HW1_Q1

November 11, 2022

1 HW1 - Q1: Linear Algebra Basics (30 points)

Notes: * Questions (a), (b), (c), and (d) need to be typewritten. * Important: * Write all the steps of the solution. * Use proper LATEX formatting and notation for all mathematical equations, vectors, and matrices.

1.0.1 (a) Given L_1 and L_2 are two lower triangular matrices of size $n \times n$, prove that $L_1 L_2$ is also a lower triangular matrix. Further, prove by induction that multiplication of $m\,(m > 2)$ lower triangular matrices $(L_1, L_2, ..., L_m)$ is also a lower triangular matrix. (6 points)

Your answer here:

1. Let the ith row and jth column of an $n \times n$ matix be:

$$i, j \in \{1...n\}$$

2. Definition of a lower triangular matrix:

$$\mathbf{L_{1_{ij}}} = \mathbf{L_{2_{ij}}} = 0 \quad \forall \quad i < j$$

3. Definition of matrix multiplication:

$$\mathbf{L_1 L_2} = \mathbf{L_{3_{ij}}} = \sum_{k=1}^{n} \mathbf{L_{1_{ik}}} \mathbf{L_{2_{kj}}}$$

4. For all k, either $\mathbf{L_{1_{ik}}}$ or $\mathbf{L_{2_{kj}}}$ is zero given $i < j$:

$$k > i \implies \mathbf{L_{1_{ik}}} = 0 k \le i < j \implies \mathbf{L_{2_{kj}}} = 0$$

5. The matrix product $\mathbf{L_1 L_2}$ satisfies the defintion of a lower triangular matrix:

$$\mathbf{L_{3_{ij}}} = \sum_{k=1}^{n} \mathbf{L_{1_{ik}}} \mathbf{L_{2_{kj}}} = 0 \quad \forall \quad i < j$$

6. The product of any number of m lower triangular matrices $(\mathbf{L_1}, \mathbf{L_2}, ..., \mathbf{L_m})$ is a lower triangular matrix. For $m = 3$, (e.g. $\mathbf{L_1}, \mathbf{L_2}, \mathbf{L_3}$):

$$\mathbf{L_{1_{ij}}L_{2_{ij}}} = \mathbf{L_{12_{ij}}} = 0 \quad \forall \quad i < j \tag{1}$$
$$\mathbf{L_{12_{ij}}L_{3_{ij}}} = \mathbf{L_{4_{ij}}} = 0 \quad \forall \quad i < j \tag{2}$$
$$\mathbf{L_{1_{ij}}L_{2_{ij}}L_{3_{ij}}} = 0 \quad \forall \quad i < j \tag{3}$$

1.0.2 (b) Use Gauss elimination to solve the following equations: (8 points)

$-4x_1 + 5x_2 - 5x_3 = -29$
$-8x_1 - 5x_2 - 3x_3 = -15$
$16x_1 - 5x_2 + 6x_3 = 45$

Your answer here:

Gaussian Elimination

$$\left[\begin{array}{ccc|c} -4 & 5 & -5 & -29 \\ -8 & -5 & -3 & -15 \\ 16 & -5 & 6 & 45 \end{array}\right] \begin{array}{c} -\mathbf{L_1}- \\ row2 - 2row1 \\ row3 + 4row1 \end{array} \left[\begin{array}{ccc|c} -4 & 5 & -5 & -29 \\ 0 & -15 & -7 & 43 \\ 0 & 15 & 14 & -71 \end{array}\right] \begin{array}{c} -\mathbf{L_2}- \\ \\ row3 + row2 \end{array} \left[\begin{array}{ccc|c} -4 & 5 & -5 & -29 \\ 0 & -15 & 7 & 43 \\ 0 & 0 & -7 & -28 \end{array}\right]$$

