LTI systems and convolution

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1 Linear time-invariant systems

So far, we have studied two signals and their respective response: the step response associated with Heaviside step function, and the impulse response associated with Dirac delta function. Yet, we do not know a general method to exhibit the system response of any input. It is impossible to answer this question in general, and we need to make assumptions on the studied system. This is why we restrict our study to the class of **linear time-invariant** (LTI) systems.

Definition 1.1
Let L be a system from $\mathcal{F}(\mathbb{R},\mathbb{K})$ to $\mathcal{F}(\mathbb{R},\mathbb{K})$.

(i) L is a **linear system** if it is a linear mapping:

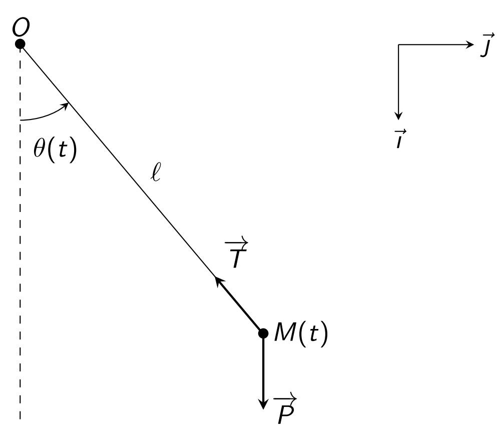
$$\forall (x_1,x_2)\in\mathcal{F}(\mathbb{R},\mathbb{K})^2 \quad L(x_1+x_2)=L(x_1)+L(x_2) \qquad \forall x\in\mathcal{F}(\mathbb{R},\mathbb{K}) \quad \forall\alpha\in\mathbb{K} \quad L(\alpha x)=\alpha L(x)$$

(ii) For any $a\in\mathbb{R}$, the **pure delay** or **shifting** system is defined as $\tau_a : x\mapsto x_a$, with $x_a : t\mapsto x(t-a)$. L is a **time-invariant system** if it commutes with any pure delay, i.e. for any $a\in\mathbb{R}$, $L\circ\tau_a=\tau_a\circ L$. In other words, if the input is delayed by a, the corresponding output is delayed by a as well.

(iii) **Linear time-invariant** (LTI) systems satisfy both properties (i) and (ii).

(iv) L is a **causal system** if for any input x and any $t\in\mathbb{R}$, the instantaneous output $L(x)(t)$ only depends on prior values of x, i.e. values $x(t-\tau)$ for $\tau\geq 0$. These systems are the ones met in practice, since systems cannot predict future input, even if computations can exhibit non-causal systems.

Example 1.1
The RC circuit and the mobile ponctual object, and more generally any system governed by a linear differential equation, are linear systems. The pendulum is an example of non-linear system.



The shaft of an engine is represented by a fixed point O in the vertical plane. We attach to this shaft a rigid rod of length ℓ and negligible mass, and at the other end of this rod a ponctual object M of mass m. Quantity $\theta(t)$ represents the angle $\left(\vec{\imath}, \overrightarrow{OM}(t)\right)$ between the vectical axis and the rode at time t. The engine in O can provide an acceleration $a_O(t)$ to the system. The second law of motion implies, among others, the equation:

$$\theta''(t) + \frac{mg}{\ell}\sin(\theta(t)) = \frac{a_O(t)}{\ell}$$

Choosing $x(t) = \dfrac{a_O(t)}{\ell}$ as input and $y(t) = \theta(t)$ as output, we get the differential equation governing the system:

$$y''(t) + \frac{mg}{\ell}\sin(y(t)) = x(t) \tag{P_1}$$

Since this differential equation is non-linear, the system is non-linear as well. However, output $y(t) = 0$ for any $t \in \mathbb{R}$ is a solution of the homogeneous equation corresponding to an equilibrium: the rod staying put and vertical. Applying small variations around this equilibrium, we can write $y(t) \approx 0$ and $\sin(y(t)) \approx y(t)$ allowing the approximation of equation (P_1) by the following linear differential equation:

$$y''(t) + \frac{mg}{\ell}y(t) = x(t) \tag{P_2}$$

We can generalize this principle to the study of any non-linear system by looking for an equilibrium and linearizing the governing equation around it, boiling down to the study of a linear system.
Likewise, systems governed by linear differential equations are time-invariant. In practice, no system is time-invariant since they wear out over time, causing a variation of their response to the same input. However we study systems on short enough periods of time to suppose them time-invariant.

