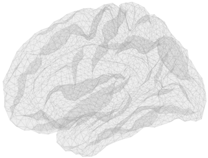
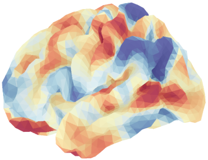
Heat Kernel Smoothing Using Laplace-Beltrami Eigenfunctions [Seo, Chung, and Vorperian, 2010]



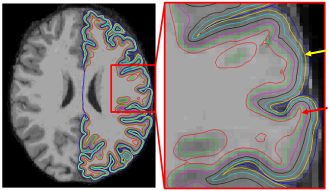




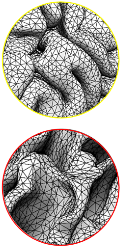

Gabriel Riegner, March 2025
DSC 291 Network Science and Graph Theory

# Cortical Surface Data

Tissue Segmentation of Anatomical MRI

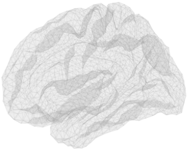



Triangular Surface Mesh

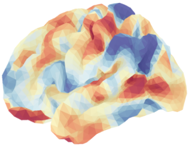


~40 k vertices
~80 k triangles
per hemisphere

FreeSurfer Population Surface



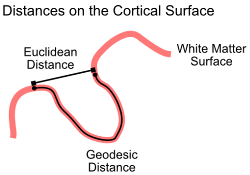
Cortical Surface Features

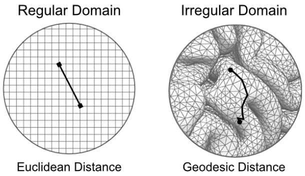


curvature,
surface area,
cortical thickness,
...

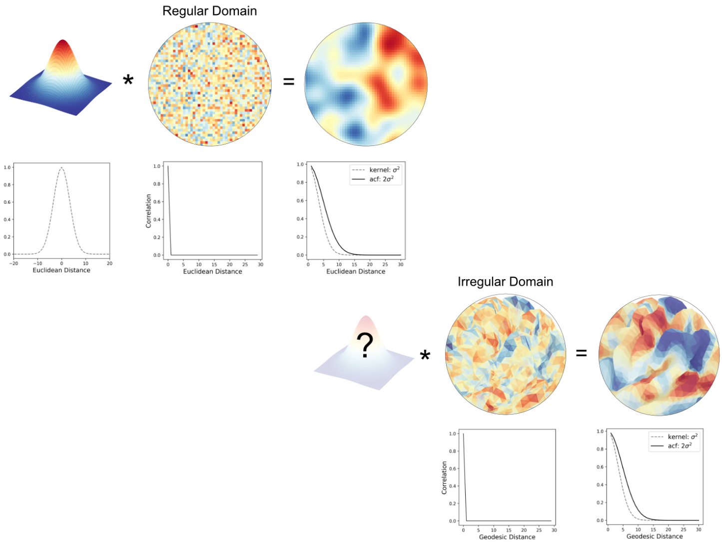
[Dale et al., 1999]

# Surface Distances



# Surface Data Smoothing




[Chung et al., 2018, Chung and Wang, 2019]

Definitions

$Y$ defined on manifold $\mathcal{M} \subset \mathbb{R}^3$:

$$Y(p) = \theta(p) + \epsilon(p), \quad \epsilon(p) \overset{\text{iid}}{\sim} \mathcal{N}(0, \mathbb{I}) \tag{1}$$

Eigenvalue problem for the Laplace-Beltrami operator $\Delta$ on $\mathcal{M}$:

$$\Delta \psi_j = -\lambda \psi_j \tag{2}$$

Heat kernel:

$$K_\sigma(p, q) = \sum_{j=0}^{\infty} e^{-\lambda_j \sigma} \psi_j(p) \psi_j(q) \tag{3}$$

Heat kernel smoothing:

$$Y_\sigma(p) \equiv K_\sigma * Y(p) = \sum_{j=0}^{\infty} e^{-\lambda_j \sigma} \beta_j \psi_j(p) \tag{4}$$

Simplification:

$$z_j \equiv \beta_j = \langle Y, \psi_j \rangle \overset{iid}{\sim} \mathcal{N}(0, 1) \tag{5}$$

[Seo et al., 2010]

