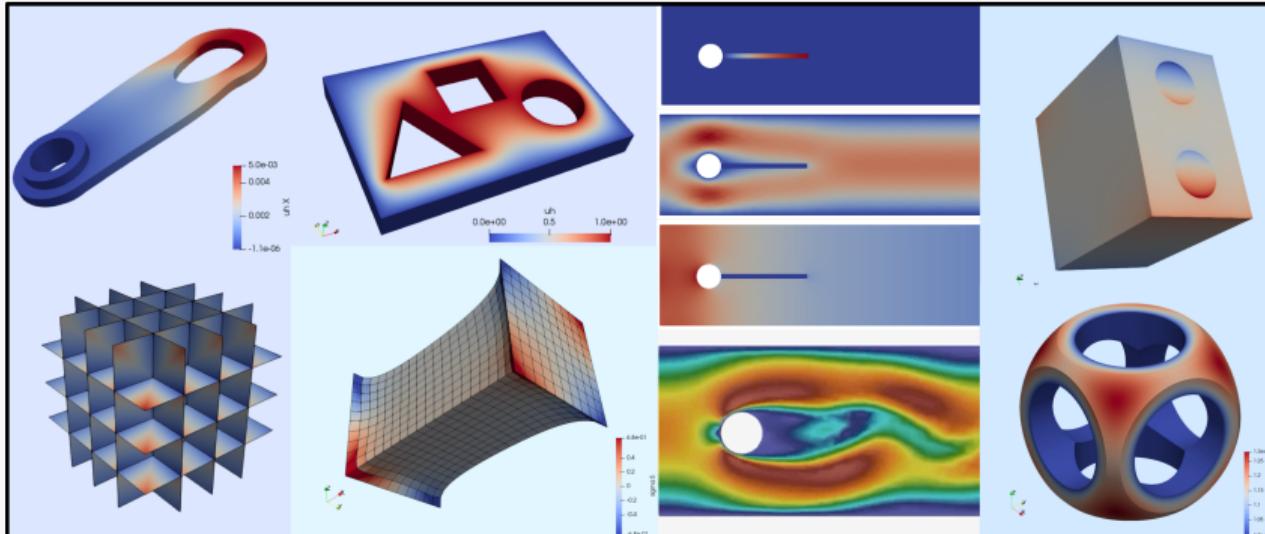


Introduction to the finite element method



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Tutorial at DD29 | Milano 22-06-2025



Introduction to FEM

Elasticity

Nonlinear problems

Time-dependent problems

Multifield problems

Div-conforming FEM

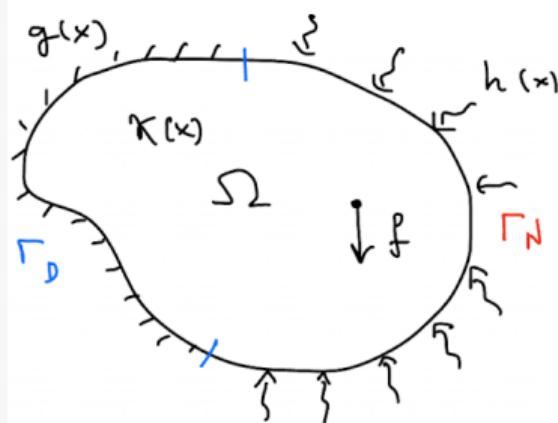
Introduction

Probably, you are familiar with the strong form of PDEs

Example: Poisson equation

$$-\nabla \cdot (\kappa \nabla u) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_D \quad -\kappa \nabla u \cdot n = h \quad \text{on } \Gamma_N$$

- ▶ $\Omega \subset \mathbb{R}^D$ is the physical domain,
- ▶ Γ_D is the Dirichlet boundary,
- ▶ Γ_N is the Neumann boundary



Weak form

PDEs can alternatively be written in weak form

Procedure:

1. Multiply the strong form by a test function v
2. Integrate by parts
3. Apply boundary conditions

$$-\int_{\Omega} v \nabla \cdot (\kappa \nabla u) = \int_{\Omega} \nabla v \cdot (\kappa \nabla u) - \int_{\partial\Omega} v \kappa \nabla u \cdot n = \int_{\Omega} \nabla v \cdot (\kappa \nabla u) - \int_{\Gamma_N} vh$$

using that $v = 0$ on Γ_D and $\kappa \nabla u \cdot n = h$ on Γ_N

Weak form (II)

Example: Poisson equation

$$\text{Find } u \in V^D : \int_{\Omega} \nabla v \cdot (\kappa \nabla u) = \int_{\Omega} vf + \int_{\Gamma_N} vh, \quad \forall v \in V^0$$

where V is a function space (crucial for well-posedness) and

$$V^D = \{v \in V : v = g \text{ on } \Gamma_D\}, \quad V^0 = \{v \in V : v = 0 \text{ on } \Gamma_D\}$$

are the **trial** and **test** spaces, respectively

The weak form is used in finite element methods

Function spaces

The weak form is a variational solution of a quadratic functional

$$u = \arg \min_{u \in V^D} J(u), \quad J(u) = \int_{\Omega} \kappa |\nabla u|^2 - \int_{\Omega} uf - \int_{\Gamma_N} uh$$

It makes sense to consider V as the space in which $J(u) < \infty$ (well-defined)

$$V = H^1(\Omega) = \{u(x) : \int_{\Omega} |\nabla u|^2 < \infty\}$$

- ▶ V is an infinite-dimensional space of functions
- ▶ We need to discretize the problem to obtain a finite-dimensional system of equations (e.g., using polynomials)

