

# Periodic signals

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## Definition 0.1 (Periodic signal, fundamental period)

A signal  $x \in \mathcal{F}(\mathbb{R}, \mathbb{K})$  is **periodic** if there exists  $T > 0$  such that for any  $t \in \mathbb{R}$ ,  $x(t + T) = x(t)$ . The smallest  $T_0 > 0$  such that  $x(t + T_0) = x(t)$  for any  $t \in \mathbb{R}$  is the **fundamental period**. We denote  $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$  the subspace of periodic signals with period  $T_0$ .

### Remarks:

- If a signal  $x$  is periodic with period  $T_0$ , then it is with period  $kT_0$  for any  $k \in \mathbb{Z}$ . This is a cornerstone property intensively used in the next part.
- In other words, a signal  $x$  is periodic with period  $T$  if it is invariant by a pure delay of  $T$ , i.e.  $\tau_T(x) = x$ .

## Definition 0.2 (Fundamental frequency, fundamental impulse)

Let  $x \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$  be a periodic signal with period  $T_0$ . The **fundamental frequency** of  $x$  is the number  $f_0 = \frac{1}{T_0}$ , and the **fundamental impulse** of  $x$  is the number  $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ .

## Definition 0.3 (Complex exponential, cosine)

The **complex exponential** of fundamental impulse  $\omega_0$ , amplitude  $A > 0$  and phase  $\varphi_0 \in [0, 2\pi[$  is the following signal in  $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{C})$ :

$$e_{\omega_0, A, \varphi_0} : t \mapsto A \exp(i(\omega_0 t + \varphi_0))$$

We simply denote  $e_{\omega_0} = e_{\omega_0, 1, 0}$  the complex exponential of amplitude  $A = 1$  and phase  $\varphi_0 = 0$ .

The **cosine** of fundamental impulse  $\omega_0$ , amplitude  $A > 0$  and phase  $\varphi_0 \in [0, 2\pi[$  is the following signal in  $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{C})$ :

$$c_{\omega_0, A, \varphi_0} : t \mapsto A \cos(\omega_0 t + \varphi_0)$$

We simply denote  $c_{\omega_0} = c_{\omega_0, 1, 0}$  the cosine of amplitude  $A = 1$  and phase  $\varphi_0 = 0$ .

### Remarks:

- With these definitions, we can check that the complex exponential and cosine of fundamental impulse  $\omega_0$  are periodic signals with period  $T_0 = \frac{2\pi}{\omega_0}$ .
- If a periodic signal with period  $T_0$  is the input of an LTI system, then the corresponding output is with period  $T_0$  as well. Indeed let an LTI system  $L$ , a periodic signal  $x$  with period  $T_0$ , et  $y = L(x)$  the corresponding output. Since LTI systems commute with pure delays,

$$\tau_{T_0}(y) = \tau_{T_0}(L(x)) = L(\tau_{T_0}(x)) = L(x) = y$$

thus  $y$  is also periodic with period  $T_0$ .

- As we are going to see in Example 0.1, we often deal in practice with signals of the form  $t \mapsto e_{\omega_0, A, \varphi_0}(t) \Upsilon(t)$  and  $t \mapsto c_{\omega_0, A, \varphi_0}(t) \Upsilon(t)$ , namely zero over  $]-\infty, 0[$  and oscillating over  $[0, +\infty[$ . These signals will be the matter of a following lecture.

If a signal  $x$  is periodic with period  $T_0$ , signal  $t \mapsto |x(t)|^2$  is clearly periodic with period  $T_0$  as well. Thus for any  $n \in \mathbb{N}^*$ ,

$$\int_{-nT_0}^{nT_0} |x(t)|^2 dt = 2n \int_0^{T_0} |x(t)|^2 dt$$

When  $n$  goes to  $+\infty$ , we note that a non-zero periodic signal has infinite energy. However, the average power of such a periodic signal  $x$  is:

$$P(x) = \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |x(u)|^2 du = \lim_{n \rightarrow +\infty} \frac{1}{2nT_0} \int_{-nT_0}^{nT_0} |x(u)|^2 du = \frac{1}{T_0} \int_0^{T_0} |x(u)|^2 du$$

We are going to define a subspace of  $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$  containing periodic signals with period  $T_0$  which are locally square integrable, i.e. they have a finite average power.