We are going to perform a computation showing how we can explicitely relate the output of an LTI system L to its input and to its impulse response $h = L(\delta)$. Let x denote the input and $y = L(x)$ the corresponding output. For any $t \in \mathbb{R}$, using the following property

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(\tau - t)d\tau = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

we obtain by linearity and time-invariance of L:

$$y(t) = L(x)(t) = \int_{-\infty}^{+\infty} x(\tau)L\left[\delta(t-\tau)\right]d\tau = \int_{-\infty}^{+\infty} x(\tau)L(\delta)(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

Thereby, the only knowledge of the input and impulse response of a system allows us to explicitely determine the corresponding output. This operation is called **convolution**.

2 Convolution

Definition 2.1 (Convolution)
Convolution $*$ defines a product in $\mathcal{F}(\mathbb{R}, \mathbb{K})$ by

$$\forall (x, y) \in \mathcal{F}(\mathbb{R}, \mathbb{K})^2 \qquad \forall t \in \mathbb{R} \qquad (x * y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du$$

The Dirac delta function δ is the identity element of the convolution, i.e. for any signal x, $x * \delta = x$.

Definition 2.2 (Convolution system)
A **convolution system** is a mapping from $\mathcal{F}(\mathbb{R},\mathbb{K})$ to $\mathcal{F}(\mathbb{R},\mathbb{K})$ of the form $x \mapsto x * h$, where h is a given function.

Remarks:

- The computation performed in the previous paragraph shows that an LTI system is a convolution system, where h appearing in the convolution is the impulse response of this system. By their definition, convolution systems provide an explicit expression of the output as a function of the input, which is thus also true for LTI systems. Conversely, any convolution system is an LTI system, as proved by properties (i) and (iii) of Proposition 2.1.

- Let L be an LTI system, thus a convolution system, with impulse response h. Let an input x and corresponding output $y = L(x) = x * h$. From the definition of convolution:

$$\forall t \in \mathbb{R} \qquad y(t) = (x*h)(t) = \int_{-\infty}^{+\infty} x(u)h(t-u)du = \int_{-\infty}^{+\infty} x(t-u)h(u)du$$

  System L is causal if the instantaneous output $y(t)$ only depends on prior input $x(t-u)$ with $u \geq 0$. It implies the following sufficient condition:

  If for any $t < 0$, $h(t) = 0$, then L is a causal system.

Example 2.1
Knowing the impulse response h of the RC circuit, we can write:

$$\forall x \in \mathcal{F}(\mathbb{R},\mathbb{K}) \qquad y = L(x) = x * h \qquad \text{with} \qquad h : t \mapsto \frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\Upsilon(t)$$

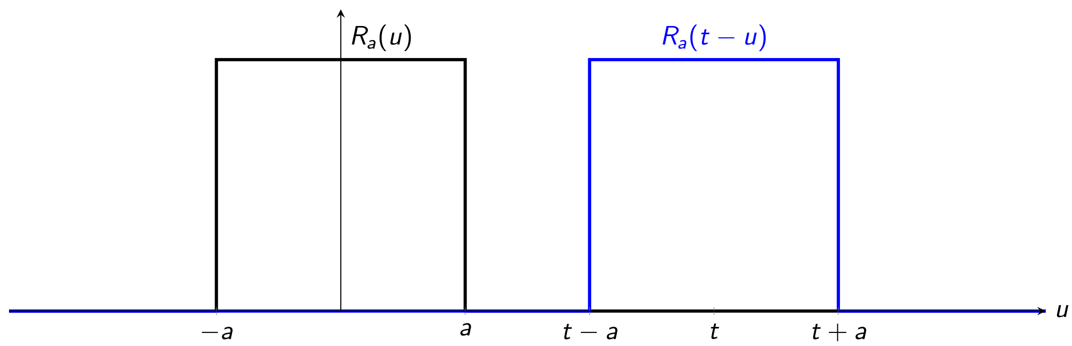
Example 2.2
Let a signal x and $a \in \mathbb{R}$. Then $x * \delta_a = x_a = \tau_a(x)$ with $x_a : t \mapsto x(t-a)$ the pure delay of x by a. Indeed, for any $t \in \mathbb{R}$,

$$(x*\delta_a)(t) = \int_{-\infty}^{+\infty} x(u)\delta_a(t-u)du = \int_{-\infty}^{+\infty} x(u)\delta(t-u-a)du = x(t-a)$$