Spectral approximation

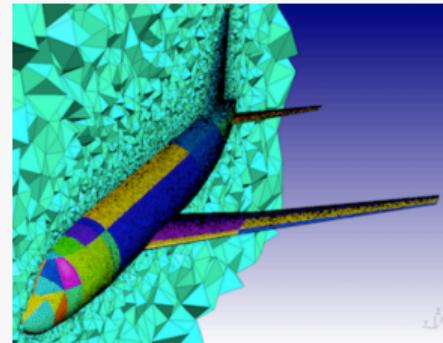
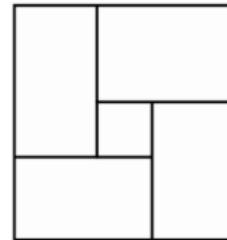
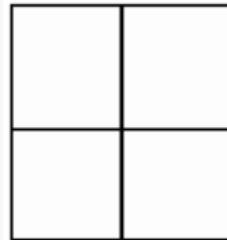
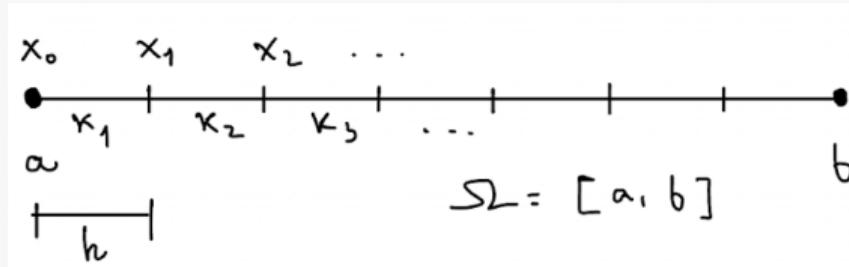
Approximate V by the polynomial space of order p

$$\mathcal{P}_p = \{1, x, x^2, \dots, x^p\}$$

- ▶ Hard to deal with geometries that are not boxes
- ▶ It exploits the smoothness of the solution (Taylor expansion)

Finite element spaces

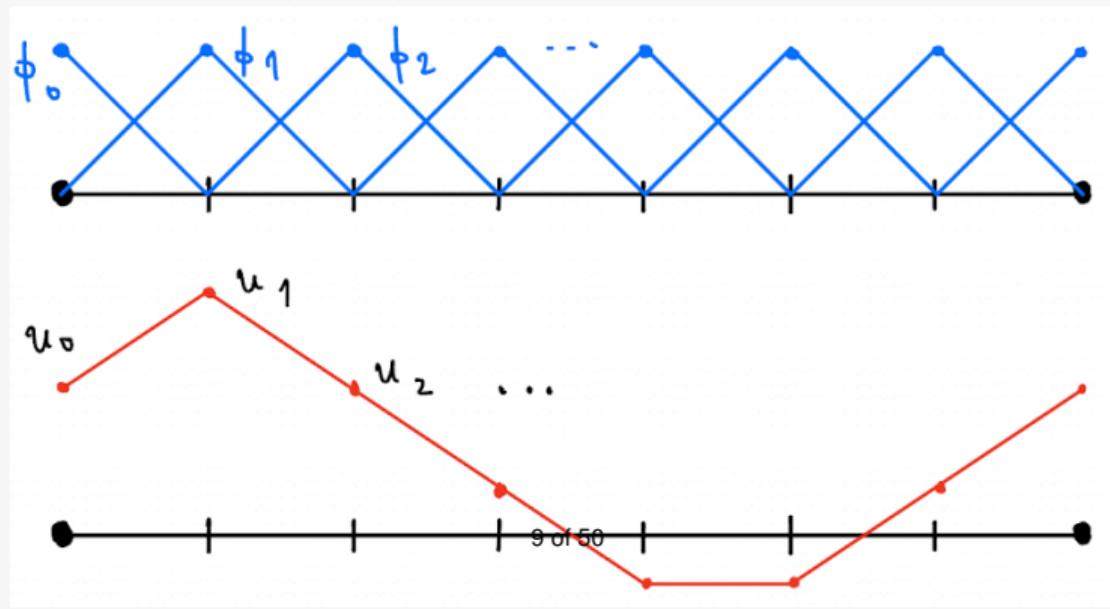
Consider a **mesh** \mathcal{M}_h , i.e., a partition of Ω into elements / cells (segments, triangles or quadrilaterals, tetrahedra or hexahedra, etc.)



FEM space

A **finite element space** $V_h \subset V$ is a space of piecewise polynomials of order p defined on \mathcal{M}_h

$$V_h \doteq \{v_h \in V : v_h|_K \in \mathcal{P}_p, \quad \forall K \in \mathcal{M}_h\}$$



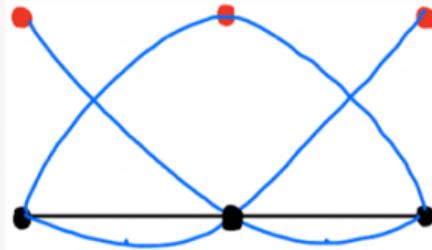
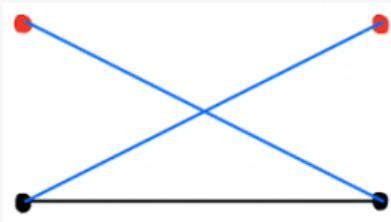
Reference FE

A **reference FE** is composed of:

- ▶ A polytope (triangle, square, etc), the reference cell \hat{K}
- ▶ A reference FE space $\hat{\mathcal{V}}$ of polynomials on \hat{K}
- ▶ The degrees of freedom (DOFs) that define the *shape functions* basis for $\hat{\mathcal{V}}$

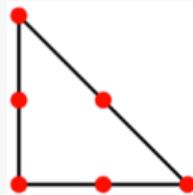
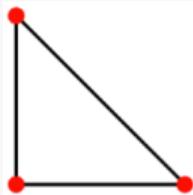
Lagrangian 1D (SEGMENT)

- ▶ $\hat{K} = [0, 1]$,
- ▶ $\hat{\mathcal{V}} = \{1, x, x^2, \dots, x^p\} = \mathcal{P}_p$,
- ▶ DOFs: Nodal values at $\{0, 1/p, 2/p, \dots, 1\}$



Lagrangian 2D (TRI)

- ▶ Triangle with vertices $(0, 0), (1, 0), (0, 1)$
- ▶ $\mathcal{P}_p = \{1, x, y, x^2, xy, y^2, \dots\}$
- ▶ DOFs: Nodal values



