Analog signals

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Version 1.0

The objective of this chapter is to provide the basic definitions necessary for the rest of this work. In the first section, we define analog signals and systems. In a second time, we introduce Heaviside step function and step response. The third part introduces the notion of Dirac delta function and impulse response. In the fourth section, we deal with the particular class of linear time-invariant systems, along with the notion of convolution. The fifth part discusses correlation. Finally the sixth section introduces periodic signals, basis for the next chapter.

1 Analog signals and systems

To clearly fix our study, we precisely define the notions of analog signal and systems. The digital counterpart will be treated in an upcoming chapter.

Definition 1.1 (Analog signal)
An **analog signal** is a function from $\mathbb{R}^n$, where $n \in \mathbb{N}^*$, to $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote $\mathcal{F}(\mathbb{R}^n, \mathbb{K})$ the vector space of analog signals.

Remarks:

- The vector space structure implies that we can add two signals or multiply a signal by a scalar, which will prove very useful in the following.
- To simplify our study, we restrict the image space to $\mathbb{R}$ or $\mathbb{C}$, although it is possible to give a more general definition dealing with a finite dimensional space. The definitions and properties presented in this chapter can be easily generalized to this perspective.
- In the following chapters, unless stated otherwise, we are interested in **univariate** analog signals, corresponding to $n = 1$.

Example 1.1
Here is a non-exhaustive list of analog signals examples met in various fields:

- the classics of signal processing: sound, speech, image, video;
- in many fields of physics, physical quantities can be studied as analog signals: in electronics, the voltage or the intensity of an electrical component, in mechanics, the position, momentum or kinetic energy of a mechanical system, in thermodynamics, the temperature or pressure in a given volume;
- in chemistry and biology, the concentration of a chemical species, the body temperature, arterial pressure, heartbeat;
- in finance and economics: the price of a commodity, a currency exchange rate, unemployment rate, inflation.

Definition 1.2 (Analog system, input, output)
An **analog system** is a mapping from $\mathcal{F}(\mathbb{R}^n, \mathbb{K})$ to $\mathcal{F}(\mathbb{R}^n, \mathbb{K})$. The argument signal of this system is called the **input signal**. The image signal is called the **output signal** or **response**. A system can be represented by the following block diagram:

$$\xrightarrow{\begin{array}{c}\text{Input}\\ x \in \mathcal{F}(\mathbb{R}^n, \mathbb{K})\end{array}} \boxed{\begin{array}{c}\text{System}\\ L : \mathcal{F}(\mathbb{R}^n, \mathbb{K}) \to \mathcal{F}(\mathbb{R}^n, \mathbb{K})\end{array}} \xrightarrow{\begin{array}{c}\text{Output}\\ y = L(x) \in \mathcal{F}(\mathbb{R}^n, \mathbb{K})\end{array}}$$

Example 1.2
We can associate the signals of the previous example with analog systems:

- in signal processing: sensors, converters, filters;
- in physics: electrical circuits, mechanical systems, thermodynamic systems;
- in biology: the human body;
- in economics: a financial market.

Remark: The expression $y = L(x)$ is misleading because it implies that knowing the input x, we can easily deduce the output y, which is generally not true. Indeed, in many occasions, y is the solution of a differential equation governing the system, whose the right member is x or a function of x. Thereby, L rarely provides an explicit definition of y as a function of x. In the following sections, we study analog system output corresponding to some particular input. Then, we develop a method to explicitly express y as a function of x for a particular class of systems.

2 Heaviside step function, step response

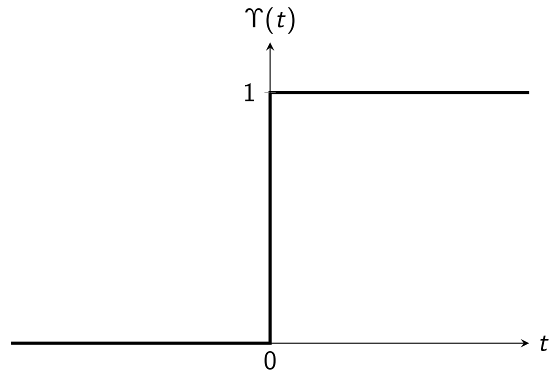
The first example of analog signal frequently appears in the study of electrical circuits, where at a moment $t = 0$, the circuit switches from an off state associated with value 0 to an on state associated with value 1. This signal is the Heaviside step function [1].

Definition 2.1 (Heaviside step function)
The **Heaviside step function** is the signal $\Upsilon \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ defined by:

$$\forall t \in \mathbb{R} \qquad \Upsilon(t) = \begin{cases} 0 & \text{si } t \in]-\infty, 0[\\ 1 & \text{si } t \in [0, +\infty[\end{cases}$$

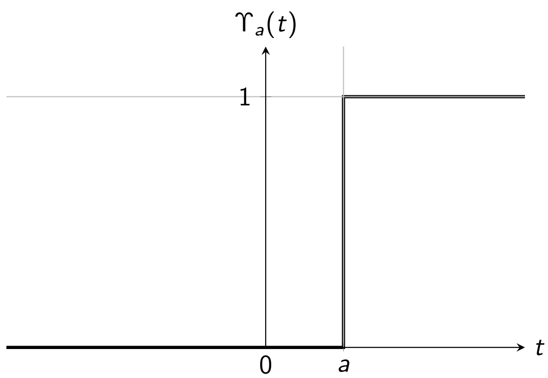
Using the notation for characteristic functions, we can write $\Upsilon = \chi_{[0,+\infty[}$ as well.

[1] Oliver Heaviside (1850-1925), British physicist

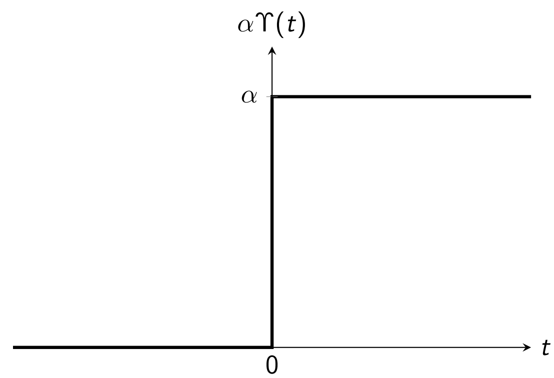


Remarks:

- An important property of Υ is the discontinuity in 0, which will cause problems when we study its differentiability in the next section.
- The Heaviside step function can be generalized by defining, for any $a \in \mathbb{R}$, the step centered in a: $\Upsilon_a : t \mapsto \Upsilon(t-a)$. By linearity, we can also define for any $\alpha \in \mathbb{R}$ the weighted step $\alpha\Upsilon : t \mapsto \alpha\Upsilon(t)$ which takes value α over $[0, +\infty[$.



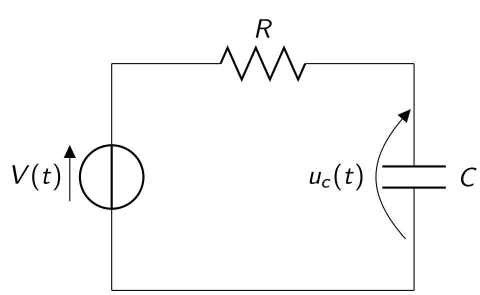
Delayed step function



Weighted step function

Definition 2.2 (Step response)
The **step response** of a system is the output corresponding to the Heaviside step function as input.

