What is $1 + 2 + 3 + 4 + ...$ equal to?

An introduction to divergent series

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1 Introduction

There have been many debates over the Internet [1], [2], [3], [4] about a very uncanny result: the sum $S = 1 + 2 + 3 + 4...$ is equal to $-\frac{1}{12}$. This result is very surprising in that summing an infinite sequence of increasing positive integers yields to a finite, negative and non-integer number. In Section 2, we develop various heuristic approaches to obtain this counter-intuive result. In Section 3, we give a formal mathematical proof showing that our intuition that the sum should be infinite is correct as well, we explain how both results are acceptable, and we give a formal proof of $S = -\frac{1}{12}$. In Section 4, we give formal proofs of the calculations done in Section 2. In Section 5, we give other examples of divergent series. In Section 6, we provide annexes about Bernoulli numbers, complex analysis and Euler gamma function.

2 Heuristic approaches to a counter-intuitive result

First and foremost, we warn the reader that, in this section, we develop approaches which had made contreversy and yield to suprising results. This section recalls these approaches but we disclaim them. In Subsection 2.4, we give some preliminary cristicisms, and in Section 3, we develop formal mathematical proofs. We present various ways to compute the sum

$$S = 1 + 2 + 3 + 4 + ... \tag{1}$$

yielding to the result $S = -\frac{1}{12}$. In these approaches, we cannot compute S directly and we need to introduce some auxiliary sums. We define the alternate sum of ones [1]:

$$S_1 = 1 - 1 + 1 - 1 + ... \tag{2}$$

and we define the alternate sum of positive integers

$$S_2 = 1 - 2 + 3 - 4 + ... \tag{3}$$

In the following subsections, we show one method to compute S_1 and S, and three methods to compute S_2.

2.1 Computation of S_1

Multiplying by -1 and adding 1, we get

$$1 - S_1 = 1 - (1 - 1 + 1 - 1 + ...) = 1 - 1 + 1 - 1 + \cdots = S_1 \tag{4}$$

thus $2S_1 = 1$ and $S_1 = \frac{1}{2}$.

[1] also known as Grandi's series

2.2 Computation of S_2

a First attempt

We have

$$\begin{array}{rcl} S_1 & = & 1-1+1-1+\dots \\ + \quad S_2 & = & 1-2+3-4+\dots \\ \hline S_1+S_2 & = & 2-3+4-5+\dots \end{array} \tag{5}$$

Then by subtracting 1 on both sides, we get $S_1 + S_2 - 1 = -1 + 2 - 3 + 4 - 5 + \dots = -S_2$. Thus $S_2 = \dfrac{1-S_1}{2} = \dfrac{1}{4}$.

b Second attempt

We define the alternate sum of odd integers:

$$S_3 = 1 - 3 + 5 - 7 + \dots \tag{6}$$

We have

$$\begin{array}{rcl} S_1 & = & 1-1+1-1+\dots \\ + \quad S_1 & = & 1-1+1-1+\dots \\ + \quad S_3 & = & 1-3+5-7+\dots \\ \hline 2S_1+S_3 & = & 3-5+7-9+\dots \end{array} \tag{7}$$

Then by multiplying both sides by -1 and adding 1, we get $1 - 2S_1 - S_3 = S_3$ thus $S_3 = \dfrac{1-2S_1}{2} = 0$. Then we compute

$$\begin{array}{rcl} S_1 & = & 1-1+1-1+\dots \\ + \quad S_3 & = & 1-3+5-7+\dots \\ \hline S_1+S_3 & = & 2-4+6-8+\dots \\ & = & 2(1-2+3-4+\dots) = 2S_2 \end{array} \tag{8}$$

Thus $S_2 = \dfrac{S_1+S_3}{2} = \dfrac{1}{4}$.

c Third attempt

This last approach is more subtle and considers the Cauchy product $S_1 \times S_1$. We have

$$\begin{aligned} S_1 \times S_1 &= (1-1+1-1+\dots) \times (1-1+1-1+\dots) \\ &= \begin{array}{c|c|c|c|c} \times & +1 & -1 & +1 & \dots \\ \hline +1 & +1 & -1 & +1 & \dots \\ \hline -1 & -1 & +1 & -1 & \dots \\ \hline +1 & +1 & -1 & +1 & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \\ &= (1 \times 1) - (1 \times 1 + 1 \times 1) + (1 \times 1 + 1 \times 1 + 1 \times 1) - \dots \\ &= 1 - 2 + 3 - \dots = S_2 \end{aligned}$$

Thus $S_2 = S_1^2 = \dfrac{1}{4}$.

2.3 Computation of S

Finally we can compute S. We have

$$\begin{array}{rcl} S & = & 1+2+3+4+\dots \\ - \quad S_2 & = & -1+2-3+4+\dots \\ \hline S - S_2 & = & 0+4+0+8+\dots \\ & = & 4(1+2+3+4+\dots) = 4S \end{array} \tag{9}$$

Thus $S = -\dfrac{S_2}{3} = -\dfrac{1}{12}$.

2.4 Criticisms

As a beginning, when we perform operations on S_1 in Equation (4), we assume that the sum $S_1 = 1-1+1-1+\dots$ exists, or more precisely, that the limit $\lim\limits_{N\to+\infty} \sum\limits_{n=0}^{N}(-1)^n$ exists and is finite, which we have not proved yet. We set for all $N \in \mathbb{N}$, $S_N = \sum\limits_{n=0}^{N}(-1)^n$. Than we easily find that

$$\forall N \in \mathbb{N} \qquad S_N = \begin{cases} 1 & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd} \end{cases}$$

Since the sequence $(S_N)_{N\in\mathbb{N}}$ oscillates between values 1 and 0, it has not limit, which makes it difficult to talk about S_1. However, before discussing other criticisms of this section, we can discuss about a way to circumvent the limit problem of S_1. More formally, we define $\mathcal{S}$ the vector space of complex-valued sequences, and $\mathcal{S}_C$ the set of convergent complex-valued series, i.e.

$$\mathcal{S}_C = \left\{ (u_n)_{n\in\mathbb{N}} \in \mathcal{S}, \lim_{N\to+\infty} \sum_{n=0}^{N} u_n \text{ exists and is finite} \right\}$$

It is clear that $\mathcal{S}_C$ is a subspace of $\mathcal{S}$. When we compute the sum of a convergent series, we simply compute the value of the linear form

$$\Sigma : \mathcal{S}_C \to \mathbb{C} \quad (u_n) \mapsto \sum_{n=0}^{+\infty} u_n$$

A cornerstone question is to determine whether it is possible to extend Σ to a mapping Σ' on $\mathcal{S}$ satisfying the properties

1) **regularity**: $\Sigma'_{|\mathcal{S}_C} = \Sigma$, i.e. the new summation method Σ' yields to the same results as Σ on $\mathcal{S}_C$;
2) **linearity**: Σ' is a linear form, which we expect from any summation method;
3) **stability**: for all $(u_n)_{n\in\mathbb{N}}$, if we define $(v_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $v_n = u_{n+1}$, then $\Sigma'((u_n)) = u_0 + \Sigma'((v_n))$. In other word, the sum of a series should be the addition of the first term and of the sum of the series of the remaining terms.

The extension is actually not unique as there are various methods to extend Σ (Cesaro summation, Abel summation, zeta function regularization, ...) but discussing extension methods is out of the scope of this article. However, our second criticism is that when we performed operations in Equations (5), (7), (8) and (9), we implicitely assumed linearity of an extended summation method, without stating which one we were using and without proving its linearity. By the same token, we can criticize that we have not proved the stability of our summation methods, in Equation (4) for instance.

3 A formal demonstration

In this section, we formally discuss the summation results yielding to S in terms of limits.

