

PROBLEM No. 3

a. 0.5p

Deriving the Lorentz transformations two-fold, we get

$$a_x = a'_x \left(\frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 + \frac{u}{c^2} v'_x} \right)^3$$

In our case $u = v_x$ and $v'_x = 0$.

$$\frac{dv_x}{dt} = a' \left(1 - \frac{v_x^2}{c^2} \right)^{\frac{3}{2}}$$

$$F_x = \frac{ma_x}{1 - \frac{v_x^2}{c^2}} = m_0 a' = \text{constant}$$

b. 0.5p

$$v_x = c \sin \alpha \Rightarrow \frac{d(c \sin \alpha)}{\left(1 - \sin^2 \alpha\right)^{\frac{3}{2}}} = a' dt \Rightarrow c \tan \alpha = a' t + C$$

At $t = 0$, $v_x = 0$, so $\alpha = 0$ and $C = 0$.

$$\frac{\frac{v_x}{c}}{\sqrt{1 - \frac{v_x^2}{c^2}}} = \frac{a' t}{c} \Rightarrow v = c \frac{\frac{a' t}{c}}{\sqrt{1 + \left(\frac{a' t}{c}\right)^2}}$$

c. 0.5p

$$dt' = dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\sqrt{1 + \left(\frac{a' t}{c}\right)^2}}; \frac{a' t}{c} = \sinh \tau \Rightarrow dt' = \frac{c}{a'} d\tau \Rightarrow t' = \frac{c}{a'} \tau + C$$

Again $C = 0$, so

$$t' = \frac{c}{a'} \operatorname{arcsinh} \left(\frac{a' t}{c} \right)$$

d. 1p

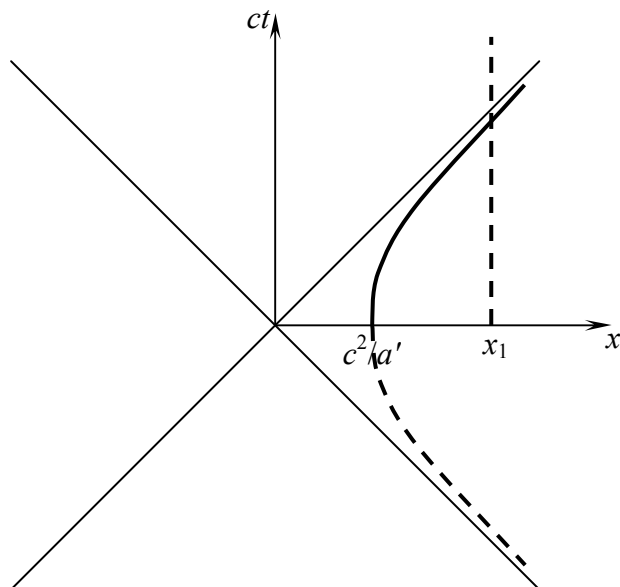
$$\left. \begin{aligned} -c^2 (dt')^2 &= -c^2 (dt)^2 + (dx)^2 \\ \frac{a' t}{c} &= \sinh \tau \Rightarrow dt = \frac{c}{a'} \cosh \tau d\tau \end{aligned} \right\} \Rightarrow (dx)^2 = \frac{c^4}{a'^2} (\cosh^2 \tau - 1) (d\tau)^2 \Rightarrow dx = \frac{c^2}{a'} \sinh \tau d\tau \Rightarrow$$

$$x = \frac{c^2}{a'} \cosh \tau + C$$

At $t = t' = 0$, $x_0 = c^2/a'$, so again $C = 0$.

e. 1p

$$ct = \frac{c^2}{a'} \sinh \tau \Rightarrow x^2 - (ct)^2 = \left(\frac{c^2}{a'}\right)^2 \Rightarrow \frac{x^2}{\left(\frac{c^2}{a'}\right)^2} - \frac{(ct)^2}{\left(\frac{c^2}{a'}\right)^2} = 1$$



f. 0.5p

$$\rho_0 = \frac{c^2}{a'} \Rightarrow \begin{cases} x = \rho_0 \cosh \tau \\ ct = \rho_0 \sinh \tau \end{cases}$$

g. 0.5p

$$\begin{cases} x = \rho \cosh \tau \\ ct = \rho \sinh \tau \end{cases} ; \begin{cases} \rho = \sqrt{x^2 - (ct)^2} \\ \tau = \operatorname{arctanh}\left(\frac{ct}{x}\right) \end{cases}$$