Example 2.1
We consider the following RC circuit as system:



We denote R the resistance and C the capacity. The considered input is the voltage $V(t)$ of the source, and the output is the voltage $u_c(t)$ of the capacitor. This electrical system is governed by the following differential equation:

$$RCu_c'(t) + u_c(t) = V(t)$$

We add the physical constraint of voltage $u_c(t)$ being continuous over time. To determine the step response of this system, we have to solve this differential equation with $V = \Upsilon$. On one hand, the corresponding homogeneous differential equation

$$RCu_c'(t) + u_c(t) = 0 \qquad \Longleftrightarrow \qquad u_c'(t) + \frac{1}{RC}u_c(t) = 0$$

admits solutions of the form $u_c(t) = K\exp\left(-\dfrac{t}{RC}\right)$, with $K \in \mathbb{R}$. Since Υ is only differentiable over $\mathbb{R}^*$, we first look for a particular solution over $\mathbb{R}^*$, that we can then extend. It is clear that the derivative of restriction $\Upsilon_{|\mathbb{R}^*}$ is the zero function $0_{\mathbb{R}^*}$ over $\mathbb{R}^*$, making $\Upsilon_{|\mathbb{R}^*}$ a particular solution of this equation. Because of the discontinuity in 0, we start with two separate solutions over $]-\infty, 0[$ and $]0, +\infty[$:

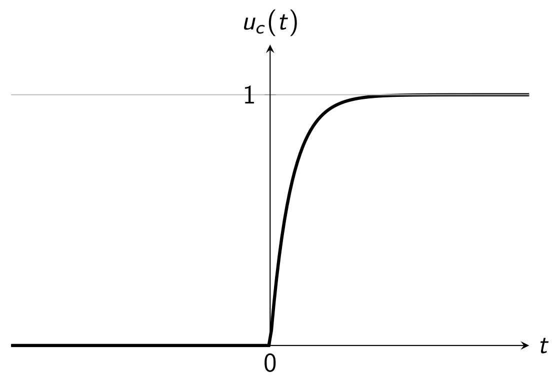
$$u_c(t) = \begin{cases} K_1 \exp\left(-\frac{t}{RC}\right) & \text{if } t < 0 \\ 1 + K_2 \exp\left(-\frac{t}{RC}\right) & \text{if } t > 0 \end{cases}$$

Now we determine constants K_1 and K_2. The electrical circuit is off for $t \in]-\infty, 0[$ and we can assume that the capacitor is initially uncharged, implying $u_c(t) = 0$ for $t < 0$, thus $K_1 = 0$. Using the continuity of $u_c(t)$ in $t = 0$,

$$\lim_{t \to 0^-} u_c(t) = 0 = \lim_{t \to 0^+} u_c(t) = 1 + K_2$$

yielding to $K_2 = -1$. We conclude that the step response of this RC circuit is:

$$u_c(t) = \left(1 - \exp\left(-\frac{t}{RC}\right)\right)\Upsilon(t)$$



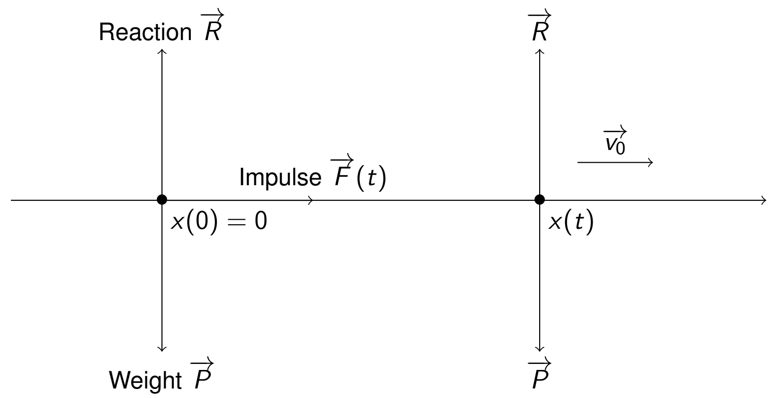
3 Dirac delta function, impulse response

3.1 Heuristic introduction

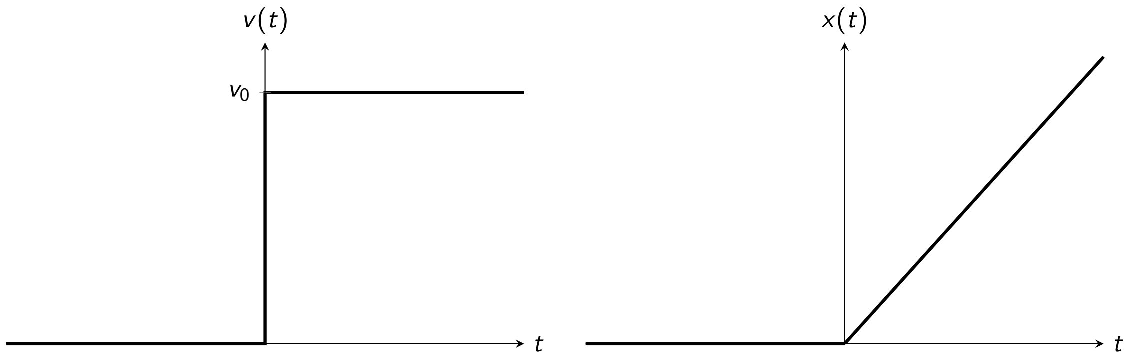
a Preliminary example

As we have seen in the previous section, the Heaviside step function is used to model the classical behavior of an electrical circuit. In mechanics, an impulse is usually applied on a mechanical system, to give him momentum for example. We detail this idea in the following example.

Example 3.1
Consider a Galilean reference and, within it, a ponctual mobile object of mass m, lying on a flat and horizontal support.



We neglect any spurious force (friction, rolling, sliding, etc.). To simplify our study, we assume that the object can only move in one direction, and we denote $x(t)$ its instantaneous position at given time t with respect to a defined origin. This position is the response of the system. We define $v(t) = x'(t)$ its instantaneous speed and $a(t) = v'(t) = x''(t)$ its instantaneous acceleration. We assume that the object is initially at rest, and weight and reaction, which are the only forces applied to this system, compensate, so that for any $t < 0$, $a(t) = v(t) = 0$ et $x(t) = 0$ is the initial position of the resting object used as origin. At time $t = 0$, we apply an impulse on the object, corresponding to a force $F(t)$ giving the direction of motion of the object, so that it instantaneously acquires speed $v_0 > 0$. For $t > 0$, no additional force is applied, so that the object is in uniform linear motion with speed $v_0 > 0$, i.e. $v(t) = v_0\Upsilon(t)$. The position, response to the input $F(t)$, is then $x(t) = v_0 t\Upsilon(t)$.

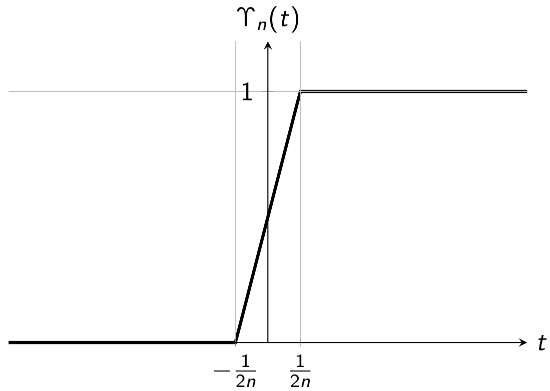


By the second law of motion, we have $ma(t) = mv'(t) = F(t)$, thus $F(t) = mv_0\Upsilon'(t)$. This example shows the necessity to carefully define the derivative of step function $\Upsilon(t)$.

b Approximating the step function with a sequence of functions

As seen in Example 2.1, function Υ is differentiable over $\mathbb{R}^*$ and for any $t \in \mathbb{R}^*$, $\Upsilon'(t) = 0$. The main issue is the discontinuity of Υ in 0, making it non differentiable at this point. First, we try to interpret this derivative by approximating Υ with a sequence of functions $(\Upsilon_n)_{n\in\mathbb{N}^*}$, and studying the behavior of the sequence of derivative functions $(\Upsilon'_n)_{n\in\mathbb{N}^*}$. We define, for any $n \in \mathbb{N}^*$, the piecewise linear function:

$$\Upsilon_n : \mathbb{R} \to \mathbb{R} \qquad t \mapsto \begin{cases} 0 & \text{if } t < -\dfrac{1}{2n} \\ nt + \dfrac{1}{2} & \text{if } -\dfrac{1}{2n} < t < \dfrac{1}{2n} \\ 1 & \text{if } t > \dfrac{1}{2n} \end{cases}$$

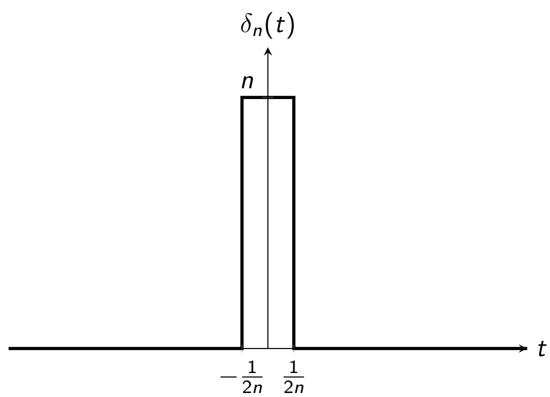


This sequence converges pointwise to Υ, i.e. for any $t \in \mathbb{R}$, $\Upsilon(t) = \lim_{n\to+\infty} \Upsilon_n(t)$. Note that for n large enough, function Υ_n is a more realistic model of the state transition from off to on by eliminating the discontinuity in 0 Recall that in order to let a limit function inherit the regularity properties (continuity, differentiability, ...) of the elements of the sequence, the convergence has to be uniform, in other words for $\varepsilon > 0$ arbitrarily small, the exist an index $N \in \mathbb{N}^*$ such that for any $n > N$, the graphical representation of Υ_n is "stuck" between the representations of $\Upsilon - \varepsilon$ and $\Upsilon + \varepsilon$, which can be mathematically translated:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N}^* \quad \forall n > N \quad \forall t \in \mathbb{R} \quad |\Upsilon_n(t) - \Upsilon(t)| < \varepsilon$$