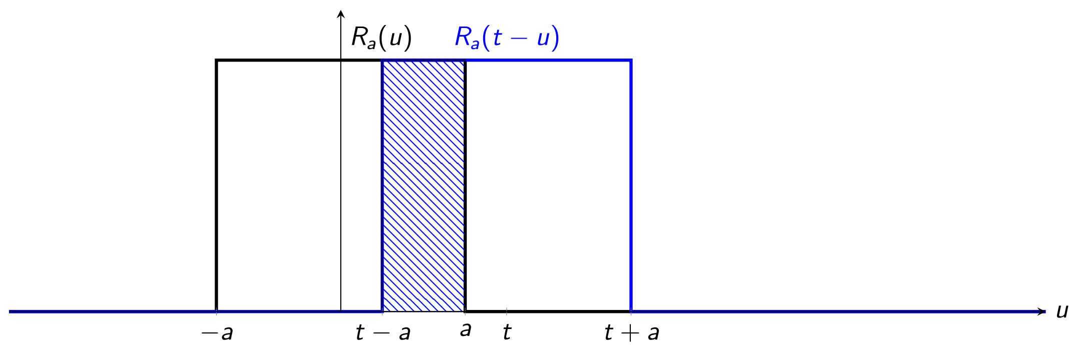
Thus the convolution of x by δ_a is the delayed signal $x_a = \tau_a(x)$. This is consistent with τ_a being an LTI system of impulse response $\tau_a(\delta) = \delta_a$, corresponding to the convolution by δ_a.

Example 2.3
We present a graphical method to compute the convolution. We look for the convolution T_a of rectangular signal $R_a = \chi_{[-a,a]}$ by itself, with $a > 0$. Let $t \in \mathbb{R}$. We treat this computation in different cases:

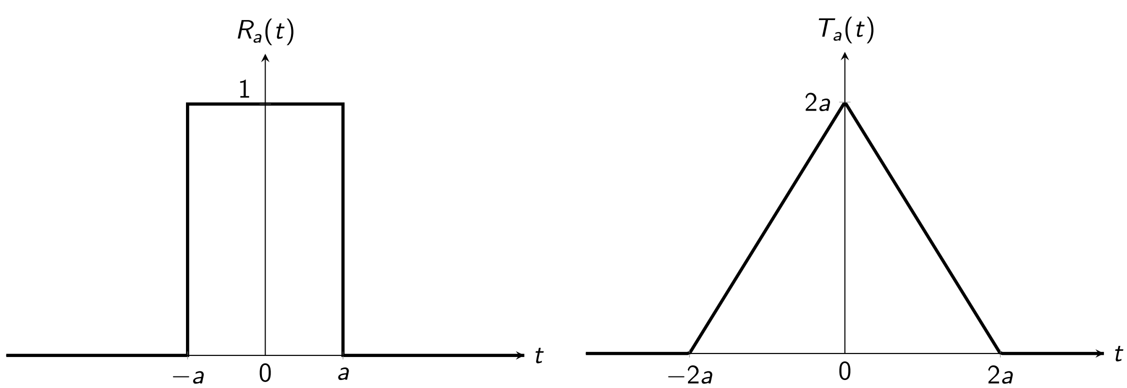
- If $|t| > 2a$ then the supports of functions $u \mapsto R_a(u)$ and $u \mapsto R_a(t-u)$ are disjoint, and the area of their common hypograph is $T_a(t) = 0$.



- If $t \in [0, 2a]$, then $u \mapsto R_a(u)$ and $u \mapsto R_a(t-u)$ have interval $[t-a, a]$ as common support, thus the area of their common hypograph is $T_a(t) = (a - (t-a)) \times 1 = 2a - t$.



- If $t \in [-2a, 0]$, then $u \mapsto R_a(u)$ and $u \mapsto R_a(t-u)$ have interval $[-a, t+a]$ as common support, thus the area of their common hypograph is $T_a(t) = (t + a - (-a)) \times 1 = t + 2a$.



Proposition 2.1

Convolution satisfies the following properties:

(i) it is bilinear: for any signals x, y and z, and any scalar $\alpha \in \mathbb{K}$, $x * (y + \alpha z) = (x * y) + \alpha(x * z)$, same for the first component;

(ii) it is associative and commutative: for any x, y and z, $x * (y * z) = (x * y) * z$ and $x * y = y * x$;

(iii) it commutes with pure delays: for any $a \in \mathbb{R}$, $\tau_a(x * y) = \tau_a(x) * y = x * \tau_a(y)$;

(iv) integration: for any signals x and y,

$$\int_{-\infty}^{+\infty}(x*y)(t)dt = \left(\int_{-\infty}^{+\infty} x(t)dt\right)\left(\int_{-\infty}^{+\infty} y(t)dt\right)$$

(v) differentiation: for any signals x and y, $(x*y)' = x'*y = x*y'$.

