MAST90138 Assignment 1

Haonan Zhong 867492

Question 1a

For the matrix Σ to be a covariance matrix, Σ needs to be positive semi-definite and symmetric. In order for Σ to be a symmetric matrix, a must be equals to 2. Therefore, we have

$$\Sigma = \Sigma^T = \begin{bmatrix} 1 & 2 \\ 2 & b \end{bmatrix}$$

And given that Σ is positive semi-definite if $X^T \Sigma X \geqslant 0$ for all $X \neq 0$, hence we need to identify values of b that makes Σ a positive semi-definite matrix. Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{aligned} X^T \Sigma X &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^2 + 4x_1x_2 + bx_2^2 \\ &= x_1^2 + 4x_1x_2 + 4x_2^2 + bx_2^2 - 4x_2^2 \\ &= (x_1 + 2x_2)^2 + (b-4)x_2^2 \geqslant 0 \end{aligned}$$

From the above equation, we can see that

$$\begin{aligned} (x_1 + 2x_2)^2 &\geqslant 0 \\ (b-4)x_2^2 &\geqslant 0 \\ \therefore b &\geqslant 4 \end{aligned}$$

Therefore, Σ is positive semi-definite when $b \geqslant 4$. Hence, Σ is a covariance matrix when $a = 2$ and $b \geqslant 4$.

Question 1b

To find the eigenvalues of Σ, we have

$$det(\Sigma - \lambda I) = det(\begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}) = |\begin{bmatrix} 13-\lambda & -4 \\ -4 & 7-\lambda \end{bmatrix}| = 0$$

Therefore, we have $(13-\lambda)(7-\lambda) - 16 = 0$

$$\therefore \lambda^2 - 20\lambda + 75 = 0$$

$$(\lambda - 5)(\lambda - 15) = 0$$

$$\therefore \lambda = 5, \ \lambda = 15$$

Thus, the eigenvalues of Σ is 5 and 15, respectively. Next, we will compute the corresponding othonormal eigenvectors.

When $\lambda = 15$,

$$(\begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix} - \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}) \times v_1 = 0$$

$$\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} \times \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$-2v_{11} - 4v_{12} = 0$$

$$-4v_{11} - 8v_{12} = 0$$

Solving the above equations we have $v_{11} = -2$ and $v_{12} = 1$, and the orthonormal eigenvector v_1 is,

$$\frac{1}{\sqrt{(-2)^2 + 1^2}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

When $\lambda = 5$,

$$(\begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}) \times v_2 = 0$$

$$\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} \times \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$8v_{21} - 4v_{22} = 0$$

$$-4v_{21} + 2v_{22} = 0$$

Solving the above equations we have $v_{21} = 1$ and $v_{22} = 2$, and the orthonormal eigenvector v_2 is,

$$\frac{1}{\sqrt{1^2 + 2^2}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

Therefore,

$$\Gamma = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \; and \; \Lambda = \begin{bmatrix} 15 & 0 \\ 0 & 5 \end{bmatrix}$$

```
Lambda <- diag(c(15, 5))
Gamma <- matrix(c(-2/sqrt(5), 1/sqrt(5), 1/sqrt(5), 2/sqrt(5)), 2, 2)
(Sigma <- Gamma %*% Lambda %*% t(Gamma))
```

```
##      [,1] [,2]
## [1,]   13   -4
## [2,]   -4    7
```

As we can see from the R output above, our calculations are indeed correct!