From this definition, there is no uniform convergence in this case since for any $n \in \mathbb{N}^*$, $\Upsilon_n(0) = \dfrac{1}{2}$, whereas $\lim_{t\to 0^-} \Upsilon(t) = 0$ and $\lim_{t\to 0^+} \Upsilon(t) = 1$. Moreover, functions Υ_n are continuous in 0 whereas Υ has a discontinuity in 0, showing that it does not inherit the continuity property from uniform convergence. Strictly speaking, we cannot consider the limit of sequence $(\Upsilon'_n)_{n\in\mathbb{N}^*}$ as the derivative of Υ. However, the study of this limit will give us an insight of the behavior of the derivative of Υ. For any $n \in \mathbb{N}^*$, we set $\delta_n = \Upsilon'_n$ and define δ as limit of this sequence, if it exists. For any $n \in \mathbb{N}^*$, function δ_n is piecewise constant and

$$\delta_n(t) = \begin{cases} n & \text{if } -\dfrac{1}{2n} < t < \dfrac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$



Then for any $t \neq 0$, $\delta(t) = 0$, since there exists an index $N \in \mathbb{N}^*$ such that $|t| > \dfrac{1}{2n}$ for any $n > N$. Moreover, $\delta(0) = \lim_{n\to+\infty} n = +\infty$, thus by extending the image set to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, we obtain the following definition:

$$\delta : \mathbb{R} \to \overline{\mathbb{R}} \qquad t \mapsto \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

However, a surprising result appears when we compute the following integral:

$$\forall n \in \mathbb{N}^* \qquad I_n = \int_{-\infty}^{+\infty} \delta_n(t)dt = \int_{-1/2n}^{1/2n} ndt = 1$$

Taking the limit, we get $\int_{-\infty}^{+\infty} \delta(t)dt = 1$. We have a function whose support is restricted to singleton $\{0\}$, the image on this support is $+\infty$ and the integral over $\mathbb{R}$ is 1, going against the rule $0 \times (+\infty) = 0$ established in measure theory. Thereby, we cannot consider δ as a classical measurable or integrable function, but as more general mathematical object called a distribution.

c Linear forms and distributions

In practice, evaluating a the image $f(x_0)$ of function f with real argument $x_0 \in \mathbb{R}$ is generally impossible, because it will require the knowledge of the infinite decimal development of both argument and image. It is more appropriate to represent this evaluation by

$$f(x_0) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} f(x)\varphi_\varepsilon(x - x_0)dx$$

where φ_ε is a function whose support is concentrated around x_0. This equality models the measure of a physical quantity f by a measuring tool φ_ε, all the more precise as ε is small. More generally, for any fixed function $g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, we notice that application

$$L_g : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \qquad f \mapsto \int_{-\infty}^{+\infty} f(t)g(t)dt$$

is a linear form, also denoted with the duality bracket $L_g(f) = \langle L_g, f\rangle = \langle g, f\rangle$. However, the space of linear forms from $\mathcal{F}(\mathbb{R}, \mathbb{R})$ to $\mathbb{R}$ contains more elements than the linear forms like L_g. We are going to express another linear form, associated with Dirac delta function.
For any $n \in \mathbb{N}^*$, we define

$$L_n : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \qquad f \mapsto \int_{-\infty}^{+\infty} f(t)\delta_n(t)dt$$

and we want to determine the limit L of this sequence. We restrict to signals f which are continuous in the vicinity of 0. We fix such a signal. Then for any $n \in \mathbb{N}^*$,

$$L_n(f) - f(0) = \int_{-\infty}^{+\infty} f(t)\delta_n(t)dt - f(0) = n\int_{-1/2n}^{1/2n} (f(t) - f(0))dt$$

Let $\varepsilon > 0$. Using the continuity of f in 0, there exists $\eta > 0$ such that for any $t \in]-\eta, \eta[$, $|f(t) - f(0)| < \varepsilon$. There exists an index $N \in \mathbb{N}^*$ such that for any $n > N$, $\frac{1}{2n} < \eta$, which gives

$$|L_n(f) - f(0)| \leq n\int_{-1/2n}^{1/2n} |f(t) - f(0)|dt < \varepsilon$$

We deduce that the sequence of real numbers $(L_n(f))_{n\in\mathbb{N}^*}$ converges to $f(0)$, thus we can define the limit linear form by:

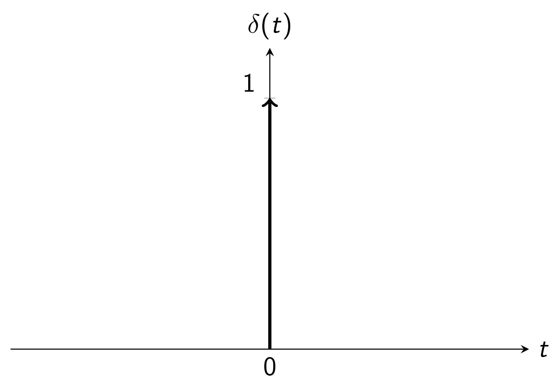
$$L : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R} \qquad f \mapsto f(0)$$

that, by abuse of notation, we write $L : f \mapsto \int_{-\infty}^{+\infty} f(t)\delta(t)dt$, while it would be better to write $L(f) = \langle L, f\rangle = \langle \delta, f\rangle$. In this context, we do not define δ as the limit of sequence of functions (δ_n) anymore, but as the distribution associated with linear form L, limit of sequence $(L_n) = (L_{\delta_n})$. Now we have all the ingredients to properly define Dirac [2] delta function. In this context, the word function is an abuse, since δ is a distribution, or sometimes also called a generalized function.

3.2 Definition of Dirac delta function

Definition 3.1 (Dirac delta function)
Dirac delta function is the distribution $\delta : \mathbb{R} \to \overline{\mathbb{R}}$ satisfying the following properties:

- For any $t \in \mathbb{R}^*$, $\delta(t) = 0$;
- $\delta(0) = +\infty$
- For any signal f defined and continuous in a vicinity of 0, $\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0)$.



Remarks:

(i) The property $\int_{-\infty}^{+\infty} \delta(t)dt = 1$ corresponds to the particular case of constant function $f : t \mapsto 1$.

(ii) It is also possible to define the Dirac delta function as the distributional derivative of the Heaviside step function through integration by parts. Indeed, assuming that the derivative Υ' exists, we consider a differentiable signal f et we want to evaluate $I = \int_{-\infty}^{+\infty} f(t)\Upsilon'(t)dt$. Setting, for any $A > 0$, $I(A) = \int_{-A}^{A} f(t)\Upsilon'(t)dt$, an integration by parts gives

$$I(A) = \int_{-A}^{A} f(t)\Upsilon'(t)dt = \Big[f(t)\Upsilon(t)\Big]_{-A}^{A} - \int_{-A}^{A} f'(t)\Upsilon(t)dt = f(A) - \int_{0}^{A} f'(t)dt = f(0)$$

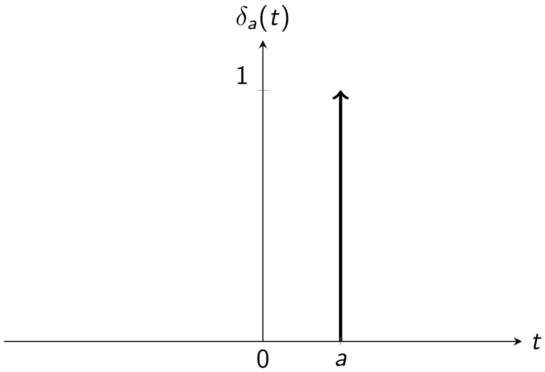
thus $I = \lim_{A\to+\infty} I(A) = f(0)$.

