3^{rd} *Romanian Master of Sciences 2010*

Physics – Theoretical Tour

NEWTONIAN COSMOLOGY

a. 1p

First of all, it is evident that **v** and **r** must have the same direction. Secondly, the expansion of any arbitrary two position vectors at any moment of time must preserve the already existing proportionality between them. That is,

$$\frac{r_1(t)}{r_2(t)} = \frac{r_1(t+\mathrm{d}t)}{r_2(t+\mathrm{d}t)} = \frac{r_1 + v_1\mathrm{d}t}{r_2 + v_2\mathrm{d}t} = \frac{v_1(t)}{v_2(t)} .$$

So Hubble's Law is

$$\vec{v}(t) = H(t)\cdot\vec{r}(t) .$$

b. 1p

Let A and B be two galaxies seen from a point in space, e.g. from Earth. According to Hubble's Law,

$$\vec{v}_\mathrm{A} = H(t)\cdot\vec{r}_\mathrm{A} ,$$
$$\vec{v}_\mathrm{B} = H(t)\cdot\vec{r}_\mathrm{B} .$$

By subtracting the two expressions, we get

$$\left(\vec{v}_\mathrm{B} - \vec{v}_\mathrm{A}\right) = H\left(\vec{r}_\mathrm{B} - \vec{r}_\mathrm{A}\right) .$$

So the relative velocity of galaxy B with respect to galaxy A is proportional to its relative position with respect to that galaxy, the proportionality factor being the same Hubble constant.

c. 0.5p

Assuming that

$$\vec{r}\left(t_0\right) = \vec{v}\left(t_0\right)\cdot t_0 ,$$

we get

$$\vec{v}\left(t_0\right) = H\left(t_0\right)\cdot\vec{v}\left(t_0\right)\cdot t_0 \Rightarrow t_0 = \frac{1}{H_0} .$$

d. 0.5p

$$\rho(t)\frac{4\pi r^3(t)}{3} = \mathrm{const} \Rightarrow \rho(t)\cdot R^3(t)\cdot r_0^3 = \mathrm{const} \Rightarrow \rho(t) = \frac{\rho_0}{R^3(t)} .$$

e. 0.5p

$$E(t) = \frac{mv^2(t)}{2} - G\frac{m\rho(t)\frac{4\pi}{3}r^3(t)}{r(t)} = \frac{mH^2(t)r^2(t)}{2} - \frac{4\pi Gm\rho(t)r^2(t)}{3} = \frac{mR^2(t)r_0^2}{2}\left(H^2(t) - \frac{8\pi G\rho(t)}{3}\right) .$$

f. 0.5p

If $\Omega > 1$, the expansion will eventually come to a halt and then the universe will start to shrink until it vanishes.

If Ω = 1, the universe will keep on expanding, approaching infinity with zero recessional velocity.
If $\Omega < 1$, the universe will expand to infinity with nonzero recessional velocity.

g. 0.5p

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \Rightarrow E(t) = \frac{mR^2(t) r_0^2 H^2(t)}{2}\left(1-\Omega(t)\right).$$

Since E = const, the sign of $1 - \Omega(t)$ does not change with time.

h. 0.5p

$$v(t) = H(t)r(t) \Rightarrow H(t) = \frac{1}{r(t)}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{R(t)}\frac{\mathrm{d}R}{\mathrm{d}t} \Rightarrow E(t) = \frac{mr_0^2}{2}\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8\pi G\rho_0}{3R(t)}\right].$$

$$\rho_0 = \Omega_0\rho_{c0} = \frac{3\Omega_0 H_0^2}{8\pi G} \Rightarrow E(t) = \frac{mr_0^2}{2}\left[\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{\Omega_0 H_0^2}{R(t)}\right] = \frac{mr_0^2 H_0^2\left(1-\Omega_0\right)}{2} \Rightarrow$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{\Omega_0 H_0^2}{R} - H_0^2\left(\Omega_0 - 1\right) = H_0^2\left(\frac{\Omega_0}{R} + 1 - \Omega_0\right).$$

