(1) RG Pratt

ETP Closed-Form Solution

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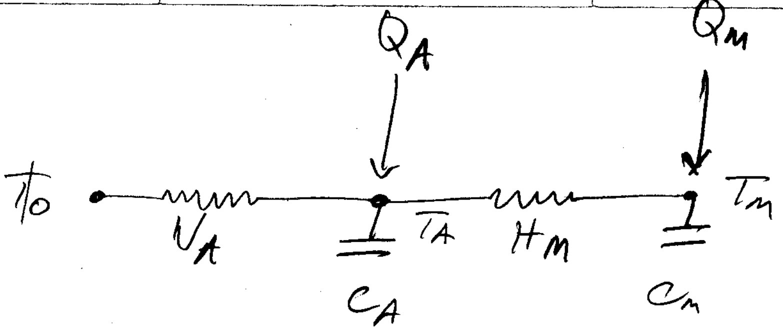

Heat balance on T_A:

$$0 = Q_A - U_A(T_A - T_o) - H_m(T_A - T_m) - C_A \frac{dT_A}{dt} \qquad (1)$$

Re-arranging (1) in the form for solving a differential equation:

$$0 = Q_A - (U_A + H_m) T_A + U_A T_o + H_m T_m - C_A \frac{dT_A}{dt} \qquad (1.1)$$

$$\frac{dT_A}{dt} + \frac{U_A + H_m}{C_A} T_A - \frac{H_m}{C_A} T_m - \frac{U_A}{C_A} T_o - \frac{Q_A}{C_A} = 0 \qquad (1.2)$$

ETP Solution (cont.)

Heat balance on T_M:

$$0 = Q_M - H_M(T_M - T_A) - C_M \frac{dT_M}{dt} \qquad (2)$$

Solving (2) for T_A:

$$0 = Q_M - H_M T_M + H_M T_A - C_M \frac{dT_M}{dt} \qquad (2.1)$$

$$T_A = \frac{C_M}{H_M} \frac{dT_M}{dt} + T_M - \frac{Q_M}{H_M} \qquad (2.2)$$

Differentiating (2.2):

$$\frac{dT_A}{dt} = \frac{C_M}{H_M} \frac{d^2 T_M}{dt} + \frac{dT_M}{dt} \qquad (2.3)$$

ETP Solution (cont.)

Solving (1.2) for T_m gives:

$$T_M = \frac{C_A}{H_m}\frac{dT_A}{dt} + \frac{U_A + H_m}{H_m} T_A - \frac{U_A}{H_m} T_0 - \frac{Q_A}{H_m} \quad (1.3)$$

Differentiating (1.3):

$$\frac{dT_m}{dt} = \frac{C_A}{H_m}\frac{d^2 T_A}{dt} + \frac{U_A + H_m}{H_m}\frac{dT_A}{dt} \quad (1.4)$$

Substituting (1.3) and (1.4) into (2.4) gives:

Re-arranging 2.1 gives:

$$0 = C_m \frac{dT_m}{dt} + H_m T_m - Q_m - H_m T_A \quad (2.4)$$

$$0 = \left[\frac{C_m C_A}{H_m}\frac{d^2 T_A}{dt} + \frac{C_m (U_A + H_m)}{H_m}\frac{dT_A}{dt}\right] + \left[C_A \frac{dT_A}{dt} + (U_A + H_m) T_A - U_A T_0 - Q_A\right] - Q_m - H_m T_A \quad (2.5)$$

$$\frac{C_m C_A}{H_m}\frac{d^2 T_A}{dt} + \left(\frac{C_m (U_A + H_m)}{H_m} + C_A\right)\frac{dT_A}{dt} + U_A T_A = Q_m + Q_A + U_A T_0 \quad (3)$$

ETP Solution (cont.)

