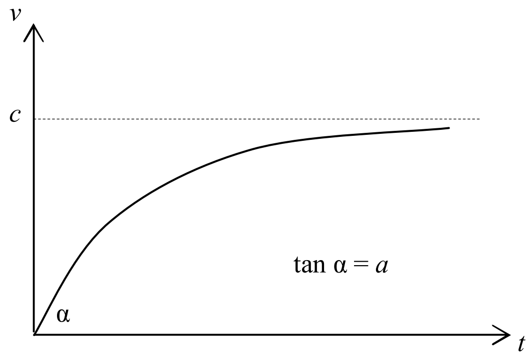
3^{rd} *Romanian Master of Sciences 2010*

Physics – Theoretical Tour

SPECIAL RELATIVITY: ACCELERATING SPACESHIP

All quantities in the rocket's reference frame will be denoted with a prime (').

a. 1p



b. 2p

The momentum of the rocket increases uniformly with time.

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = \text{const} \Rightarrow \frac{v_x}{\sqrt{1-\left(\frac{v_x}{c}\right)^2}} = at \Rightarrow v_x(t) = c\frac{\frac{at}{c}}{\sqrt{1+\left(\frac{at}{c}\right)^2}} .$$

(We took into account the fact that $v_x(0) = 0$.) The rocket's acceleration in the Earth's reference frame is

$$a_x(t) = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{a}{\left(1+\left(\frac{at}{c}\right)^2\right)^{\frac{3}{2}}} .$$

The Lorentz transformation for the accelerations on the x-axis is

$$a_x = a_x'\left(\frac{\sqrt{1-\frac{V^2}{c^2}}}{1+\frac{V}{c^2}v_x'}\right)^3 .$$

For $V = v_x$, we get $v_x' = 0$, so the astronaut's "weight" will be given by

$$a_x'(t) = \frac{a_x}{\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}} = a \Rightarrow W = ma .$$

c. 1p

$$v_x = \frac{dx}{dt} \Rightarrow dx = c\frac{\frac{at}{c}}{\sqrt{1+\left(\frac{at}{c}\right)^2}}dt \Rightarrow x(t) = \frac{c^2}{a}\left[\sqrt{1+\left(\frac{at}{c}\right)^2} - 1\right] .$$

d. 1p

The distance increases asymptotically with time:

$$\lim_{t\to\infty}\frac{x(t)}{t}=c\ ;\ \lim_{t\to\infty}\left(x(t)-ct\right)=-\frac{c^2}{a}\ .$$

The asymptote's equation is

$$x(t)=ct-\frac{c^2}{a}=c\left(t-\frac{c}{a}\right).$$

For the drawing, see **g.**

e. 1p

As seen on the diagram, in the long run the spaceship's motion gets infinitely close to the motion of a light signal emitted from Earth at Earth time c/a.

f. 1.5p

The Earth time when the first signal reaches the rocket is given by

$$x(t)=c\left(t-\frac{c}{2a}\right)\Rightarrow t=\frac{3}{4}\frac{c}{a}\ .$$

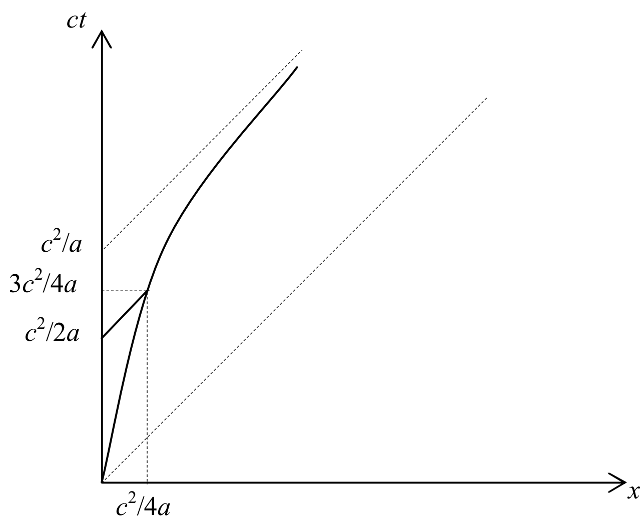
The position and the velocity of the rocket at that moment are

$$x\left(\frac{3}{4}\frac{c}{a}\right)=\frac{c^2}{4a}\ \text{ and }\ v_x\left(\frac{3}{4}\frac{c}{a}\right)=\frac{3}{5}c$$

respectively. In order to find T', for each infinitely small time interval dt in Earth's reference frame we will have to add up the corresponding time interval dt' in the rocket's reference frame.

$$\mathrm{d}t'=\mathrm{d}t\sqrt{1-\frac{v_x^2}{c^2}}=\frac{\mathrm{d}t}{\sqrt{1+\left(\frac{at}{c}\right)^2}}\Rightarrow T'=\int_0^{\frac{3}{4}\frac{c}{a}}\mathrm{d}t'=\frac{c}{a}\ln 2\approx 0.7\frac{c}{a}\ .$$

g. 0.5p



h. 1p

From the Doppler Effect formula it follows that

$$\nu' = \nu_0 \sqrt{\frac{c - v_x}{c + v_x}} = \frac{1}{2}\nu_0 \ .$$

i. 1p

Applying once again the Doppler Effect formula, we get

$$\nu = \frac{1}{4}\nu_0 \ .$$