1.

Practice - $\int 2x\cos(7x)\,dx$

Step 1 - Choose dv. u has been given to us as $u = 2x$

$\int 2x \cos(7x)\,dx$ $\quad u = 2x \quad dv = \cos(7x)\,dx$

Step 2 - since $u = 2x$, then $du = ?$

$u = 2x \rightarrow \frac{du}{dx} = 2 \rightarrow du = 2dx$

Step 3 - $v = \int \cos(7x)\,dx$. Find v (ignoring constant of integration (C)) using substitution

$\int \cos(7x)\,dx \quad w = 7x$

$\rightarrow \frac{dw}{dx} = 7 \rightarrow dw = 7\,dx \rightarrow dx = \frac{1}{7}dw$

$\int \cos(w)\frac{1}{7}dw = \frac{1}{7}\int \cos(w)\,dw \rightarrow \frac{1}{7}\sin(w) \rightarrow \frac{1}{7}\sin(7x)$

Step 4 - Now integrate by parts where $u = 2x \quad dv = \cos(7x)\,dx$

$du = 2dx \quad v = \frac{1}{7}\sin(7x)$

$\int u\,dv = uv - \int v\,du \rightarrow \int 2x\cos(7x)\,dx = (2x \cdot \frac{1}{7}\sin(7x)) - \int \frac{1}{7}\sin(7x)\,2dx$

$= \frac{2}{7}x\sin(7x) - \frac{2}{7}\int \sin(7x)\,dx$

Step 5 - we need to substitute to do the second integral. use $t = 7x$ which will give $dx = ?\,dt$ and integrate ignoring C.

$\int \sin(7x)\,dx \quad t = 7x \quad \frac{dt}{dx} = 7 \rightarrow dt = 7dx \rightarrow dx = \frac{1}{7}dt$

$\int \sin(t)\frac{1}{7}dt = \frac{1}{7}\int \sin(t)\,dt \rightarrow \frac{1}{7}[-\cos(t)] = \frac{1}{7}[-\cos(7x)]$

Step 6 - combining our results + including c

$\frac{2}{7}x\sin(7x) - \frac{2}{7}\int \sin(7x)\,dx$

$= \frac{2}{7}x\sin(7x) - \frac{2}{7}\left(\frac{1}{7}[-\cos(7x)]\right) + C$

Actual 1 - $\int 6x\cos(5x)\,dx$

S1 - $u = 6x \rightarrow dv = \cos(5x)\,dx$ S2 - $u = 6x$, so $\frac{du}{dx} = 6 \rightarrow du = 6dx$

S3 - $v = \int \cos(5x)\,dx$. Find v

$w = 5x \rightarrow \frac{dw}{dx} = 5 \rightarrow dw = 5dx \rightarrow dx = \frac{1}{5}dw \quad \int \cos(5x)\,dx \rightarrow \int \cos(w)\frac{1}{5}dw = \frac{1}{5}\int \cos(w)\,dw$

$= \frac{1}{5}\sin(w) = \frac{1}{5}\sin(5x) \rightarrow v = \frac{1}{5}\sin(5x)$

S4 - $\int u\,dv = uv - \int v\,du = 6x \cdot \frac{1}{5}\sin(5x) - \int \frac{1}{5}\sin(5x)\,6\,dx = \frac{6}{5}x\sin(5x) - \frac{6}{5}\int \sin(5x)\,dx$

S5 - $t = 5x \quad \frac{dt}{dx} = 5 \rightarrow dt = 5dx \rightarrow dx = \frac{1}{5}dt \qquad \int \sin(5x)\,dx = \int \sin(t)\frac{1}{5}dt = \frac{1}{5}\int \sin(t)\,dt = \frac{1}{5}[-\cos(t)]$

$\int \sin(5x)\,dx = \frac{1}{5}[-\cos(5x)]$

S6 - $\int 6x\cos(5x)\,dx =$

$= \frac{6}{5}x\sin(5x) - \frac{6}{5}\int \sin 5x\,dx = \frac{6}{5}x\sin(5x) - \frac{6}{5}\left(\frac{1}{5}[-\cos(5x)]\right) + C$

$2.\ \int \cos^{-1}(x)\,dx = \int \arccos(x)\,dx$

remember that dx can be integrated!

