CS 5963/6360 A2 Written Work

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QUESTIONS

1. In one sentence, explain what the following homogeneous transformation accomplishes when applied to a point (x, y, z) in terms of yaw, pitch, roll, and translation.

$$T_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Answer

   T_1 rolls the point 45 degrees, and **<u>then</u>** translates x by -1 and y by 2 because the convention is to apply rotation before translation

2. Write out the 4 x 4 homogeneous transformation T2, when applied to a point (x, y, z) in the global coordinate frame, translates the point (3, 0, 2)T, then followed by a pitch of 45 degrees. Your answer need not be simplified, and may be represented as a single matrix or the product of two or more matrices.

   Answer

   Since translation needs to come first, and order of operations matter with matrices (multiply from right to left). Then the first matrix multiplied (right in our case) should consist of a 0 rotation matrix in the top left corner, and the translation vector as the fourth column. Then, the second multiplication (left matrix in our case) should consist of a rotation matrix for pitch where $\alpha = \frac{\pi}{4}$ which when applying sin and cos gives $\frac{1}{\sqrt{2}}$. Thus, T_2 would be written as follows:

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. We would like to reverse the transformation applied by $T_2\, T_1$ that is, write out $(T_2T_1)^{-1}$. Your answer need not be simplified, and may be represented as a single matrix or the product of two or more matrices.

Answer

Since order of operations is important with matrices, T_2T_1 must be undone by reversing the action of T_2 first (since it is applied last) and then reversing T_1.

Starting with T_2, which translates and then rotates, the opposite must be the case (rotate and then translate). This is simply done by taking the Transpose of the rotation matrix block and then setting the fourth column (translation) to the negative of the translation. The result would be the matrix T_2^{-1} which would undo the action of T_2

Next, to undo T_1, which rotates and then translates, the opposite must be applied (translate and then rotate). So the first thing to be done is to grab the translation matrix (we'll call it T_t^{-1}, which would resemble the translation matrix from question 2 except the fourth column would consist of the opposite of the translations. Then, to the left of that matrix would be a matrix that performs roll (rotation around the z axis which we will call T_r^{-1}) by an inverse of the same roll value. To accomplish this, the rotation matrix block is transposed. which makes $T_1^{-1} = T_r^{-1}T_t^{-1}$

The last step is to properly arrange the matrices. Since T_2 is applied last, to reverse T_2T_1, then T_2^{-1} must be applied first. This gives the final answer of $T_1^{-1}T_2^{-1}$ which results in the following:

$$(T_2T_1)^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Write out the quaternion equivalent to the rotations in T_1 and T_2 as q_1 and q_2. Then calculate the product, that is $q_1 * q_2$ (the book is a good reference for this).

Answer

To obtain the quaternion of a rotation around a single axis, the following equation is used:

q(z-axis,θ) $= \begin{bmatrix} cos(\frac{\theta}{2}) & 0 & 0 & sin(\frac{\theta}{2}) \end{bmatrix}$

$$q(y-axis,\theta) = \begin{bmatrix} cos(\frac{\theta}{2}) & 0 & sin(\frac{\theta}{2}) & 0 \end{bmatrix}$$

$$q(z-axis,\theta) = \begin{bmatrix} cos(\frac{\theta}{2}) & sin(\frac{\theta}{2}) & 0 & 0 \end{bmatrix}$$

Applying these to the rotation matrices of T_1 and T_2 to get q_1 and q_2 respectivly gives:

$$q_1 = \begin{bmatrix} cos(\frac{\pi}{8}) & 0 & 0 & sin(\frac{\pi}{8}) \end{bmatrix}$$

$$q_2 = \begin{bmatrix} cos(\frac{\pi}{8}) & sin(\frac{\pi}{8}) & 0 & 0 \end{bmatrix}$$

Finally for quaternions of the following form: $q = [w, x, y, z]$ then the following forumla applies to $q_1 * q_2$ (this formula was found at https://www.mathworks.com/help/aeroblks/quaternionmultiplication.html):

$$q_1 * q_2 = t = \big[w, x, y, z\big]$$

$$t_w = q_{2w} * q_{1w} - q_{2x} * q_{1x} - q_{2y} * q_{1y} - q_{2z} * q_{1z}$$

$$t_x = q_{2w} * q_{1x} + q_{2x} * q_{1w} - q_{2y} * q_{1z} + q_{2z} * q_{1y}$$

$$t_y = q_{2w} * q_{1y} + q_{2x} * q_{1z} + q_{2y} * q_{1w} - q_{2z} * q_{1x}$$

$$t_z = q_{2w} * q_{1z} - q_{2x} * q_{1y} + q_{2y} * q_{1x} + q_{2z} * q_{1w}$$

Thus, applying the formula to q_1 and q_2 results in the following:

$$q_1 * q_2 = \big[cos^2(\tfrac{\pi}{8}), \quad sin(\tfrac{\pi}{8})cos(\tfrac{\pi}{8}), \quad sin^2(\tfrac{\pi}{8}), \quad sin(\tfrac{\pi}{8})cos(\tfrac{\pi}{8})\big]$$