3.1 Proof that our intuition is correct

First we have to state that the expression $S = 1 + 2 + 3 + 4 + \dots$ has no mathematical meaning. Dots in mathematical expressions are handful to deal with infinite sums, but they can hide anything that we want. For instance they could hide an infinite sum of zeros, and S would simply be equal to 10. It is clear from the context that we are trying to sum the terms of the sequence of natural integers, and it is better to write

$$S = \sum_{n=1}^{+\infty} n = \lim_{N\to+\infty} \sum_{n=1}^{N} n$$

Thus we are interested in the sum of the series $\sum n$ with associated sequence of partial sums $S_N = \sum_{n=1}^{N} n$. First we need to determine whether it is a convergent or divergent series. It is clear[2] that for any $N \in \mathbb{N}^*$, $S_N = \frac{N(N+1)}{2}$. Thus $\lim_{N\to+\infty} S_N = \lim_{N\to+\infty} \frac{N(N+1)}{2} = +\infty$. Assuming that we are playing in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we can assert that $S = +\infty$. This result confirms our intuition that an infinite sum of increasing positive numbers should be equal to infinity! Then we can legitimately ask about the result $S = -\frac{1}{12}$ found in Section 2. We transform our definition of S by noticing that

$$S = \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{1}{n^{-1}} = \lim_{N\to+\infty} \sum_{n=1}^{N} \lim_{s\to-1} \frac{1}{n^s} = \lim_{N\to+\infty} \lim_{s\to-1} \sum_{n=1}^{N} \frac{1}{n^s}$$

because we can invert a limit and a finite sum. Two questions arise: can we invert the limits and if so, do we get something different than $+\infty$, say $-\frac{1}{12}$?

3.2 Limits cannnot always be switched

Before even trying to determine the limit $\lim_{s\to-1} \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{1}{n^s}$, we may ask if it is surprising that switching limits give different results. It is not, and we can exhibit a simple example of non uniform convergence. Let us define the sequence of functions $(f_n)_{n\in\mathbb{N}}$ for all $n \in \mathbb{N}$ by $f_n : [0,1] \to [0,1] \quad x \mapsto x^n$. It is well known that this sequence of functions simply converges (but not uniformly) to the function:

$$f : [0,1] \to [0,1] \quad x \mapsto \begin{cases} 0 & \text{if } x \in [0,1[\\ 1 & \text{if } x = 1 \end{cases}$$

Then we have on one hand, using the continuity of f_n in 1,

$$\lim_{n\to+\infty} \lim_{\substack{x\to1 \\ x<1}} f_n(x) = \lim_{n\to+\infty} 1 = 1$$

[2] If not clear, we can refer to ten year-old Friedrich Gauss' proof

$$\begin{array}{rrcl} & S_N & = & 1 \;+\; 2 \;+\cdots+\; N \\ + & S_N & = & N \;+ (N-1) +\cdots+\; 1 \\ \hline & 2S_N & = & \underbrace{(N+1) + (N+1) + \cdots + (N+1)}_{N \text{ times}} \end{array}$$

and on the other hand

$$\lim_{\substack{x\to 1\\ x<1}} \lim_{n\to+\infty} f_n(x) = \lim_{\substack{x\to 1\\ x<1}} f(x) = 0$$

Therefore

$$\lim_{n\to+\infty} \lim_{\substack{x\to 1\\ x<1}} f_n(x) \neq \lim_{\substack{x\to 1\\ x<1}} \lim_{n\to+\infty} f_n(x)$$

3.3 Proof that the counter-intuition is somewhat correct as well

We want to determine, if it exists, the limit

$$\lim_{s\to -1} \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{1}{n^s}$$

We recognize Riemann's zeta function defined by

$$\zeta(s) = \lim_{N\to+\infty} \sum_{n=1}^{N} \frac{1}{n^s} = \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

We are looking for the value of $\zeta(-1)$ or, more generally, a value of $\lim\limits_{s\to -1} \zeta(s)$. We begin by defining the function zeta and giving some properties.

a Definition and properties of function zeta on the real line

Proposition 3.1
Let $s \in \mathbb{R}$. The Riemann series $\sum\limits_{n\in\mathbb{N}^*} \frac{1}{n^s}$ converges if and only if $s > 1$.

Proof : If $s < 0$, $\lim\limits_{n\to+\infty} \frac{1}{n^s} = +\infty$ and if $s = 0$, $\lim\limits_{n\to+\infty} \frac{1}{n^0} = 1$, thus by using the same reasoning as in Subsection 3.1, we can assert that the series diverges.
If $s > 0$, $\lim\limits_{n\to+\infty} \frac{1}{n^s} = 0$, thus we can use the integral test for convergence. Function $t \mapsto \frac{1}{t^s}$ is decreasing over $]0, +\infty[$, thus for all $k \in \mathbb{N}^*$ and all $t \in [k, k+1]$, $\frac{1}{(k+1)^s} \leq \frac{1}{t^s} \leq \frac{1}{k^s}$, which yields to

$$\frac{1}{(k+1)^s} = \int_k^{k+1} \frac{dt}{(k+1)^s} \leq \int_k^{k+1} \frac{dt}{t^s} \leq \int_k^{k+1} \frac{dt}{k^s} = \frac{1}{k^s}$$

Summing this inequalities for $k \in [\![1, n]\!]$,

$$\sum_{k=1}^{n} \frac{1}{(k+1)^s} \leq \int_1^{n+1} \frac{dt}{t^s} \leq \sum_{k=1}^{n} \frac{1}{k^s}$$

Setting $S_n(s) = \sum\limits_{k=1}^{n} \frac{1}{k^s}$, we can write $S_{n+1}(s) - 1 \leq \int_1^{n+1} \frac{dt}{t^s} \leq S_n(s) \leq 1 + \int_1^{n} \frac{dt}{t^s}$.

- If $s = 1$, $\int_1^{n+1} \frac{dt}{t} = \ln(n+1)$. Since $\lim\limits_{n\to+\infty} \ln(n+1) = +\infty$ and since for all $n \in \mathbb{N}^*$, $\ln(n+1) \leq S_n(1)$, we deduce that the series[3] $\sum \frac{1}{n}$ diverges.
- If $0 < s < 1$, for any $n \in \mathbb{N}^*$, $\frac{1}{n^s} > \frac{1}{n}$, and since the series $\sum \frac{1}{n}$ diverges, we deduce that the series $\sum \frac{1}{n^s}$ also diverges.

[3] also known as the harmonic series

- If $s > 1$, then $1 - s < 0$ and $\lim\limits_{n \to +\infty} n^{1-s} = 0$ thus

$$\lim_{n\to+\infty} \int_1^n \frac{dt}{t^s} = \lim_{n\to+\infty} \left[\frac{t^{1-s}}{1-s}\right]_1^n = \frac{1}{s-1}$$

Hence sequence $(S_n(s))_{n\in\mathbb{N}^*}$ is increasing and upper bounded by $1 + \dfrac{1}{s-1}$, thus sequence $(S_n(s))$ and series $\sum \dfrac{1}{n^s}$ converge. ■

Proposition 3.2
Function ζ is defined and continuous on $]1, +\infty[$. Moreover, $\lim\limits_{\substack{s\to 1\\ s>1}} \zeta(s) = +\infty$.