$u = \arccos(x)$ $\quad dv = dx$

$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$ $\quad \to v = x$

(only in $(-1, 1)$)

$du = -\frac{1}{\sqrt{1-x^2}}\,dx$

$\int u\,dv = uv - \int v\,du \to \int \arccos(x)\,dx = \arccos(x)\cdot x - \int x \cdot -\frac{1}{\sqrt{1-x^2}}\,dx$

$= x\arccos(x) - \int -\frac{x}{\sqrt{1-x^2}}\,dx = \arccos(x)\cdot x + \int \frac{x}{\sqrt{1-x^2}}\,dx$

Integrate this w/ u sub $-$ $w = 1 - x^2$

$w = 1 - x^2$ $\quad \to \frac{dw}{dx} = -2x \to dw = -2x\,dx$

$w + x^2 = 1$ $\quad \to dx = \frac{1}{-2x}\,dw$

$x^2 = 1 - w$

$x = \sqrt{1-w}$

$\int \sqrt{1-w} \cdot \frac{1}{\sqrt{w}} \cdot \frac{1}{-2x}\,dw = \int \frac{1}{\sqrt{w}} \cdot \frac{1}{-2\sqrt{1-w}} \cdot \sqrt{1-w} \cdot dw$

$\int \frac{1}{\sqrt{w}} \cdot \frac{1}{-2\sqrt{1-w}} \cdot \sqrt{1-w} \cdot dw$

$\frac{1\cdot\sqrt{1-w}}{-2\sqrt{1-w}} = \frac{1}{-2}$ $\quad \to \int \frac{1}{\sqrt{w}} \cdot \frac{1}{-2} \cdot dw =$

$\to -\frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw = -\frac{1}{2}\int w^{-1/2}\,dw$

$\int w^{-1/2}\,dw = 2w^{1/2} + C$

So we get $-\frac{1}{2} \cdot 2\sqrt{w} + C = -\sqrt{w} + C$

$= -\sqrt{1-x^2} + C$

$\arccos(x)\cdot x + -\sqrt{1-x^2} + C$

$= \arccos(x)\cdot x - \sqrt{1-x^2} + C$

3

Practice - $\int e^{7\theta}\sin(8\theta)\,d\theta$ $= I$

S1 - $u = \sin(8\theta)$, $dv = e^{7\theta}\,d\theta$

$u = \sin(8\theta)$ → $\frac{d}{d?}\sin(?) = \cos(?)$

$\frac{d}{d\theta}8\theta = 8$

$\frac{du}{d\theta}$ = chain rule

$= \cos(8\theta)\cdot 8 = 8\cos(8\theta)$

$du = 8\cos(8\theta)\,d\theta$

$dv = e^{7\theta}\,d\theta$

$v = \int e^{7\theta}\,d\theta$ . . . w sub

$w = 7\theta \rightarrow \frac{dw}{d\theta} = 7 \rightarrow d\theta = \frac{1}{7}dw$

$v = \int e^{w}\,\frac{1}{7}\,dw = \frac{1}{7}\int e^{w}\,dw$

$= \frac{1}{7}e^{w}$

$v = \frac{1}{7}e^{7\theta}$

S2 - $\int e^{7\theta}\sin(8\theta)\,d\theta = \sin(8\theta)\cdot\frac{1}{7}e^{7\theta} - \int \frac{1}{7}e^{7\theta}\,8\cos(8\theta)\,d\theta$

$= \frac{1}{7}\sin(8\theta)e^{7\theta} - \frac{8}{7}\int e^{7\theta}\cos(8\theta)\,d\theta$

S3 → use IBP on $\int e^{7\theta}\cos(8\theta)\,d\theta$ where $u = \cos(8\theta)$ and $dv = e^{7\theta}\,d\theta$

$u = \cos(8\theta)$

$dv = e^{7\theta}\,d\theta$

$\frac{du}{d\theta} = -\sin(8\theta)\cdot 8$

↳ $du = 8(-\sin(8\theta))\,d\theta$

$v = \int e^{7\theta}\,d\theta = \frac{1}{7}e^{7\theta}$

$v = \frac{1}{7}e^{7\theta}$

$\int e^{7\theta}\cos(8\theta)\,d\theta = \cos(8\theta)\cdot\frac{1}{7}e^{7\theta} - \int \frac{1}{7}e^{7\theta}\cdot 8(-\sin(8\theta))\,d\theta$