Question 1c

```
X <- read.csv("Wheat data.txt", sep = "", header = F)
# printing wheat variety vector
(wheat_variety <- X[, 8])
```

```
##   [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
##  [38] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2
##  [75] 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
## [112] 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 3 3 3 3 3 3 3 3
## [149] 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3
## [186] 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3 3
```

```
X <- as.matrix(X[, -c(8)])
# printing the dimension of X
dim(X)
```

```
## [1] 210   7
```

```
# printing the data matrix X
X
```

```
##           V1    V2     V3    V4    V5     V6    V7
##   [1,] 15.26 14.84 0.8710 5.763 3.312 2.2210 5.220
##   [2,] 14.88 14.57 0.8811 5.554 3.333 1.0180 4.956
##   [3,] 14.29 14.09 0.9050 5.291 3.337 2.6990 4.825
##   [4,] 13.84 13.94 0.8955 5.324 3.379 2.2590 4.805
##   [5,] 16.14 14.99 0.9034 5.658 3.562 1.3550 5.175
##   [6,] 14.38 14.21 0.8951 5.386 3.312 2.4620 4.956
##   [7,] 14.69 14.49 0.8799 5.563 3.259 3.5860 5.219
##   [8,] 14.11 14.10 0.8911 5.420 3.302 2.7000 5.000
##   [9,] 16.63 15.46 0.8747 6.053 3.465 2.0400 5.877
##  [10,] 16.44 15.25 0.8880 5.884 3.505 1.9690 5.533
##  [11,] 15.26 14.85 0.8696 5.714 3.242 4.5430 5.314
##  [12,] 14.03 14.16 0.8796 5.438 3.201 1.7170 5.001
##  [13,] 13.89 14.02 0.8880 5.439 3.199 3.9860 4.738
##  [14,] 13.78 14.06 0.8759 5.479 3.156 3.1360 4.872
##  [15,] 13.74 14.05 0.8744 5.482 3.114 2.9320 4.825
##  [16,] 14.59 14.28 0.8993 5.351 3.333 4.1850 4.781
##  [17,] 13.99 13.83 0.9183 5.119 3.383 5.2340 4.781
##  [18,] 15.69 14.75 0.9058 5.527 3.514 1.5990 5.046
##  [19,] 14.70 14.21 0.9153 5.205 3.466 1.7670 4.649
##  [20,] 12.72 13.57 0.8686 5.226 3.049 4.1020 4.914
##  [21,] 14.16 14.40 0.8584 5.658 3.129 3.0720 5.176
##  [22,] 14.11 14.26 0.8722 5.520 3.168 2.6880 5.219
##  [23,] 15.88 14.90 0.8988 5.618 3.507 0.7651 5.091
##  [24,] 12.08 13.23 0.8664 5.099 2.936 1.4150 4.961
##  [25,] 15.01 14.76 0.8657 5.789 3.245 1.7910 5.001
##  [26,] 16.19 15.16 0.8849 5.833 3.421 0.9030 5.307
##  [27,] 13.02 13.76 0.8641 5.395 3.026 3.3730 4.825
##  [28,] 12.74 13.67 0.8564 5.395 2.956 2.5040 4.869
##  [29,] 14.11 14.18 0.8820 5.541 3.221 2.7540 5.038
##  [30,] 13.45 14.02 0.8604 5.516 3.065 3.5310 5.097
##  [31,] 13.16 13.82 0.8662 5.454 2.975 0.8551 5.056
##  [32,] 15.49 14.94 0.8724 5.757 3.371 3.4120 5.228
##  [33,] 14.09 14.41 0.8529 5.717 3.186 3.9200 5.299
##  [34,] 13.94 14.17 0.8728 5.585 3.150 2.1240 5.012
##  [35,] 15.05 14.68 0.8779 5.712 3.328 2.1290 5.360
##  [36,] 16.12 15.00 0.9000 5.709 3.485 2.2700 5.443
```

```