Proof : Proposition 3.1 shows that ζ is defined on $]1, +\infty[$. Let show the continuity of ζ on this interval. Let $a \in]1, +\infty[$. For all $n \in \mathbb{N}^*$, function $s \mapsto \dfrac{1}{n^s}$ is continuous on $[a, +\infty[$. Moreover, for all $s \in [a, +\infty[$, $\left|\dfrac{1}{n^s}\right| = \dfrac{1}{n^s} \leq \dfrac{1}{n^a}$ with equality if and only if $s = a$. Therefore,

$$\sup\left\{\left|\frac{1}{n^s}\right|, s \in [a, +\infty[\right\} = \frac{1}{n^a}$$

Since the Riemann series $\sum\limits_{n\in\mathbb{N}^*} \dfrac{1}{n^a}$ is convergent for $a > 1$, the series of functions $s \mapsto \sum\limits_{n\in\mathbb{N}^*} \dfrac{1}{n^s}$ is normally convergent on $[a, +\infty[$, thus it is uniformly convergent on $[a, +\infty[$. Hence the sum function ζ is continuous on $[a, +\infty[$ as the uniform limit on $[a, +\infty[$ of a sequence of continuous functions on $[a, +\infty[$. Since this is true for any $a \in]1, +\infty[$, we deduce that ζ is continous on $]1, +\infty[$.
Using the proof of Proposition 3.1, we have

$$\forall s \in]1, +\infty[\qquad \frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}$$

and we can deduce the limit. ■

We see in this proposition that ζ cannot be extended by continuity on the real line because of its singularity in 1. We need to study this function on the complex plane to circumvent this singularity.

b Extension to the complex plane

This part uses some results of measure theory and complex analysis. The reader who is not familiar with these notions can skip to the conclusion in Subsection 3.4 at first reading. We recall some elements of complex analysis in Annex 6.2.
Notation: We denote the half-planes $P_0 = \{z \in \mathbb{C}, \mathrm{Re}(z) > 0\}$ and $P_1 = \{z \in \mathbb{C}, \mathrm{Re}(z) > 1\}$, and the open disc $D_{2\pi} = \{z \in \mathbb{C}, |z| < 2\pi\}$ of center 0 and radius 2π. We also denote $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}^*$ the set of nonpositive integers.

Proposition 3.3
Series $\sum \dfrac{1}{n^s}$ is absolutely convergent if and only if $s \in P_1$. Therefore, ζ is well defined on P_1.

Proof : For $s \in \mathbb{C}$, set $a = \mathrm{Re}(s) \in \mathbb{R}$ and $b = \mathrm{Im}(s) \in \mathbb{R}$, so that $s = a + ib$. Then for all $n \in \mathbb{N}^*$,

$$n^s = n^{a+ib} = n^a e^{ib\ln n} = n^a \cos(b\ln n) + in^a \sin(b\ln n)$$

and

$$\frac{1}{n^s} = \frac{\cos(b\ln n) - i\sin(b\ln n)}{n^a} \qquad \text{and} \qquad \left|\frac{1}{n^s}\right| = \frac{1}{n^a}$$

Therefore, series $\sum \dfrac{1}{n^s}$ exactly behaves like series $\sum \dfrac{1}{n^{\mathrm{Re}(s)}}$. ■

We have shown that function ζ can be extended by definition of the sum to the half-plane P_1. However, for any $s \in \mathbb{C} \setminus P_1$, the series does not converge and we cannot use the sum definition of ζ anymore. We need an alternate definition of ζ to extend its definition to $\mathbb{C} \setminus \{1\}$. In the following propositions, we use Euler gamma function Γ. We provide details on this function in Annex 6.3.

Proposition 3.4
For all $s \in P_1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Proof : Function Γ is defined on P_0 by

$$\forall s \in P_0 \qquad \Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$$

Thus for all $s \in P_1 \subset P_0$,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{+\infty} \frac{\Gamma(s)}{n^s} = \sum_{n=1}^{+\infty} \frac{1}{n^s} \int_0^{+\infty} t^{s-1} e^{-t} dt = \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(\frac{t}{n}\right)^{s-1} e^{-t} \frac{dt}{n}$$

For all $n \in \mathbb{N}^*$, function $\varphi_n :]0, +\infty[\to]0, +\infty[\quad t \mapsto \dfrac{t}{n}$ is a $\mathcal{C}^1$-diffeomorphism such that for any $t \in]0, +\infty[$, $\varphi_n'(t) = \dfrac{1}{n}$, thus by change of variable,

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{+\infty} \int_0^{+\infty} u^{s-1} e^{-nu} du$$

For any $N \in \mathbb{N}^*$, set $f_N :]0, +\infty[\to]0, +\infty[\quad u \mapsto u^{s-1} \sum_{n=1}^{N} e^{-nu}$. Then $(f_N)_{N \in \mathbb{N}^*}$ is an increasing sequence of measurable functions [4] which converges to $f :]0, +\infty[\to]0, +\infty[\quad u \mapsto u^{s-1} \dfrac{e^{-u}}{1 - e^{-u}}$. By monotone convergence theorem[5], f is measurable and

$$\int_0^{+\infty} u^{s-1} \frac{e^{-u}}{1 - e^{-u}} du = \int_0^{+\infty} f(u) du = \lim_{N \to +\infty} \int_0^{+\infty} f_N(u) du = \sum_{n=1}^{+\infty} \int_0^{+\infty} u^{s-1} e^{-nu} du$$

Thus

$$\zeta(s)\Gamma(s) = \int_0^{+\infty} u^{s-1} \frac{e^{-u}}{1 - e^{-u}} du = \int_0^{+\infty} \frac{u^{s-1}}{e^u - 1} du$$

which yields to the result. ∎

Lemma 3.5
Function $s \mapsto \dfrac{1}{\Gamma(s)} \displaystyle\int_1^{+\infty} \frac{t^{s-1}}{e^t - 1} dt$ is holomorphic over P_1.

Proof : We aim at applying the theorem of holomorphy of a parametric integral 6.3 to the function

$$(s, t) \mapsto \frac{t^{s-1}}{e^t - 1} = \frac{e^{(s-1)\ln t}}{e^t - 1}$$

[4] for all $N \in \mathbb{N}^*$ and for almost all $x \in]0, +\infty[$, $f_N(x) \leq f_{N+1}(x)$

[5] also called Beppo-Levi theorem

1) For all $s \in P_1$, $t \mapsto \dfrac{e^{(s-1)\ln t}}{e^t - 1}$ is integrable on $[1, +\infty[$.

2) For all $t \in]1, +\infty[$, $t \mapsto \dfrac{e^{(s-1)\ln t}}{e^t - 1}$ is holomorphic on P_1.

3) Let K be a compact set of P_1. There exists $M \in \mathbb{R}_+^*$ such that for all $s \in K$, $\mathrm{Re}(s) \leq M$. Since $t \in [1, +\infty[$,

$$\left| \frac{e^{(s-1)\ln t}}{e^t - 1} \right| \leq \frac{t^{M-1}}{e^t - 1}$$

and the upper bound is integrable. We used the fact that for any $s \in \mathbb{C}$, $|e^s| = e^{\mathrm{Re}(s)}$.

Therefore, by the Theorem of holomorphy of a parametric integral 6.3, $s \mapsto \displaystyle\int_1^{+\infty} \frac{t^{s-1}}{e^t - 1} dt$ is holomorphic over P_1. Moreover by Proposition 6.5, Γ is holomorphic on P_1, thus we deduce the result. ■

Proposition 3.6
For all $s \in P_1$,

$$\zeta(s) = \frac{1}{\Gamma(s)(s-1)} + \frac{1}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{B_n}{n!(n+s-1)} + \frac{1}{\Gamma(s)} \int_1^{+\infty} \frac{t^{s-1}}{e^t - 1} dt \tag{10}$$

where the B_n's are the Bernoulli numbers described in Annex 6.1. Moreover, ζ can be extended to $\mathbb{C} \setminus \{1\}$ with possible poles at $-n$ for $n \in \mathbb{N}$.