Backwards Substitution

$$-7x_3 = -28 \tag{4}$$
$$x_3 = \frac{-28}{-7} \tag{5}$$
$$\mathbf{x_3 = 4} \tag{6}$$
$$\tag{7}$$
$$-15x_2 + 7x_3 = 43 \tag{8}$$
$$-15x_2 + (7 \cdot 4) = 43 \tag{9}$$
$$-15x_2 = 43 - (7 \cdot 4) = 43 - 28 = 15 \tag{10}$$
$$x_2 = \frac{15}{-15} \tag{11}$$
$$\mathbf{x_2 = -1} \tag{12}$$
$$\tag{13}$$
$$-4x_1 + 5x_2 - 5x_3 = -29 \tag{14}$$
$$-4x_1 + (5 \cdot -1) - (5 \cdot 4) = -29 \tag{15}$$
$$-4x_1 - 5 - 20 = -29 \tag{16}$$
$$-4x_1 - 25 = -29 \tag{17}$$
$$-4x_1 = -29 + 25 = -4 \tag{18}$$
$$x_1 = \frac{-4}{-4} \tag{19}$$
$$\mathbf{x_1 = 1} \tag{20}$$
$$\tag{21}$$

Solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

1.0.3 (c) Do the LU decomposition for the matrix obtained in (b). Using the matrices L and U, do forward and backward substitution and solve for x. Match your answer with the solution obtained in (b). (8 points)

Your answer here: Given

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix}, \mathbf{U} = \begin{bmatrix} -4 & 5 & -5 \\ 0 & -15 & 7 \\ 0 & 0 & -7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -29 \\ -15 \\ 45 \end{bmatrix} \mathbf{Ax} = \mathbf{b} \xrightarrow{\mathbf{A=LU}} \mathbf{L}\underbrace{\mathbf{Ux}}_{\mathbf{y}} = \mathbf{bLy} = \mathbf{bUx} = \mathbf{y}$$

1. Solve $\mathbf{y}$ with forward substitution, given $\mathbf{L}$ and $\mathbf{b}$.

$$\mathbf{y_1} = \mathbf{-29} \tag{22}$$
$$\tag{23}$$
$$2y_1 + y_2 = -15 \tag{24}$$
$$(2 \cdot -29) + y_2 = -15 \tag{25}$$
$$y_2 = -15 + (2 \cdot 29) \tag{26}$$
$$\mathbf{y_2} = \mathbf{43} \tag{27}$$
$$\tag{28}$$
$$-4y_1 - y_2 + y_3 = 45 \tag{29}$$
$$(-4 \cdot -29) - 43 + y_3 = 45 \tag{30}$$
$$y_3 = 45 - (4 \cdot 29) + 43 \tag{31}$$
$$\mathbf{y_3} = \mathbf{-28} \tag{32}$$
$$\tag{33}$$
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -29 \\ 43 \\ -28 \end{bmatrix} \tag{34}$$

2. Solve $\mathbf{x}$ with backward substitution, given $\mathbf{U}$ and $\mathbf{y}$.

$$-7x_3 = y_3 = -28 \tag{35}$$
$$x_3 = \frac{-28}{-7} \tag{36}$$
$$\mathbf{x_3 = 4} \tag{37}$$
$$\tag{38}$$
$$-15x_2 + 7x_3 = y_2 = 43 \tag{39}$$
$$-15x_2 + (7 \cdot 4) = 43 \tag{40}$$
$$-15x_2 = 43 - 28 = 15 \tag{41}$$
$$x_2 = \frac{15}{-15} \tag{42}$$
$$\mathbf{x_2 = -1} \tag{43}$$
$$\tag{44}$$
$$-4x_1 + 5x_2 - 5x_3 = y_1 \tag{45}$$
$$-4x_1 + (5 \cdot -1) - (5 \cdot 4) = -29 \tag{46}$$
$$-4x_1 - 5 - 20 = -29 \tag{47}$$
$$-4x_1 = -29 + 25 \tag{48}$$
$$x_1 = \frac{-4}{-4} \tag{49}$$
$$\mathbf{x_1 = 1} \tag{50}$$
$$\tag{51}$$
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \tag{52}$$

1.0.4 (d) Do the QR decomposition for the matrix obtained in (b) using Gram-Schmidt algorithm. Using the decomposition, solve for x. Match your answer with the solution obtained in problem (b). (8 points)

Your answer here: Given $\mathbf{A} = \begin{bmatrix} -4 & 5 & -5 \\ -8 & -5 & -3 \\ 16 & -5 & 6 \end{bmatrix}$, solve $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} ||\mathbf{a_1}|| & \mathbf{a_2^T q_1} & \mathbf{a_3^T q_1} \\ 0 & ||\mathbf{a_2^\perp}|| & \mathbf{a_3 q_2} \\ 0 & 0 & ||\mathbf{a_3^\perp}|| \end{bmatrix}$ Let $\mathbf{a_2^\perp} = \mathbf{a_2} - (\mathbf{a_2^T q_1})\,\mathbf{q_1}$ and $\mathbf{a_3^\perp} = \mathbf{a_3} - (\mathbf{a_3^T q_1})\,\mathbf{q_1} - (\mathbf{a_3^T q_2})\,\mathbf{q_2}$