##  [37,] 16.20 15.27 0.8734 5.826 3.464 2.8230 5.527
##  [38,] 17.08 15.38 0.9079 5.832 3.683 2.9560 5.484
##  [39,] 14.80 14.52 0.8823 5.656 3.288 3.1120 5.309
##  [40,] 14.28 14.17 0.8944 5.397 3.298 6.6850 5.001
##  [41,] 13.54 13.85 0.8871 5.348 3.156 2.5870 5.178
##  [42,] 13.50 13.85 0.8852 5.351 3.158 2.2490 5.176
##  [43,] 13.16 13.55 0.9009 5.138 3.201 2.4610 4.783
##  [44,] 15.50 14.86 0.8820 5.877 3.396 4.7110 5.528
##  [45,] 15.11 14.54 0.8986 5.579 3.462 3.1280 5.180
##  [46,] 13.80 14.04 0.8794 5.376 3.155 1.5600 4.961
##  [47,] 15.36 14.76 0.8861 5.701 3.393 1.3670 5.132
##  [48,] 14.99 14.56 0.8883 5.570 3.377 2.9580 5.175
##  [49,] 14.79 14.52 0.8819 5.545 3.291 2.7040 5.111
##  [50,] 14.86 14.67 0.8676 5.678 3.258 2.1290 5.351
##  [51,] 14.43 14.40 0.8751 5.585 3.272 3.9750 5.144
##  [52,] 15.78 14.91 0.8923 5.674 3.434 5.5930 5.136
##  [53,] 14.49 14.61 0.8538 5.715 3.113 4.1160 5.396
##  [54,] 14.33 14.28 0.8831 5.504 3.199 3.3280 5.224
##  [55,] 14.52 14.60 0.8557 5.741 3.113 1.4810 5.487
##  [56,] 15.03 14.77 0.8658 5.702 3.212 1.9330 5.439
##  [57,] 14.46 14.35 0.8818 5.388 3.377 2.8020 5.044
##  [58,] 14.92 14.43 0.9006 5.384 3.412 1.1420 5.088
##  [59,] 15.38 14.77 0.8857 5.662 3.419 1.9990 5.222
##  [60,] 12.11 13.47 0.8392 5.159 3.032 1.5020 4.519
##  [61,] 11.42 12.86 0.8683 5.008 2.850 2.7000 4.607
##  [62,] 11.23 12.63 0.8840 4.902 2.879 2.2690 4.703
##  [63,] 12.36 13.19 0.8923 5.076 3.042 3.2200 4.605
##  [64,] 13.22 13.84 0.8680 5.395 3.070 4.1570 5.088
##  [65,] 12.78 13.57 0.8716 5.262 3.026 1.1760 4.782
##  [66,] 12.88 13.50 0.8879 5.139 3.119 2.3520 4.607
##  [67,] 14.34 14.37 0.8726 5.630 3.190 1.3130 5.150
##  [68,] 14.01 14.29 0.8625 5.609 3.158 2.2170 5.132
##  [69,] 14.37 14.39 0.8726 5.569 3.153 1.4640 5.300
##  [70,] 12.73 13.75 0.8458 5.412 2.882 3.5330 5.067
##  [71,] 17.63 15.98 0.8673 6.191 3.561 4.0760 6.060
##  [72,] 16.84 15.67 0.8623 5.998 3.484 4.6750 5.877
##  [73,] 17.26 15.73 0.8763 5.978 3.594 4.5390 5.791
##  [74,] 19.11 16.26 0.9081 6.154 3.930 2.9360 6.079
##  [75,] 16.82 15.51 0.8786 6.017 3.486 4.0040 5.841
##  [76,] 16.77 15.62 0.8638 5.927 3.438 4.9200 5.795
##  [77,] 17.32 15.91 0.8599 6.064 3.403 3.8240 5.922
##  [78,] 20.71 17.23 0.8763 6.579 3.814 4.4510 6.451
##  [79,] 18.94 16.49 0.8750 6.445 3.639 5.0640 6.362
##  [80,] 17.12 15.55 0.8892 5.850 3.566 2.8580 5.746
##  [81,] 16.53 15.34 0.8823 5.875 3.467 5.5320 5.880
##  [82,] 18.72 16.19 0.8977 6.006 3.857 5.3240 5.879
##  [83,] 20.20 16.89 0.8894 6.285 3.864 5.1730 6.187
##  [84,] 19.57 16.74 0.8779 6.384 3.772 1.4720 6.273
##  [85,] 19.51 16.71 0.8780 6.366 3.801 2.9620 6.185
##  [86,] 18.27 16.09 0.8870 6.173 3.651 2.4430 6.197
##  [87,] 18.88 16.26 0.8969 6.084 3.764 1.6490 6.109
##  [88,] 18.98 16.66 0.8590 6.549 3.670 3.6910 6.498
##  [89,] 21.18 17.21 0.8989 6.573 4.033 5.7800 6.231
##  [90,] 20.88 17.05 0.9031 6.450 4.032 5.0160 6.321
```

```