PROOF : We begin by breaking down the integral form of ζ in Proposition 3.4: for all $s \in P_1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{e^t - 1} dt + \frac{1}{\Gamma(s)} \int_1^{+\infty} \frac{t^{s-1}}{e^t - 1} dt$$

We aim at transforming the first integral into a series. By Proposition 6.1, for all $(s, t) \in P_1 \times [0, 1]$,

$$\frac{t^{s-1}}{e^t - 1} = \sum_{n=0}^{+\infty} \frac{B_n}{n!} t^{n+s-2}$$

In order to switch the integral and the sum, we need to apply Fubini's theorem. Note that for all $t > 0$, $t^s = |e^{s \ln t}| = e^{\mathrm{Re}(s) \ln t} = t^{\mathrm{Re}(s)}$. Then, for all $t \in]0, 1]$,

$$\sum_{n=0}^{+\infty} \left| \frac{B_n}{n!} \right| |t^{n+s-2}| = t^{\mathrm{Re}(s)-2} \sum_{n=0}^{+\infty} \frac{|B_n| t^n}{n!} = t^{\mathrm{Re}(s)-2} \left(\frac{t}{2} \left(1 - \cot\left(\frac{t}{2} \right) \right) + 2 \right)$$

For further details on this power series, we refer to the book [5]. Since $\mathrm{Re}(s) > 1$, $\mathrm{Re}(s) - 2 > -1$ and function $t \mapsto t^{\mathrm{Re}(s)-2} \left(\frac{t}{2} \left(1 - \cot\left(\frac{t}{2} \right) \right) + 2 \right)$ is integrable on $]0, 1]$. Hence

$$\int_0^1 \sum_{n=0}^{+\infty} \left| \frac{B_n}{n!} t^{n+s-2} \right| dt < +\infty$$

Therefore, by Fubini's theorem, we can switch the integral and the sum. Then

$$\begin{aligned} \int_0^1 \frac{t^{s-1}}{e^t - 1} dt &= \int_0^1 \sum_{n=0}^{+\infty} \frac{B_n}{n!} t^{n+s-2} dt = \int_0^1 t^{s-2} dt + \sum_{n=1}^{+\infty} \frac{B_n}{n!} \int_0^1 t^{n+s-2} dt \\ &= \left[\frac{t^{s-1}}{s-1} \right]_0^1 + \sum_{n=1}^{+\infty} \frac{B_n}{n!} \left[\frac{t^{n+s-1}}{n+s-1} \right]_0^1 = \frac{1}{s-1} + \sum_{n=1}^{+\infty} \frac{B_n}{n!(n+s-1)} \end{aligned}$$

Therefore, for all $s \in P_1$,

$$\zeta(s) = \frac{1}{(s-1)\Gamma(s)} + \frac{1}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{B_n}{n!(n+s-1)} + \frac{1}{\Gamma(s)} \int_1^{+\infty} \frac{t^{s-1}}{e^t - 1} dt$$

Then, let show that $f : s \mapsto \displaystyle\sum_{n=1}^{+\infty} \frac{B_n}{n!(n+s-1)}$ is meromorphic on $\mathbb{C}$ and has simple poles at $-n$ with $n \in \mathbb{N}$.

1) For all $n \in \mathbb{N}^*$, $f_n : s \mapsto \dfrac{B_n}{n!(n+s-1)}$ is meromorphic on $\mathbb{C}$ with one simple pole at $-n+1$.

2) Let K be a compact set of $\mathbb{C}$. There exists $N_K \in \mathbb{N}$ such that $K \subset \overline{D(0, N_K)}$. For all $n > N_K$, function f_n has no pole in K. Moreover, for all $s \in K$, $|n+s| \geq n - |s| \geq n - N_K$. Hence, for all $s \in K$, $|f_n(s)| \leq \dfrac{|B_n|}{n!(n-N_K)}$ thus $\displaystyle\sum_{n > N_K} f_n$ is normally convergent over K.

Therefore, by Theorem of series of meromorphic functions 6.4, f is a meromorphic function over $\mathbb{C}$ whose simple poles are non-positive integers. By Lemma 3.5, $s \mapsto \dfrac{1}{\Gamma(s)} \displaystyle\int_1^{+\infty} \frac{t^{s-1}}{e^t - 1} dt$ is holomorphic on P_1. Therefore, the developped expression of ζ in Equation (10) establishes a meromorphic continuation of ζ over $\mathbb{C}$. Moreover, the analytic continuation theorem implies that this is the unique analytic continuation of ζ on the connex open set $\mathbb{C} \setminus (\mathbb{Z}^- \cup \{1\})$. ■

Corollary 3.7
Function ζ has removable poles in $\mathbb{Z}_-$ and for all $k \in \mathbb{N}$, $\zeta(-k) = (-1)^k \dfrac{B_{k+1}}{k+1}$. Therefore, ζ is a meromorphic function over $\mathbb{C}$ with one simple pole in 1.

Proof : We study the terms in Equation (10). For $k \in \mathbb{N}$, since by Proposition 6.6 Γ has a simple pole in $-k$,

$$\lim_{\substack{s \to -k \\ s \neq -k}} \frac{1}{(s-1)\Gamma(s)} = 0 \qquad \text{and} \qquad \lim_{\substack{s \to -k \\ s \neq -k}} \frac{1}{\Gamma(s)} \int_1^{+\infty} \frac{t^{s-1}}{e^t - 1} dt = 0$$

We have to find an equivalent of the term $\dfrac{1}{\Gamma(s)} \displaystyle\sum_{n=1}^{+\infty} \frac{B_n}{n!(n+s-1)}$. By Proposition 6.7,

$$\Gamma(s) \underset{s \to -k}{\sim} \frac{(-1)^k}{k!(s+k)} \qquad \text{thus} \qquad \frac{1}{\Gamma(s)} \underset{s \to -k}{\sim} (-1)^k k!(s+k)$$

and

$$\sum_{n=1}^{+\infty} \frac{B_n}{n!(n+s-1)} = \frac{B_{k+1}}{(k+1)!(s+k)} + \sum_{\substack{n=1 \\ n \neq k+1}}^{+\infty} \frac{B_n}{n!(n+s-1)} \underset{s \to -k}{\sim} \frac{B_{k+1}}{(k+1)!(s+k)}$$

Therefore,

$$\frac{1}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{B_n}{n!(n+s-1)} \underset{s \to -k}{\sim} (-1)^k \frac{B_{k+1}}{k+1}$$

and $\displaystyle\lim_{\substack{s \to -k \\ s \neq -k}} \zeta(s) = (-1)^k \frac{B_{k+1}}{k+1}$. Hence function ζ has removable poles in $\mathbb{Z}_-$ and for all $k \in \mathbb{N}$, $\zeta(-k) = (-1)^k \dfrac{B_{k+1}}{k+1}$. ■

Consequence: Using this last property and Example 6.1, we have

$$\zeta(-1) = (-1)^1 \frac{B_2}{2} = -\frac{1}{12}.$$

3.4 Summary

As a conclusion, we cannot determine the limit

$$\lim_{s\to -1}\lim_{N\to +\infty}\sum_{n=1}^{N}\frac{1}{n^s}$$

because the expression of ζ as the Dirichlet series $\zeta(s)=\sum_{n=1}^{+\infty}\frac{1}{n^s}$ is only true for $s\in P_1$, whereas -1 is not in the closure of P_1. The best formal statement that we can make is that:

$$\boxed{\zeta \text{ is the only meromorphic function on } \mathbb{C} \text{ such that } \zeta(-1)=-\tfrac{1}{12} \text{ and for all } s\in P_1,\ \zeta(s)=\sum_{n=0}^{+\infty}\frac{1}{n^s}.}$$

In the following table, we sum up the results of our discussion in this section.

Mathematicl expression	Value
$S=1+2+3+4+\ldots$	No meaning
$S=\lim_{N\to+\infty}\lim_{s\to-1}\sum_{n=1}^{N}\frac{1}{n^s}$	$+\infty$
$S=\lim_{s\to-1}\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{1}{n^s}$	No meaning
$S=\lim_{s\to-1}\zeta(s)$ where ζ is the unique meromorphic function on $\mathbb{C}$ such that for all $s\in P_1$, $\zeta(s)=\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{1}{n^s}$	$-\frac{1}{12}$

4 Formal demonstration of the heuristic approach

We have seen in Section 3 that the equality $S=-\frac{1}{12}$ can be interpreted as the result of $\lim_{s\to 1}\zeta(s)=-\frac{1}{12}$. But we were also able in Section 2 to obtain this sum dealing with some surprising auxiliary sums. Inspiring from Section 3, we can see these sums as limits of some function of s when s tends to -1. We show in this section that calculations in Section 2 are merely legitimate operations on limits.