#### Lemma 0.1

Let  $x$  be a periodic signal with period  $T_0$ . For any  $a \in \mathbb{R}$ ,

$$\int_a^{a+T_0} x(t) dt = \int_0^{T_0} x(t) dt$$

**PROOF :** If  $a \in [0, T_0]$ , then  $T_0 \in [a, a + T_0]$ . By the change of variable  $t \mapsto t - T_0$ , we get

$$\int_a^{a+T_0} x(t) dt = \int_a^{T_0} x(t) dt + \int_{T_0}^{a+T_0} x(t) dt = \int_a^{T_0} x(t) dt + \int_0^a x(t + T_0) dt = \int_0^{T_0} x(t) dt$$

In general, let  $a \in \mathbb{R}$ . If  $b = a - \left\lfloor \frac{a}{T_0} \right\rfloor T_0$ , then  $b \in [0, T_0]$  and by the change of variable  $t \mapsto t - \left\lfloor \frac{a}{T_0} \right\rfloor T_0$ ,

$$\int_a^{a+T_0} x(t) dt = \int_b^{b+T_0} x(t) dt = \int_0^{T_0} x(t) dt \quad \blacksquare$$

This lemma indicates that the integral of a periodic signal with period  $T_0$  is identical on any interval of length  $T_0$ . Therefore, we can now define the subspace of signals with finite average power and define on this subspace a scalar product base on the average power, instead of the energy which is infinite.

#### Definition 0.4

We denote  $L_{T_0}^2(\mathbb{R}, \mathbb{K})$  the subspace of  $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$  containing the periodic signals with period  $T_0$  which are square integrable over  $[0, T_0]$ , i.e.

$$L_{T_0}^2(\mathbb{R}, \mathbb{K}) = \left\{ x \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K}), \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt < +\infty \right\}$$

#### Definition 0.5

We define a **scalar product / Hermitian product** over  $L_{T_0}^2(\mathbb{R}, \mathbb{K})$  by

$$\forall (x, y) \in L_{T_0}^2(\mathbb{R}, \mathbb{K})^2 \quad \langle x, y \rangle_{T_0} = \frac{1}{T_0} \int_0^{T_0} x(t) y^*(t) dt$$

From this scalar / Hermitian product, we can define the norm of a signal  $x$  to which we can connect the average power of the signal:

$$\forall x \in L^2_{T_0}(\mathbb{R}, \mathbb{K}) \quad P(x) = \|x\|_{T_0}^2 = \langle x, x \rangle_{T_0}$$

i.e.

$$\forall x \in L^2_{T_0}(\mathbb{R}, \mathbb{K}) \quad P(x) = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

#### Remarks:

- We defined these integrals over the interval  $[0, T_0]$  but according to the lemma, any interval of length  $T_0$  is suitable. In some cases, it is more interesting to exploit the symmetry of interval  $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$ , when we deal with odd or even signals for example.
- We can define cross-correlation and autocorrelation of periodic signals from this new scalar product.

#### Proposition 0.2

The autocorrelation of a periodic signal with period  $T_0$  is also a periodic signal with period  $T_0$ .

**PROOF :** Let  $x \in L^2_{T_0}(\mathbb{R}, \mathbb{K})$ .

$$\forall t \in \mathbb{R} \quad \gamma_x(t + T_0) = \langle x, \tau_{t+T_0}(x) \rangle = \langle x, \tau_t(x) \rangle = \gamma_x(t)$$

because the periodicity of  $x$  implies  $\tau_{t+T_0}(x) = \tau_t(\tau_{T_0}(x)) = \tau_t(x)$ . ■

Now we study the convolution of two non-zero periodic signals  $x$  and  $y$  with the same period  $T_0$ . Let  $t \in \mathbb{R}$ . Then signals  $u \mapsto y(t - u)$  and  $u \mapsto x(u)y(t - u)$  are also periodic with period  $T_0$ , thus

$$(x * y)(t) = \int_{-\infty}^{+\infty} x(u)y(t - u)du = \lim_{n \rightarrow +\infty} \int_{-nT_0}^{nT_0} x(u)y(t - u)du = \lim_{n \rightarrow +\infty} 2n \int_0^{T_0} x(u)y(t - u)du = +\infty$$

This result is not surprising, since the notions of energy and convolution are connected through correlation, and non-zero periodic signals have infinite energy. As for the scalar product, we have to adapt our definition of convolution.

#### Definition 0.6 (Circular convolution)

The **circular convolution**  $\otimes$  is a product in  $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$  defined by

$$\forall (x, y) \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})^2 \quad \forall t \in \mathbb{R} \quad (x \otimes y)(t) = \int_0^{T_0} x(u)y(t - u)du$$

#### Remarks:

- The circular convolution of two periodic signals  $x$  and  $y$  with period  $T_0$  is also periodic with period  $T_0$ . Indeed, for any  $t \in \mathbb{R}$ ,

$$(x \otimes y)(t + T_0) = \int_0^{T_0} x(u)y(t + T_0 - u)du = \int_0^{T_0} x(u)y(t - u)du = (x \otimes y)(t)$$

- As for scalar product and correlation, the integral of circular convolution can be defined on any interval of length  $T_0$ ,  $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$  for instance. **Aperiodic** signals, i.e. signals with no period, can be considered as signals with period  $T_0 = +\infty$ , yielding classical convolution:

$$(x \otimes y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du = (x * y)(t)$$

### Example 0.1

We go back to the RC circuit and we look for its response to the input  $V(t) = \sin(\omega_0 t)\Upsilon(t) = \cos\left(\omega_0 t - \frac{\pi}{2}\right)\Upsilon(t)$ , which is periodic over  $[0, +\infty[$ . We determine this response with two techniques developed so far: solving the governing differential equation and computing the convolution with the impulse response.