##  [91,] 20.10 16.99 0.8746 6.581 3.785 1.9550 6.449
##  [92,] 18.76 16.20 0.8984 6.172 3.796 3.1200 6.053
##  [93,] 18.81 16.29 0.8906 6.272 3.693 3.2370 6.053
##  [94,] 18.59 16.05 0.9066 6.037 3.860 6.0010 5.877
##  [95,] 18.36 16.52 0.8452 6.666 3.485 4.9330 6.448
##  [96,] 16.87 15.65 0.8648 6.139 3.463 3.6960 5.967
##  [97,] 19.31 16.59 0.8815 6.341 3.810 3.4770 6.238
##  [98,] 18.98 16.57 0.8687 6.449 3.552 2.1440 6.453
##  [99,] 18.17 16.26 0.8637 6.271 3.512 2.8530 6.273
## [100,] 18.72 16.34 0.8810 6.219 3.684 2.1880 6.097
## [101,] 16.41 15.25 0.8866 5.718 3.525 4.2170 5.618
## [102,] 17.99 15.86 0.8992 5.890 3.694 2.0680 5.837
## [103,] 19.46 16.50 0.8985 6.113 3.892 4.3080 6.009
## [104,] 19.18 16.63 0.8717 6.369 3.681 3.3570 6.229
## [105,] 18.95 16.42 0.8829 6.248 3.755 3.3680 6.148
## [106,] 18.83 16.29 0.8917 6.037 3.786 2.5530 5.879
## [107,] 18.85 16.17 0.9056 6.152 3.806 2.8430 6.200
## [108,] 17.63 15.86 0.8800 6.033 3.573 3.7470 5.929
## [109,] 19.94 16.92 0.8752 6.675 3.763 3.2520 6.550
## [110,] 18.55 16.22 0.8865 6.153 3.674 1.7380 5.894
## [111,] 18.45 16.12 0.8921 6.107 3.769 2.2350 5.794
## [112,] 19.38 16.72 0.8716 6.303 3.791 3.6780 5.965
## [113,] 19.13 16.31 0.9035 6.183 3.902 2.1090 5.924
## [114,] 19.14 16.61 0.8722 6.259 3.737 6.6820 6.053
## [115,] 20.97 17.25 0.8859 6.563 3.991 4.6770 6.316
## [116,] 19.06 16.45 0.8854 6.416 3.719 2.2480 6.163
## [117,] 18.96 16.20 0.9077 6.051 3.897 4.3340 5.750
## [118,] 19.15 16.45 0.8890 6.245 3.815 3.0840 6.185
## [119,] 18.89 16.23 0.9008 6.227 3.769 3.6390 5.966
## [120,] 20.03 16.90 0.8811 6.493 3.857 3.0630 6.320
## [121,] 20.24 16.91 0.8897 6.315 3.962 5.9010 6.188
## [122,] 18.14 16.12 0.8772 6.059 3.563 3.6190 6.011
## [123,] 16.17 15.38 0.8588 5.762 3.387 4.2860 5.703
## [124,] 18.43 15.97 0.9077 5.980 3.771 2.9840 5.905
## [125,] 15.99 14.89 0.9064 5.363 3.582 3.3360 5.144
## [126,] 18.75 16.18 0.8999 6.111 3.869 4.1880 5.992
## [127,] 18.65 16.41 0.8698 6.285 3.594 4.3910 6.102
## [128,] 17.98 15.85 0.8993 5.979 3.687 2.2570 5.919
## [129,] 20.16 17.03 0.8735 6.513 3.773 1.9100 6.185
## [130,] 17.55 15.66 0.8991 5.791 3.690 5.3660 5.661
## [131,] 18.30 15.89 0.9108 5.979 3.755 2.8370 5.962
## [132,] 18.94 16.32 0.8942 6.144 3.825 2.9080 5.949
## [133,] 15.38 14.90 0.8706 5.884 3.268 4.4620 5.795
## [134,] 16.16 15.33 0.8644 5.845 3.395 4.2660 5.795
## [135,] 15.56 14.89 0.8823 5.776 3.408 4.9720 5.847
## [136,] 15.38 14.66 0.8990 5.477 3.465 3.6000 5.439
## [137,] 17.36 15.76 0.8785 6.145 3.574 3.5260 5.971
## [138,] 15.57 15.15 0.8527 5.920 3.231 2.6400 5.879
## [139,] 15.60 15.11 0.8580 5.832 3.286 2.7250 5.752
## [140,] 16.23 15.18 0.8850 5.872 3.472 3.7690 5.922
## [141,] 13.07 13.92 0.8480 5.472 2.994 5.3040 5.395
## [142,] 13.32 13.94 0.8613 5.541 3.073 7.0350 5.440
## [143,] 13.34 13.95 0.8620 5.389 3.074 5.9950 5.307
## [144,] 12.22 13.32 0.8652 5.224 2.967 5.4690 5.221
```

```
## [145,] 11.82 13.40 0.8274 5.314 2.777 4.4710 5.178
## [146,] 11.21 13.13 0.8167 5.279 2.687 6.1690 5.275
## [147,] 11.43 13.13 0.8335 5.176 2.719 2.2210 5.132
## [148,] 12.49 13.46 0.8658 5.267 2.967 4.4210 5.002
## [149,] 12.70 13.71 0.8491 5.386 2.911 3.2600 5.316
## [150,] 10.79 12.93 0.8107 5.317 2.648 5.4620 5.194
## [151,] 11.83 13.23 0.8496 5.263 2.840 5.1950 5.307
## [152,] 12.01 13.52 0.8249 5.405 2.776 6.9920 5.270
## [153,] 12.26 13.60 0.8333 5.408 2.833 4.7560 5.360
## [154,] 11.18 13.04 0.8266 5.220 2.693 3.3320 5.001
## [155,] 