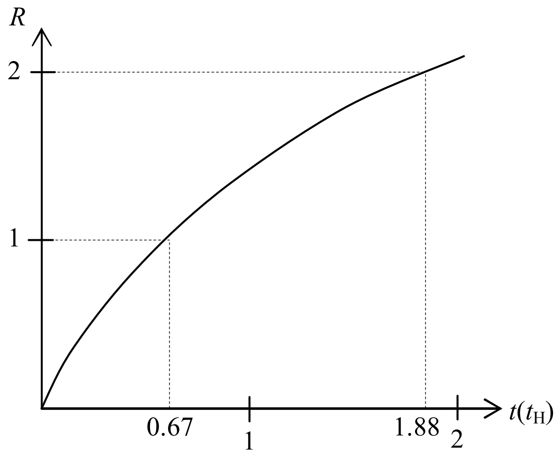
$$t \to 0 \Rightarrow R \to 0 \Rightarrow \frac{dR}{dt} \to \infty \Rightarrow RH \to \infty \Rightarrow 1-\Omega \to 0.$$

i. 0.5p

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{H_0}{\sqrt{R}} \Rightarrow \int\sqrt{R}\,\mathrm{d}R = \int H_0\,\mathrm{d}t \Rightarrow \frac{2}{3}R^{\frac{3}{2}} = H_0 t \Rightarrow R(t) = \left(\frac{3}{2}\frac{t}{t_H}\right)^{\frac{2}{3}}.$$

Since $R_0 = 1$, we get $t_0 = 2/3\ t_H$.

j. 0.5p



k. 0.5p

$$\frac{\mathrm{d}R}{\mathrm{d}t} = H_0\sqrt{\frac{\Omega_0 - \left(\Omega_0 - 1\right)R}{R}} \Rightarrow \sqrt{\frac{R}{1 - \frac{\Omega_0 - 1}{\Omega_0}R}}\,\mathrm{d}R = H_0\sqrt{\Omega_0}\,\mathrm{d}t.$$

Now

$$x=\frac{\Omega_0-1}{\Omega_0}R\Rightarrow\left(\frac{\Omega_0}{\Omega_0-1}\right)^{\frac{3}{2}}\sqrt{\frac{x}{1-x}}dx=H_0\sqrt{\Omega_0}dt\Rightarrow\arcsin\sqrt{x}-\sqrt{x(1-x)}=\frac{H_0}{\Omega_0}(\Omega_0-1)^{\frac{3}{2}}t\ .$$

(We took into account the fact that $x(0) = 0$.) So

$$t(R)=\frac{\Omega_0}{H_0(\Omega_0-1)^{\frac{3}{2}}}\left[\arcsin\sqrt{\frac{\Omega_0-1}{\Omega_0}R}-\sqrt{\frac{\Omega_0-1}{\Omega_0}R\left(1-\frac{\Omega_0-1}{\Omega_0}R\right)}\right].$$

l. 0.5p

$$\sqrt{\frac{\Omega_0-1}{\Omega_0}R}=\sin\frac{p}{2}\Rightarrow R=\frac{\Omega_0}{\Omega_0-1}\sin^2\frac{p}{2}\Rightarrow$$

$$\begin{cases}R(p)=\dfrac{1}{2}\dfrac{\Omega_0}{\Omega_0-1}(1-\cos p)\ ;\\ t(p)=\dfrac{1}{2H_0}\dfrac{\Omega_0}{(\Omega_0-1)^{\frac{3}{2}}}(p-\sin p)\ .\end{cases}$$