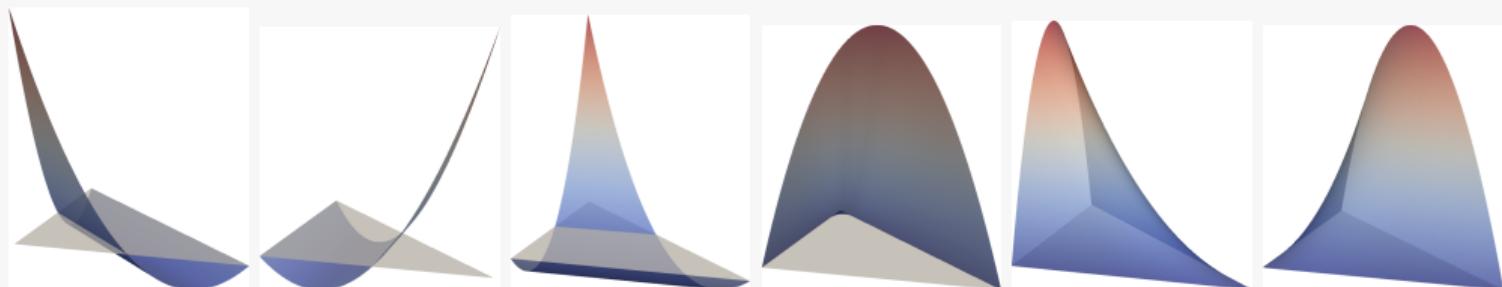
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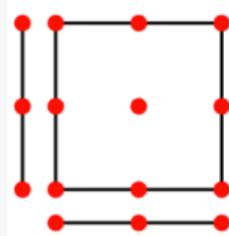
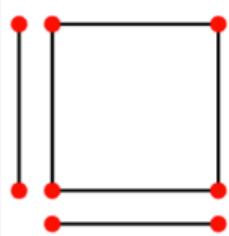
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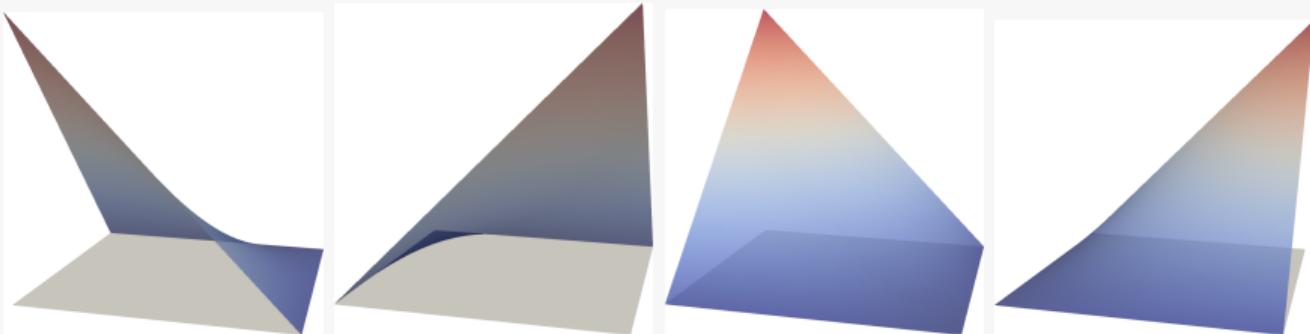
Lagrangian 2D (QUAD)

- ▶ $\hat{K} = [0, 1]^2$,
- ▶ $\mathcal{Q}_p = \{1, x, y, xy, x^2, x^2y, x^2y^2, \dots\}$
- ▶ DOFs: Nodal values



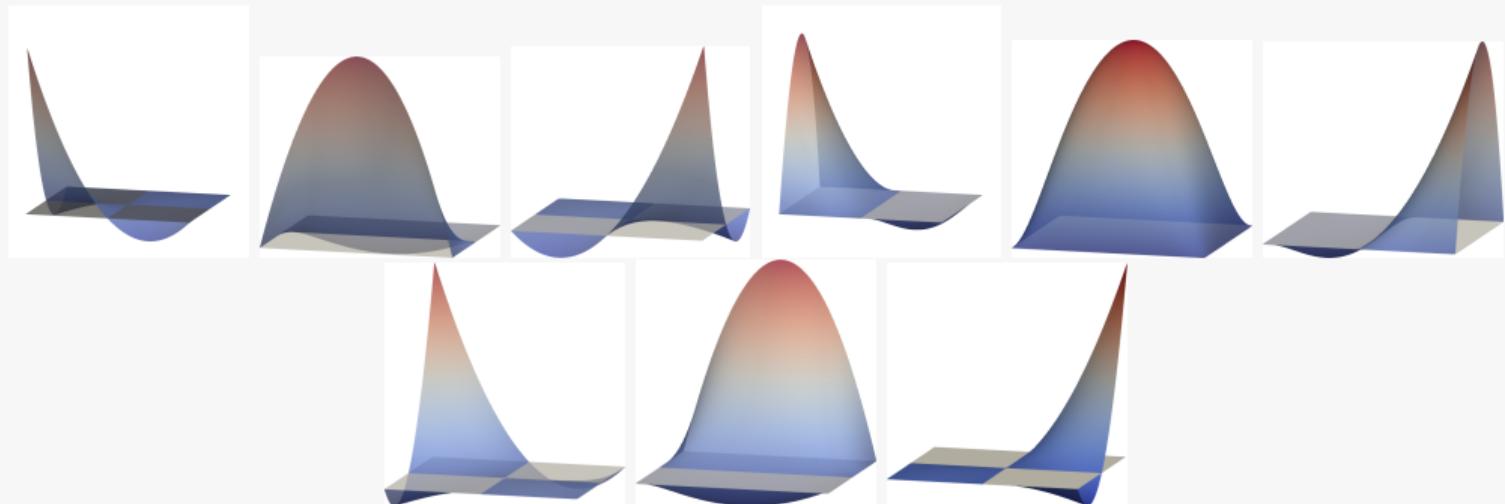
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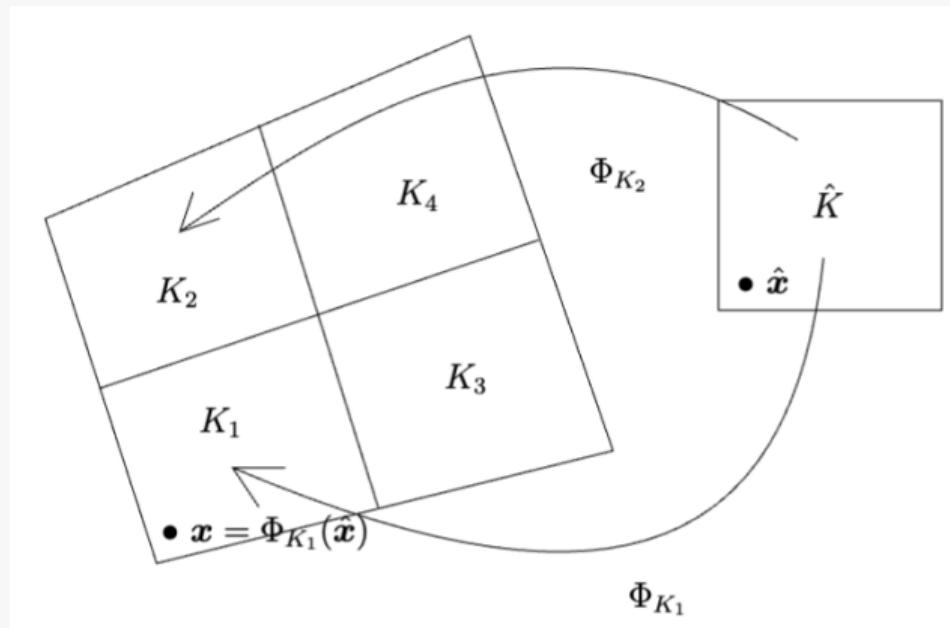
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- ▶ DOFs: Nodal values



