Solution:

a) EINSTEIN'S MIRROR

By taking $\phi = \pi / 2$ and replacing v with $-v$ in Equation (1) we obtain

$$\sin\alpha - \sin\beta = -\frac{v}{c}\sin(\alpha+\beta) \tag{3}$$

This equation can also be written in the form of

$$\left(1+\frac{v}{c}\cos\beta\right)\sin\alpha = \left(1-\frac{v}{c}\cos\alpha\right)\sin\beta \tag{4}$$

The square of this equation can be written in terms of a squared equation of $\cos\beta$, as follows,

$$\left(1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}\right)\cos^2\beta + 2\frac{v}{c}\left(1-\cos^2\alpha\right)\cos\beta + 2\frac{v}{c}\cos\alpha - \left(1+\frac{v^2}{c^2}\right)\cos^2\alpha = 0 \tag{5}$$

which has two solutions,

$$\left(\cos\beta\right)_1 = \frac{2\frac{v}{c}\cos^2\alpha - \left(1+\frac{v^2}{c^2}\right)\cos\alpha}{1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}} \tag{6}$$

and

$$\left(\cos\beta\right)_2 = \frac{-2\frac{v}{c} + \left(1+\frac{v^2}{c^2}\right)\cos\alpha}{1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}} \tag{7}$$

However, if the mirror is at rest ($v=0$) then $\cos\alpha = \cos\beta$; therefore the proper solution is

$$\cos\beta_2 = \frac{-2\frac{v}{c} + \left(1+\frac{v^2}{c^2}\right)\cos\alpha}{1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}} \tag{8}$$

b) FREQUENCY SHIFT

The reflection phenomenon can be considered as a collision of the mirror with a beam of photons each carrying an incident and reflected momentum of magnitude

$$p_f = hf / c \text{ and } p_f' = hf'/c\,, \tag{9}$$

The conservation of linear momentum during its reflection from the mirror for the component parallel to the mirror appears as

$$p_f \sin\alpha = p_f' \sin\beta \text{ or } f'\sin\beta = f'\frac{(1-\frac{v^2}{c^2})\sin\alpha}{(1+\frac{v^2}{c^2})-2\frac{v}{c}\cos\alpha} = f\sin\alpha \tag{10}$$

Thus

$$f' = \frac{(1+\frac{v^2}{c^2})-2\frac{v}{c}\cos\alpha}{(1-\frac{v^2}{c^2})} f \tag{11}$$

For $\alpha = 30^0$ and $v = 0.6\,c$,

$$\cos\alpha = \frac{1}{2}\sqrt{3},\; 1-\frac{v^2}{c^2} = 0.64,\; 1+\frac{v^2}{c^2} = 1.36 \tag{12}$$

so that

$$\frac{f'}{f} = \frac{1.36 - 0.6\sqrt{3}}{0.64} = 0.5 \tag{13}$$

Thus, there is a decrease of frequency by 50% due to reflection by the moving mirror.

c) RELATIVISTICALLY MOVING MIRROR EQUATION

Figure 3.3 shows the positions of the mirror at time t_0 and t. Since the observer is moving to the left, system is moving relatively to the right. Light beam 1 falls on point *a* at t_0 and is reflected as beam $1'$. Light beam 2 falls on point *d* at t and is reflected as beam $2'$. Therefore, $\overline{ab}$ is the wave front of the incoming light at time t_0. The atoms at point are disturbed by the incident wave front $\overline{ab}$ and begin to radiate a wavelet. The disturbance due to the wave front $\overline{ab}$ stops at time t when the wavefront strikes point *d*. As a consequence

$$\overline{ac} = \overline{bd} = c(t - t_0). \tag{14}$$

From this figure we also have $\overline{ed} = \overline{ag}$, and

$$\sin\alpha = \frac{\overline{bd} + \overline{dg}}{\overline{ag}}, \qquad \sin\beta = \frac{\overline{ac} - \overline{af}}{\overline{ag} - \overline{ef}}. \tag{15}$$

Figure 3.4 displays the beam path 1 in more detail. From this figure it is easy to show that

$$\overline{dg} = \overline{ae} = \frac{\overline{ao}}{\cos\alpha} = \frac{v(t-t_0)\sin\phi}{\cos\alpha} \tag{16}$$

and

$$\overline{af} = \frac{\overline{ao}}{\cos\beta} = \frac{v(t-t_0)\sin\phi}{\cos\beta} \tag{17}$$

From the triangles *aeo* and *afo* we have $\overline{eo} = \overline{ao}\tan\alpha$ and $\overline{of} = \overline{ao}\tan\beta$. Since $\overline{ef} = \overline{eo} + \overline{of}$, then

$$\overline{ef} = v(t-t_0)\sin\phi\left(\tan\alpha + \tan\beta\right) \tag{18}$$

By substituting Equations (14), (16), (17), and (18) into Equation (15) we obtain

$$\sin\alpha = \frac{c + v\dfrac{\sin\phi}{\cos\alpha}}{\dfrac{\overline{ag}}{t-t_0}} \tag{19}$$