Proof : (i) Bilinearity results from the linearity of the integral. Indeed, for any $t \in \mathbb{R}$,

$$\begin{aligned}(x*(y+\alpha z))(t) &= \int_{-\infty}^{+\infty} x(u)\left[(y+\alpha z)(t-u)\right]du = \int_{-\infty}^{+\infty} x(u)y(t-u)du + \alpha\int_{-\infty}^{+\infty} x(u)z(t-u)du \\ &= (x*y)(t) + \alpha(x*z)(t)\end{aligned}$$

(ii) For any $t \in \mathbb{R}$,

$$(x*(y*z))(t) = \int_{-\infty}^{+\infty} x(u)(y*z)(t-u)du = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)y(v)z(t-u-v)dvdu$$

and

$$((x*y)*z)(t) = \int_{-\infty}^{+\infty}(x*y)(u)z(t-u)du = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(v)y(u-v)z(t-u)dudv$$

By the change of variable $(u,v) \mapsto (v, u+v)$ in the first expression, we note that both expressions are equal, hence the associativity.
By the change of variable $u \mapsto t-u$, we obtain, for any $t \in \mathbb{R}$,

$$(x*y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du = \int_{-\infty}^{+\infty} x(t-u)y(u)du = \int_{-\infty}^{+\infty} y(u)x(t-u)du = (y*x)(t)$$

(iii) We use the result of Example 2.2 with associativity and commutativity of convolution to write:

$$\tau_a(x*y) = (x*y)*\delta_a = x*(y*\delta_a) = x*\tau_a(y) = (x*\delta_a)*y = \tau_a(x)*y$$

(iv) We have:

$$\int_{-\infty}^{+\infty}(x*y)(t)dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)y(t-u)dudt$$

By the change of variable $(t,u) \mapsto (u, t-u)$ and by Fubini's theorem,

$$\int_{-\infty}^{+\infty}(x*y)(t)dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)y(v)dudv = \left(\int_{-\infty}^{+\infty} x(t)dt\right)\left(\int_{-\infty}^{+\infty} y(t)dt\right)$$

(v) Let $t \in \mathbb{R}$. By the theorem of differentiation under the integral sign, we obtain:

$$(x*y)'(t) = \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}\left[x(u)y(t-u)\right]du = \int_{-\infty}^{+\infty} x(u)y'(t-u)du = (x*y')(t)$$

Then we get equality $(x*y)' = x'*y$ by commutativity of the convolution. ■

Remarks:

- With properties (i) and (iii), we show that convolution systems are linear and commute with pure delays. Thereby, the LTI systems are exactly the convolution systems.

- With property (ii), we can define chains of LTI systems. Indeed, if $L_1, \dots, L_n$ are LTI systems with respective impulse responses $h_1, \dots, h_n$, setting $L = L_1 \circ \cdots \circ L_n$, then the impulse response of L is $h = h_1 * \cdots * h_n$. For instance, we can chain pure delays, providing, for any $(a, b) \in \mathbb{R}^2$, the identity $\delta_a * \delta_b = \delta_{a+b}$.

- By recursion, property (v) can be generalized to the n-th derivative of the convolution:

$$\forall n \in \mathbb{N}^* \qquad \forall (x, y) \in \mathcal{F}(\mathbb{R}, \mathbb{K})^2 \qquad (x * y)^{(n)} = x^{(n)} * y = x * y^{(n)} = x^{(k)} * y^{(\ell)}$$

  for $k + \ell = n$. Then we can define, for any $n \in \mathbb{N}^*$, the n-th derivative of Dirac delta function:

$$\forall x \in \mathcal{F}(\mathbb{R}, \mathbb{K}) \qquad x * \delta^{(n)} = x^{(n)} * \delta = x^{(n)}$$

  Thereby, distribution $\delta^{(n)}$ corresponds to the linear form $f \mapsto f^{(n)}(0)$. In particular, distribution δ' is the impulse response of differential system D.

- By property (v), convolution systems, and incidentally LTI systems, commute with differentiation. Therefore, the impulse response of any LTI system is the derivative of its step response.