PROBLEM 3: BLACK HOLES PHYSICS

In this problem we will explore the physics of black holes – astrophysical objects so massive that no object, not even a photon, can escape from if it gets sufficiently close. Because black holes are extremely massive, in their vicinity Newton's theory of gravity breaks down and one is forced to use Einstein's general theory of relativity to obtain a correct description of their physics.

Any black hole in the physical universe is uniquely specified by exactly three quantities: its mass M, angular momentum J, and charge Q. In addition to these, a black hole also has a *space-time singularity* and an *event horizon*, which is the surface surrounding the central singularity which can only be crossed "going in". Any photon or object which falls through the event horizon will not be able to exit back out and will eventually hit the central singularity.

It is convenient to describe the spacetime of non-rotating black holes by four coordinates: t, r, θ and ϕ, with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. These can be thought of as the usual spherical coordinates plus time. Because the geometry is no longer flat, the infinitesimal spacetime interval is given by

$$(ds)^2 = -c^2 f\,(dt)^2 + \frac{(dr)^2}{g} + r^2 (d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2 \qquad (1)$$

with f and g functions of the four coordinates. For spherically symmetric uncharged black holes (known as *Schwarzschild black holes*), f and g are functions of the radial coordinate r only, and are given by

$$f(r) = g(r) = 1 - \frac{r_S}{r}, \qquad (2)$$

where r_S, the *Schwarzschild radius*, is the radial coordinate of the event horizon.

a. Write down the infinitesimal spacetime interval $(ds)^2$ for Minkowski spacetime in spherical coordinates, using the signature given above (–, +, +, +).

Suppose an observer is initially at rest ($dr/dt = 0$, $d\theta/dt = d\phi/dt = 0$) at radial coordinate $r_0 > r_S$. Under the gravitational pull of the black hole he will start falling towards the event horizon, moving only along the radial direction (and thus keeping θ and ϕ constant at all times). Let t' be the proper time measured by the observer's clock. The correct general relativistic relation between elapsed proper time dt', elapsed coordinate time dt and radial coordinate change dr is (you don't have to show this)

$$\frac{d^2 r}{dt'^2} + \frac{r_S c^2}{2r^2}\left(1 - \frac{r_S}{r}\right)\left(\frac{dt}{dt'}\right)^2 - \frac{r_S}{2r^2}\frac{1}{1 - \frac{r_S}{r}}\left(\frac{dr}{dt'}\right)^2 = 0. \qquad (3)$$

b. From equations (1) - (3) compute the proper acceleration $a \equiv d^2r/dt'^2$ in terms of c, the speed of light in empty space, r, and r_S.

Hint: The spacetime interval $(ds)^2$ is the same in all reference frames.

c. Even without resorting to any calculation, it should be expected that the Newtonian expression for a can be recovered in the large r (i.e. small M) limit of the relativistic

expression. However, now that you have done the calculation, can you make a stronger statement? Determine r_S in terms of M, the black hole's mass, c, and G, the gravitational constant.

d. Compute the proper velocity $v \equiv dr/dt'$ as a function of r, in terms of M and r_0 and G.

e. Using that

$$\int \sqrt{\frac{x}{1-x}}dx = -\sqrt{x(1-x)} - \arccos\left(\sqrt{x}\right) + C, \tag{4}$$

compute the proper time t' after which the observer reaches the event horizon. Write the result in terms of r_0, M, G and c.

The results of parts **a.** - **c.** may lead one to speculate that general relativity is not that different from Newtonian mechanics after all. This is a misleading interpretation, as the two theories differ significantly in many aspects.

f. To highlight one such aspect, calculate the coordinate time t it takes for the observer to reach the event horizon. Are you surprised?

Hint: You do not need to determine the antiderivative in order to compute what the integral equals.

We now turn towards the thermodynamical properties of black holes. If only classical physics is taken into account, black holes do not emit any form of radiation and can thus be considered to have zero temperature. However, in 1974 physicist Stephen Hawking proved that once quantum corrections are considered, black holes emit radiation according to the blackbody spectrum (you do not need to show this). The corresponding blackbody temperature is known as the *Hawking temperature*, T_H, and can be thought of as the temperature of the black hole. For a Schwarzschild black hole of mass M, the Hawking temperature is equal to

$$T_H = \frac{\hbar c^3}{8\pi G k_B M}, \tag{5}$$

where $\hbar$ is the Planck constant and k_B is the Boltzmann constant.

g. Using equation (5) compute the entropy S of a Schwarzschild black hole. Express it in terms of c, G, $\hbar$, k_B, and the *horizon area*, $A = 4\pi r_S^2$.