From reference to physical space

- ▶ A geometric map $\Phi_K : \hat{K} \rightarrow K$

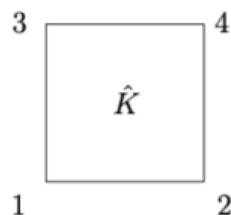


From reference to physical space

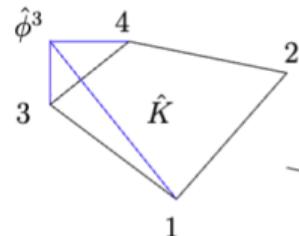
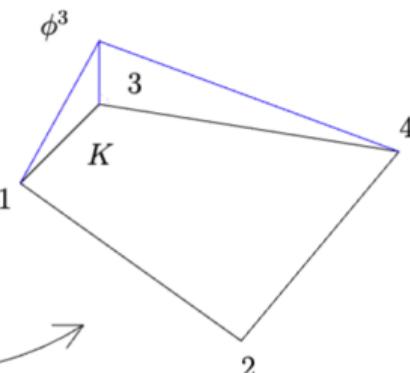
- The space to the physical cell K is

$$V_K = \{\hat{v} \circ \Phi_K^{-1} : \hat{v} \in \mathcal{P}_p(\hat{K})\}$$

$$\begin{aligned}\hat{\mathcal{V}} &\doteq \text{span} \left\{ \hat{b}^1, \hat{b}^2, \hat{b}^3, \hat{b}^4 \right\} \\ &\doteq \text{span} \left\{ 1, \hat{x}, \hat{y}, \hat{x}\hat{y} \right\}\end{aligned}$$



$$\begin{aligned}\mathcal{V} &\doteq \text{span} \{ b^1, b^2, b^3, b^4 \}, \\ \text{where } b^i(\mathbf{x}) &\doteq \hat{b}^i \circ \Phi_K^{-1}(\mathbf{x})\end{aligned}$$



Conformity

Still, there is part of the definition that is not covered:

$$V_h \doteq \{\textcolor{red}{v_h} \in V : v_h|_K \in \mathcal{P}_p, \quad \forall K \in \mathcal{M}_h\}$$

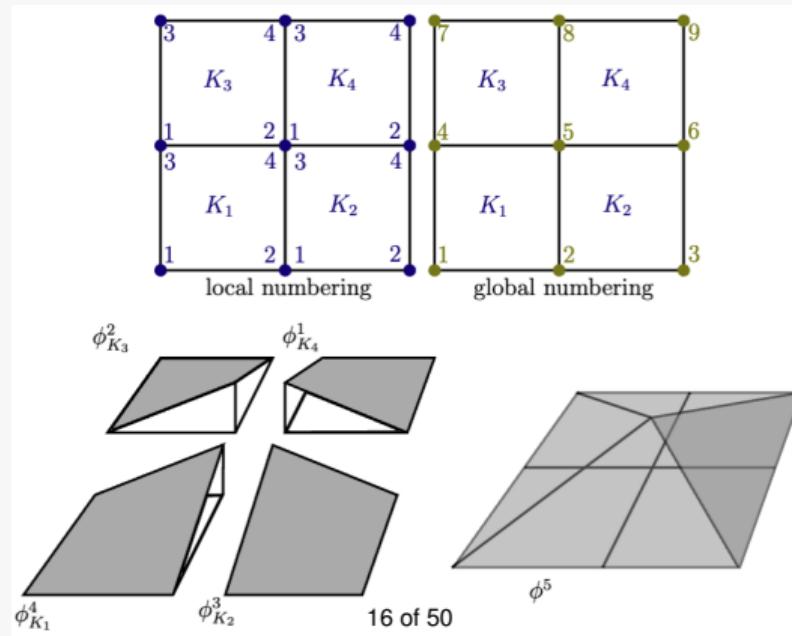
- ▶ A discontinuous piecewise polynomial is not in $H^1(\Omega)$
- ▶ $\int_{\Omega} |\nabla u_h|^2$ is not bounded
- ▶ Piecewise polynomials in $C^0(\Omega)$ are in $H^1(\Omega)$

Enforcing continuity

We must enforce continuity for $V_h \subset H^1(\Omega)$

$$V_h \doteq \{\mathbf{v}_h \in C^0(\Omega) : v_h|_K \in \mathcal{P}_p, \quad \forall K \in \mathcal{M}_h\}$$