4.1 Computing S_1 as limit of a power series

Starting with S_1, we can interpret Equation (2) as

$$S_1=\lim_{s\to-1}\sum_{n=1}^{+\infty}s^{n-1}=\lim_{s\to-1}\sum_{n=0}^{+\infty}s^{n}$$

By a ratio test, we prove that the radius of convergence of power series $\sum s^{n-1}$ is $R_1=1$. We denote f_1 the sum of this power series and $D_1=\{z\in\mathbb{C},|z|<1\}$ the open disc of center 0 and radius 1. Then

$$\forall s\in D_1 \qquad f_1(s)=\sum_{n=1}^{+\infty}s^{n-1}=\frac{1}{1-s}$$

and this function is holomorphic over D_1. Since -1 is in the closure of D_1, we can legitimately take the limit

$$S_1 = \lim_{\substack{s\to -1\\ s\in D_1}} f_1(s) = \lim_{\substack{s\to -1\\ s\in D_1}} \sum_{n=1}^{+\infty} s^{n-1} = \lim_{\substack{s\to -1\\ s\in D_1}} \frac{1}{1-s} = \frac{1}{2}$$

which yields to the result found in Section 2.
Now we can justify Equation (4) by writing

$$\forall s \in D_1 \qquad f_1(s) = \sum_{n=0}^{+\infty} s^n = 1 + s\sum_{n=0}^{+\infty} s^n = 1 + sf_1(s)$$

and by taking the limit,

$$S_1 = \lim_{\substack{s\to -1\\ s\in D_1}} f_1(s) = \lim_{\substack{s\to -1\\ s\in D_1}} (1 + sf_1(s)) = 1 - S_1.$$

4.2 Various ways to compute S_2

a Limit of a power series

We can interpret Equation (3) as

$$S_2 = \lim_{s\to -1} \sum_{n=1}^{+\infty} ns^{n-1} = \lim_{s\to -1} \sum_{n=0}^{+\infty} (n+1)s^n$$

By a ratio test, we show that the radius of convergence of this power series is 1. We denote f_2 the limit of this sum. From the properties of power series, f_2 is the derivative of f_1, i.e. for all $s \in D_1$, $f_2(s) = f_1'(s)$, thus

$$\forall s \in D_1 \qquad f_2(s) = \frac{1}{(1-s)^2}$$

Then by taking the limit,

$$S_2 = \lim_{\substack{s\to -1\\ s\in D_1}} f_2(s) = \lim_{\substack{s\to -1\\ s\in D_1}} \frac{1}{(1-s)^2} = \frac{1}{4}$$

b Using first attempt

In the first attempt, we established that $S_1 + S_2 - 1 = -S_2$. Inspiring from this expression, we have

$$\begin{aligned}\forall s \in D_1 \qquad f_1(s) + f_2(s) - 1 &= \sum_{n=1}^{+\infty} s^{n-1} + \sum_{n=1}^{+\infty} ns^{n-1} - 1 = \sum_{n=1}^{+\infty} (n+1)s^{n-1} - 1\\ &= \sum_{n=2}^{+\infty} ns^{n-2} - 1 = \frac{1}{s}\sum_{n=1}^{+\infty} ns^{n-1} - \frac{1}{s} - 1 = \frac{f_2(s)}{s} - \frac{1+s}{s}\end{aligned}$$

We can notice that, by taking the limit in this last equation, we find

$$\lim_{s\to -1} (f_1(s) + f_2(s) - 1) = -\lim_{s\to -1} f_2(s)$$

Moreover,

$$f_2(s) = \frac{s}{s-1} - \frac{s}{s-1} f_1(s) - \frac{1+s}{s-1} = \frac{1}{(1-s)^2}$$

c Using auxiliary sum S_3 (second attempt)

To compute sum S_3, we convert the computation of (7) into operations of limits. We can interpret Equation (6) as

$$S_3 = \lim_{s \to -1} \sum_{n=0}^{+\infty} (2n+1)s^n$$

The radius of convergence of this series is $R_3 = 1$. We denote f_3 the sum of this power series. Then

$$\forall s \in D_1 \qquad 2f_1(s) + f_3(s) = 2\sum_{n=0}^{+\infty} s^n + \sum_{n=0}^{+\infty}(2n+1)s^n = \sum_{n=0}^{+\infty}(2n+3)s^n = \sum_{n=1}^{+\infty}(2n+1)s^{n-1} = \frac{f_3(s)}{s} - \frac{1}{s}$$

We can notice that we are able to retrieve Calculation (7) by taking the limit in this last expression as s goes to -1, we obtain:

$$2S_1 + S_3 = \lim_{\substack{s \to -1 \\ s \in D_1}} (2f_1(s) + f_3(s)) = \lim_{\substack{s \to -1 \\ s \in D_1}} \left(\frac{f_3(s)}{s} - \frac{1}{s} \right) = -S_3 + 1$$

Finally, we have:

$$f_3(s) = \frac{2sf_1(s)}{1-s} + \frac{1}{1-s} = \frac{1+s}{(1-s)^2}$$

Hence $S_3 = \lim_{\substack{s \to -1 \\ s \in D_1}} f_3(s) = 0$. Inspiring from (8), we have

$$\forall s \in D_1 \qquad f_1(s) + f_3(s) = \sum_{n=0}^{+\infty} s^n + \sum_{n=0}^{+\infty}(2n+1)s^n = 2\sum_{n=0}^{+\infty}(n+1)s^n = 2f_2(s)$$

Thus $f_2(s) = \dfrac{f_1(s)+f_3(s)}{2} = \dfrac{1}{(1-s)^2}$.

d Cauchy product (third attempt)

Power series $\sum a_n s^n$ where $a_n = \sum_{k=0}^{n} 1 = n+1$ is the Cauchy product of $\sum s^n$ by itself. Therefore, the radius of convergence of this series is 1, and

$$\forall s \in D_1 \qquad f_2(s) = \sum_{n=0}^{+\infty}(n+1)s^n = \left(\sum_{n=0}^{+\infty} s^n \right)^2 = \frac{1}{(1-s)^2}$$

4.3 Computation of S

a Limitation of power series

We can interpret Equation (1) as

$$S = \lim_{s \to -1} \sum_{n=0}^{+\infty}(n+1)s^{2n}$$

As we have already seen, the radius of convergence of $\sum (n+1)s^{2n}$ is 1 and this series is the Cauchy product of $\sum s^{2n}$ by itself. Therefore, if we denote ℓ the sum of this power series, then

$$\forall s \in D_1 \qquad \ell(s) = \left(\sum_{n=0}^{+\infty} s^{2n} \right)^2 = \frac{1}{(1-s^2)^2}$$

Taking the limit, we obtain $\lim\limits_{\substack{s\to -1\\ s\in D_1}} \ell(s) = +\infty$. In this case, we are not able to retrieve $S = -\dfrac{1}{12}$. So far in this section, we have dealt with power series but we have not tackle with Dirichlet series yet, i.e. series of the form $\sum \dfrac{a_n}{n^s}$, whose function ζ is a particular example. Thus, we need to introduce another auxiliary function to find a relation between S_2 and S.

b Dirichlet eta function

We begin by transforming Equation (3) into a limit:

$$S_2 = \lim_{s\to -1} \lim_{N\to +\infty} \sum_{n=1}^{n} \frac{(-1)^{n-1}}{n^s}$$

We recognize Dirichlet's eta function defined by

$$\eta(s) = \lim_{N\to +\infty} \sum_{n=1}^{n} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s}$$

As an alternate series, $\sum \dfrac{(-1)^{n-1}}{n^s}$ is simply convergent for $s \in P_0$ and absolutely convergent for $s \in P_1$.