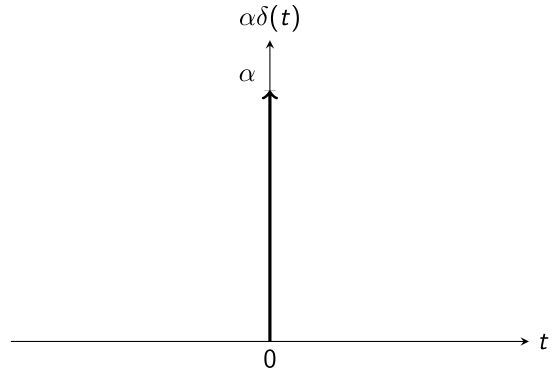
(iii) So far we have defined the delta function centered in zero, i.e. $\delta_0 = \delta$ corresponding to the linear form $f \mapsto f(0)$. We can also define for any $a \in \mathbb{R}$ the Dirac delta function centered in a by $\delta_a(t) = \delta(t-a)$, corresponding to linear form $f \mapsto f(a)$, i.e.

$$\int_{-\infty}^{+\infty} f(t)\delta_a(t)dt = \int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = \int_{-\infty}^{+\infty} f(t)\delta(a-t)dt = f(a)$$

[2] Paul Dirac (1902-1984), British mathematician and physicist

(iv) By linearity, we can define for any $\alpha \in \mathbb{R}$ the weighted Dirac delta function $\alpha\delta$ as the derivative of the weighted Heaviside step function $\alpha\Upsilon$. Then it corresponds to linear form $f \mapsto \alpha f(0)$.





(v) We can define a distribution $f\delta$, product of a regular function f by the Dirac delta function δ. Indeed, for any function h,

$$\langle f\delta, h\rangle = \int_{-\infty}^{+\infty} h(t)\left(f(t)\delta(t)\right)dt = \int_{-\infty}^{+\infty} \left(h(t)f(t)\right)\delta(t)dt = h(0)f(0)$$

This product distribution is thus the weighted delta function $f(0)\delta$. This definition has a particular interest when we look for the derivative of a function of the form $g = f\Upsilon$. The readers can convince themselves using again the integration by parts in (ii) that the derivative of this function g is the distribution $g' = f'\Upsilon + f(0)\delta$.

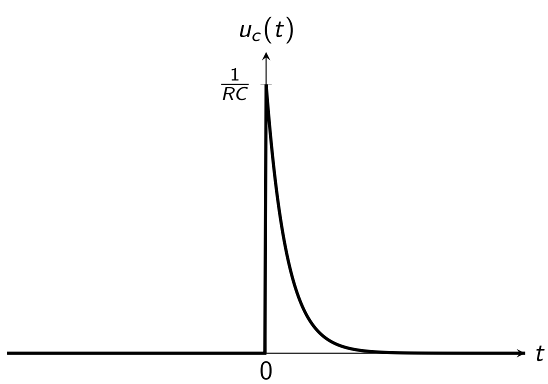
Definition 3.2 (Impulse response)
The **impulse response** of a system is its response to the Dirac delta function as input.

Example 3.2
Back to Example 3.1, the impulse response of the mechanical system, i.e. its position, is $x(t) = t\Upsilon(t)$, since delta function corresponds to speed $v_0 = 1$.

Remark: We define the differential operator $D : \mathcal{F}(\mathbb{R},\mathbb{R}) \to \mathcal{F}(\mathbb{R},\mathbb{R}) \quad f \mapsto f'$. If a system L commutes with this operator D, i.e. $L \circ D = D \circ L$, then its impulse response is the derivative of its step response.
We will see in the next section a class of systems commuting with the differential operator.

Example 3.3
The RC circuit studied in Example 2.1 commutes with differentiation. Thus its impulse response is:

$$h(t) = \frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\Upsilon(t)$$



4 Linear time-invariant systems, convolution

4.1 Linear time-invariant systems

So far, we have studied two signals and their respective response: the step response associated with Heaviside step function, and the impulse response associated with Dirac delta function. Yet, we do not know a general method to exhibit the system response of any input. It is impossible to answer this question in general, and we need to make assumptions on the studied system. This is why we restrict our study to the class of **linear time-invariant** (LTI) systems.

Definition 4.1
Let L be a system from $\mathcal{F}(\mathbb{R}, \mathbb{K})$ to $\mathcal{F}(\mathbb{R}, \mathbb{K})$.

(i) L is a **linear system** if it is a linear mapping:

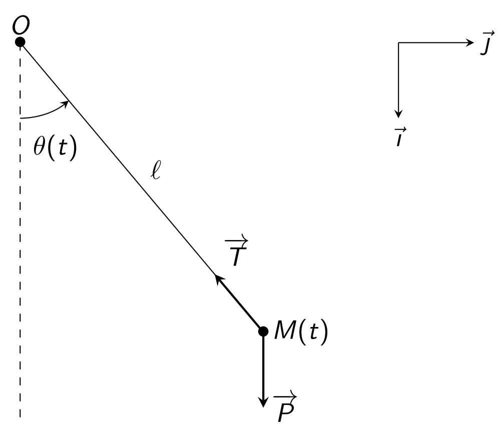
$$\forall (x_1, x_2) \in \mathcal{F}(\mathbb{R}, \mathbb{K})^2 \quad L(x_1 + x_2) = L(x_1) + L(x_2) \qquad \forall x \in \mathcal{F}(\mathbb{R}, \mathbb{K}) \quad \forall \alpha \in \mathbb{K} \quad L(\alpha x) = \alpha L(x)$$

(ii) For any $a \in \mathbb{R}$, the **pure delay** or **translation** system is defined as $\tau_a : x \mapsto x_a$, with $x_a : t \mapsto x(t - a)$. L is a **time-invariant system** if it commutes with any pure delay, i.e. for any $a \in \mathbb{R}$, $L \circ \tau_a = \tau_a \circ L$. In other words, if the input is delayed by a, the corresponding output is delayed by a as well.

(iii) **Linear time-invariant** systems satisfy both properties (i) and (ii).

(iv) L is a **causal system** if for any input x and any $t \in \mathbb{R}$, the instantaneous output $L(x)(t)$ only depends on prior values of x, i.e. values $x(t - \tau)$ for $\tau \geq 0$. These systems are the ones met in practice, since systems cannot predict upcoming input, even if computations can exhibit non-causal systems.

Example 4.1
The RC circuit (Example 2.1) and the mobile object (Example 3.1), and more generally any system governed by a linear differential equation, are linear systems. The pendulum is an example of non-linear system.



The shaft of an engine is represented by a fixed point O in the vertical plane. We attach to this shaft a rigid rod of length ℓ and negligible mass, and at the other end of this rod a ponctual object M of mass m. Quantity $\theta(t)$ represents the angle $\left(\vec{\imath}, \overrightarrow{OM}(t)\right)$ between the vectical axis and the rode at time t. The engine in O can provide an acceleration $a_O(t)$ to the system. The second law of motion implies, among others, the equation:

$$\theta''(t) + \frac{mg}{\ell} \sin(\theta(t)) = \frac{a_O(t)}{\ell}$$

Choosing $x(t) = \dfrac{a_O(t)}{\ell}$ as input and $y(t) = \theta(t)$ as output, we get the differential equation governing the system:

$$y''(t) + \frac{mg}{\ell}\sin(y(t)) = x(t) \tag{P_1}$$

Since this differential equation is non-linear, the system is non-linear as well. However, output $y(t) = 0$ for any $t \in \mathbb{R}$ is a solution of the homogeneous equation corresponding to an equilibrium: the rod staying put and vertical. Applying small variations around this equilibrium, we can write $y(t) \approx 0$ and $\sin(y(t)) \approx y(t)$ allowing the approximation of equation (P_1) by the following linear differential equation:

$$y''(t) + \frac{mg}{\ell}y(t) = x(t) \tag{P_2}$$

We can generalize this principle to the study of any non-linear system by looking for an equilibrium and linearizing the governing equation around it, boiling down to the study of a linear system.
Likewise, systems governed by linear differential equations are time-invariant. In practice, no system is time-invariant since they wear out over time, causing a variation of their response to the same input. However we study systems on short enough periods of time to suppose them time-invariant.

We are going to perform a computation showing how we can explicitely relate the output of an LTI system L to its input and to its impulse response $h = L(\delta)$. Let x denote the input and $y = L(x)$ the corresponding output. For any $t \in \mathbb{R}$, using the following property

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(\tau - t)d\tau = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau)d\tau$$

we obtain by linearity and time-invariance of L:

$$y(t) = L(x)(t) = \int_{-\infty}^{+\infty} x(\tau)L\left[\delta(t-\tau)\right]d\tau = \int_{-\infty}^{+\infty} x(\tau)L(\delta)(t-\tau)d\tau = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

Thereby, the only knowledge of the input and impulse response of a system allows us to explicitely determine the corresponding output. This operation is called **convolution**.

4.2 Convolution

Definition 4.2 (Convolution)
Convolution $*$ defines a product in $\mathcal{F}(\mathbb{R}, \mathbb{K})$ by

$$\forall (x, y) \in \mathcal{F}(\mathbb{R}, \mathbb{K})^2 \qquad \forall t \in \mathbb{R} \qquad (x * y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du$$

The Dirac delta function δ is the identity element of the convolution, i.e. for any signal x, $x * \delta = x$.