Using Lagrangian FEs, we just *assemble / glue* together DOFs of adjacent cells



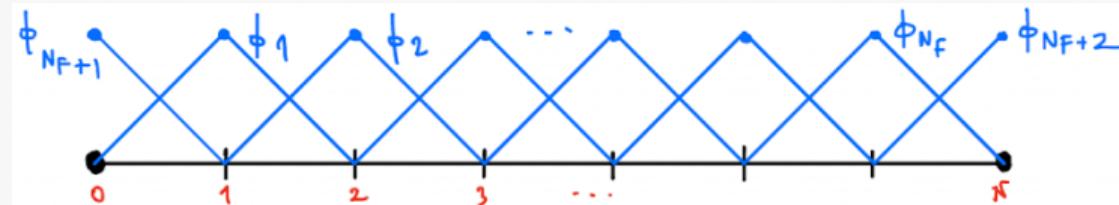
FE Basis

Let us split the Lagrangian nodes \mathcal{N} of the FE space V_h into free nodes \mathcal{N}_F (on $\Omega \subset \Gamma_D$) and Dirichlet nodes \mathcal{N}_D (on Γ_D)

For each node $i \in \mathcal{N}$, we can consider the shape functions (Lagrangian nodes)

$$\phi^i(\mathbf{x}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

It returns a basis of $V_h = \{\phi^1, \dots, \phi^{N_F}, \phi^{N_F+1}, \dots, \phi^{N_F+N_D}\}$



Galerkin + FEM

FE discretisation of the Poisson equation (using Galerkin method)

$$\text{Find } u_h \in V_h^D : \int_{\Omega} \nabla v_h \cdot (\kappa \nabla u_h) = \int_{\Omega} v_h f + \int_{\Gamma_N} v_h h, \quad \forall v_h \in V_h^0$$

where V_h is a function space (crucial for well-posedness) and

$$V_h^D = \{v_h \in V_h : v_h = g \text{ at nodes on } \Gamma_D\}, \quad V^0 = \{v_h \in V_h : v_h = 0 \text{ at nodes on } \Gamma_D\}$$

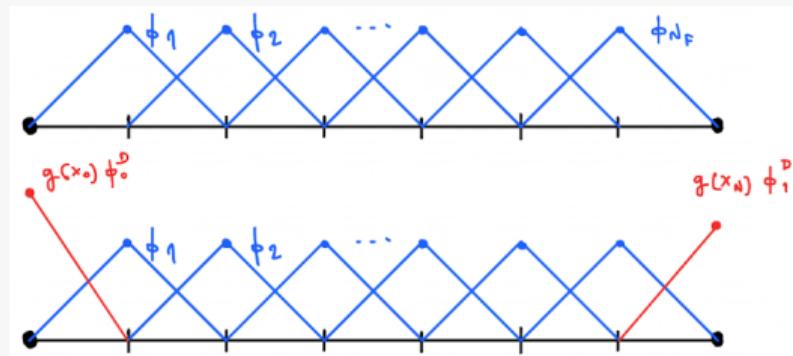
are the **trial** and **test** FE spaces, respectively

Trial/test FE basis

- ▶ Test space: A basis for $V_h^0 = \{\phi^1, \dots, \phi^{N_F}\}$
- ▶ Trial space: Any function $u_h \in V_h^D$ can be written as

$$u_h = u_h^0 + u_h^D, \quad u_h^0 = \sum_{i=1}^{N_F} \mathbf{u}^i \phi^i \in V_h^0, \quad u_h^D = \sum_{i=N_F+1}^{N_F+N_D} g(\mathbf{x}_i) \phi^i$$

where u_h^D is data (g is given) and $\mathbf{u} \in \mathbb{R}^{N_F}$ is the unknown vector of coefficients

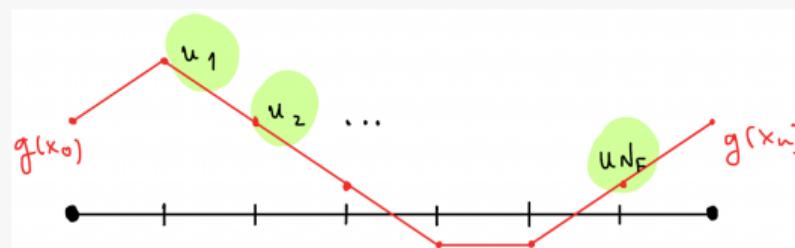


Trial/test FE basis

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Linear system (I)

- ▶ Galerkin formulation:

$$\text{Find } u_h \in V_h^D : \int_{\Omega} \nabla v_h \cdot (\kappa \nabla u_h) = \int_{\Omega} v_h f + \int_{\Gamma_N} v_h h, \quad \forall v_h \in V_h^0$$

- ▶ Using the FE basis for the test space

$$\text{Find } u_h \in V_h^D : \int_{\Omega} \nabla \phi^i \cdot (\kappa \nabla u_h) = \int_{\Omega} \phi^i f + \int_{\Gamma_N} \phi^i h, \quad \forall i = 1, \dots, N_F$$

Linear system (I)

- Galerkin formulation:

$$\text{Find } u_h \in V_h^D : \int_{\Omega} \nabla v_h \cdot (\kappa \nabla u_h) = \int_{\Omega} v_h f + \int_{\Gamma_N} v_h h, \quad \forall v_h \in V_h^0$$

- Using the decomposition $u_h = u_h^0 + u_h^D$

$$\text{Find } \mathbf{u} \in \mathbb{R}^{N_F} : \left[\int_{\Omega} \nabla \phi^i \cdot (\kappa \nabla \phi^j) \right] \mathbf{u}^j = \int_{\Omega} \phi^i f + \int_{\Gamma_N} \phi^i h - \int_{\Omega} \nabla \phi^i \cdot (\kappa \nabla u_h^D),$$

$$\forall i = 1, \dots, N_F$$

- We end up with a linear system to be solved $\mathbf{A}\mathbf{u} = \mathbf{b}$

Linear system(II)

- ▶ One can compute all the integrals using a **quadrature rule** \mathcal{Q} at the reference element, e.g.,

$$\begin{aligned}\int_{\Omega} \nabla \phi^i \cdot (\kappa \nabla \phi^j) &= \sum_{K \in \mathcal{M}_h} \int_{\hat{K}} J_K^{-T} \hat{\nabla} \hat{\phi}^i \cdot (\kappa J_K^{-T} \hat{\nabla} \hat{\phi}^j) \det(J_K) \\ &= \sum_{K \in \mathcal{M}_h} \sum_{\hat{\mathbf{x}}_{\text{gp}} \in \mathcal{Q}} J_K^{-T} \hat{\nabla} \hat{\phi}^i \cdot (\kappa \circ \Phi_K J_K^{-T} \hat{\nabla} \hat{\phi}^j) \det(J_K)|_{\hat{\mathbf{x}}_{\text{gp}}} w_{\text{gp}}\end{aligned}$$

where $J_k = \nabla \Phi_K$.