1. Solve first column of **R** and **Q**:

$$||\mathbf{a_1}|| = \sqrt{-4^2 - 8^2 + 16^2} = \sqrt{336} = 4\sqrt{21} \tag{53}$$

$$\tag{54}$$

$$\tag{55}$$

$$\mathbf{q_1} = \frac{\mathbf{a_1}}{||\mathbf{a_1}||} = \begin{bmatrix} -4 \\ -8 \\ 16 \end{bmatrix} \cdot \frac{1}{4\sqrt{21}} = \begin{bmatrix} \frac{-1}{\sqrt{21}} \\ \frac{-2}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{21}}{21} \\ \frac{-2\sqrt{21}}{21} \\ \frac{4\sqrt{21}}{21} \end{bmatrix} \tag{56}$$

$$\tag{57}$$

2. Solve second column of $\mathbf{R}$ and $\mathbf{Q}$:

$$\mathbf{a_2^T q_1} = \begin{bmatrix} 5 \\ -5 \\ -5 \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \frac{-\sqrt{21}}{21} \\ \frac{-2\sqrt{21}}{21} \\ \frac{4\sqrt{21}}{21} \end{bmatrix} = \frac{-5\sqrt{21}}{21} + \frac{10\sqrt{21}}{21} + \frac{20\sqrt{21}}{21} \tag{58}$$

$$\mathbf{a_2^T q_1} = \frac{-15\sqrt{21}}{21} \tag{59}$$

$$\tag{60}$$

$$\tag{61}$$

$$\mathbf{a_2^\perp} = \mathbf{a_2} - (\mathbf{a_2^T q_1})\mathbf{q_1} \tag{62}$$

$$\mathbf{a_2^\perp} = \begin{bmatrix} 5 \\ -5 \\ -5 \end{bmatrix} - \frac{-15\sqrt{21}}{21} \begin{bmatrix} \frac{-\sqrt{21}}{21} \\ \frac{-2\sqrt{21}}{21} \\ \frac{4\sqrt{21}}{21} \end{bmatrix} \tag{63}$$

$$\mathbf{a_2^\perp} = \begin{bmatrix} 5 \\ -5 \\ -5 \end{bmatrix} - \begin{bmatrix} \frac{15}{21} \\ \frac{30}{21} \\ \frac{-60}{21} \end{bmatrix} = \begin{bmatrix} \frac{105}{21} \\ \frac{-105}{21} \\ \frac{-105}{21} \end{bmatrix} - \begin{bmatrix} \frac{15}{21} \\ \frac{30}{21} \\ \frac{-60}{21} \end{bmatrix} \tag{64}$$

$$\mathbf{a_2^\perp} = \begin{bmatrix} \frac{90}{21} \\ \frac{-135}{21} \\ \frac{-45}{21} \end{bmatrix} \tag{65}$$

$$\tag{66}$$

$$\tag{67}$$

$$||\mathbf{a_2^\perp}|| = \sqrt{\left(\frac{90}{21}\right)^2 + \left(\frac{-135}{21}\right)^2 + \left(\frac{-45}{21}\right)^2} = \sqrt{\frac{28350}{441}} \tag{68}$$

$$||\mathbf{a_2^\perp}|| = \frac{15\sqrt{14}}{7} \tag{69}$$

$$\tag{70}$$

$$\tag{71}$$

$$\mathbf{q_2} = \frac{\mathbf{a_2^\perp}}{||\mathbf{a_2^\perp}||} \tag{72}$$

$$\mathbf{q_2} = \begin{bmatrix} \frac{90}{21} \\ \frac{-135}{21} \\ \frac{-45}{21} \end{bmatrix} \frac{7}{15\sqrt{14}} = \begin{bmatrix} \frac{630}{315\sqrt{14}} \\ \frac{-945}{315\sqrt{14}} \\ \frac{-315}{315\sqrt{14}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{-1}{\sqrt{14}} \end{bmatrix} \tag{73}$$

$$\mathbf{q_2} = \begin{bmatrix} \frac{2\sqrt{14}}{14} \\ \frac{-3\sqrt{14}}{14} \\ \frac{-\sqrt{14}}{14} \end{bmatrix} \tag{74}$$