Proposition 4.1
For all $s \in P_1$,

$$\eta(s) = (1 - 2^{1-s})\zeta(s)$$

PROOF : Inspiring from Calculation (9), we have for all $s \in P_1$,

$$\zeta(s) - \eta(s) = \sum_{n=1}^{+\infty} \frac{1 + (-1)^n}{n^s} = 2^{1-s} \sum_{n=1}^{+\infty} \frac{1}{n^s} = 2^{1-s}\zeta(s)$$

Therefore, $\eta(s) = (1 - 2^{1-s})\zeta(s)$. ■

Proposition 4.2
Function η can be extended by analytic continuation to $\mathbb{C}$, for all $s \in \mathbb{C}\setminus\{1\}$, $\eta(s) = (1-2^{1-s})\zeta(s)$, and $\lim\limits_{s\to -1} \eta(s) = \dfrac{1}{4}$.

PROOF : Functions $s \mapsto 1 - 2^{1-s}$ and ζ are holomorphic on $\mathbb{C}\setminus\{1\}$, thus function $s \mapsto (1 - 2^{1-s})\zeta(s)$ is holomorphic on $\mathbb{C}\setminus\{1\}$. By Analytic continuation theorem 6.2, for all $s \in \mathbb{C}\setminus\{1\}$,

$$\eta(s) = (1 - 2^{1-s})\zeta(s)$$

Then by taking the limit,

$$\lim_{s\to -1} \eta(s) = \lim_{s\to -1} (1 - 2^{1-s}) \times \lim_{s\to -1} \zeta(s) = (-3) \times \left(-\frac{1}{12}\right) = \frac{1}{4}$$

Hence we find $S_2 = \dfrac{1}{4}$. ■

5 Further examples of divergent series

In this section, we present other interesting and surprising divergent series. As we have done previously, we give a heuristing approach in the assignement of a value to this sums when it is possible, and we show a formal proof yielding to this result.

5.1 $1+1+1+1+\dots$

We begin our study with the sum

$$S_4 = 1+1+1+1+\dots$$

We see that this sum is similar to the Grandi's series (3). However, to the best of our knowledge, there is no conclusive heuristic approach dealing with S_1 and S_4. Inspiring from function ζ, we can write

$$S_4 = \lim_{s\to 0}\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{1}{n^s} = \lim_{s\to 0}\zeta(s)$$

Using corollary 3.7, we have

$$S_4 = \zeta(0) = (-1)^0\frac{B_1}{1} = -\frac{1}{2}$$

which is the value that we usually assign to this divergent series. Incidentally, this result provides another method to compute the value of Grandi's series S_1. Indeed we can see that

$$S_1 = \lim_{s\to 0}\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^s} = \lim_{s\to 0}\eta(s)$$

By Proposition 4.2, we have

$$\forall s\in\mathbb{C}\setminus\{1\} \qquad \eta(s) = (1-2^{1-s})\zeta(s)$$

Therefore,

$$\eta(0) = (1-2^{1-0})\zeta(0) = (-1)\times\left(-\frac{1}{2}\right) = \frac{1}{2}$$

which gives once again the result $S_1 = \dfrac{1}{2}$.

5.2 Sum of squares of positive integers

We now look at the sum of squares of positive integers:

$$S_5 = 1+4+9+16+\dots \tag{11}$$

To determine this sum, we use S_3 defined in Equation (6) and the auxiliary alternate sum of square of positive integers:

$$S_5' = 1-4+9-16+\dots \tag{12}$$

We have

$$\begin{array}{rrcl} & 1-S_3 & = & 3-5+7-9+\dots \\ + & S_5' & = & 1-4+9-16+\dots \\ \hline & 1-S_3+S_5' & = & 4-9+16-25+\dots \end{array}$$

Thus $S_3 - S_5' = 1-4+9-16+25-\dots = S_5'$, and $S_5' = \dfrac{S_3}{2} = 0$. Then we compute S_5. We have

$$\begin{array}{rrcl} & S_5 & = & 1+4+9+16+25+36+\dots \\ - & S_5' & = & -1+4-9+16-25+36-\dots \\ \hline & S_5-S_5' & = & 0+8+0+32+0+72+\dots \\ & & = & 8(1+4+9+16+\dots) = 8S_5 \end{array}$$

Thus $S_5 = -\dfrac{S_5'}{7} = 0$.
Formally, we can interpret Equation (12) as

$$S_5' = \lim_{s \to -1} \sum_{n=0}^{+\infty} (n+1)^2 s^n$$

The radius of convergence of this series is $R_5 = 1$. We denote f_5 the sum of this power series. From Subsection 4.2, we have for all $s \in D_1$,

$$f_2(s) = \sum_{n=0}^{+\infty} (n+1) s^n = \frac{1}{(1-s)^2} \qquad \text{which gives} \qquad s f_2(s) = \sum_{n=0}^{+\infty} (n+1) s^{n+1} = \frac{s}{(1-s)^2}$$

Therefore,

$$f_5(s) = \sum_{n=1}^{+\infty} (n+1)^2 s^n = (s f_2(s))' = \frac{1+s}{(1-s)^3}$$

By taking the limit, we have $S_5' = \lim_{s \to -1} f_5(s) = \dfrac{1-1}{(1+1)^3} = 0$. To relate Equations (11) and (12), we use the relation between functions ζ and η. Indeed, we can write:

$$S_5 = \lim_{s \to -2} \zeta(s) \qquad \text{and} \qquad S_5' = \lim_{s \to -2} \eta(s)$$

Using Corollary 3.7 and Proposition 4.2, we have

$$S_5 = \zeta(-2) = (-1)^2 \frac{B_3}{3} = 0 \qquad \text{and} \qquad S_5' = \eta(-2) = (1 - 2^{1-(-2)})\zeta(-2) = -7\zeta(-2) = 0$$

5.3 Sum and alternate sum of powers of 2

a Sum of powers of 2

We study the sum of powers of 2:

$$S_6 = 1 + 2 + 4 + 8 + \ldots \tag{13}$$

Multiplying by 2 and adding 1, we get $1 + 2S_6 = 1 + 2 + 4 + 8 + \cdots = S_6$. Therefore, $S_6 = -1$.
Formally, we can interpret Equation (13) as the limit:

$$S_6 = \lim_{s \to 2} \sum_{n=0}^{+\infty} s^n = \lim_{s \to 2} f_1(s)$$

We have already encountered function f_1 in Subsection 4.1, but we have determined that the radius of convergence of the power series defining f_1 is $R_1 = 1$, while 2 is not in the closure of the open disc D_1. However, as for function ζ, using the form $f_1(s) = \dfrac{1}{1-s}$, we can extend f_1 to a meromorphic function over $\mathbb{C}$ with one simple pole in 1. Then it becomes legitimate to take the limit when s goes to 2:

$$S_6 = \lim_{s \to 2} \frac{1}{1-s} = -1$$

b Alternate sum of powers of 2

We study the alternate sum of powers of 2:

$$S_6' = 1 - 2 + 4 - 8 + \ldots \tag{14}$$

To determine this sum, we use the auxiliary sum

$$S_6'' = 1 + 4 + 16 + 64 + \ldots \tag{15}$$

Inspiring from the computation of S_6, we multiply S_6'' by 4 and we add 1 to get

$$1 + 4S_6'' = 1 + 4 + 16 + 64 + \dots = S_6''$$

which gives $S_6'' = -\frac{1}{3}$. Then

$$\begin{array}{rcl} S_6 & = & 1 \ +2+4+8+16+32+\ldots \\ - \quad S_6' & = & -1+2-4+8-16+32-\ldots \\ \hline S_6 - S_6' & = & 0 \ +4+0+16+ \ 0 \ +64+\ldots \\ & = & 4(1+4+16+\ldots) = 4S_6'' \end{array} \tag{16}$$