$= \frac{1}{7}\cos(8\theta)e^{7\theta} - \frac{8}{7}\int e^{7\theta}\cdot -\sin(8\theta)\,d\theta$

$= \sin(8\theta)\cdot -1$

$= \frac{1}{7}\cos(8\theta)e^{7\theta} + \frac{8}{7}\int e^{7\theta}\sin(8\theta)\,d\theta$

S4 $I = \int e^{7\theta}\sin(8\theta)\,d\theta$

$I = \sin(8\theta)\cdot\frac{1}{7}\cdot e^{7\theta} \underbrace{-\frac{8}{7}}\Big[\underbrace{\frac{1}{7}\cos(8\theta)e^{7\theta}}_{\text{mult}} + \underbrace{\frac{8}{7}I}_{\text{mult}}\Big] + C_1$

$I = \sin(8\theta)\cdot\frac{1}{7}e^{7\theta} - \frac{8}{49}\cos(8\theta)e^{7\theta} - \frac{64}{49}I + C_1$

$\Big[-\frac{8}{7}\cdot\frac{1}{7}\cos(8\theta)e^{7\theta} + -\frac{8}{7}\cdot\frac{8}{7}\cdot I$

$= -\frac{8}{49}\cdot\cos(8\theta)e^{7\theta} - \frac{64}{49}I$

$+\frac{64}{49}I$ $+\frac{64}{49}I$

↳ $I = \frac{49}{49}I \rightarrow \frac{49}{49}I + \frac{64}{49}I = \frac{113}{49}I$

$\frac{113}{49}I = \sin(8\theta)\,\frac{1}{7}e^{7\theta} - \frac{8}{49}\cos(8\theta)e^{7\theta} + C_1$

S5- Now we have $\int e^{7\theta}\sin(8\theta)\,d\theta = I$ ... $C = \frac{49}{113}C_1$

$\frac{113}{49} I = \frac{1}{7}e^{7\theta}\sin(8\theta) - \frac{8}{49}e^{7\theta}\cos(8\theta) + C_1$

$C_1 = \frac{113}{49}C$

$\frac{113}{49} I = \frac{1}{7}e^{7\theta}\sin(8\theta) - \frac{8}{49}e^{7\theta}\cos(8\theta) + \frac{113}{49}C$

$\frac{113}{49} I - \frac{113}{49}C = \frac{1}{7}e^{7\theta}\sin(8\theta) - \frac{8}{49}e^{7\theta}\cos(8\theta)$

$\frac{113}{49}(I - C) = \left(\frac{1}{7}e^{7\theta}\sin(8\theta) - \frac{8}{49}e^{7\theta}\cos(8\theta)\right)\cdot\frac{49}{113}$

$\div \frac{113}{49}$ (aka $\times \frac{49}{113}$)

$$I - C = \frac{\left[\frac{1}{7}e^{7\theta}\sin(8\theta) - \frac{8}{49}e^{7\theta}\cos(8\theta)\right]49}{113}$$

$$I = \frac{\left[\frac{1}{7}e^{7\theta}\sin(8\theta) - \frac{8}{49}e^{7\theta}\cos(8\theta)\right]49}{113} + C$$

Actual - $\int e^{3\theta}\sin(4\theta)\,d\theta$

$u = \sin(4\theta)$ $\quad dv = e^{3\theta}\,d\theta$

$\frac{du}{d\theta} = 4\cos(4\theta)$ $\quad v = \int e^{3\theta}\,d\theta$

$du = 4\cos(4\theta)\,d\theta$ $\quad v = \frac{1}{3}e^{3\theta}$

$\int e^{3\theta}\sin(4\theta)\,d\theta = \sin(4\theta)\frac{1}{3}e^{3\theta} - \int \frac{1}{3}e^{3\theta}\,4\cos(4\theta)\,d\theta$

$= \frac{1}{3}\sin(4\theta)e^{3\theta} - \frac{4}{3}\int e^{3\theta}\cos(4\theta)\,d\theta$