- ▶ Usually, we use a Gaussian quadrature \mathcal{Q} that integrates exactly the matrix terms of the linear system
- ▶ The **degree** of the quadrature is the maximum order of a polynomial that can be integrated exactly (e.g., $2p$ for FE spaces of order p and a linear geometrical map)

Linear system (III)

- ▶ We have started with a PDE in weak form (∞ dimensional space V)
- ▶ Using a FE space V_h (finite dimensional polynomial space) we have transformed it into a linear system $\mathbf{A}\mathbf{u} = \mathbf{b}$
- ▶ This approximation comes with the price of a **numerical error**

Bounds for numerical errors

Let us define the L^2 and H^1 norms of a function u

$$\|u\|_{L^2(\Omega)} = \left(\int_{\Omega} u^2 \right)^{1/2}, \quad \|u\|_{H^1(\Omega)} = \left(\int_{\Omega} |\nabla u|^2 + \int_{\Omega} u^2 \right)^{1/2},$$

- ▶ The **discretisation error** $e_h = u - u_h$ can be bounded by

$$\|e_h\|_{H^1(\Omega)} \leq Ch^q \|u\|_{H^{q+1}(\Omega)}, \quad \|e_h\|_{L^2(\Omega)} \leq Ch^{q+1} \|u\|_{H^{q+1}(\Omega)}$$

for any $q \leq p$, where h is the mesh size and p is the order of the FE space

- ▶ The H^{p+1} norm means the L^2 -norm of all the derivatives up to $p+1$ (requires **smoothness**)

Bounds for numerical errors (II)

Assuming the solution is smooth enough ($q = p$),

$$\|e_h\|_{H^1(\Omega)} \leq C_u h^p, \quad \|e_h\|_{L^2(\Omega)} \leq C_u h^{p+1}$$

Thus,

$$\log \|e_h\|_{H^1(\Omega)} \leq C + p \log h, \quad \log \|e_h\|_{L^2(\Omega)} \leq C + (p + 1) \log h$$

We can check these bounds experimentally in the tutorials

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Linear elasticity

Linear elasticity (strong form):

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \text{ in } \Omega, \\ \mathbf{u} = \mathbf{g} \text{ on } \Gamma_D, \\ \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{h} \text{ on } \Gamma_N. \end{cases}$$

where \mathbf{u} is the displacement **vector** and $\sigma(\mathbf{u})$ is the stress 2-tensor defined as

$$\sigma(\mathbf{u}) \doteq \lambda \operatorname{tr}(\varepsilon(\mathbf{u})) I + 2\mu \varepsilon(\mathbf{u}), \quad \varepsilon(\mathbf{u}) \doteq \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$$

Weak form (elasticity)

Times \mathbf{v} the strong form and integrate by parts:

$$\begin{aligned}\int_{\Omega} \mathbf{v} \cdot (\nabla \cdot \boldsymbol{\sigma}(\mathbf{u})) &= \int_{\Omega} \nabla \mathbf{v} : \boldsymbol{\sigma}(\mathbf{u}) - \int_{\partial\Omega} \mathbf{v} \cdot \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \int_{\Omega} \nabla \mathbf{v} : \boldsymbol{\sigma}(\mathbf{u}) - \int_{\Gamma_N} \mathbf{v} \cdot \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} \\ &= \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\sigma}(\mathbf{u}) - \int_{\Gamma_N} \mathbf{v} \cdot \mathbf{h}\end{aligned}$$

Weak form (elasticity)

We get the weak form:

$$\text{Find } \mathbf{u} \in \mathbf{V}^D : \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\sigma}(\mathbf{u}) = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} + \int_{\Gamma_N} \mathbf{v} \cdot \mathbf{h}, \quad \forall \mathbf{v} \in \mathbf{V}^0$$

- ▶ $\mathbf{V} = [H^1(\Omega)]^D$ (Korn's inequality),
- ▶ $\mathbf{V}^D = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} = \mathbf{g} \text{ on } \Gamma_D\}$ is the trial space
- ▶ $\mathbf{V}^0 = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}$ is the test space

Finite element space

- ▶ We want a FE space $\mathbf{V}_h \subset \mathbf{V} = [H_0^1(\Omega)]^D$
- ▶ Same conformity, i.e., $\mathbf{V}_h \subset [C^0(\Omega)]^D$
- ▶ $\mathbf{V}_h = [V_h]^D$, where V_h is the scalar FE space of the previous section
- ▶ All the ideas in the previous section readily apply for each component

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Nonlinear problems

Let us consider a nonlinear model problem, p-Laplacian:

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$

where $p \geq 2$ is a given parameter

The weak form is

$$\text{Find } u \in V^D : \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_{\Omega} fv, \quad \forall v \in V^0$$

- ▶ Same conformity as Poisson, $V^D = H^1(\Omega)$

Newton's method

- ▶ Nonlinear problem wrt u
- ▶ One can use Newton's method to solve it
- ▶ We want to solve $f(u) = 0$ iteratively
- ▶ Using the fact that $f(u + \delta u) \approx f(u) + f'(u)\delta u$
- ▶ Given u^i

$$f'(u^i)\delta u^{i+1} = -f(u^i), \quad u^{i+1} = u^i + \delta u^{i+1}$$

till convergence

Residual and Jacobian

We better state the problem in terms of the residual:

$$u \in V^D : r(u, v) = \int_{\Omega} \nabla v \cdot (|\nabla u|^{p-2} \nabla u) \, d\Omega - \int_{\Omega} v f \, d\Omega = 0, \quad \forall v \in V^0$$