11.36 13.05 0.8382 5.175 2.755 4.0480 5.263
## [156,] 11.19 13.05 0.8253 5.250 2.675 5.8130 5.219
## [157,] 11.34 12.87 0.8596 5.053 2.849 3.3470 5.003
## [158,] 12.13 13.73 0.8081 5.394 2.745 4.8250 5.220
## [159,] 11.75 13.52 0.8082 5.444 2.678 4.3780 5.310
## [160,] 11.49 13.22 0.8263 5.304 2.695 5.3880 5.310
## [161,] 12.54 13.67 0.8425 5.451 2.879 3.0820 5.491
## [162,] 12.02 13.33 0.8503 5.350 2.810 4.2710 5.308
## [163,] 12.05 13.41 0.8416 5.267 2.847 4.9880 5.046
## [164,] 12.55 13.57 0.8558 5.333 2.968 4.4190 5.176
## [165,] 11.14 12.79 0.8558 5.011 2.794 6.3880 5.049
## [166,] 12.10 13.15 0.8793 5.105 2.941 2.2010 5.056
## [167,] 12.44 13.59 0.8462 5.319 2.897 4.9240 5.270
## [168,] 12.15 13.45 0.8443 5.417 2.837 3.6380 5.338
## [169,] 11.35 13.12 0.8291 5.176 2.668 4.3370 5.132
## [170,] 11.24 13.00 0.8359 5.090 2.715 3.5210 5.088
## [171,] 11.02 13.00 0.8189 5.325 2.701 6.7350 5.163
## [172,] 11.55 13.10 0.8455 5.167 2.845 6.7150 4.956
## [173,] 11.27 12.97 0.8419 5.088 2.763 4.3090 5.000
## [174,] 11.40 13.08 0.8375 5.136 2.763 5.5880 5.089
## [175,] 10.83 12.96 0.8099 5.278 2.641 5.1820 5.185
## [176,] 10.80 12.57 0.8590 4.981 2.821 4.7730 5.063
## [177,] 11.26 13.01 0.8355 5.186 2.710 5.3350 5.092
## [178,] 10.74 12.73 0.8329 5.145 2.642 4.7020 4.963
## [179,] 11.48 13.05 0.8473 5.180 2.758 5.8760 5.002
## [180,] 12.21 13.47 0.8453 5.357 2.893 1.6610 5.178
## [181,] 11.41 12.95 0.8560 5.090 2.775 4.9570 4.825
## [182,] 12.46 13.41 0.8706 5.236 3.017 4.9870 5.147
## [183,] 12.19 13.36 0.8579 5.240 2.909 4.8570 5.158
## [184,] 11.65 13.07 0.8575 5.108 2.850 5.2090 5.135
## [185,] 12.89 13.77 0.8541 5.495 3.026 6.1850 5.316
## [186,] 11.56 13.31 0.8198 5.363 2.683 4.0620 5.182
## [187,] 11.81 13.45 0.8198 5.413 2.716 4.8980 5.352
## [188,] 10.91 12.80 0.8372 5.088 2.675 4.1790 4.956
## [189,] 11.23 12.82 0.8594 5.089 2.821 7.5240 4.957
## [190,] 10.59 12.41 0.8648 4.899 2.787 4.9750 4.794
## [191,] 10.93 12.80 0.8390 5.046 2.717 5.3980 5.045
## [192,] 11.27 12.86 0.8563 5.091 2.804 3.9850 5.001
## [193,] 11.87 13.02 0.8795 5.132 2.953 3.5970 5.132
## [194,] 10.82 12.83 0.8256 5.180 2.630 4.8530 5.089
## [195,] 12.11 13.27 0.8639 5.236 2.975 4.1320 5.012
## [196,] 12.80 13.47 0.8860 5.160 3.126 4.8730 4.914
## [197,] 12.79 13.53 0.8786 5.224 3.054 5.4830 4.958
## [198,] 13.37 13.78 0.8849 5.320 3.128 4.6700 5.091
```

```
## [199,] 12.62 13.67 0.8481 5.410 2.911 3.3060 5.231
## [200,] 12.76 13.38 0.8964 5.073 3.155 2.8280 4.830
## [201,] 12.38 13.44 0.8609 5.219 2.989 5.4720 5.045
## [202,] 12.67 13.32 0.8977 4.984 3.135 2.3000 4.745
## [203,] 11.18 12.72 0.8680 5.009 2.810 4.0510 4.828
## [204,] 12.70 13.41 0.8874 5.183 3.091 8.4560 5.000
## [205,] 12.37 13.47 0.8567 5.204 2.960 3.9190 5.001
## [206,] 12.19 13.20 0.8783 5.137 2.981 3.6310 4.870
## [207,] 11.23 12.88 0.8511 5.140 2.795 4.3250 5.003
## [208,] 13.20 13.66 0.8883 5.236 3.232 8.3150 5.056
## [209,] 11.84 13.21 0.8521 5.175 2.836 3.5980 5.044
## [210,] 12.30 13.34 0.8684 5.243 2.974 5.6370 5.063
```