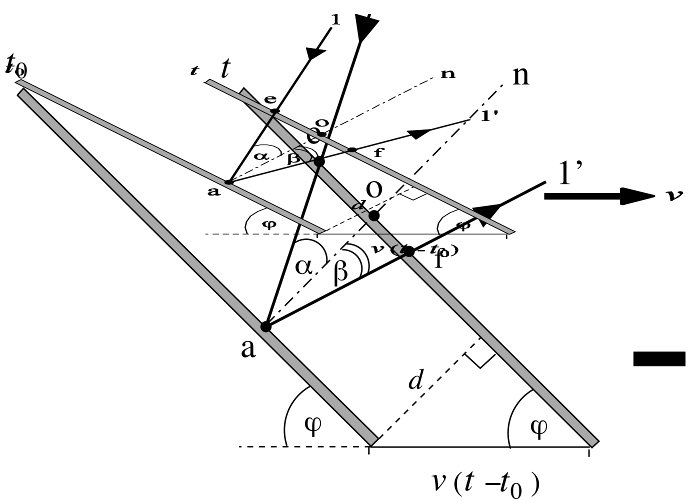

Figure 3.4.

and

$$\sin\beta = \frac{c - v\dfrac{\sin\phi}{\cos\beta}}{\dfrac{\overline{ag}}{t-t_0} - v\sin\phi\,(\tan\alpha + \tan\beta)} \tag{20}$$

Eliminating $\overline{ag}/(t-t_0)$ from the two Equations above leads to

$$v\sin\phi\,(\tan\alpha+\tan\beta)=c\left(\frac{1}{\sin\alpha}-\frac{1}{\sin\beta}\right)+v\sin\phi\left(\frac{1}{\sin\alpha\cos\alpha}+\frac{1}{\sin\beta\cos\beta}\right) \qquad (21)$$

By collecting the terms containing $v\sin\phi$ we obtain

$$\frac{v}{c}\sin\phi\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}\right)=\frac{\sin\alpha-\sin\beta}{\sin\alpha\sin\beta} \qquad (22)$$

or

$$\sin\alpha-\sin\beta=\frac{v}{c}\sin\phi\,\sin(\alpha+\beta) \qquad (23)$$

[Marking Scheme]

THEORETICAL Question 3

Relativistic Mirror

A. (3.0)	0.5	**Equation:** $\sin\alpha - \sin\beta = -\frac{v}{c}\sin(\alpha+\beta)$
	0.25	Equation $\left(1+\frac{v}{c}\cos\beta\right)\sin\alpha = \left(1-\frac{v}{c}\cos\alpha\right)\sin\beta$
	0.5	$\left(1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}\right)\cos^2\beta + 2\frac{v}{c}\left(1-\cos^2\alpha\right)\cos\beta + 2\frac{v}{c}\cos\alpha - \left(1+\frac{v^2}{c^2}\right)\cos^2\alpha = 0$
	0.75	$(\cos\beta)_1 = \dfrac{2\frac{v}{c}\cos^2\alpha - \left(1+\frac{v^2}{c^2}\right)\cos\alpha}{1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}}$ $(\cos\beta)_2 = \dfrac{-2\frac{v}{c} + \left(1+\frac{v^2}{c^2}\right)\cos\alpha}{1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}}$
	0.5	**Recognize the mirror is at rest (** $v=0$ **) then** $\cos\alpha = \cos\beta$
	0.5	$\cos\beta_2 = \dfrac{-2\frac{v}{c} + \left(1+\frac{v^2}{c^2}\right)\cos\alpha}{1-2\frac{v}{c}\cos\alpha+\frac{v^2}{c^2}}$
B(2.0)	0.25	$p_f \sin\alpha = p_f'\sin\beta$
	0.25 0.25	Know how to calculate sinβ $p_f = hf/c$
	0.75	$f' = \dfrac{(1+\frac{v^2}{c^2}) - 2\frac{v}{c}\cos\alpha}{(1-\frac{v^2}{c^2})} f$
	0.5	$\dfrac{f'}{f} = 0.5$

For part C, if the students is not able to prove the equation maximum point is 2.5.

(5.0)	1.0	Equation $\overline{ef} = v(t-t_0)\sin\phi\left(\tan\alpha + \tan\beta\right)$
	1.0	$\sin\alpha = \dfrac{c + v\dfrac{\sin\phi}{\cos\alpha}}{\dfrac{\overline{ag}}{t-t_0}}$
	0.5	$\sin\beta = \dfrac{c - v\dfrac{\sin\phi}{\cos\beta}}{\dfrac{\overline{ag}}{t-t_0} - v\sin\phi\,(\tan\alpha + \tan\beta)}$
	2.5	$\sin\alpha - \sin\beta = \dfrac{v}{c}\sin\phi\,\sin(\alpha+\beta)$

Propagation error can be considered but the maximum point is 2.5.