Definition 4.3 (Convolution system)
A **convolution system** is a mapping from $\mathcal{F}(\mathbb{R}, \mathbb{K})$ to $\mathcal{F}(\mathbb{R}, \mathbb{K})$ of the form $x \mapsto x * h$, where h is a given function.

Remarks:

- The computation performed in the previous paragraph shows that an LTI system is a convolution system, where h appearing in the convolution is the impulse response of this system. By their definition, convolution systems provide an explicit expression of the output as a function of the input, which is thus also true for LTI systems. Conversely, any convolution system is an LTI system, as proved by properties (i) and (iii) of Proposition 4.1.

- Let L be an LTI system, thus a convolution system, with impulse response h. Let an input x and corresponding output $y = L(x) = x * h$. From the definition of convolution:

$$\forall t \in \mathbb{R} \qquad y(t) = (x * h)(t) = \int_{-\infty}^{+\infty} x(u)h(t-u)du = \int_{-\infty}^{+\infty} x(t-u)h(u)du$$

System L is causal if the instantaneous output $y(t)$ only depends on prior input $x(t-u)$ with $u \geq 0$. It implies the following sufficient condition:

If for any $t < 0$, $h(t) = 0$, then L is a causal system.

Example 4.2

Using the computation in Example 3.3, we can express the output of the RC circuit (Example 2.1) as function of the input:

$$\forall x \in \mathcal{F}(\mathbb{R}, \mathbb{K}) \qquad y = L(x) = x * h \qquad \text{with} \qquad h : t \mapsto \frac{1}{RC} \exp\left(-\frac{t}{RC}\right) \Upsilon(t)$$

Example 4.3

Let a signal x and $a \in \mathbb{R}$. Then $x * \delta_a = x_a = \tau_a(x)$ with $x_a : t \mapsto x(t-a)$ the pure delay of x by a. Indeed, for any $t \in \mathbb{R}$,

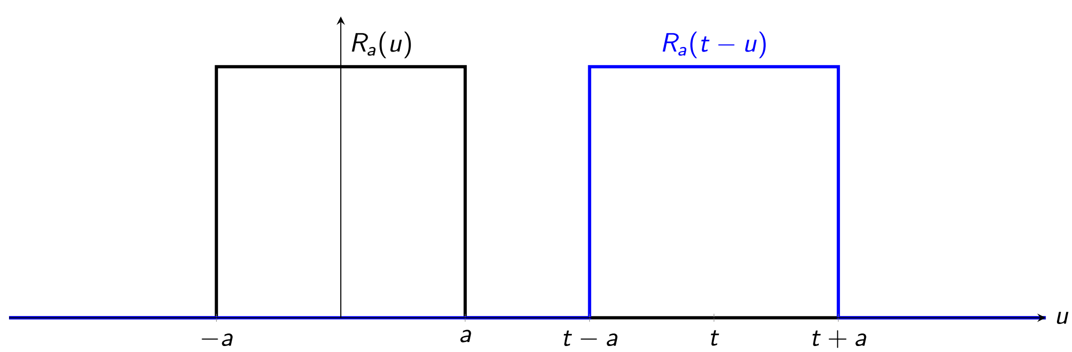
$$(x * \delta_a)(t) = \int_{-\infty}^{+\infty} x(u)\delta_a(t-u)du = \int_{-\infty}^{+\infty} x(u)\delta(t-u-a)du = x(t-a)$$

Thus the convolution of x by δ_a is the delayed signal $x_a = \tau_a(x)$. This is consistent with τ_a being an LTI system of impulse response $\tau_a(\delta) = \delta_a$, corresponding to the convolution by δ_a.

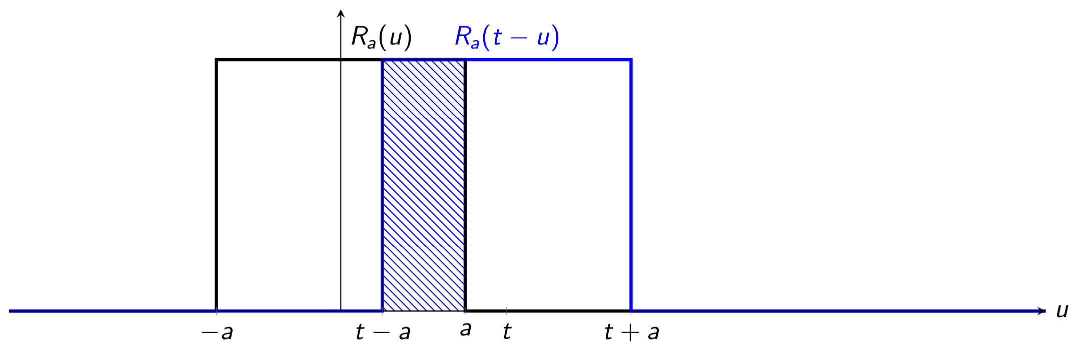
Example 4.4

We present a graphical method to compute the convolution. We look for the convolution T_a of rectangular signal $R_a = \chi_{[-a,a]}$ by itself, with $a > 0$. Let $t \in \mathbb{R}$. We treat this computation in different cases:

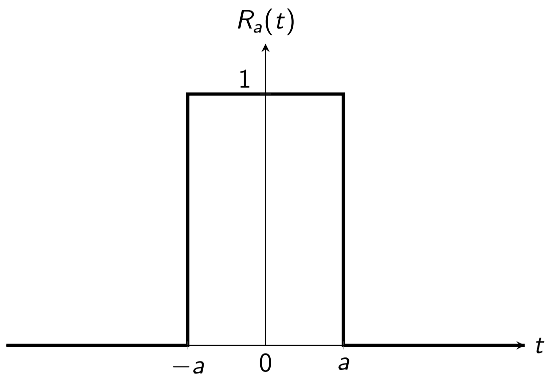
- If $|t| > 2a$ then the supports of functions $u \mapsto R_a(u)$ and $u \mapsto R_a(t-u)$ are disjoint, and the area of their common hypograph is $T_a(t) = 0$.

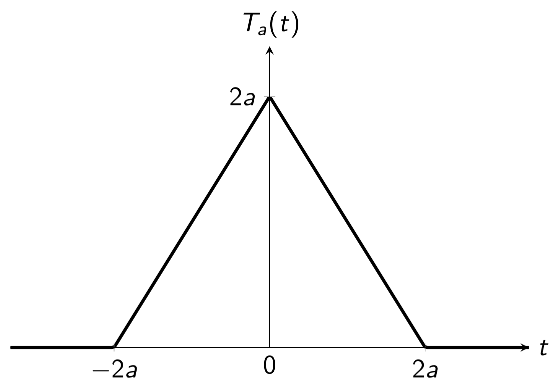


- If $t \in [0, 2a]$, then $u \mapsto R_a(u)$ and $u \mapsto R_a(t-u)$ have interval $[t-a, a]$ as common support, thus the area of their common hypograph is $T_a(t) = (a - (t-a)) \times 1 = 2a - t$.



- If $t \in [-2a, 0]$, then $u \mapsto R_a(u)$ and $u \mapsto R_a(t-u)$ have interval $[-a, t+a]$ as common support, thus the area of their common hypograph is $T_a(t) = (t + a - (-a)) \times 1 = t + 2a$.





Proposition 4.1

Convolution satisfies the following properties:

(i) it is bilinear: for any signals x, y and z, and any scalar $\alpha \in \mathbb{K}$, $x * (y + \alpha z) = (x * y) + \alpha(x * z)$, same for the first component;

(ii) it is associative and commutative: for any x, y and z, $x * (y * z) = (x * y) * z$ and $x * y = y * x$;

(iii) it commutes with pure delays: for any $a \in \mathbb{R}$, $\tau_a(x * y) = \tau_a(x) * y = x * \tau_a(y)$;

(iv) integration: for any signals x and y,

$$\int_{-\infty}^{+\infty} (x * y)(t)dt = \left(\int_{-\infty}^{+\infty} x(t)dt\right)\left(\int_{-\infty}^{+\infty} y(t)dt\right)$$

(v) differentiation: for any signals x and y, $(x * y)' = x' * y = x * y'$.

