Uncertainty Quantification Using Polynomial Chaos

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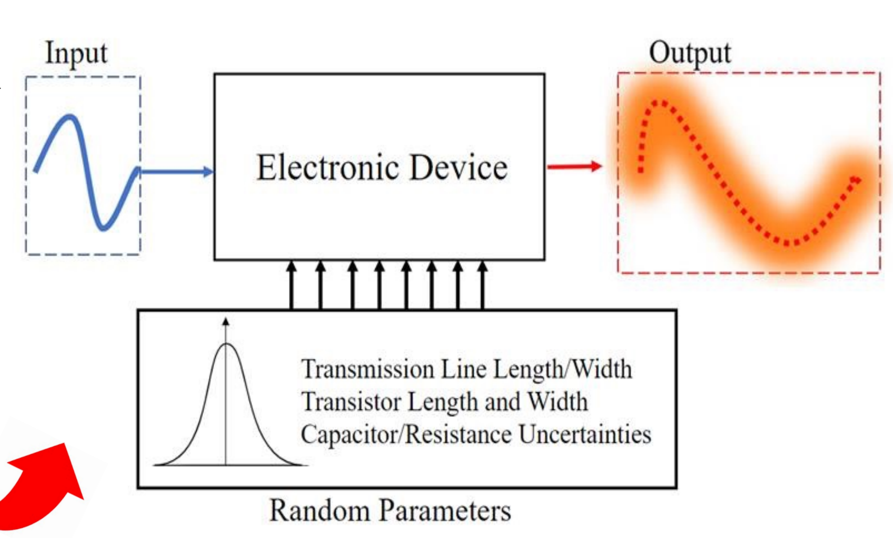
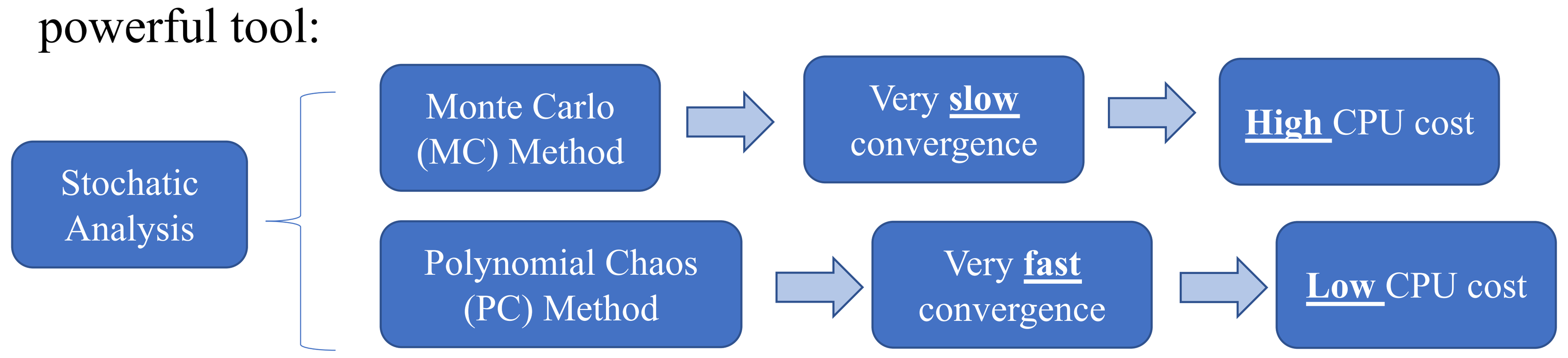
Introduction

- In today's electronic devices, the very large integration and the demand for high efficiency require the systems to be very compact and dense while also maintaining the accuracy of computation.
- In practice, the main challenge for any designer is to predict how the variability of the input parameters will affect the results of the entire system.
- During the fabrication process, it is very difficult to control the system's input, which will make the system's output uncertain.



- That is where stochastic analysis, especially polynomials chaos, comes in as a powerful tool:



- Polynomial chaos is a powerful uncertainty quantification technique that was first proposed as Homogeneous Chaos expansion by N. Weiner in 1938.
- The idea of PC is to expand a stochastic process into a series of the so-call Askey-Wiener orthogonal polynomials, including Hermite, Legendre, Laguerre polynomials, and so on.