Thus $S_6' = S_6 - 4S_6'' = -1 + \frac{4}{3} = \frac{1}{3}$.
Formally, we can interpret Equations (14) and (15) as limits:

$$S_6' = \lim_{s\to -2} \frac{1}{1-s} = \lim_{s\to 2} \frac{1}{1+s} = \frac{1}{3} \qquad \text{and} \qquad S_6'' = \lim_{s\to 4} \frac{1}{1-s} = \lim_{s\to 2} \frac{1}{1-s^2} = -\frac{1}{3}$$

If we denote $f_6 : s \mapsto \frac{1}{1+s}$ and $g_6 : s \mapsto \frac{1}{1-s^2}$, then we can interpret Calculation (16) as

$$f_1(s) - f_6(s) = \frac{1}{1-s} - \frac{1}{1+s} = \frac{2s}{1-s^2} = 2sg_6(s)$$

Then, by taking the limit as s tends to 2,

$$S_6 - S_6' = \lim_{s\to 2}(f_1(s) - f_6(s)) = \lim_{s\to 2} 2sg_6(s) = 4S_6''$$

5.4 Sum of cosines

So far, we have only dealt with numerical divergent series, but it is also possible to study functional divegent series, for instance:

$$t \mapsto \frac{1}{2} + \cos(t) + \cos(2t) + \cos(3t) + \dots$$

Harmonic analysis connoisseurs will recognize the Fourier series of the Dirac comb of period 2π and amplitude π:

$$C_{2\pi} : t \mapsto \pi \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n)$$

where $t \mapsto \delta(t - 2\pi n)$ is the Dirac mass centered on $2\pi n$. Indeed, since $C_{2\pi}$ is periodic of period 2π, we can write the partial sum of its Fourier series:

$$\forall N \in \mathbb{N} \qquad \forall t \in \mathbb{R} \qquad S_N(C_{2\pi})(t) = \sum_{n=-N}^{N} a_n(C_{2\pi}) e^{int}$$

and its Fourier coefficients

$$\forall n \in \mathbb{N} \qquad a_n(C_{2\pi}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{2\pi}(t) e^{-int} dt$$

Moreover, $S_N(C_{2\pi})$ converges to $C_{2\pi}$ as N goes to infinity. Then we have

$$\forall n \in \mathbb{N} \qquad a_n(C_{2\pi}) = \frac{1}{2} \int_{-\pi}^{\pi} \delta(t) e^{-int} dt = \frac{1}{2} e^{-in0} = \frac{1}{2}.$$

Thus we have

$$\forall t \in \mathbb{R} \qquad S_N(C_{2\pi})(t) = \frac{1}{2}\sum_{n=-N}^{N} e^{int} = \frac{1}{2} + \sum_{n=1}^{N} \cos(nt)$$

and we can write

$$\forall t \in \mathbb{R} \qquad C_{2\pi}(t) = \frac{1}{2} + \lim_{N\to+\infty}\sum_{n=1}^{N} \cos(nt) = \frac{1}{2} + \cos(t) + \cos(2t) + \cos(3t) + ...$$

Since for all $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, $C_{2\pi}(t) = 0$, we have the new numerical divergent series

$$\forall t \in \mathbb{R} \setminus 2\pi\mathbb{Z} \qquad \frac{1}{2} + \cos(t) + \cos(2t) + \cos(3t) + \cdots = 0$$

6 Annexes

6.1 Bernoulli numbers

We recall that $D_{2\pi} = \{z \in \mathbb{C}, |z| < 2\pi\}$ denotes the open disc of center 0 and radius 2π.

Proposition 6.1

There exists a sequence of complex numbers $(B_n)_{n\in\mathbb{N}}$ such that for all $t \in D_{2\pi}$,

$$\frac{t}{e^t - 1} = \sum_{n=0}^{+\infty} \frac{B_n}{n!} t^n$$

The sequence $(B_n)_{n\in\mathbb{N}}$ satisfies the recurrence relation

$$B_0 = 1 \qquad \text{and} \qquad \forall n \in \mathbb{N}^* \quad B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1} \binom{n+1}{k} B_k \tag{17}$$

The B_n's are called the **Bernoulli numbers**.

Proof : Set $f : \mathbb{C} \to \mathbb{C} \quad t \mapsto \frac{t}{e^t-1}$. The denominator of this fraction is equal to 0 if and only if there exists $k \in \mathbb{Z}$ such that $t = ik2\pi$. However, in the vicinity of 0, we have

$$\frac{t}{e^t - 1} \underset{\substack{t\to 0\\ t\neq 0}}{=} \frac{t}{t + o(t)} \underset{\substack{t\to 0\\ t\neq 0}}{=} \frac{1}{1 + o(1)}$$

thus $\lim_{t\to 0} f(t) = 1$ and 0 is a removable pole of f. Therefore, the set of poles of function f is $\mathcal{P} = \{ik2\pi, k \in \mathbb{Z}^*\}$, and f is defined on the domain $\mathbb{C} \setminus \mathcal{P}$. As a ratio of two holomorphic functions, f is a holomorphic function on $D_{2\pi}$. Hence f is an analytic function, which means that it admits a power series on $D_{2\pi}$, i.e. there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of complex numbers such that

$$\forall t \in D_{2\pi} \qquad \frac{t}{e^t - 1} = \sum_{n=0}^{+\infty} \frac{B_n}{n!} t^n$$

Since for any $t \in \mathbb{C}$, $e^t - 1 = \sum_{m=1}^{+\infty} \frac{t^m}{m!}$, we multiply both sides of last equation by $e^t - 1$, and we have the following Cauchy product

$$\forall t \in D_{2\pi} \qquad t = \left(\sum_{n=0}^{+\infty} \frac{B_n}{n!} t^n\right)\left(\sum_{m=0}^{+\infty} \frac{t^{m+1}}{(m+1)!}\right) = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} \frac{B_k}{k!(n-k+1)!}\right) t^{n+1}$$
$$= \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} \binom{n+1}{k} B_k\right) \frac{t^{n+1}}{(n+1)!}$$

By unicity of the Taylor series of $t \mapsto t$, we obtain $B_0 = 1$ and

$$\forall n \in \mathbb{N}^* \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad \Longrightarrow \quad B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k$$

which yields to the result. ■

Example 6.1
Using the recurrence relation in (17), we have $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, and so on.

6.2 Some results of complex analysis

In this annex, we remind some definitions and theorems of complex analysis. Proofs are admitted and can be found in any good textbook dealing with complex analysis. In this subsection, U denotes an open subset of $\mathbb{C}$.

Definition 6.1 (Holomorphic function)
A function $f : U \to \mathbb{C}$ is **holomorphic** on U if it is $\mathbb{C}$-differentiable, i.e. for all $z_0 \in U$, the limit $\lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is finite. In this case, f is of class $\mathcal{C}^\infty$ on U.

Definition 6.2 (Analytic function)
A function $f : U \to \mathbb{C}$ is **analytic** on U if it admits a Taylor series development at all point of U, i.e. for all $z_0 \in U$, there exists a sequence of complex numbers $(a_n)_{n \in \mathbb{N}}$ and an open set $V \subset U$ such that for all $z \in V$, $f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$.

Remark: By Morera's theorem, if f is a continuous function on U, then f is holomorphic on U if and only if f is analytic on U.

Theorem 6.2 (Analytic continuation theorem)
Let U an open and connex subset of $\mathbb{C}$, and f and f two analytic functions on U. If f and g coincide in the vicinity of a point of U, the $f = g$ on U.

Definition 6.3 (Removable singularity)
Let $a \in U$ and f a holomorphic function over $U \setminus \{a\}$. If f can be extended into a holomorphic function in the vicinity of a, then a is a **removable singularity** of f.

Definition 6.4 (Pole)
Let $a \in U$ and f a holomorphic function over $U \setminus \{a\}$. If there exists a finite sequence of complexe numbers $(a_{-1}, \dots, a_{-m})$ with $m \geq 1$ and $a_{-m} \neq 0$ such that function $z \mapsto f(z) - \sum_{k=1}^{m} \frac{a_{-k}}{(z-a)^k}$ has a removable singularity in a, then a is a **pole** of order (or multiplicity) m of f.