Question 1d

```r
# Calculating the unbiased sample covariance matrix
(S <- cov(X))
```

```
##             V1          V2            V3           V4           V5           V6
## V1  8.46635078  3.77844320  0.0418225658  1.224703671  1.066911361 -1.004355845
## V2  3.77844320  1.70552820  0.0163319511  0.562665550  0.466064932 -0.426765980
## V3  0.04182257  0.01633195  0.0005583493  0.003851826  0.006797719 -0.011776556
## V4  1.22470367  0.56266555  0.0038518256  0.196305245  0.143991709 -0.114289956
## V5  1.06691136  0.46606493  0.0067977190  0.143991709  0.142668202 -0.146542890
## V6 -1.00435584 -0.42676598 -0.0117765562 -0.114289956 -0.146542890  2.260684046
## V7  1.23513290  0.57175254  0.0026342068  0.203125110  0.139068229 -0.008187052
##              V7
## V1  1.235132905
## V2  0.571752539
## V3  0.002634207
## V4  0.203125110
## V5  0.139068229
## V6 -0.008187052
## V7  0.241553081
```

```r
# Computing the eigenvalues and eigenvectors of S
eigenval <- eigen(S)$values
eigenvec <- eigen(S)$vectors
# Constructing Lambda and Gamma
Lambda <- diag(eigenval)
Gamma <- as.matrix(eigenvec)
# Computing the spectral decomposition of S
Gamma %*% Lambda %*% t(Gamma)
```

```
##             [,1]        [,2]          [,3]         [,4]         [,5]
## [1,]  8.46635078  3.77844320  0.0418225658  1.224703671  1.066911361
## [2,]  3.77844320  1.70552820  0.0163319511  0.562665550  0.466064932
## [3,]  0.04182257  0.01633195  0.0005583493  0.003851826  0.006797719
## [4,]  1.22470367  0.56266555  0.0038518256  0.196305245  0.143991709
## [5,]  1.06691136  0.46606493  0.0067977190  0.143991709  0.142668202
## [6,] -1.00435584 -0.42676598 -0.0117765562 -0.114289956 -0.146542890
```

```
## [7,]  1.23513290  0.57175254  0.0026342068  0.203125110  0.139068229
##              [,6]         [,7]
## [1,] -1.004355845  1.235132905
## [2,] -0.426765980  0.571752539
## [3,] -0.011776556  0.002634207
## [4,] -0.114289956  0.203125110
## [5,] -0.146542890  0.139068229
## [6,]  2.260684046 -0.008187052
## [7,] -0.008187052  0.241553081
```