Objective

With knowing PC is a less expensive approach in terms of the computation cost, the objective of the project is to build the actual library using MATLAB to use the PC theory to predict the statistical behavior of a system.

Methodology

- Step 1: Choose polynomials according to the following Wiener-Askey Scheme

Random variables	Wiener-Askey Polynomials Chaos
Gaussian	Hermite-Chaos
Uniform	Legendre-Chaos
Gamma	Laguerre-Chaos

- Step 2: Determine your samples as inputs (i.e. make a combination of roots of the polynomials)
  - There are functions to compute roots for all types of polynomials
  - You can assign any type of polynomials to each dimension of your samples:

$$\begin{matrix} a1 \\ a2 \\ a3 \end{matrix} \quad \begin{matrix} b1 \\ b2 \\ b3 \end{matrix} \quad \begin{matrix} c1 \\ c2 \\ c3 \end{matrix}$$

*This is a 3-dimensional example sample

*Each column is a combination of roots

Laguerre

Legendre

Hermite

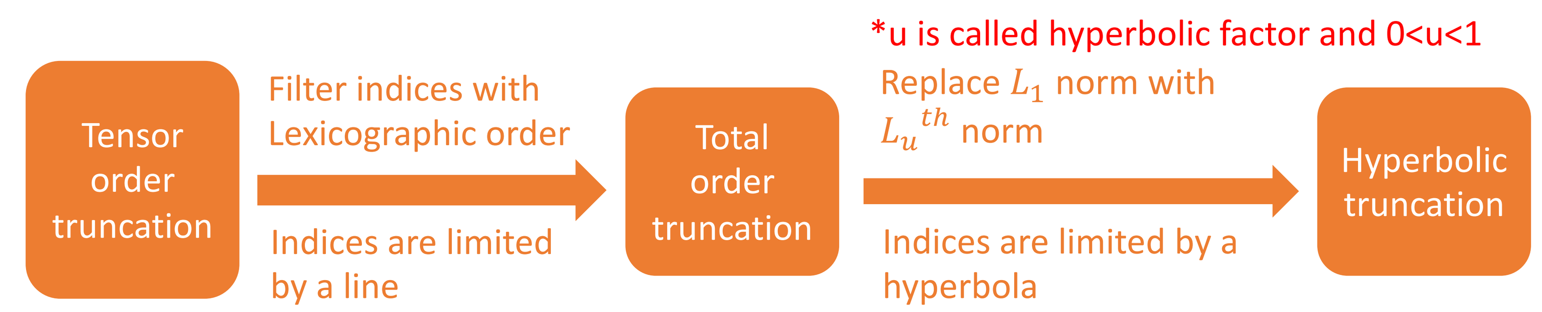
- Step 3: Evaluate the system (i.e. solve for the voltage, current,etc.) and form $\vec{b} = [\phi(x^{(1)}) \ldots \phi(x^{(k)})]$

  x: samples k: the dimension of samples

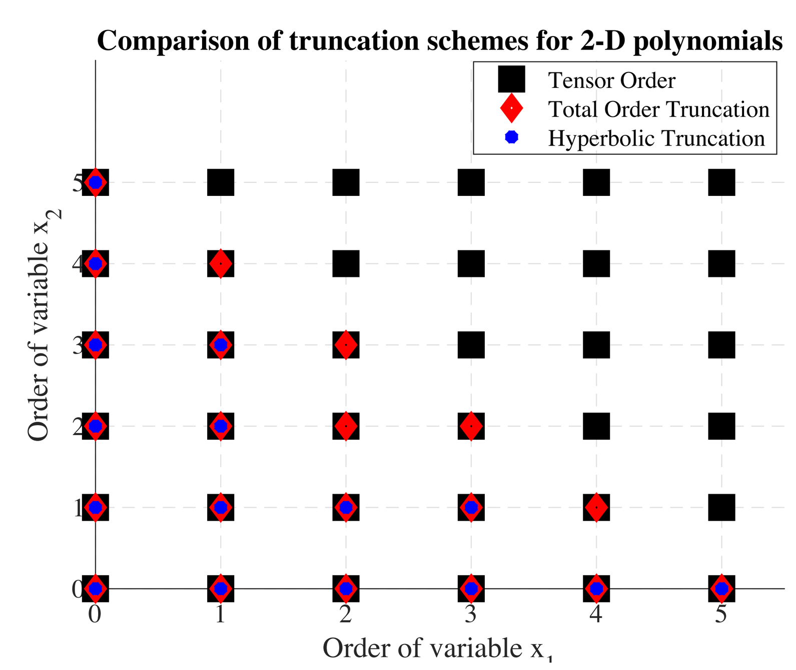
  ϕ: the ouput of the system
  - This part will take the longest time since it is evaluating the entire system
- Step 4: Form the collocation matrix $\boldsymbol{H}$ (main contribution of the library)
  - To build a collocation matrix in one dimension, I used these formulas:

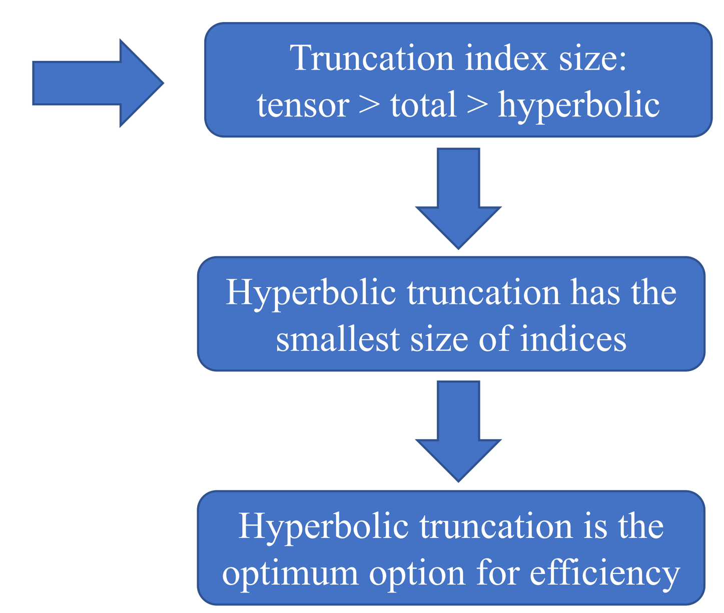
Polynomial type	Recurrence relationship
Hermite(probalist)	$H_0(x) = 1 \quad H_1(x) = x$ $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$
Legendre	$P_0(x) = 1 \quad P_1(x) = x$ $P_{n+1}(x) = \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x)$
Laguerre	$L_0^\alpha(x) = 1 \quad L_1^\alpha = 1 + \alpha - x$ $L_{n+1}^\alpha(x) = \frac{(2n+1+\alpha-x)L_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x)}{n+1}$

  - To build a multi-dimensional collocation matrix, truncation schemes are needed to specify criteria on multi-indices whose corresponding coefficients are to be included. This figure explains the relationship of 3 schemes:



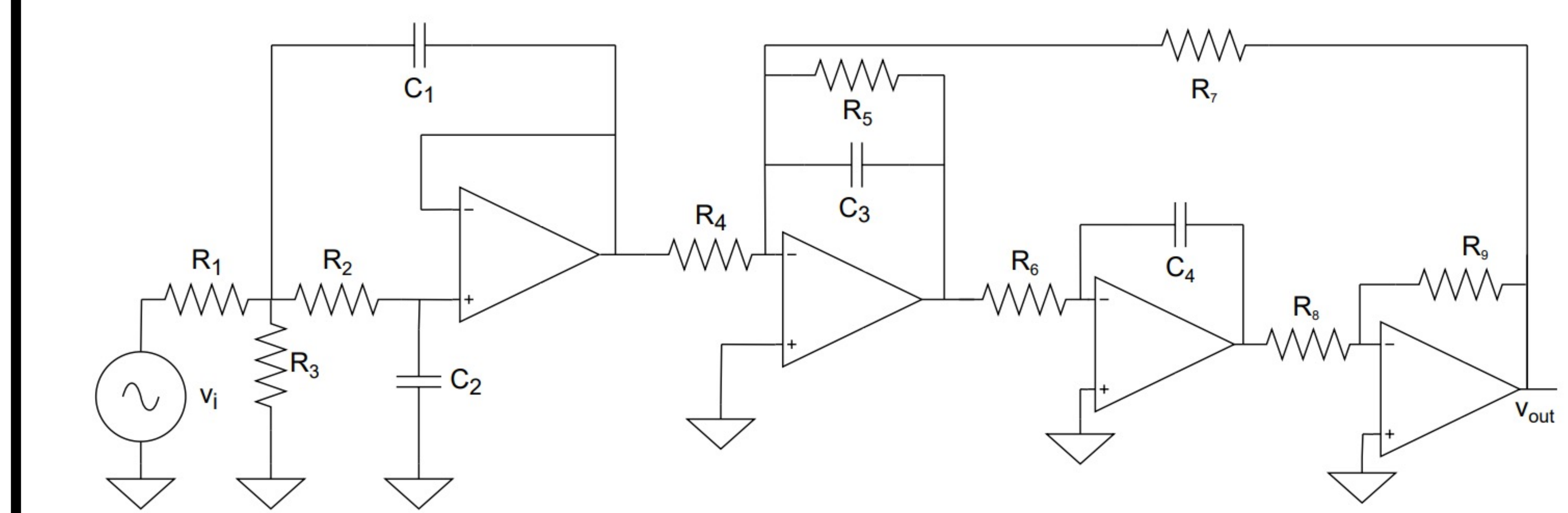
  - These 3 types of truncation schemes are available in the library. Their sizes of indices are different as shown below:





- Step 5: Solve for polynomials coefficients $\vec{a}$ with $\boldsymbol{H}\vec{a} = \vec{b}$ using techniques like LU decomposition, Cholesky decomposition, etc.
- Step 6: Use $\vec{a}$ to get statistics and PDF of the system's output