Proof : (i) Bilinearity results from the linearity of the integral. Indeed, for any $t \in \mathbb{R}$,

$$\begin{aligned}(x * (y + \alpha z))(t) &= \int_{-\infty}^{+\infty} x(u)\,[(y + \alpha z)(t-u)]\,du = \int_{-\infty}^{+\infty} x(u)y(t-u)du + \alpha \int_{-\infty}^{+\infty} x(u)z(t-u)du \\ &= (x * y)(t) + \alpha(x * z)(t)\end{aligned}$$

(ii) For any $t \in \mathbb{R}$,

$$(x*(y*z))(t) = \int_{-\infty}^{+\infty} x(u)(y*z)(t-u)du = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)y(v)z(t-u-v)dvdu$$

and

$$((x*y)*z)(t) = \int_{-\infty}^{+\infty} (x*y)(u)z(t-u)du = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(v)y(u-v)z(t-u)dudv$$

By the change of variable $(u, v) \mapsto (v, u+v)$ in the first expression, we note that both expressions are equal, hence the associativity.
By the change of variable $u \mapsto t-u$, we obtain, for any $t \in \mathbb{R}$,

$$(x*y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du = \int_{-\infty}^{+\infty} x(t-u)y(u)du = \int_{-\infty}^{+\infty} y(u)x(t-u)du = (y*x)(t)$$

(iii) We use the result of Example 4.3 with associativity and commutativity of convolution to write:

$$\tau_a(x*y) = (x*y)*\delta_a = x*(y*\delta_a) = x*\tau_a(y) = (x*\delta_a)*y = \tau_a(x)*y$$

(iv) We have:

$$\int_{-\infty}^{+\infty}(x*y)(t)dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)y(t-u)dudt$$

By the change of variable $(t, u) \mapsto (u, t-u)$ and by Fubini's theorem,

$$\int_{-\infty}^{+\infty}(x*y)(t)dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(u)y(v)dudv = \left(\int_{-\infty}^{+\infty} x(t)dt\right)\left(\int_{-\infty}^{+\infty} y(t)dt\right)$$

(v) Let $t \in \mathbb{R}$. By the theorem of differentiation under the integral sign, we obtain:

$$(x*y)'(t) = \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}\left[x(u)y(t-u)\right]du = \int_{-\infty}^{+\infty} x(u)y'(t-u)du = (x*y')(t)$$

Then we get equality $(x*y)' = x'*y$ by commutativity of the convolution. ■

Remarks:

- With properties (i) and (iii), we show that convolution systems are linear and commute with pure delays. Thereby, the LTI systems are exactly the convolution systems.
- With property (ii), we can define chains of LTI systems. Indeed, if $L_1, \dots, L_n$ are LTI systems with respective impulse responses $h_1, \dots, h_n$, setting $L = L_1 \circ \cdots \circ L_n$, then the impulse response of L is $h = h_1 * \cdots * h_n$. For instance, we can chain pure delays, providing, for any $(a, b) \in \mathbb{R}^2$, the identity: $\delta_a * \delta_b = \delta_{a+b}$.
- By recursion, property (v) can be generalized to the n-th derivative of the convolution:

  $$\forall n \in \mathbb{N}^* \qquad \forall (x, y) \in \mathcal{F}(\mathbb{R}, \mathbb{K})^2 \qquad (x*y)^{(n)} = x^{(n)}*y = x*y^{(n)} = x^{(k)}*y^{(\ell)}$$

  for $k+\ell = n$. Then we can define, for any $n \in \mathbb{N}^*$, the n-th derivative of Dirac delta function:

  $$\forall x \in \mathcal{F}(\mathbb{R}, \mathbb{K}) \qquad x*\delta^{(n)} = x^{(n)}*\delta = x^{(n)}$$

Thereby, distribution $\delta^{(n)}$ corresponds to the linear form $f \mapsto f^{(n)}(0)$. In particular, distribution δ' is the impulse response of differential system D.

- By property (v), convolution systems, and incidently LTI systems, commute with differentiation. Therefore, the impulse response of any LTI system is the derivative of its step response.

5 Correlation

We introduce another product on $\mathcal{F}(\mathbb{R}, \mathbb{K})$, the scalar product, from which we introduce the notion of correlation and relate to convolution.

Definition 5.1 (Scalar product, Hermitian product)
If E denotes a vector space over $\mathbb{R}$, a **scalar product** over E is any mapping $\langle ., . \rangle : E \times E \to \mathbb{R}$ satisfying the following properties:

- it is bilinear: for any $(x, y, z) \in E^3$ and any $(\alpha, \beta) \in \mathbb{R}^2$, $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$; same for the second component;
- it is positive: for any $x \in E$, $\langle x, x \rangle \geq 0$;
- it is definite: for any $x \in E$, $\langle x, x \rangle = 0 \Leftrightarrow x = 0_E$.

If E denotes a vector space over $\mathbb{C}$, a **Hermitian product** over E is any mapping $\langle ., . \rangle : E \times E \to \mathbb{C}$ satisfying the following properties:

- it is linear for the first component: for any $(x, y, z) \in E^3$ and any $(\alpha, \beta) \in \mathbb{C}^2$, $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;
- it is anti-linear for the second component: for any $(x, y, z) \in E^3$ and any $(\alpha, \beta) \in \mathbb{C}^2$, $\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$, where $\overline{z}$ denotes the conjugate of $z \in \mathbb{C}$;
- it is positive and definite.

In this section, we restrict our study to the subspace $L^2(\mathbb{R}, \mathbb{K})$ of $\mathcal{F}(\mathbb{R}, \mathbb{K})$ of square integrable signals.

Definition 5.2 (Scalar product over $\mathbf{L^2}(\mathbb{R}, \mathbb{R})$, Hermitian product over $\mathbf{L^2}(\mathbb{R}, \mathbb{C})$, energy)
We define a **scalar product** over $L^2(\mathbb{R}, \mathbb{R})$ by

$$\forall (x, y) \in L^2(\mathbb{R}, \mathbb{R})^2 \qquad \langle x, y \rangle = \int_{-\infty}^{+\infty} x(t) y(t) dt$$

We define a **Hermitian product** over $L^2(\mathbb{R}, \mathbb{C})$ by

$$\forall (x, y) \in L^2(\mathbb{R}, \mathbb{C})^2 \qquad \langle x, y \rangle = \int_{-\infty}^{+\infty} x(t) \overline{y(t)} dt$$

From these products, we can define the norm of a signal, from which we introduce the **energy**:

$$\forall x \in L^2(\mathbb{R}, \mathbb{K}) \qquad E(x) = \|x\|^2 = \langle x, x \rangle$$

i.e.

$$\forall x \in L^2(\mathbb{R}, \mathbb{R}) \quad E(x) = \int_{-\infty}^{+\infty} x(t)^2 dt \qquad \forall x \in L^2(\mathbb{R}, \mathbb{C}) \quad E(x) = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

Remarks:

- In other words, $L^2(\mathbb{R},\mathbb{K})$ is the subspace of $\mathcal{F}(\mathbb{R},\mathbb{K})$ of finite-energy signals.
- Cauchy-Schwarz inequality indicates that for two signals x and y of $L^2(\mathbb{R},\mathbb{K})$, $|\langle x,y\rangle| \leq \|x\|.\|y\| = \sqrt{E(x)E(y)}$, ensuring that both products are well defined over $L^2(\mathbb{R},\mathbb{K})$.
- Notation $\langle .,.\rangle$ for the scalar product is consistent with the duality bracket used in Section 3, since for any fixed signal y, mapping $x \mapsto \langle x,y\rangle$ is a linear form.
- To deal indistinctly with both products, we use notation x^* to designate $x^* = x$ for $x \in \mathbb{R}$, and $x^* = \overline{x}$ for $x \in \mathbb{C}$.
- For infinite-energy signals, we can introduce the notion of average power.

Definition 5.3 (Average power)
The **average power** of a signal $x \in \mathcal{F}(\mathbb{R},\mathbb{K})$ is the real number:

$$P(x) = \lim_{t\to+\infty} \frac{1}{2t}\int_{-t}^{t} |x(u)|^2 du$$

Remark: Finite-energy signals have a zero average power.