(3) is of the form:

$$\frac{C_m C_A}{H_m}\frac{d^2T_A}{dt} + \left(\frac{C_m U_A}{H_m} + C_m + C_A\right)\frac{dT_A}{dt} + U_A T_A =$$

$$Q_m + Q_A + U_A T_0 \qquad (3.1)$$

$$a\frac{d^2T_A}{dt} + b\frac{dT_A}{dt} + cT_A = d \qquad (3.2)$$

where:
$a \equiv \frac{C_m C_A}{H_m}$ $\left\{ \begin{array}{l} b \equiv C_m \frac{U_A}{H_m} + C_m + C_A \quad (3.3a) \\ b = C_m\left(\frac{U_A + H_m}{H_m}\right) + C_A \quad (3.3b) \end{array}\right.$

$c = U_A \quad (3.3c)$

$d \equiv Q_m + Q_A + U_A T_0 \quad (3.3d)$

let $T_A = e^{rt}$, then:

$$\frac{dT_A}{dt} = re^{rt} \qquad \frac{d^2T_A}{dt^2} = r^2 e^{rt}$$

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = d \qquad (3.4)$$

the homogeneous ($d=0$) solution to (3.4) is

$$ar^2 + br + c = 0$$

so

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (3.4.1) \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (3.4.2)$$

from 3.3b:

$$b^2 = \left[C_m\left(1+\frac{U_A}{H_m}\right)+C_A\right]^2$$

$$= C_m^2\left(1+\frac{U_A}{H_m}\right)^2 + 2C_m\left(1+\frac{U_A}{H_m}\right)C_A + C_A^2$$

$$= C_m^2 + 2C_m^2\frac{U_A}{H_m} + C_m^2\frac{U_A^2}{H_m^2} + 2C_mC_A + 2C_m\frac{U_A}{H_m}C_A + C_A^2$$

from (3.3a) + (3.3c):

$$4ac = 4\left(\frac{C_mC_A}{H_m}\right)U_A$$

so

$$b^2 - 4ac = C_m^2 + 2C_m^2\frac{U_A}{H_m} + \frac{C_m^2U_A^2}{H_m^2} + 2C_mC_A + 2\frac{C_mU_AC_A}{H_m} + C_A^2$$

$$- 4\frac{C_mC_AU_A}{H_m}$$

$$= C_m^2 + 2C_m^2\frac{U_A}{H_m} + \frac{C_m^2U_A^2}{H_m^2} + 2C_mC_A - \frac{2C_mC_AU_A}{H_m} + C_A$$

$$= C_m^2\left(1+\frac{U_A}{H_m}\right)^2 + 2C_mC_A\left(1-\frac{U_A}{H_m}\right) + C_A^2$$

$$= \left(C_m \qquad\qquad C_A\right)^2$$

So,

$$(T_A)_c = A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad (3.5)$$

and

$$T_A = (T_A)_c + (T_A)_p = A_1 e^{r_1 t} + A_2 e^{r_2 t} + (T_A)_p \qquad (3.6)$$

$$\frac{dT_A}{dt} = A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t} \qquad (3.6.1)$$

$$\frac{d^2 T_A}{dt} = A_1 r_1^2 e^{r_1 t} + A_2 r_2^2 e^{r_2 t} \qquad (3.6.2)$$

Substituting into (3.2)

$$a\, A_1 r_1^2 e^{r_1 t} + a A_2 r_2^2 e^{r_2 t} +$$
$$b\, A_1 r_1 e^{r_1 t} + b A_2 r^2 e^{r_2 t} +$$
$$c A_1 e^{r_1 t} + c A_2 e^{r_2 t} + c (T_A)_p = d \qquad (3.7)$$

$$A_1 e^{r_1 t} \cancelto{0}{(a r_1^2 + b r_2 + c)} + A_2 e^{r_2 t} \cancelto{0}{(a r_2^2 + b r_2 + c)} + c (T_A)_p = d$$

$$c (T_{A_p}) = d$$

$$T_{A_p} = \frac{d}{c} \qquad (3.7.1)$$

So the general solution is:

$$T_A = A_1 e^{r_1 t} + A_2 e^{r_2 t} + \frac{d}{c} \qquad (3.8)$$

$$T_A = A_1 e^{r_1 t} + A_2 e^{r_2 t} + \frac{Q_M + Q_A}{UA} + T_O \qquad (3.8.1)$$

ETP Solution (cont.)

with known boundary conditions at $t=0$ T_{A_0}, $\frac{dT}{dt}_0$

$$T_{A_0} = A_1 + A_2 + \frac{d}{c} \qquad (4)$$

$$0 = A_1 + A_2 + \left(\frac{d}{c} - T_{A_0}\right) \qquad (4.1)$$

$$\frac{dT_{A_0}}{dt} = A_1 r_1 + A_2 r_2 \qquad (5)$$

$$0 = A_1 r_1 + A_2 r_2 - \frac{dT_{A_0}}{dt} \qquad (5.1)$$

then A_1 and A_2 can be found as:

Subtracting (5) from (4) to eliminate A_2:

$$r_2 T_{A_0} = r_2 A_1 + r_2 A_2 + r_2\frac{d}{c} \qquad (4.2)$$

$$- \quad \frac{dT_{A_0}}{dt} = r_1 A_1 + r_2 A_2 \qquad (5)$$

$$r_2 T_{A_0} - \frac{dT_{A_0}}{dt} = (r_2 - r_1) A_1 + r_2\frac{d}{c} \qquad (6)$$

$$A_1 = \frac{r_2 T_{A_0} - \frac{dT_{A_0}}{dt} - r_2\frac{d}{c}}{r_2 - r_1} \qquad (6.1)$$

$r_2 \frac{P_4}{P_3}$

from (6.1) and (4), A_2 is

$$A_2 = T_{A_0} - \frac{d}{c} - \frac{r_2 T_{A_0} - \frac{dT_{A_0}}{dt} - r_2\frac{d}{c}}{r_2 - r_1} \qquad (7)$$

ETP Solution (cont.)

$$A_2 = T_{A_0}\left(1 + \frac{r_2}{r_2 - r_1}\right) + \frac{d}{c}\left(\frac{r_2}{r_2 - r_1} - 1\right)$$

$$- \left(\frac{r_2}{r_2 - r_1}\right)\frac{dT_{A_0}}{dt} \qquad (7.1)$$

ETP Solution (cont.)

With known boundary conditions T_{A_0}, T_{m_0} @ $t=0$:

Solving 1.2 for T_m at any time gives:

$$T_m = \frac{C_A}{H_m}\frac{dT_A}{dt} + \frac{U_A+U_m}{H_m}T_A - \frac{U_A}{H_m}T_0 - \frac{Q_A}{H_m} \qquad (1.5)$$

Substituting (3.8) and (3.6.1) for T_A and dT_A/dt, gives:

$$T_m = \frac{C_A}{H_m}\left(A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t}\right) +$$

$$\frac{U_A+H_m}{H_m}\left(A_1 e^{r_1 t} + A_2 e^{r_2 t} + \frac{Q_m + Q_A + U_A T_0}{U_A}\right)$$

$$- \frac{U_A}{H_m}T_0 - \frac{Q_A}{H_m} \qquad (1.5.1)$$

Re-arranging and simplifying:

$$T_m = \overset{K_1}{A_1}\left(\overset{1/c_2}{\frac{C_A}{H_m}} r_1 + \overset{-c_1/c_2}{\frac{U_A+H_m}{H_m}}\right)e^{r_1 t} + \overset{K_2}{A_2}\left(\overset{1/c_2}{\frac{C_A}{H_m}} r_2 + \overset{-c_1/c_2}{\frac{U_A+H_m}{H_m}}\right)e^{r_2 t}$$

$$+ \frac{U_A+H_m}{H_m U_A}Q_m + \frac{Q_A}{U_A} + T_0 \qquad (1.5.2)$$

Let

$$T_m = A_1 A_3 e^{r_1 t} + A_1 A_4 e^{r_2 t} + \frac{Q_m}{H_m} + \frac{Q_m}{U_A} + \frac{Q_A}{U_A} + T_0$$

where (1.5.3)

$$A_3 \equiv \frac{C_A}{H_m} r_1 + \frac{U_A+H_m}{H_m} \qquad (1.5.3a)$$

$$A_4 \equiv \frac{C_A}{H_m} r_2 + \frac{U_A+H_m}{H_m} \qquad (1.5.3b)$$

ETP Solution (cont.)

1.5.2 gives T_m for the next time step, i.e. at $t = t_0 + \Delta t$.

Note that the solution is discontinuous at any time T_s when Q_m, Q_A or T_0 change. So, at time t_s^+, the instant after Q_m, Q_A or T_0 change, then 1.3 can be solved for $\left.\frac{dT_A}{dt}\right|_{t_s^+}$:

$$\left.\frac{dT_A}{dt}\right|_{t_s^+} = \frac{H_m}{C_A} T_{m\,t_s} - \frac{U_A + H_m}{C_A} T_{A\,t_s} + \frac{U_A}{C_A} T_{0\,t_s} + \frac{Q_{A\,t_s^+}}{C_A} \tag{1.5.3}$$