We compute the variation of the residual wrt a given direction $\delta u \in V^0$ at $u \in V^D$

$$r(u + \delta u, v) \approx r(u, v) + \frac{\partial r(u, v)}{\partial u} \delta u$$

where $j(\delta u, u, v) = \frac{\partial r(u, v)}{\partial u}$ is the **Jacobian** evaluated at $u \in U_g$, which is the bilinear form

$$[j(u, v)\delta u] = \int_{\Omega} \nabla v \cdot (|\nabla u|^{p-2} \nabla \delta u) \, d\Omega + (p-2) \int_{\Omega} \nabla v \cdot (|\nabla u|^{p-4} (\nabla u \cdot \nabla \delta u) \nabla u) \, d\Omega.$$

Discrete problem

Using Newton + FEM:

$$\text{Find } \delta u_h^{i+1} \in V_h^0 : j(\delta u_h^{i+1}, u_h^i, v_h) = -r(u_h^i, v_h), \quad \forall v_h \in V_h^0$$

- ▶ $j(\delta u_h^{i+1}, u_h^i, v_h)$ is a linear system to be solved at each nonlinear iteration
- ▶ After linearisation, we can apply the same ideas as in the previous section
- ▶ We can compute the expression of the Jacobian by hand or using automatic differentiation

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Heat equation in weak form:

$$\text{Find } u \in V^D : \int_{\Omega} v \partial_t u + \int_{\Omega} \nabla v \cdot (\kappa \nabla u) = \int_{\Omega} vf + \int_{\Gamma_N} vh, \quad \forall v \in V^0$$

Semi-discretised problem (using ideas above, only discretise in space):

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{A}\mathbf{u} = \mathbf{b}, \quad \dot{\mathbf{u}} = -\mathbf{M}^{-1}\mathbf{A}\mathbf{u} + \mathbf{M}^{-1}\mathbf{b}$$

- ▶ $\mathbf{M}_{ij} = \int \phi^i(\mathbf{x})\phi^j(\mathbf{x})$ is the mass matrix
- ▶ $\mathbf{A}_{ij} = \int \kappa(\mathbf{x})\nabla\phi^i(\mathbf{x}) \cdot \nabla\phi^j(\mathbf{x})$ is the stiffness matrix
- ▶ \mathbf{b} is the load vector
- ▶ \mathbf{u} is the vector of unknowns

Time discretisation: Create a 1D partition of the time interval $[0, T]$,
 $\mathcal{T}_h = \{0 = t_0 < t_1 < \dots < t_N = T\}$, with $\Delta t = t_{n+1} - t_n$ and $t_n = n\Delta t$.

- ▶ Backward Euler (Implicit, 1st order)

$$\mathbf{M} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}\mathbf{u}^{n+1} = \mathbf{b}, \quad (\mathbf{M} + \Delta t \mathbf{A}) \mathbf{u}^{n+1} = \Delta t \mathbf{b} + \mathbf{M}\mathbf{u}^n$$

- ▶ Crank-Nicolson (Implicit, 2nd order)

$$\mathbf{M} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}\mathbf{u}^{n+1/2} = \mathbf{b}, \quad (\mathbf{M} + \Delta t/2 \mathbf{A}) \mathbf{u}^{n+1/2} = \Delta t/2 \mathbf{b} + \mathbf{M}\mathbf{u}^n$$

where $\mathbf{u}^{n+1/2} = 1/2(\mathbf{u}^{n+1} + \mathbf{u}^n)$

- ▶ Forward Euler (Explicit, 1st order, conditionally stable, $\Delta t < Ch^2$)

$$\mathbf{M} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{A}\mathbf{u}^n = \mathbf{b}, \quad \mathbf{M}\mathbf{u}^{n+1} = \Delta t \mathbf{b} + \mathbf{M}\mathbf{u}^n - A\mathbf{u}^n$$

- ▶ Runge-Kutta methods (implicit, explicit, IMEX), ...

Computational cost

- ▶ Solve a linear system at each time step
- ▶ Implicit methods, system matrix $\mathbf{M} + c\Delta t \mathbf{A}$
- ▶ Explicit methods, system matrix \mathbf{M} (much cheaper, better conditioned, but stringent condition for stability)

Introduction to FEM

Elasticity

Nonlinear problems

Time-dependent problems

Multifield problems

Div-conforming FEM

Stokes problem

Strong form: Find $\mathbf{u} \in V^D$ and $p \in Q$ such that

$$\begin{cases} -\nabla \cdot \mu \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_D, \\ \mu \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{n} - p \mathbf{n} = \mathbf{h} & \text{on } \Gamma_N. \end{cases}$$

Stokes problem

Testing with $\mathbf{v} \in V^0$ and integrating by parts

$$\begin{aligned}-\int_{\Omega} \mathbf{v} \nabla \cdot \mu \boldsymbol{\varepsilon}(\mathbf{u}) &= \int_{\Omega} \nabla \mathbf{v} : \mu \boldsymbol{\varepsilon}(\mathbf{u}) - \int_{\Gamma_N} \mathbf{v} \cdot \mu \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{n} \\&= \int_{\Omega} \mu \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{u}) - \int_{\Gamma_N} \mathbf{v} \cdot \mu \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{n}\end{aligned}$$

In order to have the right stresses on the Neumann boundary, we integrate by parts the pressure term

$$\int_{\Omega} \nabla p \cdot \mathbf{v} = - \int_{\Omega} p \nabla \cdot \mathbf{v} + \int_{\Gamma_N} p \mathbf{v} \cdot \mathbf{n}$$

Weak form

Adding together with mass conservation, we get the weak form: find $\mathbf{u} \in \mathbf{V}^D$ and $p \in Q$ such that

$$\int_{\Omega} \mu \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{u}) - \int_{\Omega} p \nabla \cdot \mathbf{v} + \int_{\Omega} q \nabla \cdot \mathbf{u} = \int_{\Omega} \mathbf{v} \cdot \mathbf{f} + \int_{\Gamma_N} \mathbf{v} \cdot \mathbf{h}, \quad \forall \mathbf{v} \in \mathbf{V}^0, \forall q \in Q$$

