Integration by Parts: This integral method is used to "undo" the product rule. According to the Fundamental Theorem of Calculus,

$$u(x)v(x)\Big|_a^b = \int_a^b \frac{d}{dx}\left(u(x)v(x)\right)dx = \int_a^b u(x)v'(x)dx + \int_a^b u'(x)v(x)dx.$$

FTC: $\int \frac{d}{dx} f = f$

PR: $\frac{d}{dx}(uv) = u'v + v'u$

$\downarrow$

$+ \int u(x)v'(x)dx$

Isolate as left hand side

This is often written more simply:

$$\int u\,dv = uv - \int v\,du.$$

Example:

$$\int \underbrace{x^3}_{u}\,\underbrace{\ln x\,dx}_{dv}$$

$$= \underbrace{\frac{x^4}{4}\ln(x)}_{uv} - \underbrace{\int \frac{x^4}{4}\cdot\frac{1}{x}\,dx}_{vdu} = \frac{x^4}{4}\ln(x) - \frac{1}{4}\underbrace{\int x^3 dx}_{=\frac{x^4}{4}+C} = \frac{x^4}{4}\ln(x) - \left(\frac{1}{4}\cdot\frac{1}{4}x^4 + C\right)$$

$$= \frac{x^4}{4}\ln(x) - \frac{1}{16}x^4 + C$$

Example:

$$\int (x-1)e^{5x}\,dx$$ skipped in class, practice

Example:

Easier to substitute here.

$w = x^2, \quad dw = 2x\,dx$

$\hookrightarrow dx = \frac{1}{2x}\,dw$

$$\int x^3 e^{x^2}\,dx$$

$$= \int \underbrace{x^2}_{w} e^{\overset{w}{x^2}} \cdot \frac{1}{2} \underbrace{(2x)\,dx}_{dw} = \int x^{\cancel{3}\,x^2} e^{w} \frac{1}{2\cancel{x}}\,dw = \frac{1}{2}\int w e^{w}\,dw$$

$$x^3 \cdot \frac{1}{2x} = \frac{x^3}{2x} = \frac{x^2}{2}$$

$$\int \underbrace{x^2}_{w} e^{w}\,\frac{1}{2}\,dw$$

$u = w \quad dv = e^w dw$

$du = dw \quad v = e^w$

$$\frac{1}{2}\int \underset{u}{w}\,\underset{dv}{e^{w}\,dw}$$

$$\hookrightarrow \frac{1}{2}\Big[\underset{uv}{w e^{w}} - \underset{v\,du}{\int e^{w}\,dw}\Big] = \frac{1}{2}\left[w e^{w} - e^{w} + C\right] = \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2} + C\right]$$

$$= \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2} + C\right]$$

Non-Example:

$$\int \frac{1}{x}\,dx$$

Example:

$$\int \ln x\,dx$$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}\,dx \quad v = x$

$$\int u\,dv = uv - \int v\,du \rightarrow \ln x \cdot x - \int x \cdot \frac{1}{x}\,dx$$

$$x\ln(x) - \int \frac{x}{x}\,dx = x\ln(x) - \underbrace{\int 1\,dx}_{= x + C}$$

$$= x\ln(x) - x + C$$

u should be easy to differentiate
+ v should be easy to integrate
→ If both are true . . .

You can do IBP over and over again

Example:

$$\int \cos(\ln x)\, dx$$

w

We cannot start w/ IBP here, $w = \ln(x)$ $dw = \frac{1}{x}dx$

→ $x = e^w$ → $dx = x\,dw$] Putting in terms of x

log rule

$\int \cos(\ln x)\, dx$ where $w = \ln x$ $= \int \underbrace{\cos(w)}_{\cos(\ln(x))} \underbrace{x\, dw}_{dx} = \int \cos(w) e^w\, dw$

First round of IBP . . . $I = \int e^w \cos(w)\, dw = \int \cos(w)\, e^w\, dw$

$u = \cos(w)$ $dv = e^w\, dw$

$du = -\sin(w)\, dw$ $v = \int e^w\, dw = e^w$

$I = e^w \cos(w) + \int e^w \sin(w)\, dw$ → + here bc $-\int e^w(-\sin w)dw = -\int -1 e^w (\sin w)\, dw = +1\int e^w \sin w\, dw$

Second round to make it go away

$u = \sin(w)$ $dv = e^w\, dw$

$du = \cos(w)\, dw$ $v = e^w$

→ $= e^w \cos(w) + [e^w \sin(w) - \int e^w \cos(w) dw] = e^w \cos(w) + e^w \sin(w) - I$

this is also just I

$I = e^w \cos(w) + e^w \sin(w) - I$ → $2I = e^w \cos(w) + e^w \sin(w)$

$+I$ $+I$

Just simplification

$= \frac{1}{2}[e^{\ln x} \cos(\ln(x)) + e^{\ln x} \sin(\ln(x))] + C$

$= \frac{1}{2}[x \cos(\ln(x)) + x \sin(\ln(x))] + C$

$\frac{1}{2}[e^w \cos(w) + e^w \sin(w)] + C$

Example:

$$\int_1^4 e^{\sqrt{x}}\, dx$$