3. Solve third column of $\mathbf{R}$ and $\mathbf{Q}$:

$$\mathbf{a_3^T q_1} = \begin{bmatrix} -5 \\ -3 \\ 6 \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \frac{-\sqrt{21}}{21} \\ \frac{-2\sqrt{21}}{21} \\ \frac{4\sqrt{21}}{21} \end{bmatrix} = \frac{5\sqrt{21}}{21} + \frac{6\sqrt{21}}{21} + \frac{24\sqrt{21}}{21} = \frac{35\sqrt{21}}{21} \tag{75}$$

$$\mathbf{a_3^T q_1} = \frac{5\sqrt{21}}{3} \tag{76}$$

$$\tag{77}$$

$$\mathbf{a_3^T q_2} = \begin{bmatrix} -5 \\ -3 \\ 6 \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} \frac{2\sqrt{14}}{14} \\ \frac{-3\sqrt{14}}{14} \\ \frac{-\sqrt{14}}{14} \end{bmatrix} = \frac{-10\sqrt{14}}{14} + \frac{9\sqrt{14}}{14} + \frac{-6\sqrt{14}}{14} = \frac{-7\sqrt{14}}{14} \tag{78}$$

$$\mathbf{a_3^T q_2} = \frac{-\sqrt{14}}{2} \tag{79}$$

$$\tag{80}$$

$$\mathbf{a_3^\perp} = \mathbf{a_3} - (\mathbf{a_3^T q_1})\mathbf{q_1} - (\mathbf{a_3^T q_2})\mathbf{q_2} \tag{81}$$

$$\mathbf{a_3^\perp} = \begin{bmatrix} -5 \\ -3 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{-105}{63} \\ \frac{-210}{63} \\ \frac{420}{63} \end{bmatrix} - \begin{bmatrix} \frac{-28}{28} \\ \frac{-42}{28} \\ \frac{-14}{28} \end{bmatrix} = \begin{bmatrix} \frac{-60}{12} \\ \frac{-36}{12} \\ \frac{72}{12} \end{bmatrix} - \begin{bmatrix} \frac{-20}{12} \\ \frac{-40}{12} \\ \frac{80}{12} \end{bmatrix} - \begin{bmatrix} \frac{-12}{12} \\ \frac{-18}{12} \\ \frac{-6}{12} \end{bmatrix} \tag{82}$$

$$\mathbf{a_3^\perp} = \begin{bmatrix} \frac{-28}{12} \\ \frac{-14}{12} \\ \frac{-14}{12} \end{bmatrix} \tag{83}$$

$$\tag{84}$$

$$||\mathbf{a_3^\perp}|| = \sqrt{\left(\frac{-28}{12}\right)^2 + \left(\frac{-14}{12}\right)^2 + \left(\frac{-14}{12}\right)^2} = \sqrt{\frac{1176}{144}} \tag{85}$$

$$||\mathbf{a_3^\perp}|| = \frac{7\sqrt{6}}{6} \tag{86}$$

$$\tag{87}$$

$$\tag{88}$$

$$\mathbf{q_3} = \frac{\mathbf{a_3^\perp}}{||\mathbf{a_3^\perp}||} = \begin{bmatrix} \frac{-28}{12} \\ \frac{-14}{12} \\ \frac{-14}{12} \end{bmatrix} \frac{6}{7\sqrt{6}} = \begin{bmatrix} \frac{56}{12\sqrt{14}} \\ \frac{28}{12\sqrt{14}} \\ \frac{28}{12\sqrt{14}} \end{bmatrix} = \begin{bmatrix} \frac{-168}{84\sqrt{6}} \\ \frac{-84}{84\sqrt{6}} \\ \frac{-84}{84\sqrt{6}} \end{bmatrix} \tag{89}$$

$$\mathbf{q_3} = \begin{bmatrix} \frac{-\sqrt{6}}{3} \\ \frac{-\sqrt{6}}{6} \\ \frac{-\sqrt{6}}{6} \end{bmatrix} \tag{90}$$