Definition 5.4 (Cross-correlation, autocorrelation)
Let x and y be two signals of $L^2(\mathbb{R},\mathbb{K})$. The **cross-correlation** is the function $\gamma_{xy} : \mathbb{R} \to \mathbb{K}$ defined by

$$\forall t \in \mathbb{R} \qquad \gamma_{xy}(t) = \langle x, \tau_t(y)\rangle = \int_{-\infty}^{+\infty} x(u)y^*(u-t)du$$

The **autocorrélation** of a signal $x \in L^2(\mathbb{R},\mathbb{K})$ is the cross-correlation with itself, i.e.

$$\forall t \in \mathbb{R} \qquad \gamma_x(t) = \gamma_{xx}(t) = \langle x, \tau_t(x)\rangle = \int_{-\infty}^{+\infty} x(u)x^*(u-t)du$$

Remarks:

- As a scalar product, cross-correlation measures the similarity between a signal x and a delayed version of a signal y. It enables the identification of common "patterns" between two signals. Autocorrelation enables the identification of similarities between a signal x and a delayed version of itself, which can be used to determine the periodicity of the signal for instance.
- For any signal $x \in \mathcal{F}(\mathbb{R},\mathbb{K})$, $\gamma_x(0) = \langle x,x\rangle = E(x)$, thus the energy of a signal is equal to its autocorrelation in 0.
- The convolution can be seen as a variant of cross-correlation. Indeed, let x and y be two signals of $\mathcal{F}(\mathbb{R},\mathbb{K})$. For any $t \in \mathbb{R}$,

$$(x*y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du = \int_{-\infty}^{+\infty} x(u)\tilde{y}(u-t)du = \langle x, \tau_t(\tilde{y})^*\rangle = \gamma_{x\tilde{y}^*}(t)$$

  with $\tilde{y} : t \mapsto y(-t)$. Conversely, we can write cross-correlation as a function of convolution: $\gamma_{xy} = x * \tilde{y}^*$.

- By connecting convolution to this scalar product, we can bring another proof that any LTI system is a convolution system. Indeed, let L be an LTI system of impulse response $h = L(\delta)$, $x \in \mathcal{F}(\mathbb{R}, \mathbb{K})$ an input and $y = L(x)$ the corresponding output. For any $t \in \mathbb{R}$,

$$y(t) = L(x)(t) = L(x * \delta)(t) = L\left(\langle x, \tau_t(\tilde{\delta})\rangle\right) = \langle x, L(\tau_t(\tilde{\delta}))\rangle = \langle x, \tau_t(\widetilde{L(\delta)})\rangle = \langle x, \tau_t(\tilde{h})\rangle = (x * h)(t)$$

where we use the bilinearity of L and its commutativity with operators τ_t and $x \mapsto \tilde{x}$.

> **Proposition 5.1**
> We have the following properties:
>
> (i) For any two signals x and y, cross-correlation satisfies the equality
>
> $$\forall t \in \mathbb{R} \qquad |\gamma_{xy}(t)| \leq \sqrt{E(x)E(y)}$$
>
> In particular, for any signal x, the absolute value of autocorrelation γ_x reaches its maximum $E(x)$ in 0.
>
> (ii) Autocorrelation satisfies the following symmetry property: for any signal x, for any $t \in \mathbb{R}$, $\gamma_x(-t) = \gamma_x^*(t)$.

Proof : (i) First note that any delayed signal $\tau_t(x)$ has the same energy as signal x. Indeed, by the change of variable $u \mapsto u + t$, we get

$$\forall t \in \mathbb{R} \qquad E(\tau_t(x)) = \int_{-\infty}^{+\infty} |\tau_t(x)(u)|^2 du = \int_{-\infty}^{+\infty} |x(u-t)|^2 du = \int_{-\infty}^{+\infty} |x(u)|^2 du = E(x)$$

Then by applying Cauchy-Schwarz inequality,

$$\forall t \in \mathbb{R} \qquad |\gamma_{xy}(t)| = |\langle x, \tau_t(y)\rangle| \leq \|x\|.\|\tau_t(y)\| = \sqrt{E(x)E(\tau_t(y))} = \sqrt{E(x)E(y)}$$

In the particular case of $y = x$,

$$\forall t \in \mathbb{R} \qquad |\gamma_x(t)| \leq E(x) = \gamma_x(0)$$

(ii) Let a signal x and $t \in \mathbb{R}$. By the change of variable $u \mapsto u - t$, we get:

$$\gamma_x(-t) = \int_{-\infty}^{\infty} x(u)x^*(u+t)du = \int_{-\infty}^{\infty} x(u-t)x^*(u)du = \left(\int_{-\infty}^{\infty} x(u)x^*(u-t)du\right)^* = \gamma_x^*(t) \qquad \blacksquare$$

Remark: It is consistent that the maximum of autocorrelation is in 0, since a signal has a maximum of similarity with a version of itself delayed by 0.

6 Periodic signal

In this last section, we focus on the class of periodic signals, basis for the next chapter.

> **Definition 6.1 (Periodic signal, fundamental period)**
> A signal $x \in \mathcal{F}(\mathbb{R}, \mathbb{K})$ is **periodic** if there exists $T > 0$ such that for any $t \in \mathbb{R}$, $x(t+T) = x(t)$. The smallest $T_0 > 0$ such that $x(t + T_0) = x(t)$ for any $t \in \mathbb{R}$ is the **fundamental period**. We denote $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ the subspace of periodic signals with period T_0.

Remarks:

- If a signal x is periodic with period T_0, then it is with period kT_0 for any $k \in \mathbb{Z}$. This is a cornerstone property intensively used in the next chapter.
- In other words, a signal x is periodic with period T if it is invariant by a pure delay of T, i.e. $\tau_T(x) = x$.

Definition 6.2 (Fundamental frequency, fundamental impulse)
Let $x \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ be a periodic signal with period T_0. The **fundamental frequency** of x is the number $f_0 = \dfrac{1}{T_0}$, and the **fundamental impulse** of x is the number $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T_0}$.

Definition 6.3 (Complex exponential, cosine)
The **complex exponential** of fundamental impulse ω_0, amplitude $A > 0$ and phase $\varphi_0 \in [0, 2\pi[$ is the following signal in $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{C})$:

$$e_{\omega_0,A,\varphi_0} : t \mapsto A \exp\left(i(\omega_0 t + \varphi_0)\right)$$

We simply denote $e_{\omega_0} = e_{\omega_0,1,0}$ the complex exponential of amplitude $A = 1$ and phase $\varphi_0 = 0$.
The **cosine** of fundamental impulse ω_0, amplitude $A > 0$ and phase $\varphi_0 \in [0, 2\pi[$ is the following signal in $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{C})$:

$$c_{\omega_0,A,\varphi_0} : t \mapsto A \cos(\omega_0 t + \varphi_0)$$

We simply denote $c_{\omega_0} = c_{\omega_0,1,0}$ the cosine of amplitude $A = 1$ and phase $\varphi_0 = 0$.

Remarks:

- With these definitions, we can check that the complex exponential and cosine of fundamental impulse ω_0 are periodic signals with period T_0, for $T_0 = \dfrac{2\pi}{\omega_0}$.
- If a periodic signal with period T_0 is the input of an LTI system, then the corresponding output is with period T_0 as well. Indeed let an LTI system L, a periodic signal x with period T_0, et $y = L(x)$ the corresponding output. Since LTI systems commute with translations,

  $$\tau_{T_0}(y) = \tau_{T_0}(L(x)) = L(\tau_{T_0}(x)) = L(x) = y$$

  thus y is also periodic with period T_0.
- As we are going to see in Example 6.1, we often deal in practice with signals of the form $t \mapsto e_{\omega_0,A,\varphi_0}(t)\Upsilon(t)$ and $t \mapsto c_{\omega_0,A,\varphi_0}(t)\Upsilon(t)$, namely zero over $]-\infty, 0[$ and oscillating over $[0, +\infty[$. These signals will be the matter of a following chapter.

If a signal x is periodic with period T_0, signal $t \mapsto |x(t)|^2$ is clearly periodic with period T_0 as well. Thus for any $n \in \mathbb{N}^*$,

$$\int_{-nT_0}^{nT_0} |x(t)|^2 dt = 2n \int_0^{T_0} |x(t)|^2 dt$$

When n goes to $+\infty$, we note that a non-zero periodic signal has infinite energy. However, the average power of such a periodic signal x is:

$$P(x) = \lim_{t \to +\infty} \frac{1}{2t} \int_{-t}^{t} |x(u)|^2 du = \lim_{n \to +\infty} \frac{1}{2nT_0} \int_{-nT_0}^{nT_0} |x(u)|^2 du = \frac{1}{T_0} \int_0^{T_0} |x(u)|^2 du$$

We are going to define a subspace of $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ containing periodic signals with period T_0 which are locally square integrable, i.e. they have a finite average power.