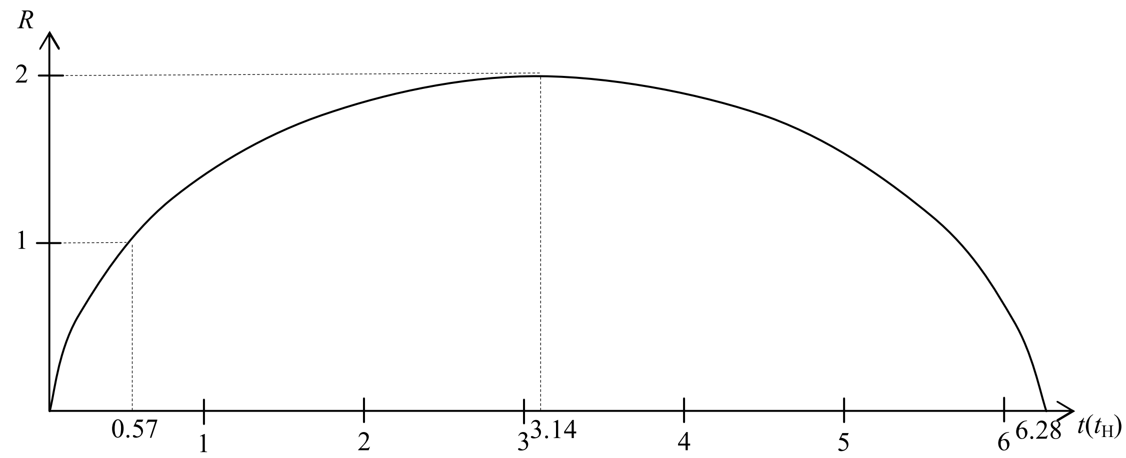
m. 0.5p

$$R(T)=0\Rightarrow 1-\cos p=0\Rightarrow p=2\pi\Rightarrow T=\pi\frac{\Omega_0}{(\Omega_0-1)^{\frac{3}{2}}}t_{\mathrm{H}}.$$

n. 0.5p

$$R_{\max}=\frac{\Omega_0}{\Omega_0-1}\ \text{for}\ p=\pi,\ \text{i.e. at}\ t=T/2\ .$$

o. 0.5p



p. 0.5p

$$\frac{\mathrm{d}R}{\mathrm{d}t}=H_0\sqrt{\frac{\Omega_0+(1-\Omega_0)R}{R}}\Rightarrow\sqrt{\frac{R}{1+\dfrac{1-\Omega_0}{\Omega_0}R}}\mathrm{d}R=H_0\sqrt{\Omega_0}\mathrm{d}t$$

Now

$$x=\frac{1-\Omega_0}{\Omega_0}R\Rightarrow\left(\frac{\Omega_0}{1-\Omega_0}\right)^{\frac{3}{2}}\sqrt{\frac{x}{1+x}}dx=H_0\sqrt{\Omega_0}dt\Rightarrow-\operatorname{arcsinh}\sqrt{x}+\sqrt{x(1+x)}=\frac{H_0}{\Omega_0}\left(1-\Omega_0\right)^{\frac{3}{2}}t\ .$$

(We took into account the fact that $x(0)=0$.) So

$$t(R)=\frac{\Omega_0}{H_0\left(1-\Omega_0\right)^{\frac{3}{2}}}\left[-\operatorname{arcsinh}\sqrt{\frac{1-\Omega_0}{\Omega_0}R}+\sqrt{\frac{1-\Omega_0}{\Omega_0}R\left(1+\frac{1-\Omega_0}{\Omega_0}R\right)}\right].$$

q. 0.5p

$$\sqrt{\frac{1-\Omega_0}{\Omega_0}R}=\sinh\frac{p}{2}\Rightarrow R=\frac{\Omega_0}{1-\Omega_0}\sinh^2\frac{p}{2}\Rightarrow$$

$$\begin{cases}R(p)=\dfrac{1}{2}\dfrac{\Omega_0}{1-\Omega_0}\left(\cosh p-1\right)\ ;\\ t(p)=\dfrac{1}{2H_0}\dfrac{\Omega_0}{\left(1-\Omega_0\right)^{\frac{3}{2}}}\left(\sinh p-p\right)\ .\end{cases}$$

r. 0.25p

$$\lim_{p\to\infty}\frac{R(p)}{t(p)}=H_0\sqrt{1-\Omega_0}\Rightarrow R(t)\propto\sqrt{1-\Omega_0}\,\frac{t}{t_{\mathrm{H}}}\ .$$

s. 0.25p