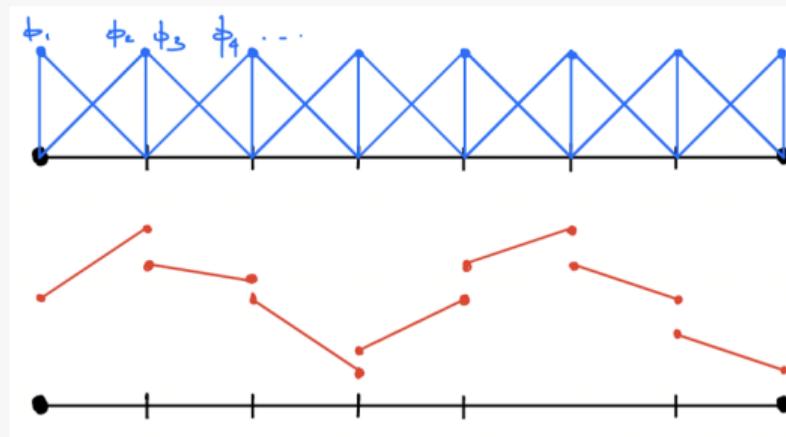
- ▶ $\mathbf{V} = [H^1(\Omega)]^D$ (Korn's inequality),
- ▶ $\mathbf{V}^D = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} = \mathbf{g} \text{ on } \Gamma_D\}$ is the trial space
- ▶ $\mathbf{V}^0 = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}$ is the test space
- ▶ $Q = L^2(\Omega)$ (no derivatives, no continuity required in FEM)

L^2 -conformity

- We need to define a FE space $Q_h \subset Q = L^2(\Omega)$

$$Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in \mathcal{P}_p(K) \text{ or } \mathcal{Q}_p(K) \ \forall K \in \mathcal{M}_h\}$$

- No inter-element continuity required by $L^2(\Omega)$
- Simplified version of the previous section (not gluing required)



Mixed FEM

- ▶ We can use discontinuous FE spaces
- ▶ We can use continuous FE spaces too
- ▶ However, we need to satisfy the so-called inf-sup stability condition

Mixed FEM (II)

Suitable spaces for the Stokes problem:

- ▶ Tris/Tets: $\mathcal{P}_k \times \mathcal{P}_{k-1}$ Taylor-Hood element, $k \geq 2$
- ▶ Quads/Hexs: $\mathcal{Q}_k \times \mathcal{Q}_{k-1}$ Taylor-Hood element, $k \geq 2$
- ▶ Quads/Hexs: $\mathcal{Q}_{k+1} \times \mathcal{P}_k^-$, $k \geq 2$

Note: \mathcal{P}_k^- means discontinuous polynomials of degree k (analogously for \mathcal{Q}_k^-)

Introduction to FEM

Elasticity

Nonlinear problems

Time-dependent problems

Multifield problems

Div-conforming FEM

Darcy equation

Strong form: Find $\mathbf{u} \in V^D$ and $p \in Q$ such that

$$\begin{cases} \mathbf{u} + \kappa \nabla p = \mathbf{0} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = f & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = g & \text{on } \Gamma_D, \\ p = g & \text{on } \Gamma_N. \end{cases}$$

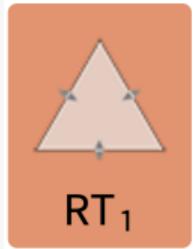
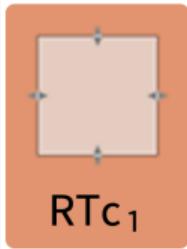
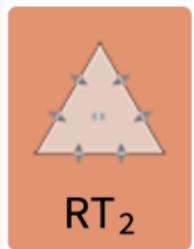
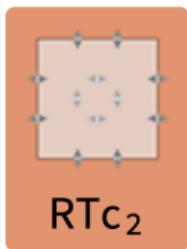
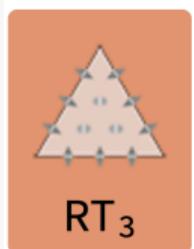
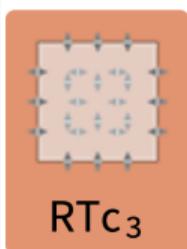
Weak form

Weak form: find $(u, p) \in H(\text{div}; \Omega) \times L^2(\Omega)$ such that $a((u, p), (v, q)) = b(v, q)$ for all $(v, q) \in H(\text{div}; \Omega) \times L^2(\Omega)$, where

$$a((u, p), (v, q)) \doteq \int_{\Omega} v \cdot u \, d\Omega - \int_{\Omega} (\nabla \cdot v) p \, d\Omega + \int_{\Omega} q (\nabla \cdot u) \, d\Omega,$$
$$b(v, q) \doteq \int_{\Omega} q f \, d\Omega - \int_{\Gamma_N} (v \cdot n) g \, d\Gamma.$$

- ▶ Only control on $\nabla \cdot \mathbf{u}$ needed
- ▶ It only implies continuity of $\mathbf{u} \cdot \mathbf{n}$ on element boundaries

Div-conforming FEM

RT₁BDM₁RTc₁RT₂BDM₂RTc₂RT₃BDM₃RTc₃

- ▶ $\mathcal{RT}_k = [\mathcal{P}_k]^D + \mathbf{x}\mathcal{P}_k$
- ▶ $\mathcal{BDM}_k = [\mathcal{P}_k]^D$
- ▶ $\mathcal{RT}_k = \mathcal{Q}_{k+1,k} \times \mathcal{Q}_{k,k+1}$
- ▶ DOFs are normal fluxes on the faces of the elements (for div-conformity) + ...

Stable pairs

Stable flux / pressure spaces:

- ▶ Tris/tets: $\mathcal{RT}_k \times \mathcal{P}_k^-$
- ▶ Tri/tets: $\mathcal{BDM}_{k+1} \times \mathcal{P}_k^-$
- ▶ Quads/hex: $\mathcal{RT}_k \times \mathcal{Q}_k^-$

Inf-sup condition

There exists $\beta > 0$ such that

$$\inf_{q \in Q_h} \sup_{\mathbf{v} \in \mathbf{V}_h^0} \frac{\int_{\Omega} q \nabla \cdot \mathbf{v}}{\|\mathbf{v}\|_{H^1(\Omega)} \|q\|_{L^2(\Omega)}} \geq \beta$$

Analogously for **curl** operator (Maxwell equations, not covered in this tutorial)