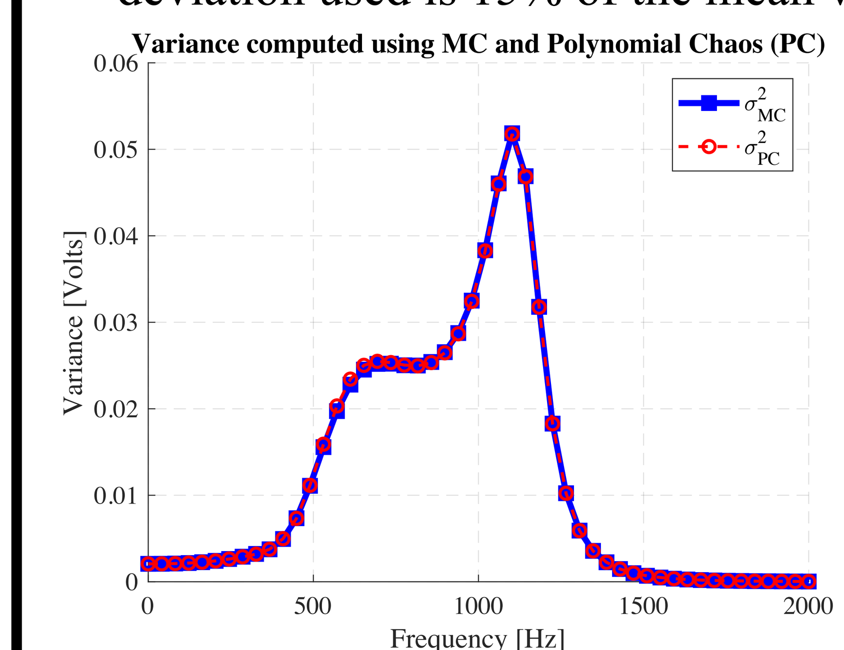
Results

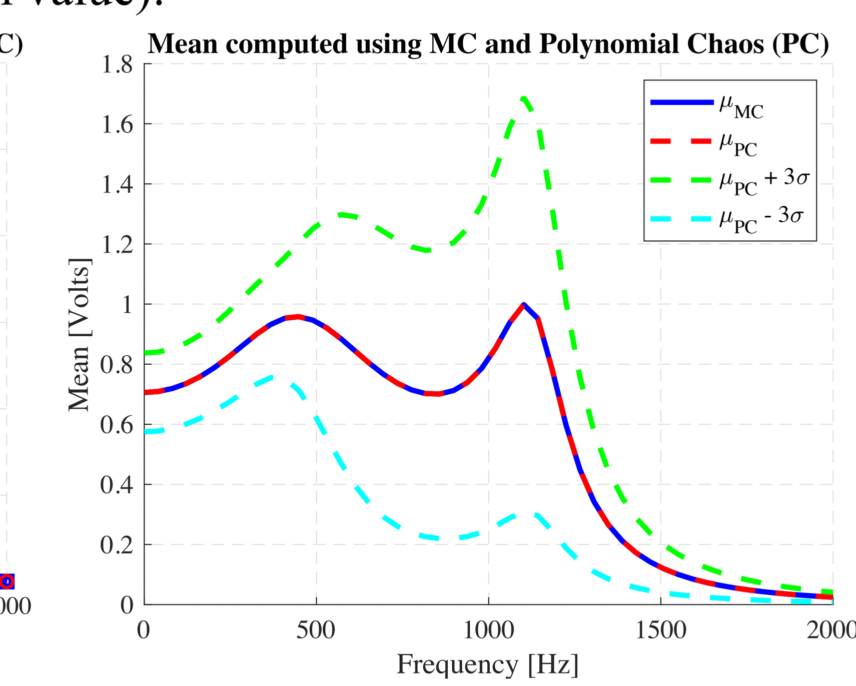


- For example, we have the following system to evaluate:

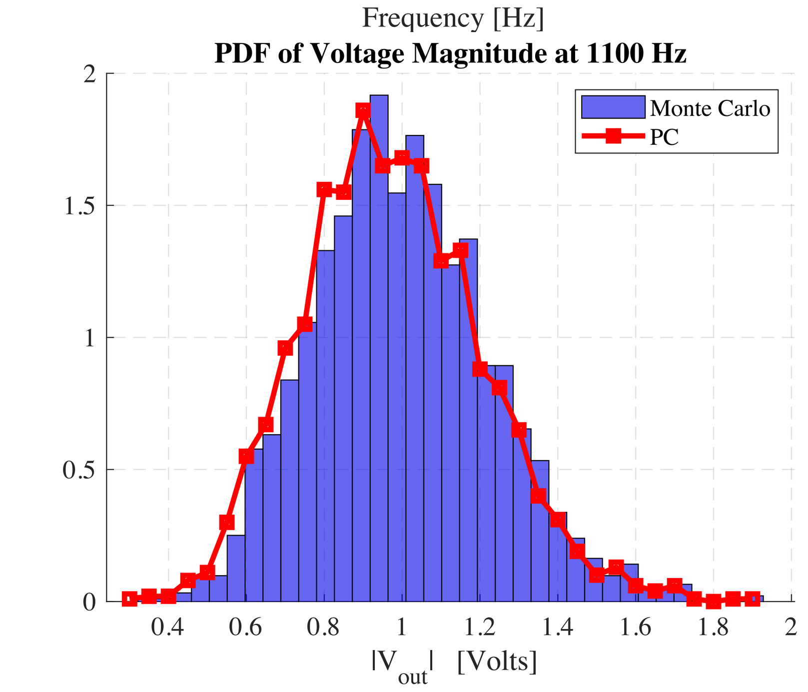
System	Distribution	Polynomial
Low-Pass Sallenkey Filter (picture above)	Gaussian	Hermite

- Following the methodology, we will get the following results (standard deviation used is 15% of the mean value):





- As shown in the figures, the variance, mean, and PDF of Polynomial Chaos match the results of Monte Carlo. However, Polynomial Chaos only simulates 35 times while Monte Carlo simulates approximately 10000 times. This contrast shows PC is generally a much better stochastic analysis method than MC.



Future work: Derivative-assisted PC Method to better improve efficiency

References

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M. Ahadi Dolatsara, "Efficient multidimensional uncertainty quantification of high speed circuits using advanced polynomial chaos approaches," Colorado State University, 2016.