These equations require that $x > 0$ and $\rho > 0$, so using these new parameters one can cover only the quadrant of spacetime characterized by $x > |ct|$.

h. 1p

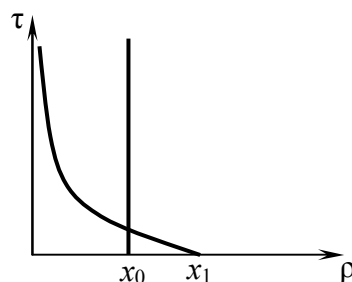
$$\left. \begin{aligned} dx &= d\rho \cosh \tau + \rho \sinh \tau d\tau \\ d(ct) &= c dt = d\rho \sinh \tau + \rho \cosh \tau d\tau \end{aligned} \right\} \Rightarrow$$

$$ds^2 = -c^2 (dt)^2 + (dx)^2 = (d\rho)^2 - \rho^2 (d\tau)^2 = -c^2 \frac{\rho^2}{c^2} (d\tau)^2 + (d\rho)^2 ; f = \frac{\rho^2}{c^2} ; g = 1$$

i. 0.5p

$$\rho = \frac{x_1}{\cosh \tau} \Leftrightarrow \tau = \operatorname{arccosh}\left(\frac{x_1}{\rho}\right)$$

$$\Delta\rho = \frac{c^2}{a'}$$



j. 0.5p

The observer will receive only those signals emitted before the beacon exits the quadrant of spacetime described by the Rindler metric.

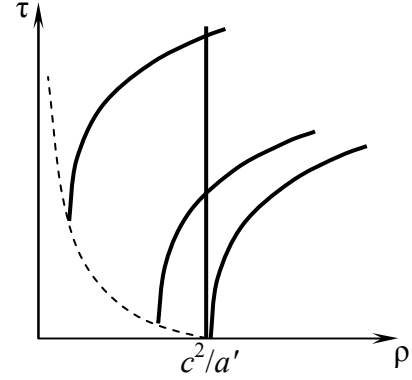
$$ct \leq x = x_0 \Rightarrow t_{\text{lim}} = \frac{x_0}{c} \Rightarrow N = \left\lfloor \frac{t_{\text{lim}}}{T_0} \right\rfloor + 1 = \left\lfloor \frac{x_0}{cT_0} \right\rfloor + 1$$

In the case of the light,

$$ds^2 = 0 \Rightarrow (d\rho)^2 = \rho^2 (d\tau)^2 \Rightarrow \frac{d\rho}{\rho} = d\tau$$

At $\tau = 0$, $\rho_0 = x_0$, so

$$\ln \frac{\rho}{\rho_0} = \tau \Rightarrow \rho = x_0 e^\tau$$



k. 1.5p

Let ρ_e and τ_e be the spacetime coordinates for the emission of a pulse.

$$\rho_e = \sqrt{x_0^2 - c^2 t^2} ; \tau_e = \text{arctanh} \left(\frac{ct}{x_0} \right)$$

$$\tanh \tau_e = \frac{e^{\tau_e} - e^{-\tau_e}}{e^{\tau_e} + e^{-\tau_e}} = \frac{ct}{x_0} \Rightarrow e^{2\tau_e} - 1 = \frac{ct}{x_0} (e^{2\tau_e} + 1) \Rightarrow e^{2\tau_e} = \frac{1 + \frac{ct}{x_0}}{1 - \frac{ct}{x_0}} \Rightarrow e^{\tau_e} = \sqrt{\frac{x_0 + ct}{x_0 - ct}}$$

$$\rho = \frac{\rho_e}{e^{\tau_e}} e^\tau = (x_0 - ct) e^\tau = \rho_0 \Rightarrow e^\tau = \frac{x_0}{x_0 - ct} \Rightarrow \tau = \ln \left(\frac{x_0}{x_0 - ct} \right)$$