$$\tag{91}$$

4. Solutions to **Q** and **R**

$$\mathbf{Q} = \begin{bmatrix} \frac{-\sqrt{21}}{21} & \frac{2\sqrt{14}}{14} & \frac{-2\sqrt{6}}{6} \\ \frac{-2\sqrt{21}}{21} & \frac{-3\sqrt{14}}{14} & \frac{-\sqrt{6}}{6} \\ \frac{4\sqrt{21}}{21} & \frac{-\sqrt{14}}{14} & \frac{-\sqrt{6}}{6} \end{bmatrix} \tag{92}$$

$$\tag{93}$$

$$\mathbf{R} = \begin{bmatrix} 4\sqrt{21} & \frac{-5\sqrt{21}}{7} & \frac{5\sqrt{21}}{3} \\ 0 & \frac{15\sqrt{14}}{7} & \frac{-\sqrt{14}}{2} \\ 0 & 0 & \frac{7\sqrt{6}}{6} \end{bmatrix} \tag{94}$$

5. Solve for $\mathbf{Ax} = \mathbf{b} \implies \mathbf{QRx} = \mathbf{b} \implies \mathbf{Rx} = \underset{\mathbf{Q^T}}{\mathbf{Q^{-1}}}\,\mathbf{b}$

$$\mathbf{Q^T b} = \begin{bmatrix} \frac{-\sqrt{21}}{21} & \frac{-2\sqrt{21}}{21} & \frac{4\sqrt{21}}{21} \\ \frac{2\sqrt{14}}{14} & \frac{-3\sqrt{14}}{14} & \frac{-\sqrt{14}}{14} \\ \frac{-2\sqrt{6}}{6} & \frac{-\sqrt{6}}{6} & \frac{-\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} -29 \\ -15 \\ 45 \end{bmatrix} \tag{95}$$

$$\tag{96}$$

$$\mathbf{Q^T b} = \begin{bmatrix} \frac{29\sqrt{21}}{21} + \frac{30\sqrt{21}}{21} + \frac{180\sqrt{21}}{21} \\ \frac{-58\sqrt{14}}{14} + \frac{45\sqrt{14}}{14} + \frac{-45\sqrt{14}}{14} \\ \frac{58\sqrt{6}}{6} + \frac{15\sqrt{6}}{6} + \frac{-45\sqrt{6}}{6} \end{bmatrix} = \begin{bmatrix} \frac{239\sqrt{21}}{21} \\ \frac{-29\sqrt{14}}{7} \\ \frac{14\sqrt{6}}{3} \end{bmatrix} \tag{97}$$

$$\tag{98}$$

$$\mathbf{Rx} = \mathbf{Q^T b} \rightarrow \begin{bmatrix} 4\sqrt{21} & \frac{-5\sqrt{21}}{7} & \frac{5\sqrt{21}}{3} \\ 0 & \frac{15\sqrt{14}}{7} & \frac{-\sqrt{14}}{2} \\ 0 & 0 & \frac{7\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{239\sqrt{21}}{21} \\ \frac{-29\sqrt{14}}{7} \\ \frac{14\sqrt{6}}{3} \end{bmatrix} \tag{99}$$

6. Solve for $\mathbf{x}$ by backwards substitution given $\mathbf{Rx} = \mathbf{Q^T b}$

$$\frac{7\sqrt{6}}{6}x_3 = \frac{14\sqrt{6}}{3} \implies x_3 \tag{100}$$

$$\tag{101}$$

$$\frac{15\sqrt{14}}{7}x_2 - \frac{\sqrt{14}}{2}(4) = \frac{-29\sqrt{14}}{7} \implies \frac{15\sqrt{14}}{7}x_2 = \frac{-15\sqrt{14}}{7} \implies x_2 \tag{102}$$

$$\tag{103}$$

$$4\sqrt{21}x_1 - \frac{5\sqrt{21}}{7}(-1) + \frac{5\sqrt{21}}{3}(4) = \frac{239\sqrt{21}}{21} \implies 4\sqrt{21}x_1 + \frac{155\sqrt{21}}{21} = \frac{239\sqrt{21}}{21} \implies 4\sqrt{21}x_1 = \frac{84\sqrt{21}}{21} \implies x_1 \tag{104}$$

$$\tag{105}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$