$\int e^{3\theta}\cos(4\theta)\,d\theta$

$u = \cos(4\theta)$ $\quad dv = e^{3\theta}\,d\theta$

$\frac{du}{d\theta} = 4(-\sin(4\theta))$ $\quad v = \int e^{3\theta}\,d\theta$

$du = 4(-\sin(4\theta))\,d\theta$ $\quad v = \frac{1}{3}e^{3\theta}$

$\cos(4\theta)\frac{1}{3}e^{3\theta} - \int \frac{1}{3}e^{3\theta}\,4(-\sin(4\theta))\,d\theta$

$\hookrightarrow = \cos(4\theta)\frac{1}{3}e^{3\theta} - \frac{4}{3}\int e^{3\theta}(-\sin(4\theta))\,d\theta = \frac{1}{3}e^{3\theta}\cos(4\theta) + \frac{4}{3}\int e^{3\theta}\sin(4\theta)\,d\theta$

$I = \frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{3}\left[\frac{1}{3}e^{3\theta}\cos(4\theta) + \frac{4}{3}I\right] + C_1$

$\left[\frac{4}{3}\cdot\frac{1}{3}\cdot e^{3\theta}\cos 4\theta\right] + \left[-\frac{4}{3}\cdot\frac{4}{3}I\right]$

$\to \frac{4}{9}e^{3\theta}\cos 4\theta + -\frac{16}{9}I$

$I = \frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta) - \frac{16}{9}I + C_1 \to \frac{9}{9}I + \frac{16}{9}I = \frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta) + C_1$

$+\frac{16}{9}I$ $\quad C = \frac{9}{25}C_1 \to C_1 = \frac{25}{9}C$ $\quad +\frac{16}{9}I$ $\quad = \frac{25}{9}I = \frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta) + C_1$

$\frac{25}{9}I = \frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta) + \frac{25}{9}C \to \frac{25}{9}I - \frac{25}{9}C = \frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta)$

$\frac{25}{9}(I - C) = \frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta) \to I - C = \left(\frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta)\right)\cdot\frac{9}{25}$

$\div 25/9$ aka $\times 9/25$

$$I - C = \frac{9\left(\frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta)\right)}{25} \to I = \frac{9\left(\frac{1}{3}e^{3\theta}\sin(4\theta) - \frac{4}{9}e^{3\theta}\cos(4\theta)\right)}{25} + C$$

4

First make a substitution and then use IBP to evaluate: $\int_0^{\pi} e^{\cos(t)} \sin(2t)\,dt$

$\int_0^{\pi} e^{\cos(t)} \sin(2t)\,dt$ → based on trig identities, $\sin(2t) = 2\sin(t)\cos(t)$

$\int_0^{\pi} e^{\cos(t)} 2\sin(t)\cos(t)\,dt = 2\int_0^{\pi} e^{\cos(t)} \sin(t)\cos(t)\,dt$

$w = \cos(t)$ $\frac{dw}{dt} = -\sin(t) \to dt = \frac{1}{-\sin(t)}dw$

$2\int_0^{\pi} e^{w} \sin(t)\, w \left(\frac{1}{-\sin(t)}\right) dw \to 2\int_0^{\pi} e^{w}\, w \frac{\sin(t)}{-\sin(t)}\, dw = 2\int_0^{\pi} e^{w} w (-1)\, dw$

$-2\int_0^{\pi} we^{w}\, dw$ $u = w$ $\frac{du}{dw} = 1$ $du = dw$ $dv = e^{w}\,dw$

$v = \int e^{w}\,dw = e^{w}$

$\int we^{w}\,dw = we^{w} - \int e^{w}\,dw$ $\int e^{w}\,dw = e^{w} + C$

$= we^{w} - e^{w} + C$

$-2\int_0^{\pi} we^{w}\,dw = -2(we^{w} - e^{w} + C)\Big|_0^{\pi} = -2(\cos(t)e^{\cos(t)} - e^{\cos(t)} + C)\Big|_0^{\pi}$

