Diagonalization

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We have introduced in the previous lecture the vector space $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ of periodic signals with signal T_0 which are locally square integrable. Before studying in details the structure of this vector space, we review some notions of linear algebra, especially diagonalization, and apply these concepts to signal processing. We denote V a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

1 Linear algrebra review

Definition 1.1 (Linear combination, generating set)
Let $v_1, \dots, v_n$ be vectors of V. Let $\text{Span}(v_1, \dots, v_n)$ be the set of **linear combinations** of $v_1, \dots, v_n$, i.e.

$$\text{Span}(v_1, \dots, v_n) = \left\{ \sum_{j=1}^{n} \lambda_j v_j, (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \right\}$$

Set $(v_1, \dots, v_n)$ is a **generating set** of V if $V = \text{Span}(v_1, \dots, v_n)$, i.e. any vector of V can be written as a linear combination of $v_1, \dots, v_n$.

Definition 1.2 (Linearly independent, linearly dependent)
A set $(v_1, \dots, v_n)$ is **linearly independent** if for any $(\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n$, relation $\lambda_1 v_1 + \dots + \lambda_n v_n = 0_V$ implies $\lambda_1 = \dots = \lambda_n = 0$.
Otherwise, i.e. if there exists $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$ such $\lambda_1 v_1 + \dots + \lambda_n v_n = 0_V$, $(v_1, \dots, v_n)$ is **linearly dependent**.

Definition 1.3 (Basis, coordinates)
A set $(v_1, \dots, v_n)$ is a **basis** of V if it is a linearly independent generating set of V. In other words, any vector of V can be written uniquely as a linear combination of $v_1, \dots, v_n$. Scalars $\lambda_1, \dots, \lambda_n$ appearing in this linear combination are called the **coordinates** of the vector.

In the previous lectures, we have equipped the spaces $L^2(\mathbb{R}, \mathbb{K})$ and $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ with a scalar product or a Hermitian product. We enrich these notions with some algebraic definitions.

Definition 1.4 (Orthogonal set, orthonormal set)
A set $(v_1, \dots, v_n)$ is **orthogonal** if for any $(j, k) \in [\![1, n]\!]^2$, with $j \neq k$, $\langle v_j, v_k \rangle = 0$. Moreover, if for any $j \in [\![1, n]\!]$, $\|v_j\| = 1$, this set is **orthonormal**.

Remark: Using these definitions, we can easily prove that any orthogonal or orthonormal set is linearly independent.

Definition 1.5 (Orthogonal basis, orthonormal basis)
An **orthogonal basis** (resp. **orthonormal basis**) of V is an orthogonal (resp. orthonormal) set which is a basis of V.

Remark: The interest in orthonormal bases is to easily express the coordinates of a vector from its scalar product with the vectors of the basis. Indeed, if $(v_1, \dots, v_n)$ is an orthonormal basis of V, then any vector x of V can be written

$$x = \sum_{j=1}^{n} \langle x, v_j \rangle v_j$$

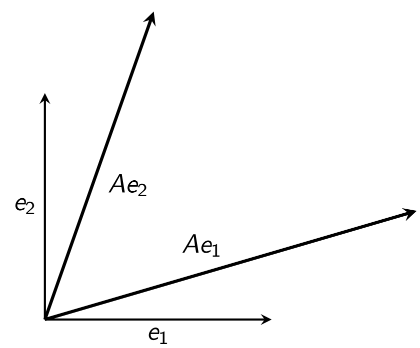
2 Diagonalization

A matrix provides the representation of a linear mapping within a vector space in a given basis, but this single data does not seem enough to "geometrically" interpret the behavior of this mapping. For instance, consider the following 2×2 matrix:

$$A = \begin{pmatrix} 1.64 & 0.48 \\ 0.48 & 1.36 \end{pmatrix}$$

We apply this matrix to vectors v_1 and v_2 of the standard basis of $\mathbb{R}^2$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Av_1 = \begin{pmatrix} 1.64 \\ 0.48 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad Av_2 = \begin{pmatrix} 0.48 \\ 1.36 \end{pmatrix}$$



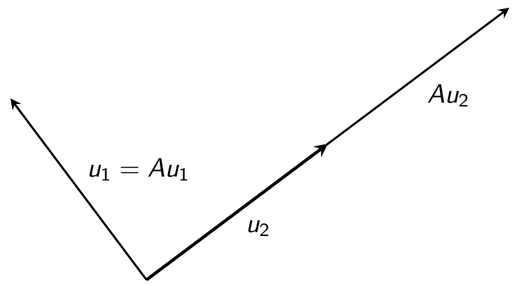
As shown on this figure, the transform rotates and scales these vectors, but it seems difficult to deduce a general behavior. The idea is to find cases for which the transform only scales the input vector. Thereby, λ is an **eigenvalue** of A if there exists a non-zero vector u such that $Au = \lambda u$. Vector u is called an **eigenvector** of A associated with eigenvalue λ. We possibly aim at determining a basis of eigenvectors, enabling the representation of the transform with a diagonal matrix only displaying the scalings in given directions. This process is called the **diagonalization** of A.
The first step consists in computing the eigenvalues, which are the roots of the characteristic polynomial:

$$\chi_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 1.64 - \lambda & 0.48 \\ 0.48 & 1.36 - \lambda \end{vmatrix} = (\lambda - 1.64)(\lambda - 1.36) - 0.48^2 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