Hint: Think of Einstein's famous formula.

The result from part **g.** suggests that at the classical level (i.e. ignoring quantum corrections) the total area of black holes involved in any physical process can never decrease. This is indeed true and has been formalized into a theorem by Hawking in 1971.

h. Using the above theorem compute the maximum amount of energy that can be radiated as gravitational waves in the merger of two Schwarzschild black holes of masses M_1 and M_2, assuming the black holes were initially at rest when far away.

We now return to the subject of Hawking temperature. While Hawking's 1974 derivation of T_H was somewhat technical, in parts **i.** - **k.** we will rederive his result using a much simpler argument. Take an infinitesimal spacetime interval of the form (1) with the functions depending only on the radial coordinate r,

$$(ds)^2 = -c^2 F(r)(dt)^2 + \frac{(dr)^2}{G(r)} + r^2 (d\theta)^2 + r^2 \sin^2\theta (d\phi)^2, \tag{6}$$

and suppose that $F(r)$ and $G(r)$ have a first order zero at r_h, that is $F(r_h) = G(r_h) = 0$, but $F'(r_h) \neq 0$ and $G'(r_h) \neq 0$. Consider only the (t, r) part of $(ds)^2$ and analytically continue the coordinate time t to "imaginary time" τ via $t \to i\tau$, so that the signature of the spacetime interval becomes Euclidean and the infinitesimal spacetime element ds becomes an infinitesimal element of ordinary length,

$$(ds)^2 = \frac{(dr)^2}{G(r)} + c^2 F(r)(d\tau)^2 \tag{7}$$

The length element ds now describes how distances are measured on an ordinary 2-dimensional plane, with the origin corresponding to $r = r_h$ and $r \geq r_h$ for any point on the plane. This coordinate system is an analogue of polar coordinates, in that r can be thought of as a radial coordinate, and τ as an angular coordinate that must be periodic with some period P.

i. Write down the distance R from the origin to a point of coordinate $r = r_h + \varepsilon$ that is infinitesimally close to the origin. Express your answer in terms of ε and $G'(r_h)$.

j. Write down the circumference L of a circle of radial coordinate $r_h + \varepsilon$ around the origin, with ε infinitesimal. Express your answer in terms of P, ε and $F'(r_h)$.

k. By imposing the condition that the plane is not singular at the origin, that is that $L = 2\pi R$, determine the period P of the τ coordinate. From field-theoretic arguments this period must be equal to $\hbar c/k_B T_H$. Solve for the Hawking temperature and recover equation (5) for $F(r) = G(r) = 1 - 2GM/(c^2 r)$.

l. Compute the black hole's heat capacity C in terms of G, c, $\hbar$, k_B and T_H.

We now consider black hole evaporation. Assuming no infalling matter or energy, a black hole will slowly radiate away its mass via *Hawking radiation photons*. Although a correct treatment of the evaporation process at high energy scales requires a theory of quantum gravity, as long as T_H is below the Planck scale the semi-classical approach we've been using so far suffices. In what follows we will obtain an estimate of the black hole evaporation timescale, ignoring Planck regime complications. Since at the semi-classical level the black hole spends most of its life below the Planck scale, this will be a lower bound on the estimate of the evaporation process duration.

m. Assuming black holes obey the blackbody law, compute the power W emitted by a black hole of mass M. Use that the Stefan-Boltzmann constant is

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}, \tag{8}$$

and express your result in terms of G, c, $\hbar$ and M.

n. Compute the evaporation time τ in terms of M, assuming the result from the previous part holds at all energy scales. Compare with the age of the universe for a black hole of mass $10M_\odot = 2\cdot10^{31}$ kg. Use that $G = 6.67\cdot10^{-11}$ m^3/(s^{2}kg), $\hbar = 1.05\cdot10^{-34}$ m^{2}kg/s, $c = 3\cdot10^8$ m/s.