As we can see from the result above, we indeed have $\Gamma\Lambda\Gamma^T = S$, where Λ and Γ is:

```
Lambda
```

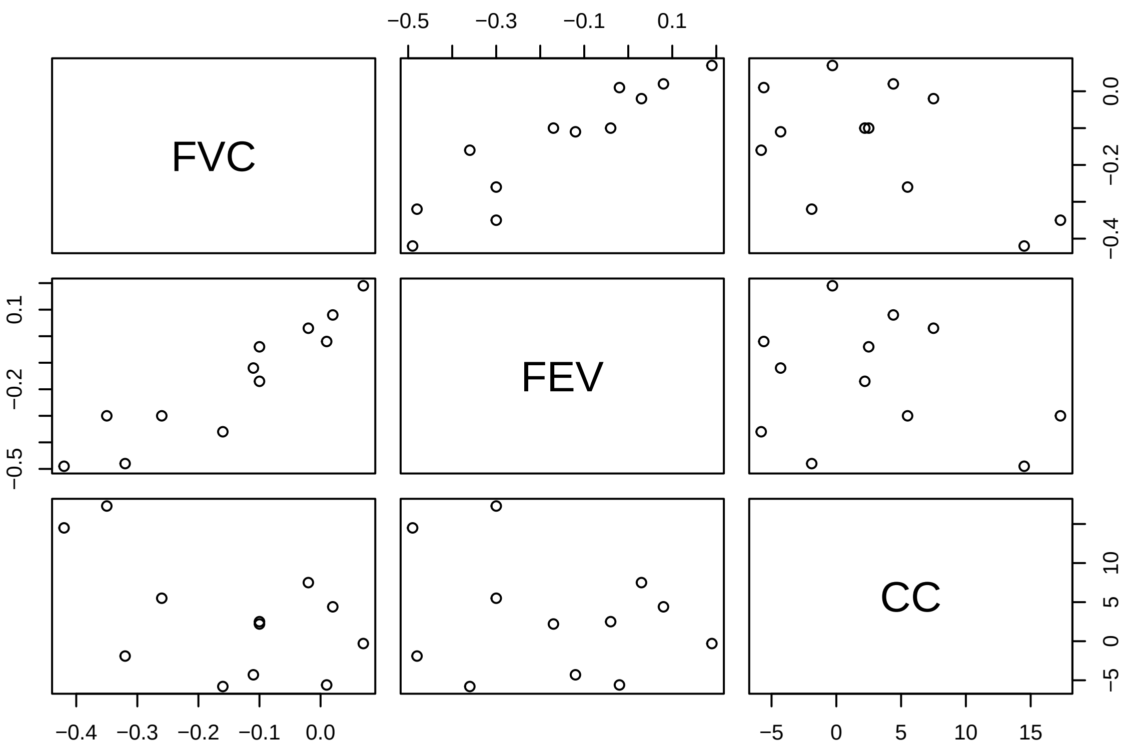
```
##          [,1]     [,2]       [,3]       [,4]        [,5]       [,6]
## [1,] 10.79333 0.000000 0.00000000 0.00000000 0.000000000 0.00000000
## [2,]  0.00000 2.129455 0.00000000 0.00000000 0.000000000 0.00000000
## [3,]  0.00000 0.000000 0.07363003 0.00000000 0.000000000 0.00000000
## [4,]  0.00000 0.000000 0.00000000 0.01288749 0.000000000 0.00000000
## [5,]  0.00000 0.000000 0.00000000 0.00000000 0.002748227 0.00000000
## [6,]  0.00000 0.000000 0.00000000 0.00000000 0.000000000 0.00157045
## [7,]  0.00000 0.000000 0.00000000 0.00000000 0.000000000 0.00000000
##              [,7]
## [1,] 0.000000e+00
## [2,] 0.000000e+00
## [3,] 0.000000e+00
## [4,] 0.000000e+00
## [5,] 0.000000e+00
## [6,] 0.000000e+00
## [7,] 2.965544e-05
```

```
Gamma
```

```
##              [,1]         [,2]        [,3]        [,4]         [,5]
## [1,]  0.884228505 -0.100805775  0.26453354 -0.19944949  0.137172970
## [2,]  0.395405417 -0.056489625 -0.28251995  0.57881686 -0.574756029
## [3,]  0.004311324  0.002894744  0.05903584 -0.05776023  0.053104536
## [4,]  0.128544478 -0.030621731 -0.40014946  0.43610024  0.786997760
## [5,]  0.111059139 -0.002372293  0.31923869 -0.23416358  0.144802899
## [6,] -0.127615624 -0.989410476  0.06429754  0.02514736  0.001575639
## [7,]  0.128966499 -0.082233392 -0.76193973 -0.61335659 -0.087653609
##              [,6]         [,7]
## [1,]  0.280639558 -0.025398239
## [2,] -0.301558638  0.065839904
## [3,] -0.045229054  0.994125646
## [4,] -0.113437606  0.001431435
## [5,] -0.896267845 -0.081549900
## [6,]  0.003287998  0.001142692
## [7,] -0.109923643  0.008971926
```