To abbreviate computations, we set  $\tau = RC$ , the time constant of the circuit. Recall that the solutions of the homogeneous differential equation are of the form  $u_c(t) = K \exp\left(-\frac{t}{\tau}\right)$ , with  $K \in \mathbb{R}$ . We look for a particular solution of the form  $u_c(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t)$  over  $[0, +\infty[$ . The derivative of such a function is  $u'_c(t) = A\omega_0 \cos(\omega_0 t) - B\omega_0 \sin(\omega_0 t)$ . Then the differential equation becomes over  $[0, +\infty[$ :

$$(A - B\tau\omega_0) \sin(\omega_0 t) + (A\tau\omega_0 + B) \cos(\omega_0 t) = \sin(\omega_0 t)$$

By identification, we get  $A - B\tau\omega_0 = 1$  and  $A\tau\omega_0 + B = 0$ , yielding

$$A = \frac{1}{1 + \tau^2\omega_0^2} \quad \text{and} \quad B = -\frac{\tau\omega_0}{1 + \tau^2\omega_0^2}$$

Thus we have the solution

$$u_c(t) = \begin{cases} 0 & \text{if } t < 0 \\ K \exp\left(-\frac{t}{\tau}\right) + \frac{1}{1 + \tau^2\omega_0^2} \sin(\omega_0 t) - \frac{\tau\omega_0}{1 + \tau^2\omega_0^2} \cos(\omega_0 t) & \text{if } t > 0 \end{cases}$$

Since  $u_c$  is continuous in  $t = 0$ ,

$$\lim_{t \rightarrow 0^-} u_c(t) = 0 = \lim_{t \rightarrow 0^+} u_c(t) = K - \frac{\tau\omega_0}{1 + \tau^2\omega_0^2}$$

for any  $t \in [0, +\infty[$ ,

$$u_c(t) = \frac{\tau\omega_0}{1 + \tau^2\omega_0^2} \exp\left(-\frac{t}{\tau}\right) + \frac{1}{1 + \tau^2\omega_0^2} \sin(\omega_0 t) - \frac{\tau\omega_0}{1 + \tau^2\omega_0^2} \cos(\omega_0 t)$$

Now we want to retrieve this result by the convolution of  $V(t)$  with impulse response  $h(t) = \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right)\Upsilon(t)$ . For  $t \in ]-\infty, 0[$ , the supports of  $u_c(u)$  and  $h(t-u)$  are disjoint, thus  $u_c(t) = (V * h)(t) = 0$ . For  $t \in [0, +\infty[$ ,

$$\begin{aligned} u_c(t) &= (V * h)(t) = \int_{-\infty}^{+\infty} \Upsilon(u) \sin(\omega_0 u) \frac{1}{\tau} \exp\left(-\frac{t-u}{\tau}\right) \Upsilon(t-u) du \\ &= \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right) \int_0^t \sin(\omega_0 u) \exp\left(\frac{u}{\tau}\right) du \end{aligned}$$

A double integration by parts (left to the interested reader) gives

$$\int_0^t \sin(\omega_0 u) \exp\left(\frac{u}{\tau}\right) du = \frac{\tau^2\omega_0}{1 + \tau^2\omega_0^2} + \frac{\tau}{1 + \tau^2\omega_0^2} \sin(\omega_0 t) \exp\left(\frac{t}{\tau}\right) - \frac{\tau^2\omega_0}{1 + \tau^2\omega_0^2} \cos(\omega_0 t) \exp\left(\frac{t}{\tau}\right)$$

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Hence we retrieve:

$$u_c(t) = \frac{\tau\omega_0}{1 + \tau^2\omega_0^2} \exp\left(-\frac{t}{\tau}\right) + \frac{1}{1 + \tau^2\omega_0^2} \sin(\omega_0 t) - \frac{\tau\omega_0}{1 + \tau^2\omega_0^2} \cos(\omega_0 t)$$

Note that  $\lim_{t \rightarrow +\infty} \exp\left(-\frac{t}{\tau}\right) = 0$  so that for  $t \gg \tau$ ,

$$u_c(t) \approx \frac{1}{1 + \tau^2\omega_0^2} \sin(\omega_0 t) - \frac{\tau\omega_0}{1 + \tau^2\omega_0^2} \cos(\omega_0 t)$$

This is called the **steady state** of the system. For  $t \approx \tau$ , we have to take into account the first term which is not negligible, corresponding to **transient state**, i.e. the transition between the off state for  $t < 0$  and the steady state for  $t \gg \tau$ .

Finally, note that the steady state is expressed as a function of  $\sin(\omega_0 t)$  and  $\cos(\omega_0 t)$ , namely the sine and cosine with the same period as the input sine  $\sin(\omega_0 t)$ . We develop this notion in the next part.