Definition 6.5 (Meromorphic function)
A function $f : U \to \mathbb{C}$ is **meromorphic** on U if there exists a locally finite subset A of U such that f is holomorphic on $U \setminus A$ and every point of A is a pole of f.

Theorem 6.3 (Holomorphy of parametric integral)
Let (X, μ) be a measured space and $f : U \times X \to \mathbb{C}$. For $s \in \mathbb{C}$, we set $F(s) = \displaystyle\int_X f(s,t) d\mu(t)$.

1) For all $s \in U$, $t \mapsto f(s,t)$ is integrable.

2) For almost all $t \in X$, $s \mapsto f(s,t)$ is holomorphic.

3) For all compact subset K of U, there exists $g \in L^1(X)$ such that for all $s \in K$ and for almost all $t \in X$, $|f(s,t)| \leq g(t)$.

Then F is holomorphic over U, and for all $s \in U$ and $n \in \mathbb{N}$, $F^{(n)}(s) = \displaystyle\int_X \frac{\partial^n f}{\partial s^n}(s,t) d\mu(t)$.

Theorem 6.4 (Series of meromorphic functions)
Let U be an open set of $\mathbb{C}$, and $(f_n)_{n\in\mathbb{N}}$ a sequence of functions $U \mapsto \mathbb{C}$.

1) Every f_n is a meromorphic function over U.

2) For all compact set K, there exists $N_K \in \mathbb{N}$ such that for all $n \geq N_K$, functions f_n have no poles in K and $\displaystyle\sum_{n \geq N_K} f_n$ uniformly converges over K.

Then $\sum f_n$ is meromorphic and we can take the derivative of the series term by term.

6.3 Properties of function Γ

We recall that P_0 denotes the half-plane $\{z \in \mathbb{C}, \mathrm{Re}(z) > 0\}$.

Proposition 6.5
For $s \in \mathbb{C}$, we set

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$$

The function Γ is defined and holomorphic on the domain P_0.

Proof : We aim at applying the Theorem of holomorphy of a parametric integral 6.3 to the function

$$(s,t) \mapsto t^{s-1} e^{-t} = e^{(s-1)\ln t} e^{-t}$$

1) For all $s \in P_0$, $t \mapsto e^{(s-1)\ln t} e^{-t}$ is integrable on $\mathbb{R}_+^*$.

2) For all $t \in \mathbb{R}_+^*$, $s \mapsto e^{(s-1)\ln t} e^{-t}$ is holomorphic on P_0.

3) Let K be a compact subset of P_0. There exists $(\varepsilon, M) \in \left(\mathbb{R}_+^*\right)^2$ such that for any $s \in K$, $\mathrm{Re}(s) \in [\varepsilon, M]$.

   - If $t \in]0,1]$,

$$\left| e^{(s-1)\ln t} e^{-t} \right| \leq e^{(\varepsilon-1)\ln t} = \frac{1}{t^{1-\varepsilon}}$$

- If $t \in [1, +\infty[$,

$$\left| e^{(s-1)\ln t} e^{-t} \right| \leq t^{M-1} e^{-t}$$

In both cases, the upper bound is integrable. We used the fact that for any $s \in \mathbb{C}$, $|e^s| = e^{\mathrm{Re}(s)}$.

Therefore, by the Theorem of holomorphy of a parametric integral, Γ is holomorphic on P_0. ∎

Proposition 6.6
For all $s \in P_0$,

$$\Gamma(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(s+n)} + \int_1^{+\infty} t^{s-1} e^{-t} dt$$

Moreover, Γ can be extended to $\mathbb{C} \setminus \mathbb{Z}^-$ with simple poles at $-n$ for $n \in \mathbb{N}$.

PROOF : We begin by breaking down the integral: for all $s \in P_0$,

$$\Gamma(s) = \int_0^1 t^{s-1} e^{-t} dt + \int_1^{+\infty} t^{s-1} e^{-t} dt$$

We aim at transforming the first integral into a series. For all $(s, t) \in P_0 \times [0, 1]$,

$$t^{s-1} e^{-t} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} t^{n+s-1}$$

In order to switch the integral and the sum, we need to apply Fubini's theorem. Note that for all $t > 0$, $t^s = |e^{s \ln t}| = e^{\mathrm{Re}(s)\ln t} = t^{\mathrm{Re}(s)}$. Then, for all $t \in]0, 1]$,

$$\sum_{n=0}^{+\infty} \left| \frac{(-1)^n}{n!} \right| |t^{n+s-1}| = t^{\mathrm{Re}(s)-1} \sum_{n=0}^{+\infty} \frac{t^n}{n!} = t^{\mathrm{Re}(s)-1} e^t$$

Since $\mathrm{Re}(s) > 0$, $\mathrm{Re}(s) - 1 > -1$ and function $t \mapsto t^{\mathrm{Re}(s)-1} e^t$ is integrable on $]0, 1]$. Hence

$$\int_0^1 \sum_{n=0}^{+\infty} \left| \frac{(-1)^n}{n!} t^{n+s-1} \right| dt < +\infty$$

Therefore, by Fubini's theorem, we can switch the integral and the sum:

$$\int_0^1 t^{s-1} e^{-t} dt = \int_0^1 \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} t^{n+s-1} dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \int_0^1 t^{n+s-1} dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left[\frac{t^{n+s}}{n+s} \right]_0^1 = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \frac{1}{n+s}$$

Therefore, for all $s \in P_0$,

$$\Gamma(s) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(s+n)} + \int_1^{+\infty} t^{s-1} e^{-t} dt \tag{18}$$

Then, let show that $f : s \mapsto \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(n+s)}$ is meromorphic on $\mathbb{C}$ and has simple poles at $-n$ with $n \in \mathbb{N}$.

1) For all $n \in \mathbb{N}$, $f_n : s \mapsto \frac{(-1)^n}{n!(n+s)}$ is meromorphic on $\mathbb{C}$ with one simple pole at $-n$.

2) Let K be a compact set of $\mathbb{C}$. There exists $N_K \in \mathbb{N}$ such that $K \subset \overline{D(0, N_K)}$. For all $n > N_K$, function f_n has no pole in K. Moreover, for all $s \in K$, $|n + s| \geq n - |s| \geq n - N_K$. Hence, for all $s \in K$, $|f_n(s)| \leq \frac{1}{n!(n - N_K)}$ thus $\sum_{n > N_K} f_n$ is normally convergent over K.

Therefore, by Theorem of series of meromorphic functions 6.4, f is a meromorphic function over $\mathbb{C}$ whose simple poles are non-positive integers. Using the proof of Proposition 6.5, we show that $s \mapsto \int_1^{+\infty} t^{s-1}e^{-t}dt$ is holomorphic on P_0. Therefore, the developped expression of Γ in Equation (18) establishes a meromorphic continuation of Γ over $\mathbb{C}$. Moreover, the Analytic continuation theorem 6.2 implies that this is the unique analytic continuation of Γ on the connex open set $\mathbb{C} \setminus \mathbb{Z}^-$. ■

Proposition 6.7
For all $k \in \mathbb{N}$,

$$\Gamma(s) \underset{s \to -k}{\sim} \frac{(-1)^k}{k!(s+k)}$$

PROOF : From Proposition 6.6, we have for all $s \in \mathbb{C} \setminus \mathbb{Z}^-$,

$$\Gamma(s) = \frac{(-1)^k}{k!(s+k)} + \sum_{\substack{n=0 \\ n \neq k}}^{+\infty} \frac{(-1)^n}{n!(s+n)} + \int_1^{+\infty} t^{s-1}e^{-t}dt$$

Since the last two terms converge to a finite value as s tends to $-k$, we find the expected result. ■

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