx \right)$$

where $g^{(m)}$ represents the mth derivative of g (and $g^{(0)} = g$). Provide example sketches of $\hat{g}$ in each of the following scenarios.

(a) $\lambda \to \infty, m = 0$.

(b) $\lambda \to \infty, m = 1$.

(c) $\lambda \to \infty, m = 2$.

(d) $\lambda \to \infty, m = 3$.

(e) $\lambda = 0, m = 3$.

3. (Exercise 3 of Chapter 7, ISLR.) Suppose we fit a curve with basis functions $b_1(X) = X$, $b_2(X) = (X-1)^2 I(X \geq 1)$. (Note that $I(X \geq 1)$ equals 1 for $X \geq 1$ and 0 otherwise.) We fit the linear regression model

$$Y = \beta_0 + \beta_1 b_1(X) + \beta_2 b_2(X) + \epsilon,$$

and obtain coefficient estimates $\hat{\beta}_0 = 1$, $\hat{\beta}_1 = 1$, $\hat{\beta}_2 = -2$. Sketch the estimated curve between $X = -2$ and $X = 2$. Note the intercepts, slopes, and other relevant information.

4. (Exercise 4 of Chapter 7, ISLR.) Suppose we fit a curve with basis functions $b_1(X) = I(0 \leq X \leq 2) - (X-1)I(1 \leq X \leq 2)$, $b_2(X) = (X-3)I(3 \leq X \leq 4) + I(4 < X \leq 5)$. We fit the linear regression model

$$Y = \beta_0 + \beta_1 b_1(X) + \beta_2 b_2(X) + \epsilon,$$

and obtain coefficient estimates $\hat{\beta}_0 = 1$, $\hat{\beta}_1 = 1$, $\hat{\beta}_2 = 3$. Sketch the estimated curve between $X = -2$ and $X = 2$. Note the intercepts, slopes, and other relevant information.

5. (Exercise 5 of Chapter 7, ISLR.) Consider two curves, $\hat{g}_1$ and $\hat{g}_2$, defined by

$$\hat{g}_1 = \arg\min_{g} \left(\sum_{i=1}^{n} (y_i - g(x_i))^2 + \lambda \int [g^{(3)}(x)]^2 dx \right)$$
$$\hat{g}_2 = \arg\min_{g} \left(\sum_{i=1}^{n} (y_i - g(x_i))^2 + \lambda \int [g^{(4)}(x)]^2 dx \right)$$

where $g^{(m)}$ represents the mth derivative of g.

(a) As $\lambda \to \infty$, will $\hat{g}_1$ or $\hat{g}_2$ have the smaller training RSS?

(b) As $\lambda \to \infty$, will $\hat{g}_1$ or $\hat{g}_2$ have the smaller test RSS?

(c) For $\lambda = 0$, will $\hat{g}_1$ or $\hat{g}_2$ have the smaller training and test RSS?

6. (COMPUTER IMPLEMENTATION) Exercise 6 of Chapter 7, ISLR (ignore the ANOVA question).

7. (COMPUTER IMPLEMENTATION) Exercise 9 of Chapter 7, ISLR.

8. (COMPUTER IMPLEMENTATION) Exercise 10 of Chapter 7, ISLR.

9. (COMPUTER IMPLEMENTATION) Exercise 11 of Chapter 7, ISLR.

10. (COMPUTER IMPLEMENTATION) Exercise 12 of Chapter 7, ISLR.