Question 2a

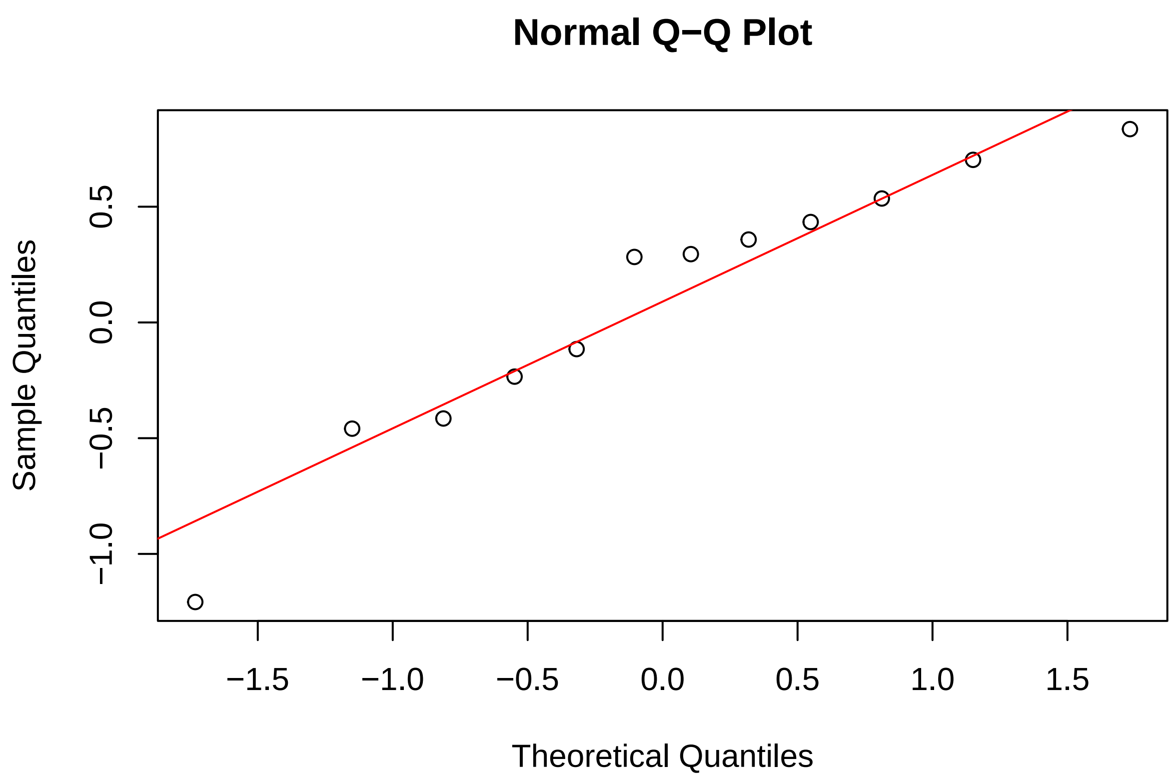
```
suppressMessages(library(ICSNP))
data(pulmonary)
plot(pulmonary)
```



As shown in scatter plot above, we can see that FVC and FEV might share a linear relationship.

Question 2b

```
maha_dist <- mahalanobis(x = pulmonary, center = colMeans(pulmonary), cov(pulmonary))
normal <- qnorm(pchisq(maha_dist, 3))
qqnorm(normal)
qqline(normal, col = 'red')
```



As we can see from the plot above, the points are relatively linear and roughly fall along a straight line, we can say that the mahalanobis distance roughly follows a χ^2 distribution with df = 3, therefore, we may conclude that the data might come from a multivariate normal distribution.

Question 2c

```
HotellingsT2(X = pulmonary, mu = c(0, 0, 0))
```

```
##
##  Hotelling's one sample T2-test
##
## data:  pulmonary
## T.2 = 3.8231, df1 = 3, df2 = 9, p-value = 0.05123
## alternative hypothesis: true location is not equal to c(0,0,0)
```

As the result above stated, the p-value is 0.05123, which is slight greater than the significance level of 0.05, hence we do not reject the null hypothesis, where the means of the three variables are all zero.

```r
# Obtain the degrees of freedom
n <- dim(pulmonary)[1]
p <- dim(pulmonary)[2]
sample_mean <- matrix(colMeans(pulmonary))
mu <- matrix(c(0, 0, 0))
(T_2 <- n * t(sample_mean - mu) %*% solve(cov(pulmonary)) %*% (sample_mean - mu))
```

```
##         [,1]
## [1,] 14.0182
```

```r
(F_stat <- (n - p)/((n - 1) * p) * T_2)
```

```
##          [,1]
## [1,] 3.823146
```

```r
pf(F_stat, p, n - p, lower.tail = FALSE)
```

```
##            [,1]
## [1,] 0.05122881
```

Using the equivalence of Hotelling's T^2 statistics and $F_{p,n-p}$ distribution, the p-value of the test statistic is equal to 0.05123, which is the same as the p-value calculated using the R function above. And since the p-value slight greater than the significance level of 0.05, hence we do not reject the null hypothesis, where the means of the three variables are all zero.