Thus the eigenvalues of A are 2 et 1.
The second step consists in determining eigenvectors u_2 and u_1, respectively associated with eigenvalues 2 and 1. Moreover, we look for vectors with norm 1 to obtain, if possible, an orthonormal basis. To find eigenvector u_2, we write:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} 1.64x + 0.48y = 2x \\ 0.48x + 1.36y = 2y \end{cases} \quad \Longrightarrow \quad x = \frac{4}{3} y$$

so that $u_2 = \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$ is an eigenvector of A associated with eigenvalue 2 and with norm 1. With the same reasoning, we find $u_1 = \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}$ as eigenvector of A associated with eigenvalue 1 and with norm 1.



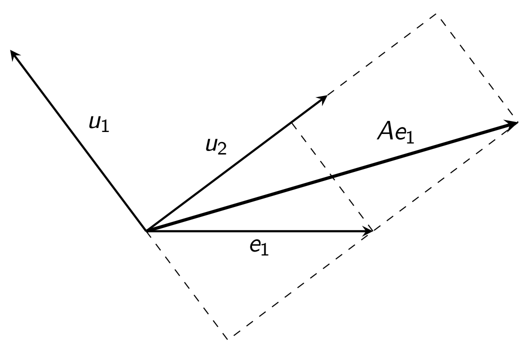
Computing $\langle u_2, u_1 \rangle = 0.8 \times (-0.6) + 0.6 \times 0.8 = 0$, we prove that (u_2, u_1) if an orthonormal basis of $\mathbb{R}^2$. We can write any vector $x \in \mathbb{R}^2$ as:

$$x = \langle x, u_2 \rangle u_2 + \langle x, u_1 \rangle u_1$$

along with its image by A:

$$Ax = \langle x, u_2 \rangle Au_2 + \langle x, u_1 \rangle Au_1 = 2\langle x, u_2 \rangle u_2 + \langle x, u_1 \rangle u_1$$

For instance e_1 can be written $e_1 = 0.8u_2 - 0.6u_1$, and its image $Ae_1 = 1.6u_2 - 0.6u_1$.



We can see on this figure that u_2 coordinate of vector e_1 is multiplied by 2, while the u_1 coordinate is unchanged (multiplied by 1).

3 Application to signal processing

We apply these ideas to our study of signals and systems by considering the signal vector space $\mathcal{F}(\mathbb{R}, \mathbb{K})$. In this case, LTI systems correspond to matrix A (since they are linear), and input signals are arguments of this mapping. In this context, we talk about **eigenfunctions** instead of eigenvectors to insist on the fact that $\mathcal{F}(\mathbb{R}, \mathbb{K})$ is a space containing functions. The following proposition provides a very important result about some eigenfunctions of an LTI system.

> **Proposition 3.1**
> For any $\omega \in \mathbb{R}$, complex exponential $e_\omega : \mathbb{R} \to \mathbb{C} \quad t \mapsto e^{i\omega t}$ is an eigenfunction of any LTI system. In other words, if L is an LTI system, then there exists $H(\omega) \in \mathbb{C}$ such that $L(e_\omega) = H(\omega)e_\omega$.

PROOF : We set $\omega \in \mathbb{R}$ and L an LTI system of impulse response $h = L(\delta)$. Then for any $t \in \mathbb{R}$,

$$L(e_\omega)(t) = (e_\omega * h)(t) = \int_{-\infty}^{+\infty} h(u)e^{i\omega(t-u)}du = e^{i\omega t}\int_{-\infty}^{+\infty} h(u)e^{-i\omega u}du$$

thus $L(e_\omega) = H(\omega)e_\omega$ with $H(\omega) = \displaystyle\int_{-\infty}^{+\infty} h(u)e^{-i\omega u}du$, therefore e_ω is an eigenfunction of L. ■

Remarks:

- **WARNING**: the complex exponentials are some eigenfunctions but not necessarily all the eigenfunctions of a given LTI system. For instance, consider the complex-valued differential system $D : \mathcal{F}(\mathbb{R},\mathbb{C}) \to \mathcal{F}(\mathbb{R},\mathbb{C}) \quad x \mapsto x'$. Then any complex number $s = a + ib \in \mathbb{C}$ is an eigenvalue of D, whose associated eigenfunctions $Ke^{st} = Ke^{at}e^{ibt}$ are the solutions of the differential equation: $D(x) = x' = sx$. In this example, complex exponential e_ω corresponds to $a = 0$ and $b = \omega$.
- We will study in details the expression of the eigenvalue $H(\omega)$ in the lecture about Fourier transform.
- We have seen in the previous lecture that if we input a sine or a cosine into the RC circuit, we obtain an ouput which is a linear combination of sine and cosine with the same fundamental impulse. This result is now explained with this proposition. We generalize this idea in the next lecture.