Lemma 6.1
Let x be a periodic signal with period T_0. For any $a \in \mathbb{R}$,

$$\int_a^{a+T_0} x(t)dt = \int_0^{T_0} x(t)dt$$

PROOF : If $a \in [0, T_0]$, then $T_0 \in [a, a+T_0]$. By the change of variable $t \mapsto t - T_0$, we get

$$\int_a^{a+T_0} x(t)dt = \int_a^{T_0} x(t)dt + \int_{T_0}^{a+T_0} x(t)dt = \int_a^{T_0} x(t)dt + \int_0^{a} x(t+T_0)dt = \int_0^{T_0} x(t)dt$$

In general, let $a \in \mathbb{R}$. If $b = a - \left\lfloor \frac{a}{T_0} \right\rfloor T_0$, then $b \in [0, T_0]$ and by the change of variable $t \mapsto t - \left\lfloor \frac{a}{T_0} \right\rfloor T_0$,

$$\int_a^{a+T_0} x(t)dt = \int_b^{b+T_0} x(t)dt = \int_0^{T_0} x(t)dt$$ ■

This lemma indicates that the integral of a periodic signal with period T_0 is identical on any interval of length T_0. Therefore, we can now define the subspace of signals with finite average power and define on this subspace a scalar product base on the average power, instead of the energy which is infinite in this case.

Definition 6.4
We denote $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ the subspace of $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ containing the periodic signals with period T_0 which are square integrable over $[0, T_0]$, i.e.

$$L^2_{T_0}(\mathbb{R}, \mathbb{K}) = \left\{ x \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K}), \, \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt < +\infty \right\}$$

Definition 6.5
We define a **scalar product** / **Hermitian product** over $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ by

$$\forall (x, y) \in L^2_{T_0}(\mathbb{R}, \mathbb{K})^2 \qquad \langle x, y \rangle_{T_0} = \frac{1}{T_0} \int_0^{T_0} x(t) y^*(t) dt$$

From this scalar / Hermitian product, we can define the norm of a signal x to which we can connect the average power of the signal:

$$\forall x \in L^2_{T_0}(\mathbb{R}, \mathbb{K}) \qquad P(x) = \|x\|^2_{T_0} = \langle x, x \rangle_{T_0}$$

i.e.

$$\forall x \in L^2_{T_0}(\mathbb{R}, \mathbb{K}) \quad P(x) = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt$$

Remarks:

- We defined these integrals over the interval $[0, T_0]$ but according to the lemma, any interval of length T_0 is suitable. In some cases, it is more interesting to exploit the symmetry of interval $\left[-\frac{T_0}{2}, \frac{T_0}{2} \right]$, when we deal with odd or even signals for example.

- We can define cross-correlation and autocorrelation of periodic signals from this new scalar product.

Proposition 6.2
The autocorrelation of a periodic signal with period T_0 is also a periodic signal with period T_0.

Proof : Let $x \in L^2_{T_0}(\mathbb{R}, \mathbb{K})$.

$$\forall t \in \mathbb{R} \qquad \gamma_x(t + T_0) = \langle x, \tau_{t+T_0}(x)\rangle = \langle x, \tau_t(x)\rangle = \gamma_x(t)$$

because the periodicity of x implies $\tau_{t+T_0}(x) = \tau_t(x)$. ∎

Now we study the convolution of two non-zero periodic signals x a y with the same period T_0. Let $t \in \mathbb{R}$. Then signals $u \mapsto y(t-u)$ and $u \mapsto x(u)y(t-u)$ are also periodic with period T_0, thus

$$(x * y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du = \lim_{n\to+\infty} \int_{-nT_0}^{nT_0} x(u)y(t-u)du = \lim_{n\to+\infty} 2n \int_0^{T_0} x(u)y(t-u)du = +\infty$$

This result is not surprising, since the notions of energy and convolution are connected through correlation, and non-zero periodic signals have infinite energy. As for the scalar product, we have to adapt our definition of convolution.

Definition 6.6 (Circular convolution)
The **circular convolution** $\otimes$ is a product in $\mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})$ defined by

$$\forall (x, y) \in \mathcal{F}_{T_0}(\mathbb{R}, \mathbb{K})^2 \qquad \forall t \in \mathbb{R} \qquad (x \otimes y)(t) = \int_0^{T_0} x(u)y(t-u)du$$

Remarks:

- The circular convolution of two periodic signals x and y with period T_0 is also periodic with period T_0. Indeed, for any $t \in \mathbb{R}$,

$$(x \otimes y)(t + T_0) = \int_0^{T_0} x(u)y(t + T_0 - u)du = \int_0^{T_0} x(u)y(t-u)du = (x \otimes y)(t)$$

- As for scalar product and correlation, the integral of circular convolution can be defined on any interval of length T_0, $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$ for instance. **Aperiodic** signals, i.e. signals with no period, can be considered as signals with period $T_0 = +\infty$, yielding to classical convolution:

$$(x \otimes y)(t) = \int_{-\infty}^{+\infty} x(u)y(t-u)du = (x * y)(t)$$

Example 6.1
We go back to the RC circuit from Example 2.1 and we look for its response to the input $V(t) = \sin(\omega_0 t)\Upsilon(t) = \cos\left(\omega_0 t - \frac{\pi}{2}\right)\Upsilon(t)$, which is periodic over $[0, +\infty[$. We determine this response with two techniques developped in this chapter: solving the governing differential equation and computing the convolution with the impulse response.
To abbreviate computations, we set $\tau = RC$, the time constant of the circuit. Recall that the solutions of the homogeneous differential equation are of the form $u_c(t) = K \exp\left(-\frac{t}{\tau}\right)$, with $K \in \mathbb{R}$. We look for a particular solution of the form $u_c(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$ over $[0, +\infty[$. The derivative of such a function is $u_c'(t) = A\omega_0\cos(\omega_0 t) - B\omega_0\sin(\omega_0 t)$. Then the differential equation becomes over $[0, +\infty[$:

$$(A - B\tau\omega_0)\sin(\omega_0 t) + (A\tau\omega_0 + B)\cos(\omega_0 t) = \sin(\omega_0 t)$$

By identification, we get $A - B\tau\omega_0 = 1$ and $A\tau\omega_0 + B = 0$, yielding to

$$A = \frac{1}{1+\tau^2\omega_0^2} \qquad \text{and} \qquad B = -\frac{\tau\omega_0}{1+\tau^2\omega_0^2}$$

Thus we have the solution

$$u_c(t) = \begin{cases} 0 & \text{if } t < 0 \\ K \exp\left(-\frac{t}{\tau}\right) + \frac{1}{1+\tau^2\omega_0^2}\sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t) & \text{if } t > 0 \end{cases}$$

Since u_c in continuous in $t = 0$,

$$\lim_{t\to 0^-} u_c(t) = 0 = \lim_{t\to 0^+} u_c(t) = K - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}$$

for any $t \in [0, +\infty[$,

$$u_c(t) = \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\exp\left(-\frac{t}{\tau}\right) + \frac{1}{1+\tau^2\omega_0^2}\sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t)$$

Now we want to retrieve this result by the convolution of $V(t)$ with impulse response $h(t) = \frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right)\Upsilon(t)$. For $t \in]-\infty, 0[$, the supports of $u_c(u)$ and $h(t-u)$ are disjoint, thus $u_c(t) = (V * h)(t) = 0$. For $t \in [0, +\infty[$,

$$\begin{aligned} u_c(t) = (V*h)(t) &= \int_{-\infty}^{+\infty} \Upsilon(u)\sin(\omega_0 u)\frac{1}{\tau}\exp\left(-\frac{t-u}{\tau}\right)\Upsilon(t-u)du \\ &= \frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right)\int_0^t \sin(\omega_0 u)\exp\left(\frac{u}{\tau}\right)du \end{aligned}$$

A double integration by parts (left to the interested reader) gives

$$\int_0^t \sin(\omega_0 u)\exp\left(\frac{u}{\tau}\right)du = \frac{\tau^2\omega_0}{1+\tau^2\omega_0^2} + \frac{\tau}{1+\tau^2\omega_0^2}\sin(\omega_0 t)\exp\left(\frac{t}{\tau}\right) - \frac{\tau^2\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t)\exp\left(\frac{t}{\tau}\right)$$

Hence we retrieve:

$$u_c(t) = \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\exp\left(-\frac{t}{\tau}\right) + \frac{1}{1+\tau^2\omega_0^2}\sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t)$$

Note that $\lim_{t\to+\infty} \exp\left(-\frac{t}{\tau}\right) = 0$ so that for $t \gg \tau$,

$$u_c(t) \approx \frac{1}{1+\tau^2\omega_0^2}\sin(\omega_0 t) - \frac{\tau\omega_0}{1+\tau^2\omega_0^2}\cos(\omega_0 t)$$

This is called the **steady state** of the system. For $t \approx \tau$,we have to take into account the first term which is not negligible, corresponding to **transient state**, i.e. the transition between the off state for $t < 0$ and the steady state for $t \gg \tau$. Finally, note that the steady state is expressed as a function of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$, namely the sine and cosine with the same period as the input sine $\sin(\omega_0 t)$. We develop this notion in the next chapter.