$-2(\cos(\pi)e^{\cos(\pi)} - e^{\cos(\pi)} + C)$ $\cos(\pi) = -1$

$-2(-1 \cdot e^{-1} - e^{-1} + C) = -2(-e^{-1} - e^{-1} + C) = -2(-2e^{-1} + C)$

$= 4e^{-1} - 2C$

$-2(\cos(0)e^{\cos(0)} - e^{\cos(0)} + C)$ $\cos(0) = 1$

$-2(1e^{1} - e^{1} + C) = -2(e - e + C) = -2C$ Now subtract

$= 4e^{-1} - 2C - (-2C)$ Finding difference for definite

$= 4e^{-1} - 2C + 2C = 4e^{-1}$

5

practice − $\int_1^4 x f''(x)\,dx$ $f(1)=4$, $f'(1)=7$, $f'(4)=5$, $f(4)=5$

f'' is continuous

$\int_1^4 x f''(x)\,dx$ $u=x$ $dv=f''(x)dx$

$du=dx$ $v=\int f''(x)dx = f'(x)$

$\to = x f'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx = ((4\cdot f'(4)) - (1\cdot f'(1)) - f(x)\Big|_1^4$

$\to f(x)\Big|_1^4$

$((4\cdot f'(4)) - f'(1)) - (f(4) - f(1))$

$((4\cdot 5) - (7)) - (5-4)$

$= 20 - 7 - 1 = \boxed{12}$ ans = 22

Actual − $\int_1^4 x f''(x)$ $f(1)=1$, $f(4)=8$

$f'(1)=7$ $f'(4)=4$

$\int_1^4 x f''(x)\,dx$ $u=x$ $dv=f''(x)dx$

$du=dx$ $v=\int f''(x)dx = f'(x)$

$\to = x f'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx = ((4\cdot f'(4)) - (1\cdot f'(1)) - f(x)\Big|_1^4$

$\to f(x)\Big|_1^4$

$((4 f'(4)) - f'(1)) - (f(4) - f(1))$

$((4\cdot 4) - 7) - (8-1)$

$(16-7) - (7) =$

$9 - 7 = \boxed{2}$ ans = 2

6

Practice $-\int p^7 \ln(p)\, dp$

$u = \ln(p)$ $\quad dv = p^7\, dp$

$\frac{du}{dp} = \frac{1}{p}$ $\quad v = \int p^7\, dp = \frac{1}{8}p^8$

$du = \frac{1}{p}\, dp$ $\quad v = \frac{1}{8}p^8$

$$\int \ln(p) p^7\, dp = \ln(p)\frac{1}{8}p^8 - \int \frac{1}{8}p^8 \frac{1}{p}\, dp$$

$$= \frac{1}{8}p^8 \ln(p) - \frac{1}{8}\int p^8 \frac{1}{p}\, dp$$

$$= \frac{1}{8}p^8 \ln(p) - \frac{1}{8}\int \frac{p^8}{p}\, dp$$

$$= \frac{1}{8}p^8 \ln(p) - \frac{1}{8}\int p^7\, dp$$

$\int p^7\, dp = \frac{1}{8}p^8 + C$

$$= \frac{1}{8}p^8 \ln(p) - \frac{1}{8}\left[\frac{1}{8}p^8 + C\right]$$

Actual $-\int p^2 \ln(p)\, dp$

$u = \ln(p)$ $\quad dv = p^2\, dp$

$\frac{du}{dp} = \frac{1}{p}$ $\quad v = \frac{1}{3}p^3$

$du = \frac{1}{p}\, dp$

$$\int \ln(p) p^2\, dp = \ln(p)\frac{1}{3}p^3 - \int \frac{1}{3}p^3 \frac{1}{p}\, dp$$

$$= \frac{1}{3}p^3 \ln(p) - \frac{1}{3}\int p^3 \frac{1}{p}\, dp$$

$$= \frac{1}{3}p^3 \ln(p) - \frac{1}{3}\int \frac{p^3}{p}\, dp$$

$$= \frac{1}{3}p^3 \ln(p) - \frac{1}{3}\int p^2\, dp$$

$= \frac{1}{3}p^3 + C$

$$= \frac{1}{3}p^3 \ln(p) - \frac{1}{3}\left[\frac{1}{3}p^3 + C\right]$$