Let t^* be the moment the observer receives the last signal.

$$v(t^*) = c \frac{\frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c} \right)^2}}$$

The frequency received is

$$\nu = \nu_0 \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} = \nu_0 \sqrt{\frac{1 - \frac{\frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c} \right)^2}}}{1 + \frac{\frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c} \right)^2}}}} = \nu_0 \sqrt{\frac{\sqrt{1 + \left(\frac{a't^*}{c} \right)^2} - \frac{a't^*}{c}}{\sqrt{1 + \left(\frac{a't^*}{c} \right)^2} + \frac{a't^*}{c}}} = \nu_0 \left[\sqrt{1 + \left(\frac{a't^*}{c} \right)^2} - \frac{a't^*}{c} \right]$$

But

$$ct^* = \frac{c^2}{a'} \sinh \tau \Rightarrow \frac{a't^*}{c} = \sinh \tau \Rightarrow \nu = \nu_0 (\cosh \tau - \sinh \tau) = \nu_0 e^{-\tau} = \nu_0 \left(1 - \frac{ct}{x_0} \right)$$

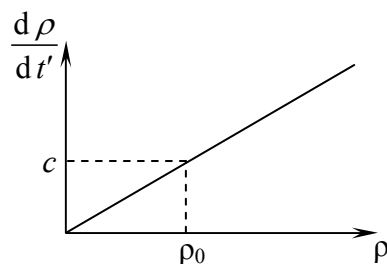
$$t = NT_0 \Rightarrow \nu = \nu_0 \left\{ 1 - \frac{cT_0}{x_0} \left(\left[\frac{x_0}{cT_0} \right] + 1 \right) \right\}$$

l. 0.5p

$$\frac{d\rho}{dt'} = \frac{d\rho}{d\tau} \frac{d\tau}{dt'} = (x_0 - ct) e^\tau \frac{a'}{c} = \frac{a'}{c} \rho$$

Upon reception,

$$e^\tau = \frac{x_0}{x_0 - ct} \Rightarrow \frac{d\rho}{dt'} = (x_0 - ct) \frac{x_0}{x_0 - ct} \frac{a'}{c} = \frac{c^2}{a'} \frac{a'}{c} = c$$



m. 1p

$$\tanh \tau = \frac{ct}{x_0} \Rightarrow \frac{1}{\cosh^2 \tau} d\tau = \frac{c}{x_0} dt \Rightarrow dt = \frac{x_0}{c} (1 - \tanh^2 \tau) d\tau = \frac{x_0}{c} \left(1 - \frac{c^2 t^2}{x_0^2} \right) d\tau$$

$$\frac{d(dt)}{dx_0} = \frac{d\tau}{c} \frac{d\left(\frac{x_0^2 - c^2 t^2}{x_0}\right)}{dx_0} = \frac{d\tau}{c} \frac{x_0^2 + c^2 t^2}{x_0^2}$$

$$\varepsilon = \frac{\frac{d\tau}{c} \frac{x_0^2 + c^2 t^2}{x_0^2} \Delta x_0}{\frac{d\tau}{c} \frac{x_0^2 - c^2 t^2}{x_0}} = \frac{x_0^2 + c^2 t^2}{x_0^2 - c^2 t^2} \frac{\Delta x_0}{x_0}$$

n. 0.5p

$$\varepsilon = \frac{\Delta x_0}{x_0} = \frac{h}{\frac{c^2}{a'}} = \frac{gh}{c^2} \approx \frac{10 \text{ m/s}^2 \cdot 360 \cdot 10^3 \text{ km}}{9 \cdot 10^{16} \text{ m}^2/\text{s}^2} = 4 \cdot 10^{-11}$$

$$\Delta t = 4 \cdot 10^{-11} \cdot 365 \cdot 24 \cdot 3600 \approx 1.26 \cdot 10^{-3} \text{ s}$$