Mean:

$$\mathbb{E}[Y_\sigma(p)] = 0 \tag{6}$$

Variance:

$$\mathbb{V}[Y_\sigma(p)] = \sum_{j=0}^{\infty} e^{-2\sigma\lambda_j} \tag{7}$$

Covariance:

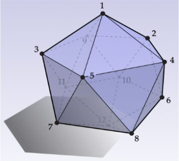
$$\mathbb{C}[Y_\sigma(p), Y_\sigma(q)] = \sum_{j=0}^{\infty} e^{-\lambda_j 2\sigma} \psi_j(p)\psi_j(q) \tag{8}$$

Asymptotics:

$$\sigma \to 0 : Y_\sigma \to Y \tag{9}$$
$$\sigma \to \infty : Y_\sigma \to 0 \tag{10}$$

[Chung and Wang, 2019]

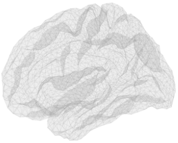
Laplace Matrix: $\Delta = D - A$



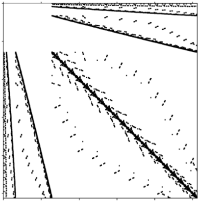

Triangle Mesh

$$\begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -5 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -5 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -5 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -5 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -5 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{bmatrix}$$

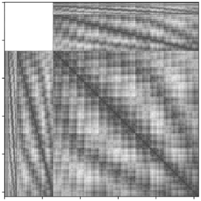
Laplace Matrix



Surface Mesh



Laplace Matrix



Geodesic Distance Matrix

[Reuter et al., 2006], [https://github.com/Deep-MI/LaPy]

# Simulation Setting

Eigenvalues:

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \qquad \text{(from 2)}$$

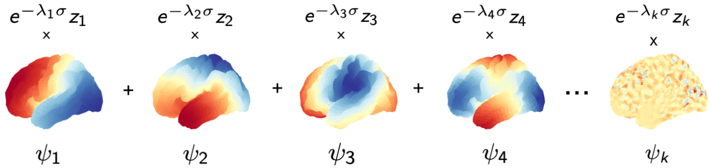
Eigenfunctions:

$$\psi_1, \psi_2, \ldots, \psi_k \qquad \text{(from 2)}$$

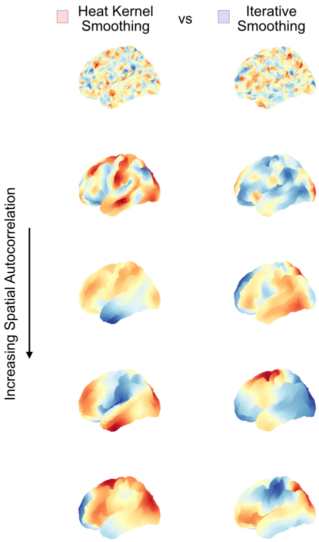
White noise:

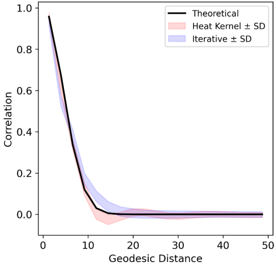
$$z_1, z_2, \ldots, z_k \overset{iid}{\sim} \mathcal{N}(0,1) \qquad \text{(from 5)}$$

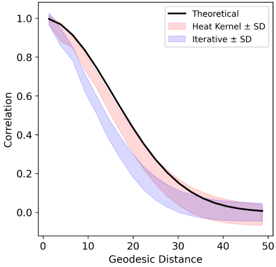
Heat Kernel Smoothing:



# Simulation Results



## Conclusions

- **Geometry-aware smoothing**: Laplace-Beltrami eigenfunctions enable random field modeling on brain surfaces
- **Spectral implementation**: Heat flow modeled via adjacency and Laplacian graphs
- **Validated accuracy**: Simulations align with theoretical autocovariance decay with geodesic distance
- **Computational efficiency**: Sparse Laplacian methods scale to large meshes (40k vertices)

## References

Seongho Seo, Moo K. Chung, and Houri K. Vorperian. Heat Kernel Smoothing Using Laplace-Beltrami Eigenfunctions. In Tianzi Jiang, Nassir Navab, Josien P. W. Pluim, and Max A. Viergever, editors, *Medical Image Computing and Computer-Assisted Intervention – MICCAI 2010*, pages 505–512, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg. ISBN 978-3-642-15711-0.

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